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A Characterization of Tight and Dual Generalized Translation Invariant Frames

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Abstract—We present results concerning generalized translation invariant (GTI) systems on a second countable locally compact abelian group G . These are systems with a family of generators $\{g_{j,p}\}_{j \in J, p \in P_j} \subset L^2(G)$, where J is a countable index set, and $P_j, j \in J$ are certain measure spaces. Furthermore, for each j we let Γ_j be a closed subgroup of G such that G/Γ_j is compact. A GTI system is then the collection of functions $\cup_{j \in J} \{g_{j,p}(\cdot - \gamma)\}_{\gamma \in \Gamma_j, p \in P_j}$. Many well known systems, such as wavelet, shearlet and Gabor systems, both the discrete and continuous types, are GTI systems. We characterize when such systems form tight frames, and when two GTI Bessel systems form dual frames for $L^2(G)$. In particular, this offers a unified approach to the theory of discrete and continuous frames and, e.g., yields well known results for discrete and continuous Gabor and wavelet systems.

I. INTRODUCTION

A *frame* for a Hilbert space \mathcal{H} with respect to a measure space M is a collection of functions $\{g_k\}_{k \in M} \subset \mathcal{H}$ such that $k \mapsto g_k, M \rightarrow \mathcal{H}$ is measurable and for $0 < A \leq B < \infty$

$$A \|f\|^2 \leq \int_M |\langle f, g_k \rangle|^2 dk \leq B \|f\|^2 \quad \forall f \in \mathcal{H}. \quad (1)$$

If M is discrete, then we say that $\{g_k\}$ is a *discrete* frame. In the literature this is the most often considered type of frame and usually the word “discrete” is omitted. This work presents results for which the measure space M is unspecified and might be discrete or continuous, we therefore use the term *frame* for either case. For literature on frame theory we refer to [1]–[6]

The constants A and B are called the lower and upper frame bound, respectively. If the upper inequality in (1) holds, we say that $\{g_k\}_{k \in M}$ is a Bessel system. If $A = B$ then $\{g_k\}_{k \in M}$ is a tight frame with respect to the measure space M . If, furthermore, $A = B = 1$, then it is a Parseval frame.

If $\{g_k\}_{k \in M}$ satisfies (1), then there exists a *dual* frame $\{h_k\}_{k \in M}$ which is also a frame with respect to the measure space M , and, more importantly, we have the *resolution-of-identity* or *reproducing* formula

$$\langle f_1, f_2 \rangle = \int_M \langle f_1, g_k \rangle \langle h_k, f_2 \rangle dk \quad \forall f_1, f_2 \in \mathcal{H}. \quad (2)$$

For a Parseval frame $\{g_k\}_{k \in M}$ equation (2) holds for $h_k = g_k$. If we consider discrete frames, then (2) holds in the strong sense, i.e., $f = \sum_k \langle f, g_k \rangle h_k$ for all $f \in \mathcal{H}$.

Given a system $\{g_k\}_{k \in M} \subset \mathcal{H}$ it is of interest to characterize when it is a Parseval or tight frame. Similarly, given two Bessel systems $\{g_k\}_{k \in M}$ and $\{h_k\}_{k \in M}$ we would like to characterize when they are dual frames. These questions have been answered for, e.g., wavelet and Gabor systems. Let us review some known results for wavelet and Gabor Parseval frame in $L^2(\mathbb{R})$. For $a, b, c \in \mathbb{R}, c \neq 0$ we define the translation, modulation and dilation operator on $L^2(\mathbb{R})$ as:

$$T_a f : x \mapsto f(x-a), E_b f : x \mapsto e^{2\pi i b x} f(x), D_c f : x \mapsto f\left(\frac{x}{c}\right) / \sqrt{|c|}.$$

Example 1. [7], [8] The discrete wavelet system $\{D_{2^j} T_k \psi\}_{j,k \in \mathbb{Z}}$ generated by $\psi \in L^2(\mathbb{R})$ satisfies the resolution-of-identity

$$f = \sum_{j,k \in \mathbb{Z}} \langle f, D_{2^j} T_k \psi \rangle D_{2^j} T_k \psi \quad \forall f \in L^2(\mathbb{R})$$

if, and only if, for all $\alpha \in \cup_{j \in \mathbb{Z}} 2^{-j} \mathbb{Z}$ and almost every ξ in the Fourier domain $\widehat{\mathbb{R}}$

$$\sum_{j \in \mathbb{Z}} \langle \alpha \in 2^{-j} \mathbb{Z} \rangle \widehat{\psi}(2^j \xi) \overline{\widehat{\psi}(2^j (\xi + \alpha))} = \delta_{\alpha,0}.$$

Example 2. [9] Let $\psi \in L^2(G)$. The continuous wavelet system $\{D_a T_b \psi\}_{b \in \mathbb{R}, a \in \mathbb{R} \setminus \{0\}}$ for all $f_1, f_2 \in L^2(\mathbb{R})$ satisfies

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}} \int_{\mathbb{R} \setminus \{0\}} \langle f_1, D_a T_b \psi \rangle \langle D_a T_b \psi, f_2 \rangle \frac{1}{|a|^2} da db$$

if, and only if, it satisfies the Calderón admissibility condition

$$\int_{\mathbb{R} \setminus \{0\}} |\widehat{\psi}(a\xi)|^2 / |a| da = 1 \quad \text{for a.e. } \xi \in \widehat{\mathbb{R}}.$$

Example 3. [10] Let $g \in L^2(\mathbb{R})$ and $a, b > 0$. The Gabor system $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is a Parseval frame, i.e.,

$$f = \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb} T_{na} g \rangle E_{mb} T_{na} g \quad \forall f \in L^2(\mathbb{R})$$

if, and only if, for all $n \in \mathbb{Z}$ and almost all $x \in \mathbb{R}$

$$\sum_{k \in \mathbb{Z}} g(x+ka) \overline{g(x+ka+n/b)} = b \delta_{n,0}.$$

Example 4. [11], [12] Let $g \in L^2(G)$. The continuous Gabor system $\{E_b T_a g\}_{a,b \in \mathbb{R}}$ satisfies

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R} \times \widehat{\mathbb{R}}} \langle f_1, E_\gamma T_x g \rangle \langle E_\gamma T_x g, f_2 \rangle dx d\gamma \quad \forall f_1, f_2 \in L^2(\mathbb{R}),$$

if, and only if, $\|g\| = 1$.

While these characterizing results of Parseval frames may seem different in these four example, they can in fact be

unified into one theory. In [13] it is shown that Examples 1 and 3 are special cases of a theory for generalized shift invariant systems in $L^2(\mathbb{R}^d)$ [14]. This theory was extended to discrete frames in the Hilbert space $L^2(G)$, where G is a second countable locally compact Abelian group [15]. In this article we give an account of the theory presented in [16], namely, that in fact all the results on the continuous and discrete frames of Examples 1 to 4 (and many more) can be unified by results on generalized translation invariant systems.

II. PRELIMINARIES

In this paper G will denote a second countable locally compact abelian group (e.g. \mathbb{R} , $[0, 1[$, \mathbb{Z} and the cyclic group $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$). To G we associate its dual group \widehat{G} which consists of all characters, i.e., all continuous homomorphisms from G into the torus $\mathbb{T} \cong \{z \in \mathbb{C} : |z| = 1\}$. Under pointwise multiplication \widehat{G} is also a locally compact abelian group – we also call \widehat{G} the Fourier domain. Throughout the paper we use addition and multiplication as group operation in G and \widehat{G} , respectively. In the examples from above, \widehat{G} can be identified with \mathbb{R} , \mathbb{Z} , $[0, 1[$ and \mathbb{Z}_n , respectively. If G is discrete, then \widehat{G} is compact, and vice versa.

The group G can be equipped with a so-called Haar measure μ_G , which is unique up to a positive constant. In the mentioned examples one often takes the Lebesgue measure on \mathbb{R} , the normalized Lebesgue measure on $[0, 1[$, and the counting measure on \mathbb{Z} and \mathbb{Z}_n . With the measure μ_G we define $L^1(G)$ and the Hilbert space $L^2(G)$ over the complex field in the usual way. $L^2(G)$ is separable, because G is assumed to be second countable. For functions $f \in L^1(G)$ we define the Fourier transform

$$\mathcal{F}f(\omega) = \widehat{f}(\omega) = \int_G f(x) \overline{\omega(x)} d\mu_G(x), \quad \omega \in \widehat{G}.$$

Here $\omega(x)$ is the action of \widehat{G} on G . In the examples from

before, we have: for $(x, \omega) \in (\mathbb{R}, \mathbb{R}) : \omega(x) = e^{2\pi i x \omega}$,
for $(x, \omega) \in ([0, 1[, \mathbb{Z}) : \omega(x) = e^{2\pi i x \omega}$,
for $(x, \omega) \in (\mathbb{Z}, [0, 1[) : \omega(x) = e^{2\pi i x \omega}$,
for $(x, \omega) \in (\mathbb{Z}_n, \mathbb{Z}_n) : \omega(x) = e^{2\pi i x \omega/n}$.

If $f \in L^1(G)$, $\widehat{f} \in L^1(\widehat{G})$ the function f can be recovered from \widehat{f} by the inverse Fourier transform

$$f(x) = \mathcal{F}^{-1}\widehat{f}(x) = \int_{\widehat{G}} \widehat{f}(\omega) \omega(x) d\mu_{\widehat{G}}(\omega), \quad x \in G. \quad (3)$$

In fact, for (3) to hold, we need the correct normalization of the measure on \widehat{G} , see [17, (31.1)]. In the examples above, it is the usual Lebesgue measure on \mathbb{R} , the counting measure on \mathbb{Z} , the normalized Lebesgue measure on $[0, 1[$ and n^{-1} times the counting measure on \mathbb{Z}_n . We assume that the measure on a group μ_G and its dual group $\mu_{\widehat{G}}$ are normalized this way. With this convention the Fourier transform \mathcal{F} extends to the usual isometric isomorphism between $L^2(G)$ and $L^2(\widehat{G})$.

For $a \in G$, we define the translation operator on $L^2(G)$ as the mapping $T_a : f \mapsto f(\cdot - a)$.

For a closed subgroup H of G , we define its annihilator H^\perp as the set $H^\perp = \{\omega \in \widehat{G} \mid \omega(x) = 1 \forall x \in H\}$. The annihilator is itself a closed subgroup in \widehat{G} , and, as topological groups,

$\widehat{H} \cong \widehat{G}/H^\perp$ and $\widehat{G}/\widehat{H} \cong H^\perp$. These relations show that for a closed subgroup H the quotient G/H is compact if and only if H^\perp is discrete. A subgroup H in G , for which G/H is compact, is called a *co-compact* subgroup.

Example 5. Let us consider a few examples of annihilators: If $H = 4\mathbb{Z}$ in $G = \mathbb{R}$, then $H^\perp = \frac{1}{4}\mathbb{Z}$ in $\widehat{G} = \mathbb{R}$. If $H = 4\mathbb{Z}$ in $G = \mathbb{Z}$, then $H^\perp = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ in $\widehat{G} = [0, 1[$. If $H = [0, 1[$ in $G = [0, 1[$ then $H^\perp = \{0\}$ in $\widehat{G} = \mathbb{Z}$.

We remind the reader of Weil's formula: Let H be a closed subgroup in G with given Haar measure μ_H and let μ_G be the Haar measure on G . Then there exists a unique Haar measure $\mu_{G/H}$ on the quotient group G/H such that for all $f \in L^1(G)$ the function $\dot{x} \mapsto \int_H f(x+h) d\mu_H(h)$, $\dot{x} = x+H$ is defined almost everywhere on G/H is integrable and for all $f \in L^1(G)$

$$\int_G f(x) d\mu_G(x) = \int_{G/H} \int_H f(x+h) d\mu_H(h) d\mu_{G/H}(\dot{x}).$$

If, furthermore, H is a co-compact subgroup of G , then we define the size of H , $d(H) := \int_{G/H} 1 d\mu_{G/H}(\dot{x})$. This definition coincides with the *lattice size* $d(H)$ for discrete, co-compact subgroups H in G , see [11], [15]. For an introduction to abstract harmonic analysis, we refer to the classical texts [17]–[19].

III. THE MAIN RESULTS

Definition III.1. Let J be a countable index set. For each $j \in J$, let P_j be some (possibly uncountable) index set and let $\{g_{j,p}\}_{p \in P_j}$ be subset of $L^2(G)$. Furthermore, let $\Gamma_j, j \in J$ be a closed, co-compact subgroup in G with Haar measure μ_{Γ_j} . The *generalized translation invariant* (GTI) system generated by $\{g_{j,p}\}_{p \in P_j, j \in J}$ with translation along closed, co-compact subgroups $\{\Gamma_j\}_{j \in J}$ is the family of functions $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$. If all Γ_j coincide, then we say $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is a translation invariant system.

We will work under the following *standing hypotheses* on the generalized translation invariant system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$. For each $j \in J$:

- (a) $(P_j, \Sigma_{P_j}, \mu_{P_j})$ is a σ -finite measure space,
- (b) the mapping $p \mapsto g_p, (P_j, \Sigma_{P_j}) \rightarrow (L^2(G), B_{L^2(G)})$ is measurable,
- (c) the mapping $(p, x) \mapsto g_p(x), (P_j \times G, \Sigma_{P_j} \otimes B_G) \rightarrow (\mathbb{C}, B_{\mathbb{C}})$ is measurable.

Here B_X is the Borel algebra for a topological space X . The reason for these assumptions are purely technical, e.g., they ensure that later manipulations go well and that integration is well defined. We note immediately that these assumptions are trivially satisfied if P_j is a discrete measure space or if P_j is itself a group with a Haar measure. For the details, see [16]. We now aim to show when a GTI system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is a Parseval frame, i.e., when the reproducing formula

$$\langle f_1, f_2 \rangle = \sum_{j \in J} \int_{P_j} \int_{\Gamma_j} \langle f_1, T_\gamma g_{j,p} \rangle \langle T_\gamma g_{j,p}, f_2 \rangle d\mu_{\Gamma_j} d\mu_{P_j} \quad (4)$$

holds for all $f_1, f_2 \in L^2(G)$ and similarly, when for two Bessel systems $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ we have that

$$\langle f_1, f_2 \rangle = \sum_{j \in J} \int_{P_j} \int_{\Gamma_j} \langle f_1, T_\gamma g_{j,p} \rangle \langle T_\gamma h_{j,p}, f_2 \rangle d\mu_{\Gamma_j} d\mu_{P_j} \quad (5)$$

for all $f_1, f_2 \in L^2(G)$.

Remark 1. We emphasize that we assume some given Haar measure on G and that the closed, co-compact subgroups Γ_j are equipped with a given Haar measure μ_{Γ_j} . This is different to the assumptions used in [16], but this change is just a matter of scaling.

Actually, in order to verify (4) and (5) it is sufficient to consider f_1, f_2 in a dense subspace of $L^2(G)$. We therefore define $\mathcal{D} := \{f : G \rightarrow \mathbb{C} \mid \text{supp } \hat{f} \text{ is compact and } \hat{f} \in L^\infty(\hat{G})\}$.

Before we can state the main results, Theorem III.3 and III.4, we need a technical definition.

Definition III.2. We say that two generalized translation invariant systems $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfy the *dual α local integrability condition* (dual α -LIC) if, for all $f \in \mathcal{D}$,

$$\sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{\hat{G}} |\hat{f}(\omega) \hat{f}(\omega\alpha)| \cdot \hat{g}_{j,p}(\omega) \hat{h}_{j,p}(\omega\alpha) |d\mu_{\hat{G}}(\omega) d\mu_{P_j}(p) < \infty. \quad (6)$$

In case $g_{j,p} = h_{j,p}$ we refer to (6) as the α *local integrability condition* (α -LIC) for the generalized translation invariant system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$.

The α -LIC should be compared to the local integrability condition for generalized shift invariant systems introduced in [13] for $L^2(\mathbb{R}^n)$ and in [15] for $L^2(G)$. For generalized translation invariant systems $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ the *local integrability conditions* (LIC) becomes that for all $\hat{f} \in \mathcal{D}$,

$$\sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{\text{supp } \hat{f}} |\hat{f}(\omega\alpha) \hat{g}_{j,p}(\omega)|^2 d\mu_{\hat{G}}(\omega) d\mu_{P_j}(p) < \infty. \quad (7)$$

The α -LIC is strictly weaker than the LIC, see [16].

We can now formulate our main result for dual generalized translation invariant frames.

Theorem III.3 ([16]). *Suppose that the two GTI systems $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ are Bessel systems satisfying the dual α -LIC. Then the following statements are equivalent:*

- (i) $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ are dual frames for $L^2(G)$, i.e., satisfy (5);
- (ii) for each $\alpha \in \cup_{j \in J} \Gamma_j^\perp$ we have that for a.e. $\omega \in \hat{G}$

$$t_\alpha(\omega) := \sum_{j \in J: \alpha \in \Gamma_j^\perp} \frac{1}{d(\Gamma_j)} \int_{P_j} \overline{\hat{g}_{j,p}(\omega)} \hat{h}_{j,p}(\omega\alpha) d\mu_{P_j}(p) = \delta_{\alpha,1}.$$

We sketch a proof in Section VII.

For Parseval frames the Bessel assumption can be omitted, and we find the following result.

Theorem III.4 ([16]). *Suppose that the generalized translation invariant system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies the α local integrability condition. Then the following assertions are equivalent:*

- (i) $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is a Parseval frame for $L^2(G)$, i.e., satisfies (4),
- (ii) for each $\alpha \in \cup_{j \in J} \Gamma_j^\perp$ we have

$$t_\alpha(\omega) := \sum_{j \in J: \alpha \in \Gamma_j^\perp} \frac{1}{d(\Gamma_j)} \int_{P_j} \overline{\hat{g}_{j,p}(\omega)} \hat{g}_{j,p}(\omega\alpha) d\mu_{P_j}(p) = \delta_{\alpha,1}.$$

IV. A CLOSER LOOK AT THE LOCALLY INTEGRABILITY CONDITIONS

In this section we take a closer look at the locally integrability conditions.

Let us first turn to sufficient conditions for a generalized translation invariant system to be a Bessel family or a frame. Proposition IV.1 is a generalization of the results in, e.g., [20] and [21], which state the corresponding result for generalized shift invariant systems in the euclidean space and locally compact abelian groups. The result is as follows:

Proposition IV.1. *Consider the generalized translation invariant system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$.*

- (i) If $B :=$

$$\text{ess sup}_{\omega \in \hat{G}} \sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} |\hat{g}_{j,p}(\omega) \hat{g}_{j,p}(\omega\alpha)| d\mu_{P_j}(p) < \infty, \quad (8)$$

then $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is a Bessel family with bound B .

- (ii) Furthermore, if also

$$A := \text{ess inf}_{\omega \in \hat{G}} \left(\sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} |\hat{g}_{j,p}(\omega)|^2 d\mu_{P_j}(p) - \sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp \setminus \{0\}} |\hat{g}_{j,p}(\omega) \hat{g}_{j,p}(\omega\alpha)| d\mu_{P_j}(p) \right) > 0,$$

then $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is a frame for $L^2(G)$ with bounds A and B .

Proof. With a few adaptations the result follows from the corresponding proofs in [20] and [21]. \square

We refer to (8) as the absolute CC-condition, see also [22].

Proposition IV.1 is useful in applications as a mean to verify that a given family indeed is Bessel, or even a frame. Moreover, in relation to the characterizing results in Theorem III.3 and III.4, the condition (8) is sufficient for the α -LIC to hold. In contrast, we remark that (8) does not imply the LIC [16].

Lemma IV.2. *If the generalized translation invariant systems $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfy*

$$\text{ess sup}_{\omega \in \hat{G}} \sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} |\hat{g}_{j,p}(\omega) \hat{h}_{j,p}(\omega\alpha)| d\mu_{P_j}(p) < \infty$$

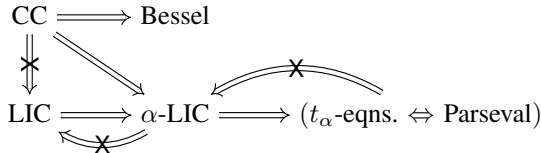
and

$$\text{ess sup}_{\omega \in \hat{G}} \sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} |\hat{g}_{j,p}(\omega\alpha) \hat{h}_{j,p}(\omega)| d\mu_{P_j}(p) < \infty,$$

then the dual α local integrability condition is satisfied. Furthermore, if $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies (8), then the α local integrability condition is satisfied.

Lemma IV.3. *If both $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfy the local integrability condition (7), then $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfy the dual α local integrability condition. In particular, if $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ satisfies the local integrability condition, then it also satisfies the α local integrability condition.*

The relationships between the various conditions considered above are summarized in the diagram below. To simplify the presentation we consider Parseval frames and not dual frames. An arrow means that the assumption at the tail of the arrow implies the assumption at the head. A crossed out arrow means that one can find a counter example for that implication; clearly, implications to the left in the top line are also not true in general.



V. SPECIAL CASES OF THE MAIN RESULT

One can show that translation invariant systems, that is, GTI systems where all Γ_j coincide, always satisfy the dual α -LIC. We therefore have the following straightforward characterization result.

Theorem V.1. *Let Γ be a closed, co-compact subgroup in G . Suppose that $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_{j,p}\}_{\gamma \in \Gamma, p \in P_j}$ are Bessel families. Then the following statements are equivalent:*

- (i) $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_{j,p}\}_{\gamma \in \Gamma, p \in P_j}$ are dual frames for $L^2(G)$,
- (ii) for each $\alpha \in \Gamma^\perp$ we have for almost all $\omega \in \widehat{G}$

$$t_\alpha(\omega) := \sum_{j \in J} \frac{1}{d(\Gamma)} \int_{P_j} \overline{\hat{g}_{j,p}(\omega)} \hat{h}_{j,p}(\omega\alpha) d\mu_{P_j}(p) = \delta_{\alpha,1}. \quad (9)$$

For TI systems with translation along the entire group $\Gamma = G$ there is only one t_α -equation in (9) since $G^\perp = \{1\}$. To be precise:

Lemma V.2. *Suppose that $\Gamma = G$. Then assertion (ii) in Theorem V.1 reduces to*

$$\sum_{j \in J} \int_{P_j} \overline{\hat{g}_{j,p}(\omega)} \hat{h}_{j,p}(\omega) d\mu_{P_j}(p) = 1 \quad a.e. \omega \in \widehat{G}.$$

For compact abelian groups all generalized translation invariant systems satisfy the local integrability condition. The characterization result is as follows.

Theorem V.3. *Let G be a compact abelian group. Suppose that $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ are Bessel families. Then the following statements are equivalent:*

- (i) $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ are dual frames for $L^2(G)$,
- (ii) for each $\alpha \in \bigcup_{j \in J} \Gamma_j^\perp$ we have for almost all $\omega \in \widehat{G}$

$$t_\alpha(\omega) := \sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} \overline{\hat{g}_p(\omega)} \hat{h}_p(\omega\alpha) d\mu_{P_j}(p) = \delta_{\alpha,1}.$$

The three results presented here also hold for Parseval frames, in which case the Bessel assumption can be omitted.

VI. EXAMPLES

Let us now look at the Gabor and wavelet systems as GTI system.

Example 6. Let $J = \{0\}$, $P_0 = \widehat{\mathbb{R}}$ with Lebesgue integration, $\Gamma_0 = \mathbb{R}$, and $\{g_{0,p}\}_{p \in P_0} = \{E_\gamma g\}_{\gamma \in \widehat{\mathbb{R}}}$ for some $g \in L^2(\mathbb{R})$. With these choices, the GTI system becomes the Gabor system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j} = \{T_x E_\gamma g\}_{x \in \mathbb{R}, \gamma \in \widehat{\mathbb{R}}}$. One can show that this is a Bessel system, in fact, by Moyal's formula $\int_{\mathbb{R} \times \widehat{\mathbb{R}}} |\langle f, T_x E_\gamma g \rangle|^2 dx d\gamma = \|f\|^2 \|g\|^2$ for all $f \in L^2(\mathbb{R})$. Similarly for a system $\{h_{0,p}\}_{p \in P_0} = \{E_\gamma h\}_{\gamma \in \widehat{\mathbb{R}}}$ for $h \in L^2(\mathbb{R})$. The GTI systems are translation invariant since there is only one Γ_j , thus the α -LIC condition is satisfied. By Lemma V.2 we have that for all $f_1, f_2 \in L^2(\mathbb{R})$

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R} \times \widehat{\mathbb{R}}} \langle f_1, T_x E_\gamma g \rangle \langle T_x E_\gamma h, f_2 \rangle dx d\gamma$$

if, and only if, for almost all $\omega \in \widehat{G}$

$$\sum_{j \in J} \int_{P_j} \overline{\hat{g}_{j,p}(\omega)} \hat{h}_{j,p}(\omega) d\mu_{P_j}(p) = \int_{\widehat{\mathbb{R}}} \overline{T_\gamma \hat{g}} T_\gamma \hat{h} d\gamma = \langle \hat{h}, \hat{g} \rangle = 1.$$

By Plancherel, this is equivalent to $\langle g, h \rangle = 1$, which is the well known criterion for the inversion of the short-time Fourier transform.

Example 7. Let $J = \{0\}$, P_0 be the multiplicative group $\mathbb{R} \setminus \{0\}$ with Haar measure $\frac{1}{|a|^2} da$. Take $\Gamma_0 = \mathbb{R}$ and consider the generators $\{g_{0,p}\}_{p \in P_0} = \{D_a \psi\}_{a \in \mathbb{R} \setminus \{0\}}$ for $\psi \in L^2(\mathbb{R})$. Then the GTI system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ becomes the continuous wavelet system $\{T_\gamma D_a \psi\}_{\gamma \in \mathbb{R}, a \in \mathbb{R} \setminus \{0\}} = \{D_a T_\gamma \psi\}_{\gamma \in \mathbb{R}, a \in \mathbb{R} \setminus \{0\}}$. By Lemma V.2, we have that

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}} \int_{\mathbb{R} \setminus \{0\}} \langle f_1, D_a T_b \psi \rangle \langle D_a T_b \psi, f_2 \rangle \frac{1}{|a|^2} da db$$

holds for all $f_1, f_2 \in L^2(\mathbb{R})$ if, and only if,

$$\sum_{j \in J} \int_{P_j} \overline{\hat{g}_p(\omega)} \hat{g}_p(\omega) d\mu_{P_j}(p) = \int_{\mathbb{R} \setminus \{0\}} |a| |\hat{\psi}(a\omega)|^2 \frac{1}{|a|^2} da = 1.$$

Which yields the Caldéron admissibility condition.

The discrete system in Example 1 and 3 can be realized in a similar fashion. For these, and more examples see [13], [15], [16].

VII. SKETCH OF THE PROOF FOR THE MAIN RESULT

In order to show the main result one needs Lemma VII.1 and Proposition VII.2 below. The proofs can be found in [16].

Lemma VII.1. *Let Γ be a closed, co-compact subgroup of G with Haar measure μ_Γ . Suppose that $f_1, f_2 \in \mathcal{D}$ and $\varphi, \psi \in L^2(G)$. Then*

$$\begin{aligned} & \int_\Gamma \langle f_1, T_\gamma \varphi \rangle \langle T_\gamma \psi, f_2 \rangle d\mu_\Gamma(\gamma) \\ &= \int_{\widehat{G}} \frac{1}{d(\Gamma)} \sum_{\alpha \in \Gamma^\perp} \widehat{f_1}(\omega) \overline{\widehat{f_2}(\omega\alpha)} \widehat{\varphi}(\omega) \widehat{\psi}(\omega\alpha) d\mu_{\widehat{G}}(\omega). \end{aligned}$$

Proposition VII.2. *If the generalized translation invariant system $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is a Bessel system with bound B , then*

$$\sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} |\widehat{g}_{j,p}(\omega)|^2 d\mu_{P_j}(p) \leq B \quad \text{for a.e. } \omega \in \widehat{G}.$$

Sketch of the proof for Theorem III.3. Let us first show that the t_α -equations are well-defined. Take B to be a common Bessel bound for the two GTI systems $\cup_{j \in J} \{T_\gamma g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{T_\gamma h_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$. By application of the Cauchy-Schwarz inequality and Proposition VII.2, we find that

$$\begin{aligned} & \sum_{j \in J: \alpha \in \Gamma_j^\perp} \frac{1}{d(\Gamma_j)} \int_{P_j} |\widehat{g}_{j,p}(\omega) \widehat{h}_{j,p}(\omega\alpha)| d\mu_{P_j}(p) \\ & \leq \left(\sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} |\widehat{g}_{j,p}(\omega)|^2 d\mu_{P_j}(p) \right)^{1/2} \\ & \quad \cdot \left(\sum_{j \in J} \frac{1}{d(\Gamma_j)} \int_{P_j} |\widehat{h}_{j,p}(\omega\alpha)|^2 d\mu_{P_j}(p) \right)^{1/2} \leq B, \end{aligned}$$

for a.e. $\omega \in \widehat{G}$. This shows that the t_α -equations are well-defined and converge absolutely.

For $f \in \mathcal{D}$, define the function $w_f : G \rightarrow \mathbb{C}$

$$x \mapsto \sum_{j \in J} \int_{P_j} \int_{\Gamma_j} \langle T_x f, T_\gamma g_{j,p} \rangle \langle T_\gamma h_{j,p}, T_x f \rangle d\mu_{\Gamma_j}(\gamma) d\mu_{P_j}(p).$$

By Lemma VII.1 and the standing hypothesis on the measure spaces P_j one can show that w_f can be expressed as a generalized Fourier series of the form

$$w_f(x) = \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^\perp} \alpha(x) \widehat{w}(\alpha), \quad (10)$$

where $\widehat{w}(\alpha) := \int_{\widehat{G}} \widehat{f}(\omega) \overline{\widehat{f}(\omega\alpha)} t_\alpha(\omega) d\mu_{\widehat{G}}(\omega)$. The dual α -LIC together with the Weierstrass M-test implies that the convergence in (10) is absolute and that w_f is the uniform limit of a generalized Fourier series and thus an almost periodic, continuous function.

One can use the continuity of w_f and the uniqueness theorem for generalized Fourier series [23, Theorem 7.12] to conclude the result. For all details we refer to [16]. \square

The proof of Theorem III.4 is similar.

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