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Direct and inverse problems of infrared tomography

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The problems of infrared tomography—direct (the modeling of measured functions) and inverse (the reconstruction of gaseous medium parameters)—are considered with a laboratory burner flame as an example of an application. The two measurement modes are used: active (ON) with an external IR source and passive (OFF) without one. Received light intensities on detectors are modeled in the direct problem or measured in the experiment whereas integral equations with respect to the absorption coefficient and Planck function (which yields the temperature profile of the medium) are solved in the inverse problem with (1) modeled and (2) measured received intensities as the input data. An axisymmetric flame and parallel scanning scheme of measurements considered in this work yield singular integral equations that are solved numerically using the generalized quadrature method, spline smoothing, and Tikhonov regularization. A software package in MATLAB has been developed. Two numerical examples—with modeled and real input data—were solved. The proposed methodology avoids the necessity of elaborate determination of the absorption coefficient by direct (point) measurements or calculation using spectroscopic databases (e.g., HITRAN/HITEMP). © 2015 Optical Society of America

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1. INTRODUCTION

A problem of infrared (IR) tomography [1–4] of a hot gaseous medium, e.g., flame, is considered in this work. IR tomography has numerous applications including

- the determination of hot gas absorption, emission, and temperature profiles in a cross section of a laboratory burner flame [5], plasma [6–9], flame inside a boiler, furnace, or steam generator [10], and hot gas flow [11], e.g., one from a rocket nozzle;
- IR spectroscopic tomography of temperature and gaseous species concentrations as the diagnostics of pulverized coal or biomass firing and burning [11];
- laser-based thermometry as the coherent anti-Stokes Raman scattering (CARS) measurements (point diagnostics) of a laboratory burner flame [5];
- stratified atmosphere parameters determination (temperature, pressure, absorption, emission, etc.) [12–14] in satellite meteorology [15];
- thermal (IR) tomography of nondestructive testing of composite materials, namely, analysis of changes in the surface

temperature of materials with time using a neuron network of the perceptrons, the thermal testing of fiber reinforced plastic in the air- and spacecraft industry, etc. [16];

- in biomedical optics, probing of biological tissues (skin) by radiation of the near IR region [17];
- IR tomography of the charge-carrier lifetime and diffusion length in semiconductor-grade silicon ingots [18].

Obviously, the scope of the IR tomography is broad and diverse. This work is focused on the IR tomography of a hot gaseous medium, which is one of the problems of combustion physics [4,5,19,20].

In IR tomography, absorption and emission of light (without scattering) by a gaseous medium are considered. Hence the absorption coefficient k and emission coefficient ϵ (related to the Planck function B and the temperature profile of the medium T_g) are the two unknown functions. In the ideal case (e.g., as in [4]), the two modes of the measurement are necessary: active, when an external IR source is illuminating the medium (referred to as ON in [4]), and passive, when there is no external IR radiation (referred to as OFF in [4]) and

Fig. 1. Light beam traveling through some z cross section of a gaseous medium, e.g., flame.

the medium is the source itself. That yields two integral equations (IE) with respect to k and B . However, often only the ON mode is considered [2,3], which yields only one equation, either differential or integral, with respect to those two unknown functions. In that case, it is often assumed that the coefficient k is already measured in some way or determined using the spectroscopic databases, e.g., HITRAN/HITEMP [21,22] or others.

The two modes of measurement—ON and OFF—are used in this work and two IEs with respect to the two unknown functions k and B are obtained. Let us note that in [1,4] and other works, the problem of emission-absorption tomography is also considered based on measurements with an external source [4] (or mirror [1]) and without that (Fig. 1).

It is noted in [4] that the difference of the ON and OFF measurements enables the determination of the absorption coefficient k , Planck function B , and temperature T_g . The expressions for the intensities in the form of the path integrals and the Ramachandran–Lakshminarayanan and Shepp–Logan reconstruction algorithms are given there. However, the technique in [4] is presented somewhat briefly; the authors do not consider the application of experimental data smoothing, regularization, and other methods that could give a new impact to this problem. (These issues are discussed in the other publications, see below.)

Note also that although both the active (ON) and passive (OFF) measurement modes were used in [2–4,6–10], the problem was not reduced to the system of two IEs.

These issues are considered in this work all together as well as from a new perspective.

2. BASIC RELATIONS

In order to estimate the solution error for the inverse problem (the reconstruction of k , B , and T_g), first, the direct problem is solved: the modeling of measurable functions at some given values of k , B , and T_g . Then the inverse problem is solved:

the reconstruction of k , B , and T_g from the modeled values of the measurable functions.

A. Radiative Transfer Equation

Let us consider a stable laminar flame supported by a laboratory burner [5]. Let a beam of monochromatic radiation from a black body (the external source) be traveling through some z cross section of the flame parallel to the y axis (Fig. 1):

$$B(T_0) = \frac{2hc^2\nu^3}{e^{h\nu/k_B T_0} - 1}, \quad (1)$$

where h is the Planck constant, c is the velocity of light in vacuum, $\nu = 1/\lambda$ is the wave number, λ is the wavelength of the monochromatic radiation, k_B is the Boltzmann constant, and T_0 is the black body source temperature.

This radiation from the black body is subject to absorption with absorption coefficient $k = k(x, y)$ as it travels from the source to the detector through the gaseous medium. On the other hand, the medium itself radiates with emission coefficient $\varepsilon = \varepsilon(x, y) = \varepsilon(T_g(x, y))$, where $T_g = T_g(x, y)$ is the gaseous medium temperature profile in a z cross section.

Assuming local thermodynamic equilibrium at every point (x, y) , Kirchhoff's law can be used, i.e., $\varepsilon/k = B(T_g)$, where $B(T_g)$ is the Planck function at the temperature T_g of the medium:

$$B(T_g) = B(T_g(x, y)) = \frac{2hc^2\nu^3}{e^{h\nu/k_B T_g(x, y)} - 1}, \quad (2)$$

or $\varepsilon(x, y) = k(x, y) B(T_g(x, y))$, so the radiative transfer equation (being a differential equation) can be written as (cf. [2,9,12])

$$\frac{dI(x, y)}{dy} = k(x, y)[B(T_g(x, y)) - I(x, y)], \quad (3)$$

where $I(x, y)$ is the unknown intensity of the radiation along the beam, and furthermore $k(x, y(x)) = 0$ for $y(x) < y_1(x)$ (the first boundary condition) and $y(x) > y_2(x)$ (the second boundary condition), where $y_1(x)$ and $y_2(x)$ are the boundaries of the medium (which are, generally speaking, not clearly defined, see Fig. 1). Coordinate x in Eq. (3) is a parameter, i.e., Eq. (3) is valid at any fixed x .

B. Solution of the Radiative Transfer Equation

The analytical solution of Eq. (3) for every point (x, y) of every beam in the case of the ON measurement mode is (cf. [1,2,4,9])

$$I(x, y) = B(T_0) \exp\left(-\int_{y_1(x)}^y k(x, y') dy'\right) + \int_{y_1(x)}^y k(x, y') B(T_g(x, y')) \exp\left(-\int_{y'}^y k(x, y'') dy''\right) dy'. \quad (4)$$

Further in this work, only the following values are of interest: received intensity on the detector in the ON mode

$$I_R(x) = B(T_0) \exp \left(- \int_{y_1(x)}^{y_2(x)} k(x, y) dy \right) + \int_{y_1(x)}^{y_2(x)} k(x, y) B(T_g(x, y)) \exp \left(- \int_y^{y_2(x)} k(x, y') dy' \right) dy \quad (5)$$

and that in the OFF mode [cf. [1,4] and Eq. (5)]

$$I_g(x) = \int_{y_1(x)}^{y_2(x)} k(x, y) B(T_g(x, y)) \exp \left(- \int_y^{y_2(x)} k(x, y') dy' \right) dy. \quad (6)$$

C. Problem Formulation

In the **direct problem**, the received intensities on the detectors $I_R(x)$ and $I_g(x)$ are calculated at some given values of $k(x, y)$ and $T_g(x, y)$ according to Eqs. (5) and (6). In the **inverse problem**, $k(x, y)$ and $T_g(x, y)$ are reconstructed from some given (or measured) values of $I_R(x)$ and $I_g(x)$ treating Eqs. (5) and (6) as IEs.

However k , B , and T_g are the functions of two variables, x and y , whereas I_R and I_g are the functions of only one variable x . It is therefore necessary as, e.g., in x-ray computerized tomography [8,23], to take the measurements at several projection angles ϕ , which yields the functions of two variables $I_R(x, \phi)$ and $I_g(x, \phi)$, or to assume that a gaseous medium has an axisymmetric profile of its properties. This can be a reasonable assumption in case the gaseous medium is, first, a flame (hence, it has somewhat sharp boundaries with respect to the ambient air) that is, second, supported by a burner designed for axisymmetric flames. (This case is considered in Sections 3–6.)

The problem can also be simplified by introducing $I_T(x) = I_R(x) - I_g(x)$:

$$I_T(x) = B(T_0) \exp \left(- \int_{y_1(x)}^{y_2(x)} k(x, y) dy \right). \quad (7)$$

This results, in the inverse problem, in two independent IEs that are solved sequentially instead of the system of the two IEs: first, Eq. (7) is solved with respect to $k(x, y)$; then, Eq. (6) is solved with respect to $B(T_g(x, y))$ using the obtained value of $k(x, y)$. This avoids the necessity of elaborate determination of $k(x, y)$ by direct (point) measurements or calculation using spectroscopic databases.

The *temperature profile* of the flame is then calculated using the obtained value of $B(T_g(x, y)) = B(x, y)$ as [see Eq. (2)]

$$T_g(x, y) = \frac{h\nu/k_B}{\ln \left(\frac{2hc^2\nu^3}{B(x, y)} + 1 \right)}. \quad (8)$$

As previously mentioned, functions I_T and I_g must be obtained at several projection angles ϕ , i.e., $I_T(x, \phi)$ and $I_g(x, \phi)$, or the assumption that the flame has an axisymmetric profile of its properties must be made. In the latter case, one projection is enough and $I_T = I_T(x)$, $I_g = I_g(x)$, $k = k(r)$, and $B = B(T_g(r))$, where r is the distance of a point (x, y) from the z axis of symmetry (Fig. 2).

3. AXIAL SYMMETRY AND PARALLEL SCANNING

Let us consider the case of axial symmetry, being rather widespread, when, for each z cross section of the flame, the isolines of k and T_g (as well as those of ε) are circles (Fig. 2). The parallel scanning scheme of measurements (i.e., when the measurement beams are parallel to each other) is also considered in this work. This case of axial symmetry and parallel scanning is considered in a number of works ([1,4,8,9,19,20,24] and others). This work presents further development of this case.

A. Determination of the Absorption Coefficient k

In the case of the axial symmetry of the flame, Eq. (7) takes the form of Abel's singular IE (SIE) [20]:

$$2 \int_x^R \frac{r}{\sqrt{r^2 - x^2}} k(r) dr = q(x), \quad 0 \leq x \leq R, \quad (9)$$

where $R = r_{\max}$ is a boundary value at which $k(R) = 0$ and

$$q(x) = - \ln \frac{I_T(x)}{B(T_0)}. \quad (10)$$

B. Numerical Algorithms

A number of methods and algorithms have been developed for the numerical solution of Eq. (9), including (cf. [25])

- an algorithm employing a shift $\Delta \in [0, h/2]$ ($h = \text{const}$ is the grid spacing) of the node grids along r and x [26], which enables avoiding certain issues (e.g., singularities) of the numerical methods, e.g., those of the quadrature method;
- several variants of the quadrature method: the method of discrete vortices [26] in which the left rectangle formula (top-left corner approximation) is used with a shift $\Delta = h/2$; the onion-peeling method [19] in which the integral

$$\int_{r_j - h/2}^{r_j + h/2} \frac{r dr}{\sqrt{r^2 - x^2}}$$

over an annular element $[r_j - h/2, r_j + h/2]$ (see Fig. 2) is taken analytically with $\Delta = 0$; a similar method (the generalized quadrature method) is described in [25]; these methods are reduced to systems of linear algebraic equations (SLAE) with respect to k_j ;

Fig. 2. Axial symmetry and parallel scanning of a flame. The sources and detectors can be arranged (a) in a line or (b) on the edge of the flame. Both arrangements are equivalent with respect to the methods considered in this work.

– an algorithm [6] employing polynomial approximation of $k(r)$ and/or $q(x)$.

The abovementioned methods and algorithms possess an intrinsic quality of *self-regularization* and the shift Δ plays the role of the regularization parameter. Furthermore, all these algorithms result in a SLAE whose coefficient matrix is diagonally dominant (but not infinite).

Let us also mention a number of algorithms. Equation (9) has an analytical solution [6,8,19,20,24,25],

$$k(r) = -\frac{1}{\pi} \int_r^R \frac{q'(x)}{\sqrt{x^2 - r^2}} dx, \quad 0 \leq r \leq R, \quad (11)$$

that, however, contains the derivative $q'(x)$ of the experimental, and hence noisy, function $q(x)$, and the problem of differentiation is ill-posed [27]. Furthermore, the integral in Eq. (11) is improper (singular). Nevertheless, a number of algorithms are proposed for the computation of the solution according to Eq. (11), including

- a two/three point interpolation in the vicinity of every point r_j is applied in [24] to compute the derivative $q'(x)$, and the integral in Eq. (11) over every segment is computed analytically; a similar algorithm is proposed in [25];
- approximation of $q(x)$ by smoothing polynomials or splines on the whole interval $x \in [0, R]$ with their subsequent differentiation to compute $q'(x)$ [7,28,29];
- an algorithm avoiding $q'(x)$ by transforming Eq. (11) into an expression without the derivative [30] (see also [6–8,19]):

$$k(r) = -\frac{1}{\pi} \left\{ \frac{q(R) - q(r)}{\sqrt{R^2 - r^2}} + \int_r^R \frac{x[q(x) - q(r)]}{\sqrt{(x^2 - r^2)^3}} dx \right\}, \quad 0 \leq r \leq R. \quad (12)$$

This algorithm was also implemented in [7] together with the approximation of $q(x)$ by a cubic spline [31];

– Tikhonov regularization [27,32–35] is used in a number of works ([19,20,32] and others) to increase the stability of the algorithms.

In this work, the solution of Eq. (9) with respect to the unknown function $k(r)$ is considered using an integrated approach that includes preliminary smoothing of the experimental functions I_R and I_g and application of the (modified) generalized quadrature method to solve Eq. (9), as well as Tikhonov regularization.

4. GENERALIZED QUADRATURE METHOD

Let us consider Eq. (9). The onion-peeling method was proposed in [24] and implemented in [19] for the numerical solution of Eq. (9) on uniform and coinciding node grids along r and x with analytical computation of the integral [Eq. (9)] on every j th annular element assuming $k_j = \text{const}$. A similar method is described in [25]—the generalized quadrature method. Let us introduce a modification of this method.

A. Computation of the Function $k(r)$

Let us consider the solution of Eq. (9) on some node grids that are, generally speaking, nonuniform (but coinciding) and have the same length:

$$0 = x_1 = r_1 < x_2 = r_2 < \dots < x_i = r_i < \dots < x_n = r_n = R. \quad (13)$$

Assuming $k(r) = k(r_j) = k_j = \text{const}$ on each annular element $[r_j, r_{j+1})$, $j = 1, \dots, n-1$, one can get

$$\int_{r_j}^{r_{j+1}} \frac{r}{\sqrt{r^2 - x^2}} k(r) dr = \left(\sqrt{r_{j+1}^2 - x^2} - \sqrt{r_j^2 - x^2} \right) k_j, \quad j = 1, \dots, n-1, x \leq r_j < r_{j+1} \leq R. \quad (14)$$

Equation (14) is the generalized quadrature formula of the left rectangles for the singularity $r/\sqrt{r^2 - x^2}$, and $(\sqrt{r_{j+1}^2 - x^2} - \sqrt{r_j^2 - x^2})$ are the quadrature coefficients of this formula. The advantage of Eq. (14) is that the singular integral

$$\int_{r_j}^{r_{j+1}} \frac{r dr}{\sqrt{r^2 - x^2}}$$

can be computed analytically and does not possess any peculiarities.

The integral in Eq. (9) is the sum of the integrals in Eq. (14):

$$\int_{x_i}^R \frac{r}{\sqrt{r^2 - x_i^2}} k(r) dr = \sum_{j=i}^{n-1} \left(\sqrt{r_{j+1}^2 - x_i^2} - \sqrt{r_j^2 - x_i^2} \right) k_j.$$

Consequently, Eq. (9) transforms into a SLAE

$$\sum_{j=i}^{n-1} p_{ij} k_j = q_i/2, \quad i = 1, \dots, n-1, \quad (15)$$

where

$$p_{ij} = \sqrt{r_{j+1}^2 - x_i^2} - \sqrt{r_j^2 - x_i^2}. \quad (16)$$

In this case Eq. (15) is a SLAE with an upper triangular matrix of dimensions $(n-1) \times (n-1)$ with respect to k_j , $j = 1, \dots, n-1$. Equation (15) has the following solution expressed by a recurrence relation:

$$\left. \begin{aligned} k_{n-1} &= \frac{q_{n-1}/2}{p_{n-1,n-1}}, \\ k_i &= \frac{q_i/2 - \sum_{j=i+1}^{n-1} p_{ij} k_j}{p_{ii}}, \quad i = n-2, n-3, \dots, 1, \\ k_n &= k_{n-2} + \left(\frac{r_{n-1} - r_{n-2}}{r_{n-1} - r_{n-2}} \right) \cdot (k_{n-1} - k_{n-2}), \end{aligned} \right\} \quad (17)$$

where $k_i = k(r_i)$, $q_i = q(x_i)$. Let us note that in this work, Eq. (17) also contains the computation of k_n by linear extrapolation [36].

Formulas similar to Eq. (14) are given also in [19]; however, this is only true for the case of uniform and coinciding node grids along r and x and using the top-left corner and midpoint approximations, whereas formulas similar to Eq. (17) are not given there.

Equations (13)–(16) are rather more general than the formulas given in [19], and Eq. (17) is rather more explicit.

A method for solving Eq. (9) according to Eqs. (13)–(17) is designated in [25], the generalized quadrature method for solving Eq. (9).

The modification of the method in this work compared to that in [25] consists in the nonuniformity of the grids [Eq. (13)], as well as new calculation of k_n , which enhances

the effectiveness of the method, as well as extends its applicability (see also Sections 4.B and 6 for the other enhancements).

The method given above can also be applied for the numerical computation of the solution $k(r)$ according to Eq. (11) (cf. [24,25]). However, this work is focused on the numerical computation of $k(r)$ by solving Eq. (9) according to Eqs. (13), (17).

B. Estimation of the Solution Errors in Case the Exact Solution is Unknown

The estimations of the errors Δk_i of the solution k of Eq. (9) by the generalized quadrature method according to Eq. (17) were obtained in [25,37]. Specificity of these estimations consists in the fact that they are obtained without knowledge of the exact solution $\bar{k}$ (cf. [33]). Let us write the obtained estimations as

$$\left. \begin{aligned} \Delta k_n &= \Delta k_{n-1} = \frac{\Delta \varepsilon_{n-1,n-1}}{p_{n-1,n-1}}, \\ \Delta k_i &= \frac{\varepsilon_i - \sum_{j=i+1}^{n-1} p_{ij} \Delta k_j}{p_{ii}}, \quad i = n-2, n-3, \dots, 1, \end{aligned} \right\} \quad (18)$$

where $\varepsilon_i = \sum_{j=i}^{n-1} \Delta \varepsilon_{ij}$ and

$$\Delta \varepsilon_{ij} = \begin{cases} \frac{k'(\xi_j)}{2} r_j^2 \text{ at } i = j = 1, \\ \frac{k'(\xi_j)}{2} \left[(r_{j+1} - 2r_j) \sqrt{r_{j+1}^2 - x_i^2} + r_j \sqrt{r_j^2 - x_i^2} \right. \\ \left. + x_i^2 \ln \frac{r_{j+1} + \sqrt{r_{j+1}^2 - x_i^2}}{r_j + \sqrt{r_j^2 - x_i^2}} \right], & \text{otherwise.} \end{cases}$$

Here $\xi_j \in [r_j, r_{j+1})$. Approximately,

$$k'(\xi_j) = 0.5 \left(\frac{k_{j+1} - k_j}{r_{j+1} - r_j} \right), \quad j = 1, 2, \dots, n-1.$$

Equations (18) are the estimations of the errors of the solution k caused only by the quadrature error of the generalized quadrature method without taking into account the errors in the right-hand side q of Eq. (9). Another specificity of these estimations consists in the fact that they give the values Δk_i of the solution errors with regard to the sign (cf. [33, p. 42, 122]), which is contrasting with the majority of the works where they are given as $|\Delta k_i|$ or $\|\Delta k_i\|$. As a result, the estimations [Eq. (18)] are not overstated.

The estimations of the solution error are obtained in [37], also taking into account the errors δ in the right-hand side q of Eq. (9), not only for the quadrature error. However these estimations are overstated since they are given in the form $|\Delta k_i| \leq \dots$, for which reason they are not applied in this work.

The estimations [Eq. (18)] are particularly helpful when dealing with real experimental data (see Example 2 in Section 6.B), i.e., when the exact solution is unknown. In this case the estimations Δk_i enable estimating the difference between the obtained and exact solutions and even to obtain a refined (corrected) solution:

$$(k_{\text{cor}})_i = k_i + \Delta k_i, \quad i = 1, 2, \dots, n.$$

In contrast to [25], where the original generalized quadrature method was presented, in this work the nonuniform node grids are introduced and the formulas are obtained not only for k_i and Δk_i but also for B_i and ΔB_i , and spline approximation

and regularization are applied (see Section 6.B). This enhances the effectiveness of the generalized quadrature method.

C. Computation of the Function $B(T_g(x, y))$

Let us consider Eq. (6) as an IE with respect to $B(x, y) = B(T_g(x, y))$, provided that $k(x, y)$ was already obtained. Equation (6) can be written as

$$\int_{y_1(x)}^{y_2(x)} \tilde{K}(x, y) B(x, y) dy = I_g(x), \quad -R \leq x \leq R, \quad (19)$$

where

$$\tilde{K}(x, y) = k(x, y) \exp \left(- \int_y^{y_2(x)} k(x, y') dy' \right) \quad (20)$$

is the kernel and $B(x, y) = B(T_g(x, y))$ is the unknown function.

As follows from the appendix, taking into account Eq. (49), Eq. (19) can be written as a SIE being Abel's IE:

$$\int_x^R K(x, r) B(r) dr = Q(x), \quad 0 \leq x \leq R, \quad (21)$$

where

$$K(x, r) = \frac{rk(r)}{\sqrt{r^2 - x^2}} (e^{u(x, r)} + e^{-u(x, r)}), \quad (22)$$

$$Q(x) = I_g(x) e^{u(x, R)}, \quad (23)$$

$$u(x, r) = \int_x^r \frac{r' k(r')}{\sqrt{r'^2 - x^2}} dr', \quad 0 \leq x \leq R, x \leq r \leq R. \quad (24)$$

It can be seen that Eq. (21), as well as the integral in Eq. (24), is singular. Let us also apply to this case the generalized quadrature method, as in the case of Eq. (9). Consequently, the singular integral in Eq. (24) is approximated by

$$u(x_i, r_j) = \begin{cases} 0, j = i, \\ \sum_{j'=i}^{j-1} \left(\sqrt{r_{j'+1}^2 - x_i^2} - \sqrt{r_{j'}^2 - x_i^2} \right) k_{j'}, j > i, \\ i = 1, \dots, n-1, j \geq i, \end{cases} \quad (25)$$

on the node grids [Eq. (13)]. Further, taking into account Eqs. (22)–(25), Eq. (21) is transformed into a SLAE:

$$\sum_{j=i}^{n-1} P_{ij} B_j = Q_i, \quad i = 1, \dots, n-1, \quad (26)$$

where

$$P_{ij} = \left(\sqrt{r_{j+1}^2 - x_i^2} - \sqrt{r_j^2 - x_i^2} \right) \cdot k_j \cdot (e^{u(x_i, r_j)} + e^{-u(x_i, r_j)}), \quad (27)$$

$$Q_i = I_g(x_i) \cdot e^{u(x_i, R)}. \quad (28)$$

SLAE (26) contains an upper triangular matrix $P_{(n-1) \times (n-1)}$, and it has the following recurrence solution [cf. Eq. (17)]:

$$\left. \begin{aligned} B_{n-1} &= \frac{Q_{n-1}}{P_{n-1,n-1}}, \\ B_i &= \frac{Q_i - \sum_{j=i+1}^{n-1} P_{ij} B_j}{P_{ii}}, \quad i = n-2, n-3, \dots, 1, \\ B_n &= B_{n-2} + \left(\frac{r_n - r_{n-2}}{r_{n-1} - r_{n-2}} \right) \cdot (B_{n-1} - B_{n-2}), \end{aligned} \right\} \quad (29)$$

where $B_i = B(r_i)$, $Q_i = Q(x_i)$.

Similar to the given above estimations of the errors Δk_i of the solution k of Eq. (9), the estimations of the errors ΔB_i of the solution B of Eq. (21) and Eq. (26) can also be written

$$\begin{aligned} \Delta B_n &= \Delta B_{n-1} = \frac{\Delta E_{n-1,n-1}}{P_{n-1,n-1}}, \\ \Delta B_i &= \frac{E_i - \sum_{j=i+1}^{n-1} P_{ij} \Delta B_j}{P_{ii}}, \quad i = n-2, n-3, \dots, 1, \end{aligned}$$

where $E_i = \sum_{j=i}^{n-1} \Delta E_{ij}$,

$$\begin{aligned} \Delta E_{ij} &= \frac{B'(\zeta_j)}{2} \left[(r_{j+1} - 2r_j) \sqrt{r_{j+1}^2 - x_i^2} + r_j \sqrt{r_j^2 - x_i^2} \right. \\ &\quad \left. + x_i^2 \ln \frac{r_{j+1} + \sqrt{r_{j+1}^2 - x_i^2}}{r_j + \sqrt{r_j^2 - x_i^2}} \right], \quad \zeta_j \in [r_j, r_{j+1}), \\ B'(\zeta_j) &= 0.5 \left(\frac{B_{j+1} - B_j}{r_{j+1} - r_j} \right), \quad j = 1, 2, \dots, n-1. \end{aligned}$$

Having obtained $B(r)$, the *temperature profile* is calculated as [cf. Eq. (8)]

$$T_g(r) = \frac{hcv/k_B}{\ln \left(\frac{2hc^2v^3}{B(r)} + 1 \right)}. \quad (30)$$

5. USING REGULARIZATION

The problem of solving IEs of the first kind [Eqs. (7) and (6)] and their special cases [Eqs. (9) and (21)] is, strictly speaking, *ill-posed*. However, Eqs. (9) and (21) are singular, but SIEs, as is known [26], possess an intrinsic quality of natural regularization, or *self-regularization*, owing to numerical solutions by quadrature methods, resulting in a SLAE having a diagonally dominant coefficient matrix. Besides, the Volterra IEs of the first kind, as are Eqs. (9) and (21), take the intermediate position between the Fredholm IEs of the first kind and Volterra IEs of the second kind [33]. In other words, the problem of solving the Volterra IEs of the first kind can be well- or ill-posed depending on which methods are used to solve it and in which spaces.

It is studied in this work to what extent the generalized quadrature method (without regularization, but with smoothing of experimental data, see the examples in Section 6) makes the problem of solving Eqs. (9) and (21) stable and to what extent Tikhonov regularization enhances the solution stability (cf. [19,20,27,32–35]).

Tikhonov regularization was chosen in this work instead of the other stable methods since it is one of the most effective methods for solving ill-posed problems and often applied to

the problems of IR tomography ([19,20,32] and others). Other stable methods (e.g., the Wiener or Kalman filtering [33,34,36]) require much additional information, and the iteration methods (e.g., those of Landweber, Friedman, or others [33,34]) are rather sensitive to the initial approximation, the parameters of the method, and the number of iterations.

A. Regularization for the Case of the Function k

For this task, Eq. (15) can be rewritten as

$$Ak = f, \quad (31)$$

where

$$A_{ij} = \begin{cases} p_{ij}, & j \geq i, \\ 0, & j < i, \end{cases} \quad f_i = \frac{q_i}{2}, \quad i, j = 1, \dots, n-1. \quad (32)$$

According to Tikhonov regularization, the following SLAE is solved instead of Eq. (31):

$$(\alpha I + A^T A) k_\alpha = A^T f, \quad (33)$$

where $\alpha > 0$ is the regularization parameter, I is the identity matrix, and A^T is a transposed matrix. Often [14,19,20,32–34] the regularized SLAE is written as

$$(\alpha LL^T + A^T A) k_\alpha = A^T f \quad (34)$$

or

$$(\alpha ALL^T + A^T A) k_\alpha = A^T f, \quad (35)$$

where matrix $L_{(n-1) \times n}$ is a discrete Laplacian [32]. Equations (34) and (35) have a higher order of regularization than Eq. (33).

The question of *choosing the regularization parameter* α should also be addressed. In the case of modeling, when an exact value of the absorption coefficient $\bar{k}$ is given, α is chosen to minimize the relative error of the solution (cf. [32]):

$$\varepsilon_\alpha = \frac{\|k_\alpha - \bar{k}\|}{\|\bar{k}\|}. \quad (36)$$

In the case of solving a real problem, i.e., when $\bar{k}$ is unknown, α is usually chosen with the aid of the discrepancy principle, L-curve criterion, cross validation, and other methods [7,20,33–36,38]. In this work, the Morozov discrepancy principle [38] is used: α is chosen according to

$$\|Ak_\alpha - f\| = \delta, \quad (37)$$

where δ is an upper estimate of the noise contamination by norm: $\delta = \|\delta f\|_2$. Here $f = q/2$; hence [see Eq. (10)] $|\delta f| = |\delta q|/2 = |\delta I_T|/2$ or

$$\delta = \frac{1}{2} \left\{ \sum_{i=1}^n \left(\frac{\delta I_T}{I_{T,s}} \right)_i^2 \right\}^{1/2}, \quad (38)$$

where $|\delta I_T| = |I_{T,m} - I_{T,s}|$, $I_{T,m}$ is the value of I_T obtained by measurement, and $I_{T,s}$ is the approximation of $I_{T,m}$ by a smoothing spline (see Section 6.B), which is proposed to be considered as the “exact” value of I_T since the exact value is unknown in the case of real measurements (as opposed to modeling).

The discrepancy $\|Ak_\alpha - f\|$ can be written as

$$\|Ak_\alpha - f\|_2 = \left\{ \sum_{i=1}^{n-1} \left[\sum_{j=i}^{n-1} A_{ij} k_{\alpha j} - f_i \right]^2 \right\}^{1/2}. \quad (39)$$

Let us denote by α_{opt} the value of the regularization parameter minimizing the relative error ε_α of the solution [Eq. (36)] and by α_d the value chosen with the aid of the discrepancy principle [Eq. (37)].

B. Regularization for the Case of the Function B

For this case, Eq. (26) is written as

$$AB = Q, \quad (40)$$

where

$$A_{ij} = \begin{cases} P_{ij}, & j \geq i, \\ 0, & j < i, \end{cases} \quad i, j = 1, \dots, n-1. \quad (41)$$

According to Tikhonov regularization, the following SLAE is solved instead of Eq. (40):

$$(\alpha LL^T + A^T A) B_\alpha = A^T Q. \quad (42)$$

6. NUMERICAL ILLUSTRATIONS

A toolbox as a set of functions was developed in MATLAB implementing the computation of the absorption coefficient k and temperature profile T_g in both the direct and inverse problems from the parallel scanning measurements of the laboratory burner flame in the ON and OFF modes assuming axial symmetry of the flame. Let us present two numerical examples processed with the aid of this toolbox.

A. Example 1—Reconstructing from Modeled Input Data

The values of the absorption coefficient $\bar{k}(r)$ were set on the node grid $r = x = 0$ (0.2) 3.8 cm, or $r_i = x_i = 0.2(i-1)$ cm, $i = 1, \dots, n$, in some z cross section of the flame (Fig. 3).

According to [35, p. 139] the model and the right-hand sides of the equations describing the model must be in agreement. To determine the degree of this agreement, Example 1 is

Fig. 3. Absorption coefficient [cm^{-1}]: $\bar{k}(r)$, the exact (set) values; $k(r)$, the values computed (reconstructed) by the generalized quadrature method (GQM) according to Eq. (17), as proposed in this work; and $k_\alpha(r)$, the values computed using Tikhonov regularization method (TRM) according to Eq. (34), both corresponding to the cases of no noise and that of moderate level in I_T .

considered in two variants: (1) $r_{\text{max}} = 3.4$ cm ($n = 18$) and (2) $r_{\text{max}} = 3.8$ cm ($n = 20$).

The exact temperature profile $\bar{T}_g(r)$ was also set (Fig. 4).

The shape of $\bar{k}(r)$ characterized by a certain depression at around the axis of symmetry was taken to be similar to that in, e.g., [4,19,20], and the temperature profile was set similar to that in [5].

Furthermore, the temperature of the blackbody radiation source, which was used as an external IR source in the ON measurement mode, was set to be $T_0 = 1168$ K, and the calculations were performed for the wavenumber $\nu = 2271.5 \text{ cm}^{-1}$. These values were also used in the experiment with the laboratory burner (see also Example 2) carried out at the Technical University of Denmark, Department of Chemical and Biochemical Engineering (before 1 January 2012 Risø DTU, Optical Diagnostics Group), within the joint project [11,39].

First, the *direct problem* was solved. Function $I_g(x)$ (obtained in the OFF mode) is calculated according to [see Eqs. (23)–(28)]

$$I_g(x_i) = Q(x_i) e^{-u(x_i, R)} = e^{-u(x_i, R)} \sum_{j=i}^{n-1} P_{ij} \bar{B}_j, \quad i = 1, \dots, n-1, \quad (43)$$

$$I_g(x_n) = I_g(x_{n-1}), \quad \bar{B}_j = B(\bar{T}_g(r_j)),$$

and $I_R(x)$ (the ON mode) is calculated as [see Eq. (5)]

$$I_R(x_i) = B(T_0) \exp\left(-2 \sum_{j=i}^{n-1} p_{ij} \bar{k}_j\right) + I_g(x_i), \quad i = 1, \dots, n-1, \quad (44)$$

$$I_R(x_n) = I_R(x_{n-1}), \quad \bar{k}_j = \bar{k}(r_j).$$

Then

$$I_T(x_i) = I_R(x_i) - I_g(x_i), \quad i = 1, \dots, n. \quad (45)$$

Functions I_g , I_R , and I_T without simulated measurement noise and with it are shown in Fig. 5. Normal error (noise) ΔI_g was added to I_g using m-function randn.m with the relative standard deviation (RSD) of 0.0005 and 0.005, which corresponds to the relative error,

$$\sigma = \frac{\|\Delta I\|}{\|I\|}, \quad (46)$$

Fig. 4. Temperature profile [K]: $\bar{T}_g(r)$, the exact (set) values; $T_g(r)$, the values calculated from $B(r)$ (see Fig. 7) and $T_{g\alpha}(r)$ from $B_\alpha(r)$ (see Fig. 7), both according to Eq. (30). See Fig. 7 for the comments on the text labels of this figure.

Fig. 5. Direct problem. The modeled functions $I_g(x)$, $I_R(x)$ and $I_T(x)$ without noise contamination (–) and with that of moderate level (+), [W m^{-1}].

[cf. Eq. (36)], of $\sigma_1 = 0.0028 = 0.28\%$ (moderate noise level) and $\sigma_2 = 0.031 = 3.1\%$ (heightened noise level), respectively. Function I_R was contaminated by noise also with $\text{RSD} = 0.0005$ ($\sigma_1 = 0.0025 = 0.25\%$) and $\text{RSD} = 0.005$ ($\sigma_2 = 0.020 = 2.0\%$).

This gave rise to noise in $I_T = I_R - I_g$ with $\sigma_1 = 0.0076 = 0.76\%$ and $\sigma_2 = 0.122 = 12.2\%$, as well as in q [see Eq. (10)] with $\sigma_1 = 0.0154 = 1.54\%$ and $\sigma_2 = 0.292 = 29.2\%$. It can be seen that moderate noise levels in I_g and I_R give rise to heightened noise level in I_T and q , but k depends on I_T and q .

Second, the *inverse problem* was solved. The node grid used in the inverse problem was the same as the one in the direct problem, i.e., $r = x = 0$ (0.2) 3.8 cm.

Function $k(r)$ was computed by solving Eq. (9) by the generalized quadrature method according to Eq. (17) without noise in I_T (the resulting solution error $\sigma_0 = 0.0016$) and with moderate noise in I_T (the solution error $\sigma_1 = 0.0555$). The solutions $k(r)$ are shown in Fig. 3. It can be seen that practically the exact solution was obtained in the case without noise and a moderately unstable solution in the case of moderate noise in I_T .

Besides that, the solution $k(r)$ on the shortened node grid ($r_{\max} = 3.4$ cm) does not reach zero at $r = r_{\max}$, i.e., the solution is inadequate and the situation referred to as “inverse crime” occurs [35]. However, in the case of $r_{\max} = 3.8$ cm, the solution $k(r)$ does reach zero, which makes the model adequate and consistent.

Function $k(r)$ was also determined using Tikhonov regularization by solving Eq. (34) (denoted in this case by $k_\alpha(r)$) at different values of the regularization parameter α .

Note that Tikhonov regularization was also applied, e.g., in [19,20] for the reconstruction of axisymmetric flame properties. In this work, the modified version of the generalized quadrature method (proposed in [25] to solve Abel’s SIEs and published only in [25]) is proposed to solve the inverse problem of IR tomography [see Eq. (17) and the discussion immediately following it].

Relative errors ε_α [Eq. (36)] of the regularized solutions $k_\alpha(r)$ as functions of α are shown in Fig. 6 for the following cases: no noise in I_T (curve 1) and when I_T is contaminated

Fig. 6. Relative error ε_α [Eq. (36)] of the regularized solution $k_\alpha(r)$ ($B_\alpha(r)$) as a function of α : 1 (for $k_\alpha(r)$, 4 for $B_\alpha(r)$)—corresponding to no noise in I_T (I_g and $k_\alpha(r)$ for $B_\alpha(r)$); 2 (5)—moderate noise in I_T (I_g and $k_\alpha(r)$); 3 (6)—heightened noise in I_T (I_g and $k_\alpha(r)$).

by noise of moderate level (curve 2) and a heightened level (curve 3). For each of the three cases, the optimal values α_{opt} minimizing ε_α were found from the curves. Shown in Fig. 3 are the regularized solutions $k_\alpha(r)$ at $\alpha = \alpha_{\text{opt}}$ in the case of no noise in I_T ($\varepsilon_\alpha = 0.0020$), as well as that of moderate level ($\varepsilon_\alpha = 0.0412$). It can be seen that the regularization somewhat reduced the solution error ε_α in the case of moderate noise.

Further, $Q(x_i) = I_g(x_i) \exp(u(x_i, R))$ is calculated [see Eqs. (28) and (25)], and Eq. (21) is solved with respect to $B(r)$ with $Q(x)$ on the right-hand side by the generalized quadrature method (GQM) according to Eq. (29), and regularized $B_\alpha(r)$ was obtained applying Tikhonov regularization [Eq. (42)] at $\alpha = \alpha_{\text{opt}}$ (see Fig. 6). Solutions $B(r)$ and $B_\alpha(r)$ are shown in Fig. 7, both corresponding to the cases of no noise and that of moderate level in I_g and $k(r)$ ($k_\alpha(r)$ for $B_\alpha(r)$).

Note that in contrast to the case of computing $k(r)$ and $k_\alpha(r)$ in the presence of noise when only the right-hand side q of Eq. (9) was contaminated by noise due to the noisy $I_T (= I_R - I_g)$, in the case of $B(r)$, and $B_\alpha(r)$ noise is present

Fig. 7. Inverse problem. The Planck function [W m^{-1}]: $\bar{B}(r)$, the exact values; $B(r)$, the values reconstructed by the generalized quadrature method (GQM) according to Eq. (29), as proposed in this work; and $B_\alpha(r)$, the values computed using the Tikhonov regularization method (TRM) according to Eq. (42) (TRM is also used, e.g., in [19,20]), both corresponding to the cases of no noise and that of moderate level in I_g and $k(r)$ ($k_\alpha(r)$ for $B_\alpha(r)$).

not only in the right-hand side Q of Eq. (21) due to the noisy I_g but is also present now in the kernel [Eq. (22)]. This is due to the kernel contains $k(r)$ ($k_\alpha(r)$ for $B_\alpha(r)$), being noisy now not only due to the noisy I_T but also being contaminated, in addition to that, by the computation noise as a result of solving Eq. (9).

Finally, the temperature profile $T_g(r)$ of the medium (in this case, the axisymmetric laboratory burner flame) was calculated from $B(r)$ [which, in its turn, was reconstructed by the generalized quadrature method for the cases of no noise and with that of moderate level in I_g and $k(r)$], and $T_{g\alpha}(r)$ was obtained from $B_\alpha(r)$ [reconstructed using Tikhonov regularization also in the cases of no noise and with that of moderate level in I_g and $k_\alpha(r)$], both according to Eq. (30), see Fig. 4.

It can be seen that the solution of the inverse problem (reconstructing k , B and T_g) obtained by the generalized quadrature method in the case of no noise in the measurable functions I_g , I_R , and I_T is close to the exact values, and in the presence of moderate noise ($\sim 1\%$) the application of Tikhonov regularization slightly improves the results. In the case of considerable noise level ($\sim 10\%$) in the measurable functions the inverse problem requires the application of special stable methods.

It should be stressed, however, that it is only in the case of modeling that the exact solution is known so the optimal value α_{opt} of the regularization parameter that minimizes the solution error ε_α can be found. In other words, the regularization parameter is found in a way so as to guarantee the best results. In the case of real measurements, the optimal value of the regularization parameter is, generally speaking, an arguable question [e.g., it is proposed in this work to use a spline approximation of measured I_T as the exact I_T , see Eq. (38) and Section 6.B]. The generalized quadrature method, as it is formulated and proposed in this work, does not use any regularization parameter or any quantity that plays its role.

The results obtained in this section confirm the fact that the problem of solving SIEs is moderately ill-posed and possesses the quality of self-regularization.

The following example—reconstructing from real measurements—demonstrates, in addition to the generalized quadrature method and Tikhonov regularization method, the application of one more technique, namely, spline approximation of experimental data that is contaminated by real measurement noise, containing, in addition to random, also systematic measurement errors and measured on a coarse node grid.

B. Example 2—Reconstructing from Real Input Data

Functions $I_{R,m}(x)$, $I_{g,m}(x)$, and $I_{T,m}(x) = I_{R,m}(x) - I_{g,m}(x)$, shown in Fig. 8, were measured on the node grid

$$x = 0, 0.4, 0.8, 1.2, 1.6, 2, 2.4, 2.8, 3.2, 3.6, 3.8 \text{ cm}, \quad (47)$$

($n = 11$) at the height above the burner plate of 1.2 cm. As mentioned in the previous example, $I_{R,m}(x)$ was obtained with the blackbody radiation source at $T_0 = 1168 \text{ K}$, and the values of $I_{R,m}(x)$, $I_{g,m}(x)$, and $I_{T,m}(x)$ at $\nu = 2271.5 \text{ cm}^{-1}$ were taken for further processing. The measurements were carried out at the Technical University of Denmark, Department of Chemical and Biochemical Engineering (before 1 January 2012 Risø DTU, Optical Diagnostics Group) within the joint project [11,39].

Fig. 8. Measured functions $I_{R,m}(x)$ (obtained in the ON measurement mode), $I_{g,m}(x)$ (the OFF measurement mode), and $I_{T,m}(x) = I_{R,m}(x) - I_{g,m}(x)$, [W m^{-1}]. The measurements were carried out at the Technical University of Denmark, Department of Chemical and Biochemical Engineering (before 1 January 2012 Risø DTU, Optical Diagnostics Group) within the joint project [11,39].

It is stressed in [35, p. 139] that in the case of processing real data it is necessary to guarantee the agreement between the mathematical model represented by matrices A , Eqs. (32) and (41), and the right-hand sides f and Q that, in their turn, depend on experimental data [$I_{R,m}(x)$ and $I_{g,m}(x)$ in this example]. Otherwise the situation referred to as “inverse crime” occurs [35]. Hence, the value of $x_{\text{max}} = 3.8 \text{ cm}$ was selected so as to guarantee the functions $k(r)$, $B(r)$, and $T_g(r)$ being close to zero at $r \geq x_{\text{max}}$.

The absorption coefficient $k_m(r)$, obtained by the numerical solution of Abel’s Eq. (9) by the generalized quadrature method according to Eq. (17) from the measured values $I_{T,m}(x)$ on the node grid [Eq. (47)] with $r_i = x_i$, is shown in Fig. 9 by a short dashed line.

It can be seen that the solution $k_m(r)$ exhibits certain fluctuations. This is due to coarse grid spacing along x (or, in other words, small n), as well as measurement errors in $I_{R,m}(x)$ and $I_{g,m}(x)$ [and, hence, $I_{T,m}(x)$].

The specificity of Example 2 consists in the fact that here the exact solution $\bar{k}(r)$ is unknown, which makes it difficult to judge the verisimilitude of the computed solution $k_m(r)$ and, in particular, fluctuations in it. Nevertheless, first, the

Fig. 9. Absorption coefficient [cm^{-1}]: $k_m(r)$, without regularization (short-dashed line); $k_{m\alpha}(r)$, with the regularization; $k_m(r) + \Delta k_m(r)$, corrected $k_m(r)$ using the solution error estimation $\Delta k_m(r)$ [Eq. (18)].

Fig. 10. Discrepancy $\|Ak_\alpha - f\|$: the short-dashed line, without spline smoothing of the measured function $I_{T,m}(x)$; the continuous line, with the spline smoothing of $I_{T,m}(x)$.

estimations $\Delta k_m(r)$ of the solution error obtained taking into account the sign [Eq. (18)] enable correcting the solution $k_m(r) + \Delta k_m(r)$, which decreased the fluctuations, as Fig. 9 shows. Second, the results, e.g., in [1–4,19,20,24,32] and, in particular, those obtained using a different technique (CARS) [5], do not exhibit significant fluctuations in $k_m(r)$ like those in Fig. 9. The above considerations support the conclusion that the fluctuations in $k_m(r)$ in Fig. 9 are unrealistic.

Also shown in Fig. 9 are the values of $k_{m\alpha}(r)$ (the continuous curve) that are the solutions of Eq. (31) by Tikhonov regularization, according to Eq. (34). The value of the regularization parameter α was chosen with the aid of the discrepancy principle according to Eq. (37). The norm δ of the upper estimate of the noise contamination was obtained according to Eq. (38) in which the approximation of $I_{T,m}$ by a smoothing spline was used as the exact values of I_T . The resultant values are $\delta = 0.037$ and $\alpha = 10^{-0.09} = 0.813$ (Fig. 10). Figure 9 shows that the regularization method somewhat smoothed out the solution $k_m(r)$.

Spline approximation [7,31,36] was applied in this work to moderately smooth out the fluctuations in $I_{R,m}(x)$, $I_{g,m}(x)$, and $I_{T,m}(x)$, which also enables chopping the grid spacing along x . The approximations $I_{R,s}(x)$, $I_{g,s}(x)$, and $I_{T,s}(x)$ of these functions by cubic smoothing splines, obtained in MATLAB by calling the m-function csaps.m with the smoothing parameter

Fig. 11. Measured functions $I_{R,m}(x)$, $I_{g,m}(x)$, and $I_{T,m}(x)$ (opened circles $\circ$) and their spline approximations $I_{R,s}(x)$, $I_{g,s}(x)$, and $I_{T,s}(x)$ (continuous curves) with the highlighted (small solid circles $\bullet$) values at $x = 0$ (0.2) 3.8 cm used in the following calculations, $[\text{W m}^{-1}]$.

Fig. 12. Absorption coefficient $[\text{cm}^{-1}]$: $k(r)$, without regularization; $k_\alpha(r)$, with the regularization; $\Delta k(r)$, solution error estimations for $k(r)$ [Eq. (18)].

$p = 0.97$ (corresponding to moderate smoothing), are shown in Fig. 11.

The values of the spline $I_{T,s}(x)$ at the 20 nodes $x = 0$ (0.2) 3.8 cm were used instead of the measured values $I_{T,m}(x)$ in the same procedure as before, i.e., again, Eq. (9) is solved by the generalized quadrature method according to Eq. (17) (see the solution $k(r)$ in Fig. 12), and Eq. (31) is solved by Tikhonov regularization, according to Eq. (34).

The value of the regularization parameter α was again chosen with the aid of the discrepancy principle: $\delta = 0.003$, $\alpha = 10^{-2.1} = 0.00794$ (see also Fig. 10), in which the approximation of the already smooth $I_{T,s}$ by a slightly smoother spline was used as the exact values of I_T in Eq. (38) (see the solution $k_\alpha(r)$ in Fig. 12).

Figures 11 and 12 show that the application of the spline approximation, first of all, simply enabling chopping the grid spacing along x , also results in a (moderate) smoothing out of the solution $k(r)$ as well as $k_\alpha(r)$. Furthermore, the solutions $k(r)$ without regularization and $k_\alpha(r)$ with the regularization are practically the same (as in Example 1, see Fig. 3).

The next step is the determination of the function B and temperature profile T_g . The right-hand side $Q(x)$ of Eq. (21) is computed according to Eqs. (23) and (24) using the spline $I_{g,s}(x)$ as $I_g(x)$ instead of $I_{g,m}(x)$ (see Fig. 11, $n = 20$).

Accordingly, the function $B(r)$ without regularization is obtained by solving Eq. (21) by the generalized quadrature

Fig. 13. Planck function $[\text{W m}^{-1}]$: $B(r)$, without regularization; $B_\alpha(r)$, with the regularization ($\alpha = 10^{-2.5}$).

Fig. 14. Temperature profile [K]: $T_g(r)$, without regularization; $T_{g\alpha}(r)$, with the regularization.

method according to Eq. (29), as well as the function $B_\alpha(r)$ with the regularization obtained by solving SLAE (42) (see Fig. 13 and cf. Fig. 7).

Finally, the values $T_g(r)$ as well as $T_{g\alpha}(r)$ of the temperature profile of the medium (in this case, the axisymmetric laboratory burner flame) were calculated according to Eq. (30), respectively, from $B(r)$ (which, in its turn, was reconstructed without regularization), as well as $B_\alpha(r)$, which was reconstructed with the regularization [see Fig. 14].

It can be seen that the results without regularization (but with smoothing and generalized quadratures) and those with the regularization are quite close. Once again, this confirms the fact that the problem of solving SIEs is moderately ill-posed and possesses the quality of self-regularization.

7. CONCLUSIONS

The IR tomography of a hot gaseous medium, already extensively studied in the literature, was elaborated from a new perspective.

The active (ON) and passive (OFF) modes of measurement yield the two new independent IEs that are solved sequentially instead of the system of the two IEs: the one is with respect to the absorption coefficient k and the other one with respect to the Planck function B (which, in its turn, is directly related to the temperature profile T_g of the medium). This enables avoiding elaborate determination of k by direct (point) measurements or calculation using spectroscopic databases.

The case of the axial symmetry of the flame (as well as the parallel scanning scheme of measurements) considered in this work is governed by Abel's SIEs (the two independent Abel's SIEs with respect to k and B), which were solved by the newly formulated generalized quadrature method.

The two numerical examples—with modeled and real input data, respectively—were solved. In the first example, the SIEs were solved by the modified generalized quadrature method, as proposed in this work, as well as using Tikhonov regularization, already studied, e.g., in [19,20], for the reconstruction of axisymmetric flame properties. In the second example, the experimental input data were moderately smoothed out by the spline approximation, and the SIEs with the smoothed input data

were solved again by the generalized quadrature method as well as by Tikhonov regularization.

The results of both examples confirm the fact that SIEs possess the quality of self-regularization and the problem of solving those equations is moderately ill posed. Furthermore, the application of the generalized quadrature method with preliminary spline-smoothing of the experimental input data yields practically the same results as those obtained by Tikhonov regularization in the case of moderate noise level ($\sim 1\%$) in the input data. In the case of high noise level ($\sim 10\%$) it is necessary to use the generalized quadrature method, spline smoothing, and Tikhonov regularization (as well as the methods of Wiener, Kalman, Landweber, and others). Regularization is necessary in the presence of noise but not to the same extent as, e.g., when solving the Fredholm IE of the first kind, since Abel's SIE has the quality of self-regularization.

In the future, the case of elliptic isolines (i.e., when for each z cross section of the flame the isolines of k and T_g are ellipses, not circles) and a fan (sweep) scanning scheme of measurements (i.e., when the measurement beams constitute a kind of a hand-held fan) will be considered.

APPENDIX A: TRANSFORMING THE INTEGRALS INTO POLAR COORDINATES

Taking into account the axial symmetry and parallel scanning of the flame (Fig. 15), as well as to preserve the way the integrals in Eqs. (19) and (20) are written when transforming into the polar coordinates (with r being the distance of a point (x, y) taken from the z axis of symmetry), the integral in Eq. (19) is rewritten as the sum of the two integrals:

$$\int_{y_1(x)}^{y_2(x)} \tilde{K}(x, y) B(x, y) dy = \int_{y_1(x)}^0 \tilde{K}(x, y) B(x, y) dy + \int_0^{y_2(x)} \tilde{K}(x, y) B(x, y) dy. \quad (\text{A1})$$

Taking into account that for the first integral in the right-hand side of Eq. (A1) (i.e., at $y \leq 0$) $y = -(r^2 - x^2)^{1/2}$, $dy = -rdr/(r^2 - x^2)^{1/2}$ and for the second integral (at $y \geq 0$)

Fig. 15. Axial symmetry and parallel scanning of the flame: (a) $y \in [y_1(x), 0]$ and (b) $y \in [0, y_2(x)]$.

$y = (r^2 - x^2)^{1/2}$, $dy = r dr / (r^2 - x^2)^{1/2}$, the sum of these integrals [with regard to Eq. (20)] is

$$\begin{aligned} & \int_x^R \frac{rk(r)}{\sqrt{r^2 - x^2}} \exp \left\{ - \left(\int_x^R + \int_x^r \right) \frac{r'k(r')}{\sqrt{r'^2 - x^2}} dr' \right\} B(r) dr \\ & + \int_x^R \frac{rk(r)}{\sqrt{r^2 - x^2}} \exp \left\{ - \left(\int_x^R - \int_x^r \right) \frac{r'k(r')}{\sqrt{r'^2 - x^2}} dr' \right\} B(r) dr \\ & = \exp \left\{ - \int_x^R \frac{r'k(r')}{\sqrt{r'^2 - x^2}} dr' \right\} \cdot \int_x^R \frac{rk(r)}{\sqrt{r^2 - x^2}} \\ & \times \left[\exp \left(\int_x^r \frac{r'k(r')}{\sqrt{r'^2 - x^2}} dr' \right) + \exp \left(- \int_x^r \frac{r'k(r')}{\sqrt{r'^2 - x^2}} dr' \right) \right] B(r) dr, \end{aligned}$$

or

$$\begin{aligned} & \int_{y_1(x)}^{y_2(x)} \tilde{K}(x, y) B(x, y) dy \\ & = e^{-u(x, R)} \cdot \int_x^R \frac{rk(r)}{\sqrt{r^2 - x^2}} (e^{u(x, r)} + e^{-u(x, r)}) B(r) dr = I_g(x), \end{aligned} \quad (A2)$$

where $u(x, r)$ is expressed in Eq. (24).

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