



Aspects of the Tutte polynomial

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Aspects of the Tutte polynomial

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Kongens Lyngby 2015

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Summary (English)

This thesis studies various aspects of the Tutte polynomial, especially focusing on the Merino-Welsh conjecture.

We write $T(G; x, y)$ for the Tutte polynomial of a graph G with variables x and y . In 1999, Merino and Welsh conjectured that if G is a loopless 2-connected graph, then

$$T(G; 1, 1) \leq \max\{T(G; 2, 0), T(G; 0, 2)\}.$$

The three numbers, $T(G; 1, 1)$, $T(G; 2, 0)$ and $T(G; 0, 2)$ are respectively the numbers of spanning trees, acyclic orientations and totally cyclic orientations of G .

First, I extend Negami's splitting formula to the multivariate Tutte polynomial. Using the splitting formula, Thomassen and I found a lower bound for the number of spanning trees in a k -edge-connected graph. Our bound is tight for k even, but for k odd we give a slightly better lower bound which we believe is not tight. We prove that the minimum number of spanning trees in a 3-edge-connected graph with n vertices is, not surprisingly, significantly smaller than the minimum number of spanning trees in a 4-edge-connected graph. However, we conjecture that the minimum number of spanning trees of a 5-edge-connected graph is actually obtained by a 6-edge-connected graph asymptotically.

Thomassen proved the following partial result for the Merino-Welsh conjecture. Assume the graph G is loopless, bridgeless and has n vertices and m edges.

If $m \leq 1.066n$ then $T(G; 1, 1) \leq T(G; 2, 0)$.

If $m \geq 4n - 4$ then $T(G; 1, 1) \leq T(G; 0, 2)$.

I improve in this thesis Thomassen's result as follows:

If $m \leq 1.29(n - 1)$ then $T(G; 1, 1) \leq T(G; 2, 0)$.

If $m \geq 3.58(n - 1)$ and G is 3-edge-connected then $T(G; 1, 1) \leq T(G; 0, 2)$.

Strengthening Thomassen's idea that acyclic orientations dominate spanning trees in sparse graphs, I conjecture that the ratio $T(G; 2, 0)/T(G; 1, 1)$ increases as G gets sparser. To support this conjecture, I prove a variant of the conjecture for series-parallel graphs.

The Merino-Welsh conjecture has a stronger version claiming that the Tutte polynomial is convex on the line segment between $(2, 0)$ and $(0, 2)$ for loopless 2-connected graphs. Chávez-Lomelí et al. proved that this holds for coloopless paving matroids, and I provide a shorter proof of their theorem. I also prove it for minimally 2-edge-connected graphs. As a general statement for the convexity of the Tutte polynomials, I show that the Tutte polynomial of a sparse paving matroid is almost surely convex in the first quadrant. In contrast, I conjecture that the Tutte polynomial of a sparse paving matroid with fixed rank is almost never convex in the first quadrant.

The following multiplicative version of the Merino-Welsh conjecture was considered by Noble and Royle:

$$T(G; 1, 1)^2 \leq T(G; 2, 0) T(G; 0, 2).$$

Noble and Royle proved that this multiplicative version holds for series-parallel graphs, using a computer algorithm that they designed. Using a property of the splitting formula which I found, I improve their algorithm so that it is applicable to the class of graphs with bounded treewidth (or pathwidth). As an application, I verify that the multiplicative version holds for graphs with pathwidth at most 3.

Preface

This thesis was prepared at DTU Compute in fulfilment of the requirements for acquiring a Ph.D. in Applied Mathematics.

The thesis deals with a conjecture called the Merino-Welsh conjecture about the Tutte polynomial in graph theory.

The thesis presents new results for three variations of the conjecture and suggests new ways of investigating the Tutte polynomial.

Lyngby, 30-September-2015



Seongmin Ok

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Introduction

This chapter presents the terminologies for graphs and matroids we will use throughout. We follow mostly Diestel [Die10] for graphs and Oxley [Oxl11] for matroids. We list some graph examples at the end of this section.

1.1 Graphs and directed graphs

We list the definitions, especially focusing on introducing the notion of spanning tree, acyclic orientation and totally cyclic orientation.

A **graph** G has a nonempty set of **vertices** $V(G)$, a set of **edges** $E(G)$, and a set of **incidences** $I(G) \subseteq V(G) \times E(G)$. A vertex $v \in V(G)$ is **incident** with an edge $e \in E(G)$ if $(v, e) \in I(G)$. If v is incident with e then v is an **end** of e . An edge has at most two ends. An edge with only one end is called a **loop**. Two edges with same set of ends are **parallel**. If the ends of e are u and v then we write $e = uv$. If a graph G has no parallel pair of edges then we may consider e as a subset of $V(G)$ of size 2 and write $e = \{u, v\}$. A **simple** graph is a graph without parallel

edges and loops. We shall always consider the incidence set $I(G)$ as implicit and we write $G = (V(G), E(G))$.

A **subgraph** of a graph $G = (V, E)$ is a graph $H = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. Possibly $H = G$. The subgraph $G' = (V, E \setminus \{e\})$ for an edge e is denoted by $G - e$ and called the graph obtained from G by **deleting** (or **removing**) e . If $v \in V$ is a vertex of $G = (V, E)$ but not the only vertex, then $G - v$ denotes the subgraph $(V \setminus \{v\}, E'')$ where E'' is the set of edges in E not incident with v . We call $G - v$ also the **deletion** (or **removal**) of v from G . The graph G/e , the **contraction** of a non-loop edge e , is obtained from G by removing e and then identifying the ends of e . If e is a loop, then $G/e = G - e$. For a sequence of edges and vertices, say x_1, x_2, \dots, x_k , we write $G - \{x_1, x_2, \dots, x_k\} = ((G - x_1) - x_2 \cdots) - x_k$. For $F \subseteq E$, we write G/F for the graph obtained from G by a sequence of contractions of edges in F . The operations of deletion and contraction are interchangeable, in the sense that $(G - e)/f = (G/f) - e$ for any pair of edges e and f . A graph obtained from G by a sequence of deletion and contractions is called a **minor** of G .

A subgraph of $G = (V, E)$ with vertex set V is called **spanning**. For $X \subseteq V$, the graph $G - (V \setminus X)$ is the subgraph of G **induced by** X , denoted by $G[X]$. If $Y \subseteq E$, then the subgraph of G **induced by** Y is obtained from $G = (V, Y)$ by removing the vertices without incident edges.

A sequence of edges e_1, e_2, \dots, e_k , $e_i = u_i v_i$ is a **path** if $v_i = u_{i+1}$ for each i and the vertices $u_1, u_2, \dots, u_k, v_k$ are distinct. We say that the path is from u_1 to v_k and write it as $u_1 u_2 \cdots u_k v_k$. A **cycle** is a sequence of edges e_1, e_2, \dots, e_k , $e_i = u_i v_i$ such that e_1, e_2, \dots, e_{k-1} is a path and $v_k = u_1$. We denote it by $u_1 u_2 \cdots u_k u_1$. The **length** of a path or a cycle is the number of edges in it. A loop is considered as a cycle of length 1. If a graph G has a path between each pair of its vertices, then G is **connected**. A graph is **disconnected** if it is not connected. An inclusion-wise maximal connected subgraph is called a **connected component** or simply a **component**. A graph without a cycle is a **forest** and a connected forest is a **tree**. It is easy to see that a connected graph has a spanning tree. An edge e of a graph G is called **bridge** if $G - e$ has more components than G . A graph G is **k -connected** if $|V(G)| \geq k + 1$ and G remains connected after removing k vertices arbitrarily. A graph G is **k -edge-connected** if $|E(G)| \geq k$ and G stays connected after removing k edges arbitrarily.

A **directed graph** (or a **digraph**) $G' = (V, E)$ has a **directed incidence set** $D(G') \subseteq V \times V \times E$ instead of the incidence set. Each edge appears precisely once in $D(G')$. If $(u, v, e) \in D(G')$ and $u \neq v$ then u is the **tail** of e and v is the **head** of e . We write $e = uv$ or $e = (u, v)$. If $(v, v, e) \in D(G')$ then v is both the head and tail of e . We say a digraph G' is an **orientation** of a graph $G = (V, E)$ if for each edge, the two incident vertices (or one if the edge is a loop) in G is the same as the head and tail in G' . In this case, we say G is the **base graph** of G' . A path (a cycle) e_1, e_2, \dots, e_k , $e_i = u_i v_i$ of G is a **directed path** (a **directed cycle**) in G' if $e_i = (u_i, v_i)$ in G' also. An orientation of G is **acyclic** if it has no directed cycle. A **totally cyclic orientation** is an orientation such that each edge is in a directed cycle. It is easy to show that a loopless graph has an acyclic orientation and a bridgeless graph has a totally cyclic orientation.

Now we list some simple graphs that will be used later.

The **complete graph** on n vertices, denoted by K_n , is the graph with n vertices such that each pair of vertices is connected by an edge. The **wheel graph** on n vertices, $n \geq 4$, is obtained from a cycle of length $n - 1$ by adding a new vertex called the **center** adjacent to all vertices in the cycle. The **ladder graph** is obtained from two paths of equal length, say $u_1 u_2 \cdots u_k$ and $v_1 v_2 \cdots v_k$, by adding edges $u_i v_i$ for each i . The **prism graph** is obtained from two cycles of equal length, say $u_1 u_2 \cdots u_k u_1$ and $v_1 v_2 \cdots v_k v_1$, by adding edges $u_i v_i$ for each i .

1.2 Matroids

Let E be a finite nonempty set. A **matroid on E** , say M , is a set system (E, \mathcal{I}) where \mathcal{I} is a nonempty set of subsets of E satisfying the following:

1. If $I \subset J$ and $J \in \mathcal{I}$ then $I \in \mathcal{I}$.
2. If $I, J \in \mathcal{I}$ and $|I| < |J|$ then there exists a $x \in J \setminus I$ such that $I \cup \{x\} \in \mathcal{I}$.

The set E is called the **ground set** of M . The subsets of E in \mathcal{I} are called **independent** in M . A set is **dependent** if it is not independent. An inclusion-wise maximal independent set is a **basis** of M . By (1), the set of bases completely define a matroid. By (2), all bases have the same size, which is the **rank** of M .

and denoted by $r(M)$. The **rank function** r_M is defined by $r_M(A) = \max_{J \in \mathcal{I}, J \subseteq A} |J|$ for each $A \subseteq E$. It is easy to see that $0 \leq r_M(A) \leq |A|$ for each $A \subseteq E$ and $|A| = r_M(A)$ if and only if A is independent. Hence the rank function completely defines a matroid. We often write simply $r(A)$ instead of $r_M(A)$ when we consider a fixed matroid M . In Chapter 7, we write a matroid M as $M = (E, r)$ where E is the ground set and r is the rank function. A **loop** is an element not in any independent set. A **coloop** is an element contained in every basis.

For two matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ with $E_1 \cap E_2 = \emptyset$, the **direct sum** of M_1 and M_2 , denoted by $M_1 \oplus M_2$, is the matroid $(E_1 \cup E_2, \mathcal{I})$ where $\mathcal{I} = \{I \subseteq E_1 \cup E_2 : I = I_1 \cup I_2 \text{ for some } I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$.

A **circuit** is an inclusion-wise minimal dependent set. Thus a set is dependent if and only if it contains a circuit. It follows that the set of circuits also completely defines a matroid. A matroid is **paving** if every circuit has size at least the rank of the matroid.

The **restriction** of a matroid $M = (E, \mathcal{I})$ to a set $S \subseteq E$ is the matroid on S , denoted by $M - (E \setminus S)$, whose rank function is the same as r_M . In other words, all the independent sets of M contained in S are precisely the independent sets of $M - (E \setminus S)$. If $E \setminus S = \{e\}$ is a singleton then we write $M - e$. The **dual matroid**, namely M^* , of a matroid M with rank function r is defined by using the following function r^* as its rank function:

$$r^*(A) = r(E \setminus A) + |A| - r(M).$$

The bases of M^* are precisely the complements of the bases of M . The matroid $M/e = (M^* - e)^*$ is called the **contraction** of e from M . A matroid obtained from another matroid M by a series of deletions and contractions is a **minor** of M . A matroid is **sparse paving** if itself and its dual are both paving.

Matroids can be thought of as a generalization of graphs. Given a graph $G = (V, E)$, the set system (E, \mathcal{I}) where $X \subseteq E$ is in \mathcal{I} if and only if X has no cycle of G becomes a matroid, called the **cycle matroid** of G . If M is the cycle matroid of a graph G and e is an edge of G , then $M - e$ and M/e are respectively the cycle matroids of $G - e$ and G/e .

Matroids will be considered substantially in Chapters 6 and 7.

The Tutte Polynomial

2.1 Introduction

In this chapter we shall introduce the definition of the Tutte polynomial and the multivariate Tutte polynomial, together with their motivations and applications. In Section 2.2 we show how to find the Tutte polynomial of the edge-disjoint union of two graphs from certain information about the two edge-disjoint subgraphs separately. The method was found by Negami [Neg87] and is named the **splitting formula**. We shall use an approach different from [Neg87], starting with the multivariate Tutte polynomial. The splitting formula for the multivariate Tutte polynomial (Theorem 2.2.3) has not been mentioned explicitly in the literature as far as I know. After that, we also prove a variant of the splitting formula which shall be useful in chapter 5.

The Tutte polynomial, defined by Tutte [Tut54], has become one of the most popular polynomial invariant of graphs. It has two variables and we shall use the notation $T(G; x, y)$ for the Tutte polynomial of a graph G with variables x, y . The Tutte polynomial has several equivalent definitions and we give two of them which are appropriate for our purpose. The first definition uses the **rank** of an edge set

$A \subseteq E(G)$, denoted by $r(A)$, which is the number of edges in a maximal forest contained in A .

$$T(G; x, y) = \sum_{A \subseteq E(G)} (x-1)^{r(E(G))-r(A)} (y-1)^{|A|-r(A)}. \quad (2.1)$$

Equation (2.1) can be immediately used to define the Tutte polynomial of a matroid, using the matroid rank function. The function $r(A)$ used above coincides with the rank function of the cycle matroid of G .

The Tutte polynomial can also be defined recursively, as follows. Let $T(\bar{K}_n; x, y) = 1$ when \bar{K}_n is the graph on n vertices with no edges. The above definition can be used to show that the Tutte polynomial satisfy the following.

1. $T(G; x, y) = xT(G - e; x, y)$ if e is a bridge.
2. $T(G; x, y) = yT(G - e; x, y)$ if e is a loop.
3. $T(G; x, y) = T(G - e; x, y) + T(G/e; x, y)$ if e is neither a bridge nor a loop.

On the other hand, it can also be shown that the Tutte polynomial is well-defined using (1), (2) and (3) starting from $T(\bar{K}_n; x, y) = 1$. The equation (3) is also called the **deletion-contraction formula**.

The invention of the Tutte polynomial was motivated by the chromatic polynomial [Tut54]. The **chromatic polynomial**, denoted by $P(G; \lambda)$, is a polynomial whose values at positive integer k is the number of vertex- k -colorings of the graph G . The chromatic polynomial satisfies the following for every edge e .

$$P(G; \lambda) = P(G - e; \lambda) - P(G/e; \lambda). \quad (2.2)$$

Equation (2.2) can be thought of as a deletion-contraction formula since

$$(-1)^n P(G; \lambda) = (-1)^n P(G - e; \lambda) + (-1)^{n-1} P(G/e; \lambda)$$

where n is the number of vertices of G . Tutte considered also the so-called **flow polynomial**, which is dual to the chromatic polynomial, and generalized both concepts into the Tutte polynomial. The chromatic polynomial is a specialization of the Tutte polynomial in the following sense:

$$P(G; \lambda) = (-1)^{n-\gamma(G)} \lambda^{\gamma(G)} T(G; 1 - \lambda, 0)$$

where $n = |V(G)|$ and $\gamma(G)$ is the number of connected components of G . It is easy to prove this by induction on the number of edges using (2.2). Similarly, it can be shown that the flow polynomial (which we do not consider in the present work) is a specialization of the Tutte polynomial.

In fact, any invariant satisfying the deletion-contraction formula is a specialization of the Tutte polynomial as shown by the following theorem, proved by Oxley and Welsh which they called the **recipe theorem**.

Theorem 2.1.1 (the recipe theorem [Wel99]). *Let \mathcal{C} be a class of matroids which is closed under direct sums and the taking minors, and suppose that F is a function defined on \mathcal{C} and satisfies*

$$F(M) = aF(M - e) + bF(M/e) \quad \text{for } e \in M, \quad \text{and}$$

$$F(M_1 \oplus M_2) = F(M_1)F(M_2).$$

where $M_1 \oplus M_2$ is the direct sum of two matroids M_1 and M_2 . Then F is given by

$$F(M) = a^{|E|-r(E)} b^{r(E)} T(M; \frac{x_0}{b}, \frac{y_0}{a}),$$

where x_0 and y_0 are the values of F on a coloop and a loop, respectively.

Some of the graph invariants satisfying the deletion-contraction formula are listed below.

- the number of spanning trees, $t(G) = T(G; 1, 1)$
- the number of acyclic orientations, $a(G) = T(G; 2, 0)$
- the number of totally cyclic orientations, $c(G) = T(G; 0, 2)$

There are also many polynomial invariants of graphs which are specializations of the Tutte polynomial. We saw that the Tutte polynomial, when restricted to the x -axis, specializes to the chromatic polynomial. Some other examples are listed below, with the corresponding curve on which the Tutte polynomial specializes.

- the flow polynomial : the y -axis
- the all-terminal reliability polynomial : the line $x = 1$
- the Jones polynomial : the curve $xy = 1$
- the q -state Potts model partition function : the curve $(x - 1)(y - 1) = q$

For more thorough theory of the Tutte polynomial and its applications, see [BO92].

As the Tutte polynomial contains enormous information about graphs, there have been various attempts to generalize the Tutte polynomial. I shall consider the **multivariate Tutte polynomial** in particular.

The multivariate Tutte polynomial of a graph G is the polynomial

$$Z(G; q, \mathbf{v}) = \sum_{A \subseteq E(G)} q^{\gamma(A)} \prod_{e \in A} v_e \quad (2.3)$$

where q and $\mathbf{v} = \{v_e\}_{e \in E(G)}$ are commuting variables, and $\gamma(A)$ is the number of connected components of the graph $(V(G), A)$. In the late 1960s, Fortuin and Kasteleyn introduced (2.3) as an equivalent form of the q -state Potts model partition function which plays an important role in statistical mechanics. For an excellent presentation of the history of the multivariate Tutte polynomial and its connection between graph theory and statistical mechanics, see [Sok05].

It is easy to see from (2.3) that the following generalized version of the deletion-contraction formula holds for $Z(G; q, \mathbf{v})$, where the vector \mathbf{v}' is \mathbf{v} with v_e omitted.

$$Z(G; q, \mathbf{v}) = v_e Z(G/e; q, \mathbf{v}') + Z(G - e; q, \mathbf{v}')$$

In particular, if we set $v_e = w$ for a constant w for all edges e , then the two-variable polynomial $Z(G; q, w)$ satisfies

$$Z(G; q, w) = wZ(G/e; q, w) + Z(G - e; q, w)$$

and by the recipe theorem (Theorem 2.1.1), we get

$$T(G; x, y) = (x - 1)^{-\gamma(G)} (y - 1)^{-|V(G)|} Z(G; (x - 1)(y - 1), y - 1),$$

where $q = (x - 1)(y - 1)$ and $v_e = y - 1$ for all e . Therefore, the multivariate Tutte polynomial is indeed a generalization of the Tutte polynomial.

The indeterminates $\mathbf{v} = \{v_e\}_{e \in E(G)}$ can be thought of an edge-weight vector. One advantage of the multivariate Tutte polynomial is that, as explained in Lemma 2.2.4 below, we can replace a component of a 2-cut by a single edge with an appropriate edge-weight. In the following, we use the notation Lemma (2.2.4) rather than Lemma 2.2.4 to indicate that the result itself is formulated later but stated here for better readability.

Lemma (2.2.4). *Let G be a graph with two vertices, say x, y , such that $G' = G - \{x, y\}$ is not connected. Let H be a component of G' . Then for each q and $\mathbf{v} = \{v_e\}_{e \in E(G)}$, there are w and σ , described roughly below, such that*

$$Z(G; q, \mathbf{v}) = \sigma Z(G - H + xy; q, \mathbf{v}')$$

where \mathbf{v}' is the edge-weight vector on $G - H + xy$ which coincides with \mathbf{v} on $G - H$, and $v'_{xy} = w$. The edge-weight w and the factor σ are defined in terms of q and the edge-weights in the subgraph of G obtained from H by adding x and y with their edges to H .

Lemma 2.2.4 has been a key tool in analyzing the computational complexity of the Tutte polynomial. In a recent paper [GJ12], Goldberg and Jerrum used Lemma 2.2.4 extensively to classify the computational complexity of determining the sign of the Tutte polynomial at almost every point in the real plane. In the paper [JVV90] showing that the Tutte polynomial evaluation is computationally hard, the authors' idea was the same as Lemma 2.2.4 but executed on the Tutte polynomial without using the multivariate Tutte polynomial. Jackson and Sokal [JS09] investigated the zeroes of the Tutte polynomial using the so-called serial and parallel reduction, and again the idea of replacing a 2-cut component by an edge is proven to be useful.

2.2 Splitting formula

The main idea of this chapter is to express the Tutte polynomial of a graph from some Tutte polynomials of 'fragments' which are defined below.

Let us consider a graph G being expressed as the union of two edge-disjoint subgraphs, say H and K . When $|V(H) \cap V(K)| = k$ and the vertices in $V(H) \cap V(K)$ are distinctively labelled, we call H and K k -fragments. In other words, a **k -fragment** is a graph with k labelled vertices. Unless otherwise stated we shall use $[k] = \{1, 2, \dots, k\}$ for the set of labels. Given two k -fragments, say H and K , with the same set of labels, we define the graph $H \oplus K$ as the union of H and K where the vertices of the same labels are identified. The graph $H \oplus K$ can also be considered as a k -fragment using the labels inherited from H and K . Note that according to the above formulation, we have $G = H \cup K = H \oplus K$. But since we

are going to often ‘replace’ K with another k -fragment, we shall use the operation \oplus instead of the union \cup .

S. Negami was the first to prove that when $G = H \oplus K$, the Tutte polynomial $T(G)$ can be calculated using the information of the k -fragments H and K separately. The information we need is of course the Tutte polynomials, but not only that of H and K but also the Tutte polynomials of the graphs obtained from H and K by identifying some of the labelled vertices. To express formally, let us discuss beforehand the concept of **partition lattice**.

Let $[k] = \{1, 2, \dots, k\}$. A **partition** of $[k]$ is a set of pairwise disjoint nonempty subsets of $[k]$ whose union is $[k]$. Each set in a partition is called a **block**. The number of blocks of a partition P is denoted by $|P|$. We shall write $\Gamma(k)$ to denote the set of all partitions of $[k]$. A common way to give a partial order to $\Gamma(k)$ is to use the refinement order. A partition P is a **refinement** of another partition P' , denoted by $P \geq_R P'$, if for each block B of P , the partition P' has a block B' such that $B \subseteq B'$. The order \geq_R is called the **refinement order**.

A partially ordered set (S, \geq) is a **lattice** if for each pair of elements $u, v \in S$, there are the greatest lower bound (g.l.b.) $u \wedge v$ and the least upper bound (l.u.b.) $u \vee v$, which are defined as the following:

1. an element $s \in S$ is the g.l.b. of u and v if $u \geq s, v \geq s$ and for each element s' such that $u \geq s'$ and $v \geq s'$, it holds that $s \geq s'$.
2. an element $t \in S$ is the l.u.b. of u and v if $u \leq t, v \leq t$ and for each element t' such that $u \leq t'$ and $v \leq t'$, it holds that $t \leq t'$.

Let P_1, P_2 be two partitions of $[k]$. It is easy to check that the following two partitions are indeed $P_1 \vee P_2$ and $P_1 \wedge P_2$ in $(\Gamma(k), \geq_R)$.

- $P_1 \vee P_2 = \{B_1 \cap B_2 : B_1 \in P_1, B_2 \in P_2 \text{ and } B_1 \cap B_2 \neq \emptyset\}$,
- two integers $n, m \in [k]$ are in the same block of $P_1 \wedge P_2$ if and only if there is a sequence n_1, n_2, \dots, n_s such that $n_1 = n, n_s = m$ and for each i , the pair $\{n_i, n_{i+1}\}$ is in a block of P_1 or P_2 .

Thus $(\Gamma(k), \geq_R)$ is a lattice, and we call $\Gamma(k)$ the **partition lattice**.

A k -fragment H naturally induces a partition on its labelled vertices, or on the set of used labels, such that vertices in the same component of H form a block. We shall write $H \vdash P$ when $P \in \Gamma(k)$ and H is a k -fragment with labels from $[k]$ such that the vertices with labels i, j are in the same component of H if and only if i, j are in the same block of P .

Now we illustrate the needed information of H to find $T(H \oplus K)$. Let H be a k -fragment and $P \in \Gamma(k)$. For each block B of P , we identify the vertices of H with labels in B into a single vertex. Let us denote the resulting graph by H_P . To find the Tutte polynomial of $H \oplus K$ where K is another k -fragment, the information we need from H is the **Tutte vector** $T_v(H)$, indexed by $\Gamma(k)$, and whose entries are $T(H_P)$ for $P \in \Gamma(k)$. We shall use $\Gamma(k)$ as the index set of vectors and matrices in the rest of this chapter. Also, the vectors are assumed to be column vectors unless otherwise stated.

The following theorem is from Negami [Neg87]. A precise description of the matrix N shall be given in Theorem 2.2.5.

Theorem 2.2.1 (Negami's splitting formula). *Let $k > 0$ and H, K be two k -fragments. Then*

$$T(H \oplus K) = T_v(H)^t N T_v(K)$$

where N is a $\Gamma(k) \times \Gamma(k)$ -matrix whose entries are fractional functions of two variables x, y . The entry corresponding to $(P, P') \in \Gamma(k) \times \Gamma(k)$ depends on the number of connected components of H_P and $K_{P'}$.

Negami's proof of Theorem 2.2.1 started with introducing his own polynomial invariant of graphs, denoted by $f(G; t, x, y)$. It has three variables, t, x and y , and is defined recursively via the following rules.

1. $f(\bar{K}_n; t, x, y) = t^n$,
2. $f(G; t, x, y) = xf(G/e; t, x, y) + yf(G - e; t, x, y)$ for each edge e .

Note that unlike the standard Tutte polynomial, the second rule does not depend on whether e is a bridge or a loop.

The following is from [Neg87].

Lemma 2.2.2. *The polynomial $f(G)$ is well-defined for each graph G .*

Proof. We shall use induction on $|E(G)|$. Suppose that the statement is true for $|E(G)| < m$. Let G be a graph with m edges. We may assume that $m \geq 2$, and let e, g be two distinct edges of G . We shall compare the deletion-contraction of e after g with that of g after e .

$$\begin{aligned} & xf(G/e) + yf(G - e) \\ &= x(xf(G/e/g) + yf(G/e - g)) + y(xf(G - e/g) + yf(G - e - g)), \text{ and} \\ & xf(G/g) + yf(G - g) \\ &= x(xf(G/g/e) + yf(G/g - e)) + y(xf(G - g/e) + yf(G - g - e)). \end{aligned}$$

Since the order of deletion and contraction of two edges is interchangeable, $f(G)$ is well-defined by induction. \square

Negami first found a splitting formula for his polynomial $f(G)$, and then obtained the splitting formula for the Tutte polynomial using the following equation:

$$(x - 1)^{\gamma(G)}(y - 1)^{|V(G)|}T(G; x, y) = f(G; (x - 1)(y - 1), y - 1, 1),$$

which is not hard to prove by either induction on the number of edges or the recipe theorem (Theorem 2.1.1).

We shall take a different approach which starts from the multivariate Tutte polynomial. The splitting formula that I provide here has not appeared in the literature, and it extends Negami's splitting formula for the Tutte polynomial, Theorem 2.2.1. It should be noted that if we assign the same edge weight w to all edges of a graph G , then the multivariate polynomial satisfies

1. $Z(\bar{K}_n; q, w) = q^n$ and
2. $Z(G; q, w) = wZ(G/e; q, w) + Z(G - e; q, w)$

which implies $Z(G; q, w) = f(G; q, w, 1)$. Thus, we may expect that the splitting formula for the multivariate Tutte polynomial would be similar to that of Negami's polynomial. We shall use the **multivariate Tutte vector** $Z_v(H) = (Z(H_P))_{P \in \Gamma(k)}$, an analogue of the Tutte vector $T_v(H)$.

Theorem 2.2.3. *Let H, K be two k -fragments. Then*

$$Z(H \oplus K) = Z_v(H)^t \hat{N} Z_v(K),$$

where \hat{N}^{-1} is the $\Gamma(k) \times \Gamma(k)$ -matrix whose (P_i, P_j) -entry is $q^{|P_i \wedge P_j|}$.

We shall point out that if we replace $Z(H)$ with $f(H)$ and replace q with t in Theorem 2.2.3, then it is exactly Negami's splitting formula for his polynomial $f(H)$ in [Neg87].

For brevity, a $\Gamma(k) \times \Gamma(k)$ -matrix whose (P_i, P_j) -entry is $A(P_i, P_j)$ shall be written as $(A(P_i, P_j))_{P_i, P_j \in \Gamma(k)}$. For example, $\hat{N}^{-1} = (q^{|P_i \wedge P_j|})_{P_i, P_j \in \Gamma(k)}$.

Proof of Theorem 2.2.3. Recall that a k -fragment H induces a partition $P \in \Gamma(k)$, denoted by $H \vdash P$, if two labelled vertices of H are in the same connected component precisely when they are in the same block of P . Let $Z_P(G)$ be the partial summation of $Z(G)$ defined by

$$Z_P(G) = \sum_{S \subseteq E(G), S \vdash P} q^{\gamma(S)} \mathbf{v}^S.$$

Thus, $Z(G) = \sum_{P \in \Gamma(k)} Z_P(G)$.

We start by finding a splitting formula in terms of the vector $Z'_v(G) = [Z_P(G)]_{P \in \Gamma(k)}$ instead of $Z_v(G) = [Z(G_P)]_{P \in \Gamma(k)}$. The vector $Z'_v(G)$ provides a simpler proof for the splitting formula. However, the individual entries of $Z_v(G)$ are easier to calculate and hence the statement is expressed using $Z_v(G)$. We shall see later that there is an invertible matrix M such that $Z_v(G) = M Z'_v(G)$.

Firstly, we shall find a matrix \hat{N}' such that

$$Z(H \oplus K) = Z'_v(H)^t \hat{N}' Z'_v(K)$$

for all k -fragments H and K .

Let us divide the summation

$$Z(H \oplus K) = \sum_{S \subseteq E(H), T \subseteq E(K)} q^{\gamma(S \cup T)} \mathbf{w}^S \mathbf{w}^T$$

up to the partitions S and T induce. That is,

$$Z(H \oplus K) = \sum_{P_1, P_2 \in \Gamma(k)} \left[\sum_{\substack{S \subseteq E(H) \\ \bar{S} \vdash P_1}} \sum_{\substack{T \subseteq E(K) \\ \bar{T} \vdash P_2}} q^{\gamma(S \cup T)} \mathbf{w}^S \mathbf{w}^T \right] \quad (2.4)$$

Now we compare $\gamma_{H \oplus K}(S \cup T)$ with $\gamma_H(S) + \gamma_K(T)$. Let $S \subseteq E(H)$ and $T \subseteq E(K)$ such that $S \vdash P_1$ and $T \vdash P_2$ for some $P_1, P_2 \in \Gamma(k)$. We may consider S and T as k -fragments themselves. When we compare the number of connected components of the disjoint union of S and T with that of $S \oplus T$, the only difference is between the components of S and T containing the labelled vertices. The fragments S and T have respectively $|P_1|$ and $|P_2|$ such components. In $S \oplus T$, the condition that vertices with labels i, j are in the same component is precisely the condition that i, j are in the same block of $P_1 \wedge P_2$, so that $|P_1 \wedge P_2|$ components of $S \oplus T$ contain the labelled vertices. Therefore,

$$\gamma(S) + \gamma(T) - \gamma(S \oplus T) = |P_1| + |P_2| - |P_1 \wedge P_2|,$$

and

$$\gamma_{H \oplus K}(S \cup T) = \gamma(S \oplus T) = \gamma(S) + \gamma(T) - |P_1| - |P_2| + |P_1 \wedge P_2|.$$

Putting it to Equation (2.4), we get

$$\begin{aligned} Z(H \oplus K) &= \sum_{P_1, P_2 \in \Gamma(k)} \left[\sum_{\substack{S \subseteq E(H) \\ \bar{S} \vdash P_1}} \sum_{\substack{T \subseteq E(K) \\ \bar{T} \vdash P_2}} q^{\gamma(S \cup T)} \mathbf{w}^S \mathbf{w}^T \right] \\ &= \sum_{P_1, P_2 \in \Gamma(k)} \left[\sum_{\substack{S \subseteq E(H) \\ \bar{S} \vdash P_1}} \sum_{\substack{T \subseteq E(K) \\ \bar{T} \vdash P_2}} q^{\gamma(S) + \gamma(T) - |P_1| - |P_2| + |P_1 \wedge P_2|} \mathbf{w}^S \mathbf{w}^T \right] \\ &= \sum_{P_1, P_2 \in \Gamma(k)} \left[\sum_{\substack{S \subseteq E(H) \\ \bar{S} \vdash P_1}} q^{\gamma(S)} \mathbf{w}^S \sum_{\substack{T \subseteq E(K) \\ \bar{T} \vdash P_2}} q^{\gamma(T)} \mathbf{w}^T \right] q^{|P_1 \wedge P_2| - |P_1| - |P_2|} \\ &= \sum_{P_1, P_2 \in \Gamma(k)} Z_{P_1}(H) Z_{P_2}(K) q^{|P_1 \wedge P_2| - |P_1| - |P_2|} \end{aligned}$$

Thus, we have

$$Z(H \oplus K) = Z'_v(H)^t \hat{N}' Z'_v(K) \quad (2.5)$$

where $\hat{N}' = (q^{|P_i \wedge P_j| - |P_i| - |P_j|})_{P_i, P_j \in \Gamma(k)}$.

Now we find a matrix M such that $Z_v(H) = M Z'_v(H)$. Let us fix $P_i \in \Gamma(k)$. For ease of writing, we shall write $H_i = H_{P_i}$ within the following formula and also $H_j = H_{P_j}$ where P_j is an auxiliary partition in $\Gamma(k)$. Since H_i has the same set of edges as H ,

$$\begin{aligned} Z(H_i) &= \sum_{S \subseteq E(H_i)} q^{\gamma_{H_i}(S)} \mathbf{w}^S \\ &= \sum_{\substack{S \subseteq E(H) \\ \bar{S} \vdash P_j}} q^{\gamma_{H_i}(S)} \mathbf{w}^S \\ &= \sum_{P_j \in \Gamma(k)} \sum_{\substack{S \subseteq E(H) \\ \bar{S} \vdash P_j}} q^{\gamma_{H_i}(S)} \mathbf{w}^S \end{aligned} \quad (2.6)$$

Let us compare $\gamma_{H_i}(S)$ with $\gamma_H(S)$ when $S \subseteq E(H)$ and $\bar{S} \vdash P_j$. The graph H_i is obtained from H by identifying some of the labelled vertices, so the difference between $\gamma_{H_i}(S)$ and $\gamma_H(S)$ is from only the components containing at least one of the labelled vertices. The graph $(V(H), S)$ has $|P_j|$ such components, whereas in $(V(H_i), S)$, the vertices with labels s, t lie in the same component if and only if s, t are in the same block of $P_i \wedge P_j$, analogous to the previous situation. Thus,

$$\gamma_H(S) - \gamma_{H_i}(S) = |P_j| - |P_i \wedge P_j|,$$

and

$$\gamma_{H_i}(S) = \gamma_H(S) - |P_j| + |P_i \wedge P_j|.$$

Putting it into Equation (2.6), we get

$$\begin{aligned} Z(H_i) &= \sum_{P_j \in \Gamma(k)} \sum_{\substack{S \subseteq E(H) \\ \bar{S} \vdash P_j}} q^{\gamma_{H_i}(S)} \mathbf{w}^S \\ &= \sum_{P_j \in \Gamma(k)} \sum_{\substack{S \subseteq E(H) \\ \bar{S} \vdash P_j}} q^{\gamma_H(S) - |P_j| + |P_i \wedge P_j|} \mathbf{w}^S \\ &= \sum_{P_j \in \Gamma(k)} \left[\sum_{\substack{S \subseteq E(H) \\ \bar{S} \vdash P_j}} q^{\gamma_H(S)} \mathbf{w}^S \right] q^{|P_i \wedge P_j| - |P_j|} \\ &= \sum_{P_j \in \Gamma(k)} Z_{P_j}(H) q^{|P_i \wedge P_j| - |P_j|} \\ &= M'_j \cdot Z'_v(H) \end{aligned}$$

where M'_i is the vector indexed by $\Gamma(k)$, whose P_j -entry is $q^{|P_i \wedge P_j| - |P_j|}$.

Thus the multivariate Tutte vector $Z_v(H) = (Z(H_P))_{P \in \Gamma(k)}$ can be obtained from $Z'_v(H) = (Z_P(H))_{P \in \Gamma(k)}$ in the following way:

$$Z_v(H) = M Z'_v(H),$$

where $M = (q^{|P_i \wedge P_j| - |P_j|})_{P_i, P_j \in \Gamma(k)}$. The matrix M is invertible if and only if the matrix $(q^{|P_i \wedge P_j|})_{P_i, P_j \in \Gamma(k)}$ is invertible. The latter is invertible because when we consider the determinant, the highest degree of q appears only at the main diagonal.

Thus M^{-1} exists, and $Z'_v(H) = M^{-1} Z_v(H)$. Putting it into Equation (2.5), we get

$$Z(H \oplus K) = Z_v(H)^t (M^{-1})^t \hat{N}' M^{-1} Z_v(K), \quad (2.7)$$

where $\hat{N}' = (q^{|P_i \wedge P_j| - |P_i| - |P_j|})_{P_i, P_j \in \Gamma(k)}$.

Since \hat{N}' is obtained from M by multiplying P_i -row with $q^{-|P_i|}$, we have $\hat{N}' M^{-1} = (q^{-|P_i|} \delta(P_i, P_j))_{P_i, P_j \in \Gamma(k)}$ where $\delta(P_i, P_j) = 1$ if $P_i = P_j$ and 0 otherwise.

On the other hand, $(M^{-1})^t = (M^t)^{-1} = (q^{|P_i \wedge P_j| - |P_i|})_{P_i, P_j \in \Gamma(k)}^{-1}$. In the following equations, we shall omit the subscript $P_i, P_j \in \Gamma(k)$ but still the terms refer to the matrices. Since

$$(q^{|P_i \wedge P_j| - |P_i|}) (q^{|P_i \wedge P_j|})^{-1} = (q^{-|P_i|} \delta(P_i, P_j)),$$

we have

$$(q^{|P_i \wedge P_j| - |P_i|})^{-1} = (q^{|P_i \wedge P_j|})^{-1} (q^{|P_i|} \delta(P_i, P_j)),$$

and

$$(M^{-1})^t \hat{N}' M^{-1} = (q^{|P_i \wedge P_j|})^{-1} (q^{|P_i|} \delta(P_i, P_j)) (q^{-|P_i|} \delta(P_i, P_j)) = (q^{|P_i \wedge P_j|})^{-1}.$$

Therefore, by Equation (2.7),

$$Z(H \oplus K) = Z_v(H)^t \hat{N} Z_v(K),$$

where $\hat{N} = (q^{|P_i \wedge P_j|})_{P_i, P_j \in \Gamma(k)}^{-1}$. □

Using Theorem 2.2.3, it is now easy to show the following useful property of the multivariate Tutte polynomial which is mentioned in Section 2.1.

Lemma 2.2.4. *Let G be a graph with two vertices, say x, y , such that $G' = G - \{x, y\}$ is not connected. Let H be a component of G' . Then for each q and $\mathbf{v} = \{v_e\}_{e \in E(G)}$, there are w and σ defined by q and the weights of the edges not in $G - H$ such that*

$$Z(G; q, \mathbf{v}) = \sigma Z(G - H + xy; q, \mathbf{v}')$$

where \mathbf{v}' is the edge-weight vector on $G - H + xy$ which coincides with \mathbf{v} on $G - H$, and $v'_{xy} = w$.

Proof. Let us consider $H' = G - H$ as a 2-fragment with x, y being the labelled vertices, and let K be the 2-fragment such that $G = H' \oplus K$. Then by Theorem 2.2.3,

$$Z(G) = Z_v(H')^t \hat{N} Z_v(K).$$

On the other hand, the multivariate Tutte polynomial of $G - H + xy$ is

$$Z(G - H + xy) = Z_v(H')^t \hat{N} Z_v(K_2)$$

where K_2 is a single edge with weight w . Let

$$Z_v(K) = \begin{bmatrix} F_1(q, \mathbf{v}|_{\mathbf{K}}) \\ F_2(q, \mathbf{v}|_{\mathbf{K}}) \end{bmatrix}.$$

where $\mathbf{v}|_{\mathbf{K}}$ is the restriction of \mathbf{v} to K . Since

$$Z_v(K_2) = \begin{bmatrix} q(w + q) \\ q(w + 1) \end{bmatrix},$$

by solving the linear equation

$$\frac{q(w + q)}{q(w + 1)} = \frac{F_1(q, \mathbf{v}|_{\mathbf{K}})}{F_2(q, \mathbf{v}|_{\mathbf{K}})}$$

with respect to w and setting $\sigma = \frac{F_1(q, \mathbf{v}|_{\mathbf{K}})}{q(w + q)}$, we get

$$Z(G; q, \mathbf{v}) = \sigma Z(G - H + xy; q, \mathbf{v}').$$

□

Combining Theorem 2.2.3 with the following expression for the Tutte polynomial

$$T(G; x, y) = (x - 1)^{-\gamma(G)} (y - 1)^{-|V(G)|} Z(G; (x - 1)(y - 1), y - 1),$$

we get the splitting formula for the standard Tutte polynomial.

$$\frac{1}{q(q-1)(q-2)} \begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ -1 & q-1 & 1 & 1 & -q \\ -1 & 1 & q-1 & 1 & -q \\ -1 & 1 & 1 & q-1 & -q \\ 2 & -q & -q & -q & q^2 \end{pmatrix}$$

Figure 2.1: The matrix \hat{N} for the multivariate splitting formula when $k = 3$.

$$\frac{1}{(q-1)(q-2)} \begin{pmatrix} (y-1)^2 & 1-y & 1-y & 1-y & 2 \\ 1-y & q-1 & 1 & 1 & 1-x \\ 1-y & 1 & q-1 & 1 & 1-x \\ 1-y & 1 & 1 & q-1 & 1-x \\ 2 & 1-x & 1-x & 1-x & (x-1)^2 \end{pmatrix}$$

Figure 2.2: The matrix N for the splitting formula when $k = 3$ and H, K are both connected. $q = (x-1)(y-1)$. See Theorem 2.2.5.

Theorem 2.2.5 (Negami's splitting formula). *Let H, K be two k -fragments. Then*

$$T(H \oplus K) = T_v(H)^t N T_v(K),$$

where $N = \left((x-1)^{\gamma(H_{P_i}) + \gamma(K_{P_j}) - \gamma(H \oplus K)} (y-1)^{|P_i| + |P_j| - |V(H \oplus K)|} \hat{N}_{ij} \right)_{P_i, P_j \in \Gamma(k)}$
and \hat{N}_{ij} is the (P_i, P_j) -entry of $(q^{|P_i \wedge P_j|})_{P_i, P_j \in \Gamma(k)}^{-1}$.

To illustrate, the matrix \hat{N} in Theorem 2.2.3 is shown in Figure 2.1 and the matrix N in Figure 2.2 when $k = 3$. The first row and column correspond to the partition with 3 blocks. The last row and column correspond to the partition with a single block.

Theorem 2.2.5 immediately suggests the existence of the vector $T_f(K)$ for each k -fragment K such that

$$T(H \oplus K) = T_v(H)^t T_f(K)$$

for every connected k -fragment H , where $T_f(K)$ is of dimension $\Gamma(k)$ and its entries are rational functions of x and y . Here we prove a uniqueness property of $T_f(K)$ and prove also that each entry of $T_f(K)$ is a two-variable polynomial with positive integer coefficients. The latter property will be used in Chapter 5.

Theorem 2.2.6. *Let K be a k -fragment. There is a unique vector $T_f(K)$ such that for each connected k -fragment H , the following holds:*

$$T(H \oplus K) = T_v(H)^t T_f(K). \quad (2.8)$$

Moreover, each entry of $T_f(K)$ is a polynomial in x, y with positive integer coefficients.

The existence of a vector $T_f(K)$ satisfying (2.8) may seem to be a natural consequence of Theorem 2.2.1, as we can set $T_f(K) = N T_v(K)$. However, the matrix $N = N(H, K)$ varies depending on the number of connected components of both H and K , so that the vector $N T_v(K)$ may change according to whether H is connected or not. The need for the assumption that H is connected in Theorem 2.2.6 is explained below with concrete examples.

Let K be the 2-fragment with two vertices and a single edge. If H is a connected 2-fragment, then $H \oplus K$ is a graph obtained from H by adding an edge, say e , which is neither a bridge nor a loop. Thus $T(H \oplus K) = T(H \oplus K - e) + T(H \oplus K/e) = T(H_{P_1}) + T(H_{P_2})$, where P_1 has two blocks and P_2 has the single block $\{1, 2\}$. That is, because of the uniqueness stated in Theorem 2.2.6, $T_f(K) = [1, 1]^t$. But if we choose H to be a graph which is not connected, for example a graph without any edge, then $H \oplus K$ has a single nonloop edge and $T(H \oplus K) = x$, whereas $T(H_{P_1}) = T(H_{P_2}) = 1$ and $T(H \oplus K) \neq T(H_{P_1}) + T(H_{P_2})$. Therefore, we need to assume H to be connected in Theorem 2.2.6.

The uniqueness of $T_f(K)$ is also interesting for the following reason. In Section 3.1, we shall consider the splitting formula for the number of spanning trees, which is essentially evaluating Negami's splitting formula (Theorem 2.2.5) at the point $(1, 1)$. But in that case, we will see that there are infinitely many matrices $N(1, 1)$ satisfying

$$t(H \oplus K) = t_v(H)^t N(1, 1) t_v(K)$$

for all k -fragments H and K , when $k \geq 4$. Also, we completely characterize these matrices $N(1, 1)$ in terms of so-called **generalized inverse matrix**.

Now we prove Theorem 2.2.6.

Proof of Theorem 2.2.6. The existence of $T_f(K)$ satisfying (2.8) follows from Theorem 2.2.1. We start by proving that $T_f(K)$ is unique. I shall use (2.8) with various trees for H , and then express $T_f(K)$ in terms of these trees and K .

Let $P_i \in \Gamma(k)$. We shall construct a graph G^{P_i} from k independent vertices, labelled by $1, 2, \dots, k$. For each block of P_i , say B , we introduce a new vertex v_B which has an edge to each element of B . After that, we add a new vertex u_{P_i} which has an edge to v_B for each block B of P_i . We call this tree as G^{P_i} . For another partition $P_j \in \Gamma(k)$, the graph $G_{P_j}^{P_i}$ obtained from G^{P_i} by identifying the labelled vertices using P_j , can be constructed also in the following way.

We have a vertex u_{P_i} , together with vertices v_B for each block B of P_i and vertices $w_{B'}$ for each block B' of P_j . The sets $\{v_B : B \text{ a block of } P_i\}$ and $\{w_{B'} : B' \text{ a block of } P_j\}$ are both independent. The vertex u_{P_i} has an edge to each v_B but not to $w_{B'}$ for any B' . Lastly, for each block B of P_i and each block B' of P_j , we have $|B \cap B'|$ edges between v_B and $w_{B'}$. We shall use these vertex notations when referring to the vertices of $G_{P_j}^{P_i}$.

Let us consider the $\Gamma(k) \times \Gamma(k)$ -matrix, namely $M_{(k)}$, whose P_i -row is $T_v(G^{P_i})^t$, the transpose of the Tutte vector of G^{P_i} . In other words, the (P_i, P_j) -entry of $M_{(k)}$ is the polynomial $T(G_{P_j}^{P_i})$. Fix a k -fragment K . We know $T_f(K)$ exists and we also know that the P_i -entry of $M_{(k)}T_f(K)$ is the polynomial $T(G^{P_i} \oplus K)$, where G^{P_i} is considered as a k -fragment using the labels on its leaves. That is, $M_{(k)}T_f(K)$ is a polynomial vector completely determined when K is chosen. Let us denote $V_K = M_{(k)}T_f(K)$. Once we show that the determinant of $M_{(k)}$ is a nonzero polynomial, we can find its inverse $M_{(k)}^{-1}$ whose entries are fractional functions of x and y . By applying $M_{(k)}^{-1}$ on the left of both sides of $M_{(k)}T_f(K) = V_K$, we get $T_f(K) = M_{(k)}^{-1}V_K$, which means that such a vector $T_f(K)$ satisfying $T(H \oplus K) = T_v(H)^t T_f(K)$ for any connected k -fragment H is unique.

Now we show that the determinant of $M_{(k)}$ is not zero. For a polynomial $p(x, y)$ with two variables x and y , let us call the monomial $x^i y^j$ in $p(x, y)$ with nonzero coefficient such that i is the highest among which j is the highest, as the (y, x) -**term** of p . We claim that the (y, x) -term of $\det M_{(k)}$ has coefficient 1. We shall use the following simple fact which is easy to prove.

Proposition 2.2.7. *Let G be a connected graph. If G has n vertices and m edges among which k edges are bridges, then the largest degree of y in $T(G)$ is $m - n + 1$. Moreover, $x^k y^{m-n+1}$ is the only monomial in $T(G)$ obtaining the highest degree of y and it has coefficient 1.*

The above proposition implies that for any graph G , the (y, x) -term of $T(G)$ has

coefficient 1. Let $s(P_i, P_j)$ and $t(P_i, P_j)$ be the integers such that $x^{s(P_i, P_j)}y^{t(P_i, P_j)}$ is the (y, x) -term of $T(G_{P_j}^{P_i})$. The determinant $\det M_{(k)}$ is the sum of the products $\text{sign}(\sigma) \prod_{P_j \in \Gamma(k)} T(G_{P_j}^{\sigma(P_j)})$, where σ ranges over all permutations on $\Gamma(k)$. To prove our claim, it is enough to show that the identity permutation is the only one contributing to the (y, x) -term of $\det M_{(k)}$.

Let us focus only on the (y, x) -term of $T(G_{P_j}^{P_i})$ and set $\hat{M}_{(k)} = (x^{s(P_i, P_j)}y^{t(P_i, P_j)})_{P_i, P_j \in \Gamma(k)}$ be the matrix consisting of the (y, x) -terms of $T(G_{P_j}^{P_i})$. The graph $G_{P_j}^{P_i}$ is connected for every $P_i, P_j \in \Gamma(k)$, and it has $|P_i| + |P_j| + 1$ vertices and $|P_i| + k$ edges. By Proposition 2.2.7, we have $t(P_i, P_j) = k - |P_j|$, which is independent of P_i . That is, for any permutation σ on $\Gamma(k)$, the corresponding summand of $\det(\hat{M}_{(k)})$ has the same y -degree. Thus it is enough to consider the permutations maximizing the x -degree, which counts the bridges of corresponding graphs by Proposition 2.2.7.

Let σ be a permutation on $\Gamma(k)$ and let $b_\sigma(P_j)$ be the number of bridges in $G_{P_j}^{\sigma(P_j)}$. We are interested in finding σ which maximizes $\sum_{P_j \in \Gamma(k)} b_\sigma(P_j)$. Recall that $G_{P_j}^{\sigma(P_j)}$

has three types of vertices, the vertex $w_{B'}$ for each block B' of P_j , the vertex v_B for each block B of $\sigma(P_j)$, and the special vertex $u_{\sigma(P_j)}$. We shall consider the edges between $w_{B'}$ and v_B first. For each block B' of P_j , the edges incident with $w_{B'}$ are bridges if and only if $|B'| = 1$. Thus these edges do not change their contribution to $\sum_{P_j \in \Gamma(k)} b_\sigma(P_j)$ when σ varies. Therefore, to maximize $\sum_{P_j \in \Gamma(k)} b_\sigma(P_j)$, we need

to maximize the number of bridges among the edges between $u_{\sigma(P_j)}$ and $\{v_B : B \text{ is a block of } \sigma(P_j)\}$. Since $G_{P_j}^{\sigma(P_j)}$ has $|\sigma(P_j)|$ edges incident with $u_{\sigma(P_j)}$ and σ is a permutation, the total number of edges to consider is $\sum_{P_i \in \Gamma(k)} |P_i|$, which is

independent of σ . If we choose σ to be the identity permutation, then all of those edges are bridges and the upper bound $\sum_{P_i \in \Gamma(k)} |P_i|$ is realized. Let us show that

there is no other permutation σ such that all the edges incident with $u_{\sigma(P_j)}$ in $G_{P_j}^{\sigma(P_j)}$ are bridges for every P_j . Note that if some $w_{B'}$ has at least two neighbors in $G_{P_j}^{\sigma(P_j)}$, then the edges from $u_{\sigma(P_j)}$ to the neighbors of $w_{B'}$ are not bridges. That is, if in $G_{P_j}^{\sigma(P_j)}$ all edges incident with $u_{\sigma(P_j)}$ are bridges, then for each block B' of P_j , the vertex $w_{B'}$ has only one neighbor, which implies that $\sigma(P_j)$ has a block B containing B' . Hence P_j is a refinement of $\sigma(P_j)$. Since $\Gamma(k)$ forms a finite lattice under the refinement order, the only permutation on $\Gamma(k)$ which maps each

partition to its refinement is the identity. Therefore, the (y, x) -term of $\det M_{(k)}$ is $\prod_{P_i \in \Gamma(k)} x^{s(P_i, P_i)} y^{t(P_i, P_i)}$ with coefficient 1, which completes the proof that $T_f(K)$ is unique.

Now we prove that the entries of $T_f(K)$ are polynomials in x, y with positive integer coefficients. We shall use induction on $|E(K)|$ and k , the latter being the number of labelled vertices of K .

For $k = 1$, we have $T(H \oplus K) = T(H)T(K)$ and $T_f(K) = [T(K)]$. Obviously $T(K)$ is a polynomial in x, y with positive integer coefficients. If $|E(K)| = 0$, then $T(H \oplus K) = T(H)$ so that all entries of $T_f(K)$ are 0 but the entry for the partition whose blocks are all singletons, which is 1. These are the base cases for our induction.

Let K be an arbitrary k -fragment, and assume that we can find $T_f(K')$ with nonnegative entries for any $(k - 1)$ -fragment K' with at most $|E(K)|$ edges, and also for any k -fragment K' with at most $|E(K)| - 1$ edges. Suppose that K has an edge, say e , such that at least one end of e is not labelled. With the natural labelling inherited from K , the graphs K/e and $K - e$ can be considered as k -fragments. Note that $(H \oplus K)/e = H \oplus (K/e)$ and $(H \oplus K) - e = H \oplus (K - e)$. If e is a bridge of $H \oplus K$, then $T(H \oplus K) = xT((H \oplus K)/e) = xT(H \oplus (K/e)) = xT_v(H)^t T_f(K/e)$. That is, $T_f(K) = xT_f(K/e)$ and each entry of $T_f(K)$ is a polynomial in x, y with positive integer coefficients. The same argument works when e is a loop, resulting in $T_f(K) = yT_f(K - e)$. If e is neither a loop nor a bridge of $H \oplus K$, then

$$\begin{aligned} T(H \oplus K) &= T((H \oplus K) - e) + T((H \oplus K)/e) = T(H \oplus (K - e)) + T(H \oplus (K/e)) \\ &= T_v(H)^t T_f(K - e) + T_v(H)^t T_f(K/e) = T_v(H)^t (T_f(K - e) + T_f(K/e)), \end{aligned}$$

and hence $T_f(K) = T_f(K - e) + T_f(K/e)$, whose entries are the polynomials with positive integer coefficients because of the induction hypothesis.

Thus we may assume that all edges of K are connecting two labelled vertices, and furthermore, all vertices of K are labelled. By a way similar to above, we may also assume that K has no loop but $E(K) \neq \emptyset$. Let H be a connected k -fragment and let e be an edge of K . Since H is connected, the edge e is not a bridge in $H \oplus K$, so that $T(H \oplus K) = T((H \oplus K) - e) + T((H \oplus K)/e)$. By the induction hypothesis, $T((H \oplus K) - e) = T(H \oplus (K - e)) = T_v(H)^t T_f(K - e)$, where $T_f(K - e)$ is a vector whose entries are polynomials in x, y with positive integer coefficients.

Now we look at $T((H \oplus K)/e)$. Note that $(H \oplus K)/e$ is equal to $H_P \oplus (K/e)$ where $P \in \Gamma(k)$ has a single block of size two containing the ends of e and all other blocks of P are singletons. Since K/e is a $(k-1)$ -fragment with $|E(K)| - 1$ edges, by the induction hypothesis, we have $T(H_P \oplus (K/e)) = T_v(H_P)^t T_f(K/e)$. On the other hand, each entry of $T_v(H_P)$ is also an entry of $T_v(H)$, so that there is a $(0, 1)$ -matrix M_P of dimension $\Gamma(k-1) \times \Gamma(k)$, depending only on P , which satisfies $T_v(H_P) = M_P T_v(H)$ for any k -fragment H . Thus $T(H_P \oplus (K/e)) = T_v(H_P)^t T_f(K/e) = T_v(H)^t M_P^t T_f(K/e)$ and we get

$$T(H \oplus K) = T(H \oplus (K-e)) + T(H_P \oplus (K/e)) = T_v(H)^t T_f(K-e) + T_v(H)^t (M_P^t T_f(K/e)).$$

Since M_P is a $(0, 1)$ -matrix, the entries of $M_P^t T_f(K/e)$ are also polynomials in x, y with positive integer coefficients, and we can set $T_f(K) = T_f(K-e) + M_P^t T_f(K/e)$, proving the second property of $T_f(K)$ as stated. \square

So far we have proved Theorem 2.2.6. Our proof suggests an algorithm to find $T_f(K)$ based on the deletion-contraction formula, which results in an exponential time algorithm. But if we assume that the k -fragment K has bounded treewidth, then we can find the Tutte polynomial of K as well as all other entries of the Tutte vector of K , in polynomial time; c.f. [And98, Nob98]. Once we know $T_v(K)$, we have $T_f(K) = N T_v(K)$ where N is a $\Gamma(k) \times \Gamma(k)$ -matrix defined in Theorem 2.2.5, so that $T_f(K)$ can be computed in time polynomial of $|V(K)|$.

Spanning Trees

This chapter is about the number of spanning trees, denoted by $t(G)$ for a graph G , which is equal to the value of the Tutte polynomial $T(G)$ at $(1, 1)$. We start by investigating the splitting formula for spanning trees in Section 3.1, which is a technique to find the number of spanning trees of a graph from the number of some forests in two edge-disjoint subgraphs.

The splitting formula for spanning trees can be used to find asymptotic behaviour of the number of spanning trees when we consider graphs with repeating structures, for example the so-called wheel graphs, the ladder graphs, or the prism graphs and so on. A practical application of the splitting formula is given in Section 3.2.5, which leads us to the following counter-intuitive conjecture.

Conjecture (3.2.26). *If $k \geq 5$ is an odd number, then each k -regular k -edge-connected graph on n vertices has more spanning trees than the n -cycle with edge-multiplicity $(k + 1)/2$, which is $(k + 1)$ -edge-connected $(k + 1)$ -regular.*

Section 3.2 is mostly from a paper by Ok and Thomassen [OT]. We investigate the minimum number of spanning trees when the edge-connectivity, say k , is fixed.

When k is even we present a tight lower bound, namely $n \left(\frac{k}{2}\right)^{n-1}$ where n is the number of vertices, only attained by the cycle with edge-multiplicity $k/2$. When k is odd, however, we do not know the precise answer and show the following results.

Theorem (3.2.9). *Let $k > 1$ be an odd number and let G be a k -edge-connected graph on n vertices. Then*

$$t(G) \geq \left(\frac{kc_k}{2}\right)^{n-1},$$

where $c_k = \sqrt{1 + \frac{4}{(k+3)^2 - 4}} > 1$. In particular, $t(G) > 1.59^{n-1}$ if G is 3-edge-connected and $t(G) > 2.58^{n-1}$ if G is 5-edge-connected.

Theorem (3.2.10). *Let G be a 3-edge-connected graph on n vertices. Then $t(G) > 1.77^{n-1}$.*

Theorem (3.2.18). *Let G be a 5-regular 5-edge-connected graph on n vertices. Then $t(G) \geq 7.6^{(n-1)/2} \approx 2.7568^{n-1}$.*

3.1 A splitting matrix for the spanning trees

The purpose of this section is to find a splitting formula for spanning trees. We characterize completely the matrices that can be used in that formula. They turn out to be so-called the **generalized inverses** of a matrix arising from a variation of the splitting formula.

Let G be a graph with two edge-disjoint connected subgraphs H and K meeting at k vertices. The splitting formula (Theorem 2.2.5) says that there is a fixed matrix N_k such that the Tutte polynomial $T(G)$ can be obtained by

$$T(G) = T_v(H)^t N_k T_v(K) \tag{3.1}$$

where $T_v(H)$ depends only on H and $T_v(K)$ depends only on K .

Since the number of spanning trees is the value of the Tutte polynomial at $(1, 1)$, we may expect that there must be a fixed matrix, say $N_k(1, 1)$, such that an equation similar to Equation (3.1) holds for spanning trees. But since the matrix N_k is obtained from the inverse of a matrix whose entries are powers of $(x-1)(y-1)$, we cannot obtain such a matrix $N_k(1, 1)$ from (3.1).

But still, there is such a matrix $N_k(1,1)$ that $t(G) = t_v(H)^t N_k(1,1) t_v(K)$ although it is not unique when $k \geq 4$. Here $t_v(H)$ denotes the evaluation of $T_v(H)$ at $(1,1)$. In this section we describe how to find $N_k(1,1)$ for each $k \geq 4$.

We begin with establishing an equation similar to Equation (3.1) using the vector $t'_v(H) = [t_P(H)]_{P \in \Gamma(k)}$ where $t_P(H)$ is the number of spanning forests of H such that

1. each component contains a labelled vertex, and
2. two labelled vertices are in the same component if and only if they are in the same block of P .

A forest, say F , satisfying 1,2 is said to **induce the partition** P and we write $F \vdash P$. Recall that a similar concept was used when we found the splitting formula for the multivariate Tutte polynomial in Chapter 2.

It is easy to see that there is a $|\Gamma(k)| \times |\Gamma(k)|$ -matrix T_k whose entries are either 0 or 1 such that

$$t(G) = t'_v(H)^t T_k t'_v(K) \tag{3.2}$$

whenever H and K are k -fragments and $G = H \oplus K$. Specifically, the (P_i, P_j) -entry of T_k is 1 if and only if $\text{fr}(P_i) \oplus \text{fr}(P_j)$ is a tree where $\text{fr}(P_i)$ is the forest with k leaves, namely $\{1, 2, \dots, k\}$, and additionally one vertex for each block B of P_i which is adjacent to each element of B . The forest $\text{fr}(P)$ may be thought of as a representative of the forests inducing the partition P .

Recall that the size of a partition P is the number of blocks of P and denoted by $|P|$. The forest $\text{fr}(P)$ for $P \in \Gamma(k)$ has $k + |P|$ vertices and k edges. Thus, if the (P_i, P_j) -entry of T_k is 1 then $\text{fr}(P_i) \oplus \text{fr}(P_j)$ is a tree with $k + |P_i| + |P_j|$ vertices and $2k$ edges implying that $|P_i| + |P_j| = k + 1$. The matrix T_k is not invertible when $k \geq 4$. As an example, T_4 is given in Figure 3.1. The two 1s outside the two boxes correspond to the partition into singletons and the partition with a single block. The two submatrices of dimensions 6×7 and 7×6 correspond to the partitions with two blocks and the partitions with three blocks. Because of these two non-square submatrices, the matrix T_4 is not invertible. For $k > 4$, note that $\Gamma(k)$ contains $2^{k-1} - 1$ partitions of size 2 and $\binom{k}{2}$ partitions of size $k - 1$. Using this it is easy to see that T_k is not invertible for $k \geq 4$.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	0	
0	0	0	0	0	0	0	0	0	1	0	1	1	0	1	0	
0	0	0	0	0	0	0	0	1	0	0	1	1	1	0	0	
0	0	0	0	0	0	0	0	0	1	1	0	1	1	0	0	
0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	
0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0	
0	0	0	1	0	1	1	1	0	0	0	0	0	0	0	0	
0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	
0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	
0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	
0	1	0	1	1	0	1	0	0	0	0	0	0	0	0	0	
0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	

Figure 3.1: An auxiliary splitting matrix T_4 for spanning trees. All 1-s (except two) are inside the boxes.

Now we express the vector $t_v(H) = [t(H_P)]_{P \in \Gamma(k)}$ in terms of $t'_v(H)$. Recall that the graph H_P is obtained from H by, for each block B of P , identifying the labelled vertices of H in B into a single vertex. The set of edges of a spanning tree of H_P forms a forest in H such that every component contains a labelled vertex. Thus, it is a forest counted in $t_{P'}(H)$ for some $P' \in \Gamma(k)$.

Let T be a spanning tree of H_P and $E(T)$ be its edge set, considered in H . If we take $E(T) \oplus \text{fr}(P)$ and then contract the edges in $\text{fr}(P)$ then we get T back. Thus, to find $t(H_P)$, it is enough to sum up the numbers $t_{P'}(H)$ for the partitions P' such that $\text{fr}(P) \oplus \text{fr}(P')$ is a tree. That is,

$$t(H_P) = \sum_{\substack{P' \in \Gamma(k) \\ \text{fr}(P) \oplus \text{fr}(P') \text{ is a tree}}} t_{P'}(H),$$

and using the matrix T_k defined above, we have

$$t_v(H) = T_k t'_v(H).$$

The initial goal of this section is to find a matrix N_k (more precisely $N_k(1, 1)$) such

that

$$t(G) = t_v(H)^t N_k t_v(K) \quad \text{for all } H \text{ and } K$$

which is equivalent to, by the above formula,

$$t(G) = t'_v(H)^t T_k^t N_k T_k t'_v(K).$$

By Equation (3.2) and the fact that T_k is symmetric for all k , it is sufficient to find a matrix N_k such that

$$T_k = T_k N_k T_k. \quad (3.3)$$

The N_k satisfying Equation (3.3) is precisely what is called the **generalized inverse matrix** of T_k . It is known that for each non-regular square matrix, in particular our T_k , there are infinitely many generalized inverse matrices of T_k . By the above argument, each such matrix N_k satisfies the desired equation

$$t(G) = t_v(H) N_k t_v(K)^t. \quad (3.4)$$

I would like to point out that it is common to choose the so-called **Moore-Penrose pseudoinverse** as a ‘representative’ of the generalized inverse matrices. I have calculated the Moore-Penrose pseudoinverse of T_4 . All entries are $0, 1, \pm \frac{3}{14}$ and $\pm \frac{4}{14}$. It is not clear if that matrix has any particular significance compared to other generalized inverses in our context, and therefore I omit it.

We have now shown the existence of $N_k = N_k(1, 1)$. We now turn to uniqueness (up to generalized inverses).

Recall that we showed the following.

1. T_k satisfies $t(H \oplus K) = t'_v(H)^t T_k t'_v(K)$ for all H, K .
2. T_k satisfies $t_v(H) = T_k t'_v(H)$ for all H .

I shall now show that if M is a matrix such that

$$t(H \oplus K) = t'_v(H)^t M t'_v(K) \quad \text{for all } H, K \quad (3.5)$$

then $M = T_k$, and also if M' is a matrix such that

$$t_v(H) = M' t'_v(H) \quad \text{for all } H \quad (3.6)$$

then $M' = T_k$. It will then follow, by the above discussion, that N_k satisfies $t(H \oplus K) = t_v(H)^t N_k t_v(K)$ for all H and K if and only if $T_k N_k T_k = T_k$.

Both claims will follow if we find a sequence of k -fragments, $\{F_P\}_{P \in \Gamma(k)}$, indexed by $\Gamma(k)$ such that the $\Gamma(k) \times \Gamma(k)$ -matrix, whose P -column is $t'_v(F_P)$ for each $P \in \Gamma(k)$, is invertible. Suppose that we found such a sequence $\{F_P\}_{P \in \Gamma(k)}$ and let N be the invertible matrix whose column vectors are $t'_v(F_P)$. Let us assume that a matrix M satisfies both (3.5) and (3.6). Then the (P, P') -entry of the matrix product $N^t M N$ is $t'(F_P)^t M t'(F_{P'}) = t(F_P \oplus F_{P'})$ by (3.5). Hence, if we set L be the $\Gamma(k) \times \Gamma(k)$ -matrix whose (P, P') -entry is $t(F_P \oplus F_{P'})$, then we have

$$(N^t)^{-1} L N^{-1} = M \tag{3.7}$$

and we know already that

$$(N^t)^{-1} L N^{-1} = T_k \tag{3.8}$$

so that $M = T_k$. The claim for (3.6) can be settled likewise.

Thus we aim at finding a sequence $\{F_P\}_{P \in \Gamma(k)}$ such that the $\Gamma(k) \times \Gamma(k)$ -matrix $[t'_v(F_P)]_{P \in \Gamma(k)}$ is invertible. I shall set $F_P = \text{fr}(P)$, using the forest $\text{fr}(P)$ defined earlier in this section and below again. For $P \in \Gamma(k)$, the forest $\text{fr}(P)$ has k leaves, labelled using $\{1, 2, \dots, k\}$, and additionally one vertex for each block B of P which is adjacent to each element of B .

I shall show that $N = [t'_v(F_P)]_{P \in \Gamma(k)}$ is invertible by describing a Gaussian elimination from N to the identity matrix. Let P_{triv} be the partition into singletons. The graph $\text{fr}(P_{\text{triv}})$ is a matching with k edges, and the P_{triv} -entry of $t'_v(\text{fr}(P_{\text{triv}}))$ is 1 and all other entries are 0. Thus, by column operations, we can ignore the row and column of N corresponding to P_{triv} . In the remaining submatrix of N , we consider the partitions in which all blocks are singletons but one block with two elements. Since we excluded the row for P_{triv} , the columns of N corresponding to these partitions have precisely one nonzero entry at the main diagonal which is 1. Again by column operations, we remove all nonzero entries in the rows corresponding to these blocks except those 1's in the main diagonal. This process can inductively continued until N becomes the identity matrix, which means N is invertible. Therefore, by (3.7) and (3.8), T_k is the unique matrix satisfying (3.5) and (3.6) so that the following holds.

Theorem 3.1.1. *Let $k \geq 2$ be an integer and let T_k be the matrix defined after*

Equation (3.2). A matrix N_k satisfies

$$t(H \oplus K) = t_v(H)^t N_k t_v(K) \quad \text{for all } k\text{-fragments } H \text{ and } K$$

if and only if N_k is one of the generalized inverse matrices of T_k , i.e. $T_k N_k T_k = T_k$.

To sum up this section, we found that the analogue of the Tutte polynomial splitting formula holds for the number of spanning trees, although the splitting matrix is not unique in general. We also completely characterized the matrices which can be used in the splitting formula in terms of generalized inverse matrix. However, I do not know whether there is an analogue for the number of acyclic orientations or totally cyclic orientations. I believe that the matrix N_k in Theorem 2.2.1 is invertible when we evaluate it at the points $(2, 0)$ and $(0, 2)$, so that the splitting formulas for acyclic orientations and totally cyclic orientations follows easily from Theorem 2.2.1, Negami's splitting formula.

3.2 Minimum number of spanning trees in k -edge-connected graphs

In this section we shall consider the minimum number of spanning trees in a k -edge-connected graph on fixed number of vertices. The contents are mostly from the paper by Ok and Thomassen [OT], but the proofs of the Lemmas 3.2.4 and 3.2.5 are modified since we use the splitting formula here.

The main tool we use is Theorem 3.2.1 below, called **Mader's lifting theorem**. Firstly, we define the lifting operation.

Given a vertex s with its two neighbors u and v , **lifting** the edges su and sv is removing both edges su and sv and then adding uv . Note that we allow graphs to have parallel edges, and we remove one edge from each parallel class containing su and sv respectively and then add a single edge. A **lifting at s** is a lifting of two edges incident with s . Lifting two parallel edges is simply removing those two edges.

Theorem 3.2.1 (Mader's lifting theorem [Mad78]). *Let $G = (V, E)$ be a graph and let $s \in V$. For each pair $a, b \in V - s$, let $\lambda_G(a, b)$ be the maximum number*

of pairwise edge-disjoint paths between a and b in G . If s is not incident with a bridge and if the degree of s is not 3, then G has a lifting at s such that for the graph G' resulting from the lifting,

$$\lambda_G(a, b) = \lambda_{G'}(a, b)$$

for each pair $a, b \in V - s$.

The following two corollaries of the Mader's lifting theorem is of interest to us.

Corollary 3.2.2. *Let G be a k -edge-connected graph, $k \geq 2$. Let s be a vertex of G . If the degree of s is at least $k + 2$, then G has a lifting at s resulting in another k -edge-connected graph.*

Corollary 3.2.3. *Let G be a graph and let s be a vertex of G of even degree, say $2d$. If G has at least $k \geq 2$ pairwise edge-disjoint paths between any two vertices except s , then G has a sequence of d liftings at s which result in a k -edge-connected graph after removing the isolated vertex s .*

In the following sections, we prove that a k -edge-connected graph on n vertices has at least $n \left(\frac{k}{2}\right)^{n-1}$ spanning trees, which is tight when k is even (Theorem 3.2.7). However for k odd, we prove in Theorem 3.2.9 that there are more than $\left(\frac{k c_k}{2}\right)^{n-1}$ spanning trees where c_k is a constant such that $c_k > 1$. I do not know the precise minimum number of spanning trees for odd edge-connectivity, and the investigation is focused on the following number τ_k :

$$\tau_k = \liminf_{n \rightarrow \infty} \min_{\substack{G \text{ a graph} \\ |V(G)|=n \\ G \text{ } k\text{-edge-connected}}} t(G)^{1/n}.$$

The discussion above says that $\tau_k = \frac{k}{2}$ for k even and $\tau_k \geq \frac{k c_k}{2}$ for k odd. We shall prove that $\tau_3 > 1.59$ and $\tau_5 > 2.58$. See Theorem 3.2.9.

In section 3.2.3 we show that $1.77 < \tau_3 < 1.932$. For 5-edge-connected graphs, we confine our consideration to 5-regular graphs and show that a 5-regular 5-edge-connected graph on n vertices has more than 2.7568^{n-1} spanning trees. In the last section, Section 3.2.5, we establish an upper bound for c_k (k odd) by finding the number of spanning trees in a class of graphs we call multiprisms. This class leads us to the following counter-intuitive conjectures.

Conjecture (3.2.25). *If $k \geq 5$ is an odd number, then $\tau_k = \tau_{k+1}$.*

Conjecture (3.2.26). *If $k \geq 5$ is an odd number, then each k -regular k -edge-connected graph on n vertices has more spanning trees than the n -cycle with edge-multiplicity $(k+1)/2$, which is $(k+1)$ -edge-connected $(k+1)$ -regular.*

3.2.1 Lifting a pair of edges

We shall consider here how the lifting operation changes the number of spanning trees.

Let $e = vu, f = vw$ be two adjacent edges of a graph. **Lifting** e, f is the operation of replacing e, f by an edge uw if $u \neq w$. If $u = w$ we simply remove both edges e, f . By **lifting at v** we mean that we lift a pair of edges incident with v . A **complete lifting** at a vertex v with even degree is a sequence of liftings at v until no edges are left at v . Then we remove v .

For the following lemma, we define a constant h_d depending on a positive integer d as follows:

$$h_d = \min_{d_1, d_2, \dots, d_k} \min_H \frac{\prod_{i=1}^k d_i}{t(H)},$$

where the minimum is taken over all sequences of positive integers d_1, d_2, \dots, d_k with varying length k such that $\sum_{i=1}^k d_i = 2d$, and over all connected graphs H on k vertices with degree sequence d'_1, d'_2, \dots, d'_k such that $d'_i \leq d_i$ for each i .

In the above definition of h_d , the graph H has at most d edges, so $h_1 = 1$. Furthermore, $h_2 = 2$, $h_3 = 8/3$ and $h_4 = 18/5 = 3.6$, which are attained by a 2-cycle, a 3-cycle, and a 3-cycle plus a parallel edge, respectively.

Lemma 3.2.4. *Let G be a graph with a vertex v of degree $2d$. Let G' be a graph obtained from G by a complete lifting at v . Then $t(G) \geq h_d t(G')$, where h_d is defined as above.*

Proof. We shall use the splitting formula to both G and G' with the common subgraph $H = G - v$ and its complements in G and G' . Let K be the subgraph of G consisting of v and its incident edges, and let K' be the subgraph of G' induced

by the edges not in H . We shall denote the neighbors of v in G by v_1, v_2, \dots, v_{2d} which are not necessarily distinct so that the edges vv_i are all the edges incident with v . We may assume that the labelling is chosen such that the complete lifting to obtain G' produced the edges $v_{2i-1}v_{2i}$ for $i = 1, 2, \dots, d$ except those i where $v_{2i-1} = v_{2i}$.

The splitting formula for spanning trees (Theorem 2.2.6) says that

$$t(G) = t_v(K) \cdot t_f(H), \quad t(G') = t_v(K') \cdot t_f(H),$$

where $t_f(H)$ is a vector with nonnegative integer entries determined by H . Thus it is enough to show that for each appropriate partition $P \in \Gamma(2d)$, the entries of $t_v(K) = [t(K_P)]$ and $t_v(K') = [t(K'_P)]$ satisfy the following:

$$t(K_P) \geq h_d t(K'_P) \tag{3.9}$$

The graph K_P is obtained from the star graph K which may have parallel edges by identifying some of the non-center vertices, thus it is still a star graph with parallel edges. Let us say that K_P has vertices v, v_1, v_2, \dots, v_k and d_i edges between v and v_i for $i = 1, 2, \dots, k$. We know $\sum_{i=1}^k d_i = 2d$ and $t(K_P) = \prod_{i=1}^k d_i$.

Since the vertex set of K' is that of K without v and the graphs K_P and K'_P are obtained by identifying vertices according to the same rule P , the graph K'_P has vertices v_1, v_2, \dots, v_k . Also, K' is obtained from K by a complete lifting at v , so that the vertex degrees of K' is not more than that of K , which holds also between K'_P and K_P . That is, the degree d'_i of v_i in K'_P is at most d_i . If K'_P is not connected then Equation (3.9) is trivial. Otherwise, $t(K_P)/t(K'_P)$ is one of the fractions in the definition of h_d which the constant h_d is minimizing. Thus, Equation (3.9) holds and so does the lemma. \square

Lemma 3.2.5. *Let G be a graph with a vertex v of degree $d \geq 3$. Let G' be a graph resulting from lifting edges vu, vw in G . Then $t(G) \geq (1 + \frac{4}{d^2 - 4})t(G')$.*

Proof. We repeat the proof of Lemma 3.2.4. Let $H = G - v$ be the common subgraph of G and G' , and let K and K' be respectively the subgraphs of G and G' induced by the edges not in H . From the splitting formula for spanning trees (Theorem 2.2.6), we have

$$t(G) = t_v(K) \cdot t_f(H), \quad t(G') = t_v(K') \cdot t_f(H)$$

where $t_f(H)$ is a vector with nonnegative integer entries determined by H . Thus it is enough to show that

$$t(K_P) \geq \frac{d^2}{d^2 - 4} t(K'_P) \quad (3.10)$$

where K_P and K'_P are obtained from K and K' respectively by identifying some of the neighbors of v according to a partition P .

Since K is a star graph possibly with parallel edges, K_P is also such a graph and K'_P is obtained by lifting two edges in K_P . Let us say K has vertices v, v_1, v_2, \dots, v_k with d_i edges between v and v_i for $i = 1, 2, \dots, k$ and K' is obtained by lifting vv_1 and vv_2 . If $d_1 = d_2 = 1$ then $t(K'_P) = 0$ and Equation (3.10) is trivial. Otherwise

$$\frac{t(K_P)}{t(K'_P)} = \frac{d_1 d_2}{(d_1 - 1)(d_2 - 1) + (d_1 - 1) + (d_2 - 1)} = \frac{d_1 d_2}{d_1 d_2 - 1} \geq \frac{d^2}{d^2 - 4}$$

where the last inequality comes from

$$d_1 d_2 \leq \left(\frac{d_1 + d_2}{2} \right)^2 \leq \left(\frac{d_1 + d_2 + \dots + d_k}{2} \right)^2 = \frac{d^2}{4}.$$

□

3.2.2 k -edge-connected graphs

In this subsection we show that a k -edge-connected graph on n vertices has at least $n \binom{k}{2}^{n-1}$ spanning trees, which is tight when k is even. When k is odd, we show that there is a constant $c_k > 1$ such that every k -edge-connected graph on n vertices has more than $\left(\frac{kc_k}{2} \right)^{n-1}$ spanning trees, implying that the former bound is not tight.

Let G be a connected graph with n vertices and m edges. Consider the pairs (e, T) where $e \in E(G)$ and T a spanning tree of G containing e . For each $e \in E(G)$ we have $t(G/e)$ such pairs and for each T , we have $n - 1$ such pairs. Therefore $(n - 1)t(G) = \sum_{e \in E(G)} t(G/e)$. Hence, G has an edge e such that $\frac{t(G/e)}{t(G)} \leq \frac{n - 1}{m}$.

We restate this conclusion as the following observation.

Observation 3.2.6. *Let G be a connected graph with $n > 1$ vertices and m edges. Then G has an edge e such that $t(G) \geq \frac{m}{n - 1} t(G/e)$.*

Theorem 3.2.7. *Let G be a k -edge-connected graph on n vertices. Then G has at least $n \left(\frac{k}{2}\right)^{n-1}$ spanning trees. Moreover, G has more than $n \left(\frac{k}{2}\right)^{n-1}$ spanning trees unless k is even and G is a cycle whose edge-multiplicities are all $\frac{k}{2}$.*

Proof. We shall use induction on n . Since G is k -edge-connected, the minimum degree of G is at least k and thus $m \geq \frac{kn}{2}$. By Observation 3.2.6, G has an edge e such that $t(G) \geq \frac{m}{n-1}t(G/e) \geq \frac{kn}{2(n-1)}t(G/e)$. By the induction hypothesis, $t(G/e) \geq (n-1)(k/2)^{n-2}$ so that $t(G) \geq n(k/2)^{n-1}$. If equality holds, then k is even, $m = \frac{kn}{2}$, and G/e is a cycle where all edge-multiplicities are $k/2$. Moreover, any edge can play the role of e . This implies that all edge-multiplicities in G are $k/2$. If H denotes the subgraph of G obtained by replacing every multiple edge by a single edge, then H has the property that the contraction of any edge results in a cycle. Then also H is a cycle. \square

For k even Theorem 3.2.7 is tight. However, for k odd we present a lower bound for the number of spanning trees in a k -edge-connected graph of the form c^{n-1} with $c > k/2$.

By Mader's lifting theorem (Theorem 3.2.1) and Menger's Theorem, given a k -edge-connected graph and a vertex of degree $\geq k+2$, we can find a lifting without decreasing the edge-connectivity. By Lemma 3.2.5, a lifting always decreases the number of spanning trees and hence the minimum number of spanning trees of a k -edge-connected graph on n vertices must be obtained by a graph whose degrees are only k or $k+1$. We state this as an observation for later use.

Observation 3.2.8. *If G is a k -edge-connected graph on n vertices with minimum $t(G)$, then each vertex of G has either k or $k+1$ incident edges.*

Now we prove the following lower bound for odd edge-connectivity.

Theorem 3.2.9. *Let $k > 1$ be an odd number and let G be a k -edge-connected graph on n vertices. Then $t(G) \geq \left(\frac{kc_k}{2}\right)^{n-1}$, where $c_k = \sqrt{1 + \frac{4}{(k+3)^2 - 4}} > 1$*

Proof. Let e be an edge for which $\frac{t(G)}{t(G/e)}$ is maximum. By Observation 3.2.6 we

know $\frac{t(G)}{t(G/e)} \geq \frac{k}{2}$. If the vertex of G/e resulting from the contraction of e , say v , has degree bigger than $k+1$, then using Theorem 3.2.1 we can lift a pair of edges at v such that G/e after the lifting is still k -edge-connected. We do the lifting at v until the degree of v is at most $k+1$. Let H be the resulting graph. If $\frac{t(G)}{t(H)} \geq \frac{kc_k^2}{2}$ then we call e a good edge. Note that, if $H \neq G/e$, then by applying Lemma 3.2.5 at the last lifting, we see that e is good. Also, if e has multiplicity at least $\frac{k+1}{2}$, then $\frac{t(G)}{t(H)} \geq \frac{t(G)}{t(G/e)} \geq \frac{k+1}{2} > \frac{kc_k^2}{2}$ so that e is good. If one of the ends of e has degree at least $k+1$, then either e has multiplicity at least $(k+1)/2$, or the vertex obtained by the contraction of e has degree at least $k+2$, so that e is good. Thus e is not good only if the ends of e both have degree precisely k . In particular, both ends of e have odd degree.

Now we repeat the contractions of an edge with maximum $\frac{t(G)}{t(G/e)}$, followed by liftings whenever possible, until only two vertices are left. Because of parity, among the $n-2$ contractions, at most $\lceil (n-2)/2 \rceil$ of them are edges whose ends both have odd degree. Thus at least $\lfloor (n-2)/2 \rfloor$ times we get an additional factor of c_k^2 , so that

$$t(G) \geq k \cdot \left(\frac{k}{2}\right)^{n-2} c_k^{2\lfloor (n-2)/2 \rfloor} > \left(\frac{kc_k}{2}\right)^{n-1}.$$

□

Theorem 3.2.9 shows that although Theorem 3.2.7 is tight for even edge-connectivity, it is not for any odd edge-connectivity. In the following two subsections we focus on k -edge-connected graphs where $k = 3, 5$.

3.2.3 3-edge-connected graphs

Let G be a 3-edge-connected graph on n vertices. By Theorem 3.2.9, the lower bound $t(G) \geq n \left(\frac{3}{2}\right)^{n-1}$ is not tight. Kostochka [Kos95] showed that a cubic simple 2-connected graph on n vertices has at least $8^{n/4} \approx 1.68^n$ spanning trees. This result is essentially best possible because of the cubic 2-connected graphs obtained by a collection of K_4 's minus an edge by adding a matching. In this section, we prove the following theorem.

Theorem 3.2.10. *Let G be a 3-edge-connected graph on n vertices. Then $t(G) > 1.77^{n-1}$.*

Kreweras [Kre78] showed that the prism graph on n vertices has approximately 1.93^n spanning trees; see Section 3.2.5. Thus $1.77 < c_3 < 1.93$. By Observation 3.2.8, a 3-edge-connected graph on n vertices with minimum number of spanning trees has vertex degrees only 3 and 4. Thus by Lemma 3.2.4, the following Theorem 3.2.11 is enough to prove Theorem 3.2.10. Note that a cubic graph with more than two vertices has the same connectivity and edge-connectivity.

Theorem 3.2.11. *Let G be a 3-connected cubic graph on n vertices. Then $t(G) > 1.77^{n-1}$.*

An often used operation to construct a 3-connected cubic graph is to **join** two edges, i.e. for non-parallel edges e, f , we replace each edge by a path of length 2 and connect the two new vertices of degree 2 by an edge. Note that joining two non-parallel edges in a 3-connected cubic graph results in another 3-connected cubic graph. The following lemma explains how the number of spanning trees changes after joining.

Lemma 3.2.12. *Let G be a graph with two non-parallel edges e and f . Let G' be the graph obtained from G by joining e and f . Then $t(G') \geq (4 - r)t(G)$, where $r = t(G/e/f)/t(G) \leq 1$.*

Proof. We only consider the case when e, f are not adjacent, but the other case can be done likewise. Let $e = ab$ and $f = cd$. Let T be a spanning tree of G . Then $T - e - f$ is a spanning forest of G in which each component contains at least one of a, b, c and d . We shall consider how many ways $T - e - f$ can be extended to a spanning tree in G and G' respectively. For example, if $T - e - f$ has two components such that one of them contains a, c and the other contains b, d , then we can extend $T - e - f$ in two ways to a spanning tree of G , whereas there are eight ways for G' . In fact, there are at least four times as many extensions in G' as extensions in G , unless T contains both e and f , in which case we have a factor 3. Thus, $t(G') \geq 4(t(G) - t(G/e/f)) + 3t(G/e/f) = (4 - r)t(G)$. \square

To prove Theorem 3.2.11, we shall consider the following two operations to construct 3-connected cubic graphs.

1. Let v be a vertex v in a graph such that $\deg(v) = 3$ and all three neighbors of v are distinct. Then the **blow-up** of v is obtained by joining two of the incident edges of v .
2. Select three edges, which may not be pairwise distinct, but not all the same, and subdivide each of them so that we have three new vertices of degree 2. Add a new vertex v and an edge from v to each of the three vertices of degree 2. We call this a **vertex-addition**.

Since a blow-up is a join of two non-parallel edges, we get the following observation by Lemma 3.2.12.

Observation 3.2.13. *Let G be a graph with a vertex v of degree 3 whose neighbors are all distinct. Let G' be the graph obtained from G by a blow-up of v . Then $t(G') \geq 3t(G)$.*

Barnette and Grünbaum [BG69] and independently Titov [Tit75] gave a characterization of 3-connected graphs which implies that every 3-connected cubic graph can be obtained from K_4 by successively joining edges. We shall here prove a stronger result for cubic graphs.

Theorem 3.2.14. *Let G be a 3-connected cubic graph with more than two vertices. Then G can be constructed from K_4 or $K_{3,3}$ by blow-ups and vertex-additions, such that blow-ups are never used consecutively.*

Proof. Our proof consists of two parts. We show that if G has no induced subgraph which is a subdivision of another 3-connected graph, then G is one of K_4 , $K_{3,3}$ or the prism on 6 vertices defined in Section 3.2.5. Then we assume that G has a maximal induced subgraph, say H , which is a subdivision of another 3-connected graph H^* , and we show that G can be obtained from H^* by a vertex addition, possibly followed by a blow-up.

Suppose that G has no proper induced subgraph which is a subdivision of a 3-connected cubic graph. Let C be a cycle in G of minimum length so that C has no chord. Let v be a vertex in $G - V(C)$. Since G is 3-connected, Menger's Theorem implies that G has three paths P_1, P_2, P_3 where $P_i = vu_1^i u_2^i \dots u_{k_i}^i u_i$, $C \cap P_i = \{u_i\}$ for each i and the paths P_1, P_2, P_3 share only v . Let v be such a vertex with $k_1 + k_2 + k_3$ being smallest. Note that some k_i may be 0, implying that P_i is an edge. If G has an edge between the non-endvertices of two P_i 's, say

$u_i^1 u_j^2$, then by taking $v = u_i^1$ instead and using $P_1 \cup P_3$ and $u_i^1 u_j^2 u_{j+1}^2 \dots u_{k_2}^2$, we get a smaller sum of the lengths of the paths unless u_j^2 is the neighbor of v in P_2 . Similarly, we deduce that u_i^1 is also the neighbor of v in P_1 . In this case, $vu_1^1 u_1^2$ is a triangle and hence C must also be a triangle, so that the vertex set of $C \cup P_1 \cup P_2 \cup P_3$, say V , induces a subgraph of G which is a subdivision of the prism graph. Thus by the assumption, G itself is the prism graph.

Hence we may assume that G has no edge between the non-endvertices of P_i 's. Denote by $G[V]$ the subgraph of G induced by V . Suppose $k_1 \geq 1$ and some u_i^1 has a neighbor on C different from u_1 . Because of the minimality of $k_1 + k_2 + k_3$, we have $i = k_1$ and by taking $v = u_{k_1}^1$ and using its two neighbors on C , we see $k_2 = k_3 = 0$. Therefore $G[V]$ is a subdivision of either the prism graph or $K_{3,3}$, so that again G itself is either the prism graph or $K_{3,3}$. The remaining case leaves no other edge in $G[V]$ than $C \cup P_1 \cup P_2 \cup P_3$, which is a subdivision of K_4 . Thus in this case G itself is K_4 . This completes the first part.

Now we assume that G has an induced proper subgraph which is a subdivision of a 3-connected cubic graph. Let H be a maximal such subgraph. Let us call a path in H suspended if its ends both have degree 3 in H and all other vertices in the path have degree 2 in H . Suspended paths intersect only at their ends. By replacing each suspended path of H by an edge between its ends, we get a 3-connected cubic graph, which we denote H^* . Since G is 3-connected, H has at least two suspended paths. If G has a vertex, say v , outside H which has neighbors in at least two distinct suspended paths of H , then the subgraph of G induced by $V(H) \cup \{v\}$ is a subdivision of a 3-connected graph, which must be G because of the maximality of H . Then G can be obtained from H^* by the vertex-addition of v . Thus we may assume that for each vertex in $V(G) \setminus V(H)$, its neighbors in H , if any, are in a single suspended path of H . Also, we may assume that $|V(G) \setminus V(H)| > 1$. If $V(G) \setminus V(H) = \{u, v\}$, then u and v are adjacent, and they have neighbors in distinct suspended paths. Thus we can obtain G from H^* by first vertex-adding u and then a blow-up to make v . Therefore, we assume that $|V(G) \setminus V(H)| > 2$.

Since G is 3-connected, at least one component of $G - V(H)$ has edges to two distinct suspended paths of H . Thus G has a path of length > 1 between distinct suspended paths of H which intersects H at only its ends. Let $P = v_0 v_1 \dots v_k$ be such a path with smallest length. Since P has no chord, the subgraph of G induced by $H \cup P$ is a subdivision of a 3-connected graph, so that $V(H) \cup V(P) = V(G)$, implying $k \geq 4$. By assumption, the neighbors of v_1 and v_{k-1} , respectively, are in

different suspended paths of H . Let v be the neighbor of v_2 in H . Then either $v_0v_1v_2v$ or $vv_2v_3\dots v_k$ contradicts the minimality of P , a contradiction which completes the proof. \square

Let c be the positive real solution of the equation $x^6 - 3x^4 = 3$ which is approximately $c \approx 1.8108$. Note that a vertex-addition is equivalent to a joining of two edges and then joining the new edge with an edge.

Lemma 3.2.15. *Let G_0 be a 3-connected graph and let G be a graph obtained from G_0 by joining two non-parallel edges of G_0 , where e denotes the joining edge. Let G' be a graph obtained from G by joining e with another edge f . Then either $t(G') \geq c^2t(G)$ or $t(G') \geq c^4t(G_0)$.*

Proof. Let $r = t(G/e/f)/t(G)$ be as in Lemma 3.2.12. Let $r' = t(G/e)/t(G)$ so that $t(G)/t(G-e) = 1/(1-r')$. Since $r' \geq r$, Lemma 3.2.12 implies $t(G') \geq (4-r)t(G) \geq (4-r')t(G)$. If $4-r' \geq c^2$ then we are done. Thus we may assume that $4-r' < c^2$, equivalently $1-r' < c^2-3$. Since $r' \leq 1$, $t(G') \geq 3t(G)$. By modifying the equation for c , we get $3/(c^2-3) = c^4$, so that

$$t(G') \geq 3t(G) = \frac{3t(G)}{t(G_0)}t(G_0) \geq \frac{3t(G)}{t(G-e)}t(G_0) = \frac{3}{1-r'}t(G_0) > \frac{3}{c^2-3}t(G_0) = c^4t(G_0).$$

\square

Proof of Theorem 3.2.11. We shall prove $t(G) \geq (3c^2)^{(n-1)/4}$ by induction on $n = |V(G)|$, where c is the constant used in Lemma 3.2.15. We may assume that $n \geq 8$ because K_4 , $K_{3,3}$ and the prism on 6 vertices have 16, 81 and 75 spanning trees, respectively. By Theorem 3.2.14, G can be obtained from K_4 or $K_{3,3}$ by repeatedly applying vertex-additions and blow-ups. If the last operation is a vertex-addition, then by Lemma 3.2.15, $t(G) \geq c^2t(G')$ or $t(G) \geq c^4t(G'')$ for some 3-connected cubic graph G' with $n-2$ vertices or G'' with $n-4$ vertices, so we are done. Otherwise, G can be obtained from a 3-connected cubic graph using a vertex-addition and then a blow-up. By Observation 3.2.13, a blow-up multiplies the number of spanning trees by at least 3, so that using Lemma 3.2.15, $t(G) \geq 3c^2t(G')$ or $t(G) \geq 3c^4t(G'')$ for some 3-edge-connected cubic graph G' with $n-4$ vertices or G'' with $n-6$ vertices. By the induction hypothesis, $t(G) \geq (3c^2)^{(n-1)/4} > 1.77^{n-1}$. \square

3.2.4 5-regular 5-edge-connected graphs

Let G be a 5-regular 5-edge-connected graph. A **5-cut** is a set of edges E with $|E| = 5$ such that $G - E$ is disconnected. If one of the components of $G - E$ is a single vertex, then we call E **trivial**. Otherwise we call E **nontrivial**. A **5-side** is a set $X \subseteq V(G)$ such that $\delta(X)$ (that is, the set of edges with precisely one end in X) is a nontrivial 5-cut. If a 5-side X has the property that no nontrivial 5-cut contains an edge with both ends in X , then X is called **minimal**.

Lemma 3.2.16. *Let G be a 5-regular 5-edge-connected graph. If G has a nontrivial 5-cut, then G has a minimal 5-side.*

Proof. Let A be a 5-side which is not minimal. Then some nontrivial 5-cut $S = \delta(B)$ contains an edge uv with $u \in A \cap B$ and $v \in A \cap B^c$. Let $T = \delta(A)$. One of the sets $A \cap B$, $A \cap B^c$, $A^c \cap B$ or $A^c \cap B^c$ is empty because G is 5-edge-connected, S, T are 5-cuts and 5 is odd. Since $u \in A \cap B$ and $v \in A \cap B^c$, either $A^c \cap B$ or $A^c \cap B^c$ is empty, so that either $A \cap B$ or $A \cap B^c$ is a 5-side strictly smaller than A . If it is not minimal, then we repeat the argument until we eventually find a minimal 5-side. \square

Lemma 3.2.17. *Let G be a connected graph with a connected subgraph H . If G' is the graph obtained by contracting H into a single vertex, then $t(G) \geq t(H)t(G')$.*

Proof. For each pair S, T of spanning trees of H, G' , we can expand the contracted vertex of G' using S to get a spanning tree of G . \square

Theorem 3.2.18. *Let G be a 5-regular 5-edge-connected graph on n vertices. Then $t(G) \geq 7.6^{(n-1)/2} \approx 2.7568^{n-1}$.*

Proof. We shall use induction on n . Being 5-regular and 5-edge-connected, G has no edge of multiplicity at least 3. If G has a nontrivial 5-cut, then by Lemma 3.2.16, we can find a minimal 5-side, and we let $e = uv$ be an edge inside that minimal side. Otherwise let $e = uv$ be an arbitrary edge.

Suppose firstly that e has multiplicity 1. Then G/e has a vertex of degree 8, which we can completely lift using Theorem 3.2.1. Denote the resulting 5-regular 5-edge-connected graph by G' . By Lemma 3.2.4, $t(G/e) \geq 3.6t(G')$. Now we consider

$G - e$. Since e is not contained in any nontrivial 5-cut, $G - e$ has at least 5 edge-disjoint paths between any pair of vertices distinct from the ends of e . Thus by Theorem 3.2.1, we can completely lift u, v in $G - e$ so that the resulting graph, say G'' , is 5-edge-connected and 5-regular. By Lemma 3.2.4, $t(G - e) \geq 4t(G'')$ and by the induction hypothesis,

$$t(G) = t(G/e) + t(G - e) \geq 3.6t(G') + 4t(G'') \geq 7.6^{(n-1)/2}.$$

Now we may assume that every edge of G with multiplicity 1 is contained in a nontrivial 5-cut. Let X be a minimal 5-side. Since the edges inside X are not contained in any nontrivial 5-cut, every edge inside X must be a double edge. Hence every vertex in X is incident with $\delta(X)$, so that X is the 5-double-cycle which has 80 spanning trees. By Lemma 3.2.17, $t(G) \geq 80t(G/X)$, and by the induction hypothesis, $t(G) \geq 7.6^{(n-1)/2}$. \square

3.2.5 The number of spanning trees of the multiprisms

In this section we investigate a class of $(2s + 1)$ -regular $(2s + 1)$ -edge-connected graphs that we believe may have the minimum number of spanning trees.

The **prism** on $2n$ vertices, PR_{2n} , is the Cartesian product of C_n and K_2 . In other words, it has the vertex set

$$V(PR_{2n}) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$$

and the edge set

$$E(PR_{2n}) = \{u_i u_{i+1} : 1 \leq i \leq n\} \cup \{v_i v_{i+1} : 1 \leq i \leq n\} \cup \{u_i v_i : 1 \leq i \leq n\},$$

where $u_{n+1} = u_1, v_{n+1} = v_1$.

The **multiprism** $MP_{2n}(s)$ is the $(2s + 1)$ -regular graph on $2n$ vertices defined as follows:

1. Let v_1, v_2, \dots, v_{2n} be the vertices, and add s edges between v_i and v_{i+1} for each i , and also between v_1 and v_{2n} .
2. Add edges $v_1 v_4, v_3 v_6, v_5 v_8, \dots, v_{2n-1} v_2$.

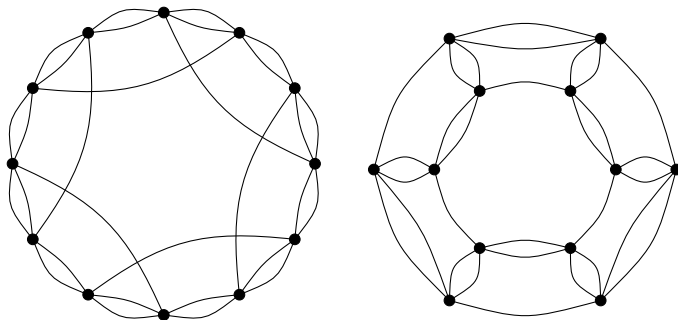


Figure 3.2: Two drawings of the multiprism $MP_{12}(2)$

Note that when n is even, the multiprism $MP_{2n}(s)$ can be obtained from the prism PR_{2n} by adding parallel edges along a Hamiltonian cycle. See Figure 3.2.

We shall firstly use the splitting formula for spanning trees to find $t(MP_{2n}(s))$ asymptotically. The technique can be used to find the asymptotic behaviour of the number of spanning trees of graphs with repeating structures, such as the wheel graphs, the ladder graphs, the prism graphs and so on. Thereafter, we find a closed expression of $t(MP_{2n}(s))$ using an idea of M. Rubey [Rub00]. We mentioned the formula without proof in [OT].

We start by explaining why we can remove a constant number, in this case $s + 1$, of edges from $MP_{2n}(s)$ for each n without changing $\lim_{n \rightarrow \infty} t(MP_{2n}(s))^{1/(2n)}$. We shall use the **Rayleigh's monotonicity law** from electrical network theory.

A graph G can be considered as an electrical network where each edge represents a resistance of 1 ohm. Then the effective resistance between two vertices s and t , denoted by $r_G(s, t)$, can be measured using the number of spanning trees in the following way; c.f. [Tho90].

$$r_G(s, t) = \frac{t(G/st)}{t(G)},$$

where G/st is the graph obtained from G by identifying the vertices s and t .

From the electrical point of view, it is intuitively clear that if we add a new resistance between two points in an electrical network, then the effective resistance between any pair of points cannot increase, since we made a possibly new way for the electricity to flow. This property, called the **Rayleigh's monotonicity law**,

can be stated formally using graph theoretical terms as in Theorem 3.2.19 below. See [BSST40, Tho90] for proofs.

Theorem 3.2.19 (Rayleigh's monotonicity law). *Let G be a graph and let s, t be two distinct vertices of G . Let e be an edge connecting two vertices of G which does not belong to G . Then*

$$r_{G+e}(s, t) \leq r_G(s, t).$$

Lemma 3.2.20. *Let G be a connected graph on n vertices. Then for every pair s, t of two vertices of G , we have*

$$r_G(s, t) \leq n - 1.$$

Proof. Let us choose a path P between s and t in G . Then $r_{st}(P) \leq n - 1$. We shall build up G from P by adding edges one by one. By Theorem 3.2.19, the effective resistance $r_{st}(P)$ never increases and thus

$$r_G(s, t) \leq r_P(s, t) = n - 1.$$

□

Proposition 3.2.21. *Let G be a graph with n vertices. If e is an edge of G such that $G - e$ is connected, then*

$$t(G - e) \geq \frac{t(G)}{n}.$$

Proof. We may assume that e is not a loop. Let $e = st$. By Lemma 3.2.20,

$$r_{G-e}(s, t) \geq n - 1,$$

which means

$$\frac{t(G/e)}{t(G - e)} \geq n - 1.$$

Hence

$$\frac{t(G - e)}{t(G)} = \frac{t(G - e)}{t(G - e) + t(G/e)} = \frac{1}{1 + \frac{t(G/e)}{t(G - e)}} \geq \frac{1}{n}.$$

□

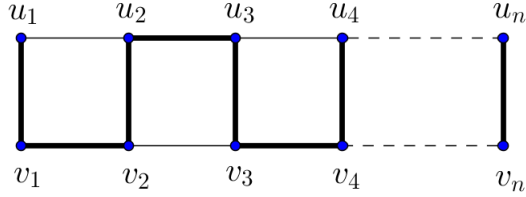


Figure 3.3: The graph L_n obtained from $MP_{2n}(s)$ by removing $s+1$ edges. The bold edges represent s parallel edges.

Proposition 3.2.21 implies that if we remove $s+1$ edges from $MP_{2n}(s)$, then the number of spanning trees changes by at most n^{s+1} , a polynomial in s .

Recall the following two theorems.

Theorem 3.2.22 ([Alo90]). *Let G be a k -regular connected simple graph with n vertices. Then*

$$t(G) \geq (k(1 - o(1)))^n.$$

Theorem 3.2.23 ([Kos95]). *Let G be a connected simple graph on n vertices with degree sequence $1 < k = d_1 \leq d_2 \leq d_3 \leq \dots \leq d_n$. Let $d(G) = \prod_{i=1}^n d_i$. Then*

$$t(G) \geq d(G)k^{-nO(\log k/k)}.$$

Theorems 3.2.22 and 3.2.23 imply that, for fixed minimum degree, the number of spanning trees is at least an exponential function of the number of vertices. Clearly, the number of spanning trees of a regular graph is bounded above by an exponential function of the number of vertices.

Thus, to find the asymptotic behaviour of $t(MP_{2n}(s))$, we consider the graph L_n in Figure 3.3 obtained from $MP_{2n}(s)$ by removing $s+1$ edges using the observation before Theorem 3.2.22. It is easy to see that

$$\begin{aligned} \limsup_{n \rightarrow \infty} t(MP_{2n}(s))^{1/(2n)} &= \limsup_{n \rightarrow \infty} t(L_n)^{1/(2n)}, \\ \liminf_{n \rightarrow \infty} t(MP_{2n}(s))^{1/(2n)} &= \liminf_{n \rightarrow \infty} t(L_n)^{1/(2n)}. \end{aligned}$$

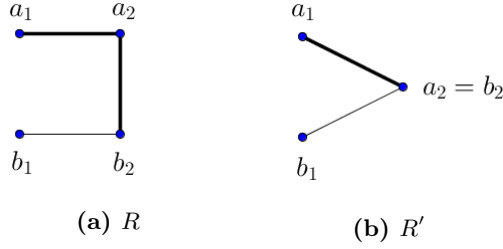


Figure 3.4: Two graphs R and R' to be added to L_n to form L_{n+1} and $L_{n+1}/u_{n+1}v_{n+1}$. The bold edges represent s parallel edges.

The graph L_n has $2n$ vertices, u_1, u_2, \dots, u_n and v_1, v_2, \dots, v_n , and edges $u_i v_i$ for $1 \leq i \leq n$, $u_i u_{i+1}$ for $1 \leq i < n$ and $v_i v_{i+1}$ for $1 \leq i < n$. The edges in the path $u_0 v_0 v_1 u_1 u_2 \dots$ are replaced by s parallel edges.

To find the number $t(L_n)$, I shall use the splitting formula (Theorem 2.2.6). Note that the graph L_{n+1} can be obtained from the 2-fragment L_n with labelled vertices u_n, v_n by adding the 2-fragment R in Figure 3.4 with labelled vertices a_1, b_1 . The splitting formula for spanning trees, obtained from Theorem 2.2.6 by setting $x = 1$ and $y = 1$, states that

$$t(L_{n+1}) = t_v(L_n)^t t_f(R)$$

where

$$t_v(L_n) = \begin{bmatrix} t(L_n) \\ t(L_n/u_n v_n) \end{bmatrix}, \quad t_f(R) = N_2(1, 1) t_v(R) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t(R) \\ t(R/a_1 b_1) \end{bmatrix}.$$

But to find $t(L_{n+2})$ again from L_{n+1} , we need to know the Tutte vector

$$t_v(L_{n+1}) = \begin{bmatrix} t(L_{n+1}) \\ t(L_{n+1}/u_{n+1}v_{n+1}) \end{bmatrix}$$

which contains the number $t(L_{n+1}/u_{n+1}v_{n+1})$ also. Similarly to above, the graph $L_{n+1}/u_{n+1}v_{n+1}$ can be obtained from L_n by adding R' , and from the splitting formula we get

$$t(L_{n+1}/u_{n+1}v_{n+1}) = t_v(L_n)^t t_f(R')$$

where

$$t_f(R') = N_2(1, 1) t_v(R') = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t(R') \\ t(R'/a_1 b_1) \end{bmatrix}.$$

Therefore, the Tutte vector $t_v(L_{n+1})$ is

$$t_v(L_{n+1})^t = [t(L_{n+1}) \ t(L_{n+1}/u_{n+1}v_{n+1})] = t_v(L_n)^t [t_f(R) \ t_f(R')].$$

Since

$$t_f(R) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t(R) \\ t(R/a_1b_1) \end{pmatrix} = \begin{pmatrix} s^2 + 2s \\ s^2 \end{pmatrix}$$

and

$$t_f(R') = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t(R') \\ t(R'/a_1b_1) \end{pmatrix} = \begin{pmatrix} s + 1 \\ s \end{pmatrix},$$

the Tutte vector $t_v(L_{n+1})$ is equal to

$$t_v(L_{n+1})^t = t_v(L_n)^t \begin{pmatrix} s^2 + 2s & s + 1 \\ s^2 & s \end{pmatrix}.$$

Using $t_v(L_1) = (s, 1)$,

$$t_v(L_n) = (s, 1) \begin{pmatrix} s^2 + 2s & s + 1 \\ s^2 & s \end{pmatrix}^{n-1}. \quad (3.11)$$

The eigenvalues of the matrix

$$\begin{pmatrix} s^2 + 2s & s + 1 \\ s^2 & s \end{pmatrix}$$

are

$$\lambda_{\pm} = \frac{1}{2}s \left(s + 3 \pm \sqrt{s^2 + 6s + 5} \right),$$

and asymptotically $\lambda_+ = \left(s + \frac{3}{2} + O\left(\frac{1}{s}\right) \right)^2$. So far we have shown the following.

Theorem 3.2.24. *Let $MP_{2n}(s)$ be the multiprism on $2n$ vertices, which is $(2s+1)$ -regular and $(2s+1)$ -edge-connected. Then*

$$\lim_{n \rightarrow \infty} [t(MP_{2n}(s))]^{1/(2n)} = \lim_{n \rightarrow \infty} t(L_n)^{1/(2n)} = \lim_{n \rightarrow \infty} \lambda_+^{1/2} = s + \frac{3}{2} + O\left(\frac{1}{s}\right).$$

Recall that our interest was in the following number for odd k , $k \geq 3$:

$$\tau_k = \liminf_{n \rightarrow \infty} \min_{\substack{|V(G)|=n \\ G \text{ } k\text{-edge-connected}}} t(G)^{1/n}.$$

Theorem 3.2.9 showed that $\tau_k > \frac{k}{2}$ for each odd $k \geq 3$, but the lower bound in Theorem 3.2.9 tends to $\frac{k}{2}$ as k tends to infinity. On the other hand, the cycle of length n with edge-multiplicity $\frac{k+1}{2}$ gives $\tau_k \leq \frac{k+1}{2}$. The multiprisms we investigated above gives asymptotically $s + \frac{3}{2} = \frac{k}{2} + 1$ which is bigger than $\frac{k+1}{2}$. We believe that the multiprisms have asymptotically minimum number of spanning trees among k -edge-connected k -regular graphs for odd k , and hence we conjecture the following.

Conjecture 3.2.25. *If $k \geq 5$ is odd, then $\tau_k = \tau_{k+1} = \frac{k+1}{2}$.*

Conjecture 3.2.26. *Let*

$$\tau'_k = \liminf_{n \rightarrow \infty} \min_{\substack{|V(G)|=n \\ G \text{ } k\text{-edge-connected} \\ G \text{ } k\text{-regular}}} t(G)^{1/n}.$$

If $k \geq 5$ is odd, then $\tau'_k > \tau'_{k+1} = \frac{k+1}{2}$.

In other words, Conjecture 3.2.26 claims that if $k \geq 5$ is odd, then each k -edge-connected k -regular graph on n vertices has more spanning trees than the cycle of length n with edge-multiplicity $\frac{k+1}{2}$, which is $(k+1)$ -edge-connected $(k+1)$ -regular.

For the sake of completeness we find an explicit formula for $t(L_n)$ and $t(MP_{2n}(s))$. The number $t(L_n)$ is an entry of the vector $t_v(L_n)$ given by Equation (3.11).

$$t(L_n) = \frac{\lambda_+^n - \lambda_-^n}{\sqrt{s^2 + 6s + 5}}.$$

Now we find the exact formula of $t(MP_{2n}(s))$. The formula is complicated, but we now try to explain the ideas leading to the formula. Rubey [Rub00] used the following method to calculate the exact number of spanning trees of prisms, and we apply it to the multiprisms. We do the calculation only when n is even, although the same method can be applied to when n is odd resulting in a slightly different exact formula.

Let us start with drawing $MP_{2n}(s)$ as in Figure 3.5, where the bold lines represent s parallel edges.

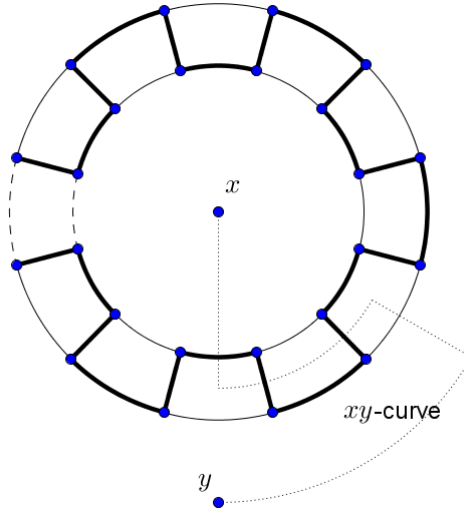


Figure 3.5: Curves from inside to outside of $MP_{2n}(s)$.

The drawing has two faces, the inner and the outer, with boundary cycle of length n . We choose a point x in the inner face and another point y in the outer face. Let us consider a simple curve C on the plane from x to y not containing the vertices of $MP_{2n}(s)$. We also require that the curve C visits each face of the drawing at most once. We shall call such a curve an *xy-curve*. The set of edges of $MP_{2n}(s)$ intersecting with C is completely determined by an edge of the inner boundary cycle, another edge of the outer boundary cycle and the orientation, either clockwise or anticlockwise unless the two chosen edges are on the same quadrangle. Thus there are precisely $2n(n-1) + n$ distinct sets of edges obtained from such curves.

For each of such set of edges, say $E(C)$ for a *xy-curve* C , the complement $MP_{2n}(s) - E(C)$ is a spanning connected subgraph of $MP_{2n}(s)$ so that it contains spanning trees of $MP_{2n}(s)$. Since a tree drawn on the plane has only a single face, each spanning tree of $MP_{2n}(s)$ avoids a curve connecting x and y , implying that by summing up $t(MP_{2n}(s) - E(C))$ over all possible $E(C)$ we count each spanning tree of $MP_{2n}(s)$ at least once. Moreover, if C_1, C_2 are two *xy-curves* such that $E(C_1) \neq E(C_2)$, then the closed curve $C_1 \cup C_2$ separates the vertices of $MP_{2n}(s)$ so that no spanning tree of $MP_{2n}(s)$ is contained in both $MP_{2n}(s) - E(C_1)$ and

$MP_{2n}(s) - E(C_2)$. Therefore, the summation

$$\sum_{E(C)} t(MP_{2n}(s) - E(C))$$

is precisely $t(MP_{2n}(s))$. The rest of this section is to find individual $t(MP_{2n}(s) - E(C))$ and then sum up the results.

Let L_n be the graph in Figure 3.3 on $2n$ vertices and denote $l_n = t(L_n)$. If a xy -curve C passes through f quadrangular faces for an odd number f , then

$$t(MP_{2n}(s) - E(C)) = s^{f-1}l_{n-f}$$

and there are $2n$ such sets $E(C)$ if $f > 1$ and n sets if $f = 1$.

If f is even then $t(MP_{2n}(s) - E(C)) = s^{f-2}l_{n-f}$ or $s^f l_{n-f}$ and each case happens half the times out of $2n$ possibilities of $E(C)$. Hence

$$\begin{aligned} \sum_{E(C)} t(MP_{2n}(s) - E(C)) &= nl_{n-1} + \sum_{\substack{3 \leq f \leq n-1 \\ f \text{ odd}}} 2ns^{f-1}l_{n-f} + \sum_{\substack{2 \leq f \leq n \\ f \text{ even}}} (ns^{f-2}l_{n-f} + ns^f l_{n-f}) \\ &= nl_{n-1} + 2n \sum_{f=1}^{n/2-1} s^{2f} l_{n-2f-1} + n(1+s^2) \sum_{f=0}^{n/2-1} s^{2f} l_{n-2f-2}. \end{aligned}$$

Using

$$t(L_n) = \frac{\lambda_+^n - \lambda_-^n}{\sqrt{s^2 + 6s + 5}}$$

we get

$$t(MP_{2n}(s)) = \frac{1}{\sqrt{s^2 + 6s + 5}} \left[\begin{aligned} &n\lambda_+^{n-1} + 2n \sum_{f=1}^{n/2-1} s^{2f} \lambda_+^{n-2f-1} + n(1+s^2) \sum_{f=0}^{n/2-1} s^{2f} \lambda_+^{n-2f-2} \\ &-n\lambda_-^{n-1} - 2n \sum_{f=1}^{n/2-1} s^{2f} \lambda_-^{n-2f-1} - n(1+s^2) \sum_{f=0}^{n/2-1} s^{2f} \lambda_-^{n-2f-2} \end{aligned} \right]$$

Since the summations are geometric,

$$t(MP_{2n}(s)) = \frac{n}{\sqrt{s^2 + 6s + 5}} \left[\begin{aligned} &\lambda_+^{n-1} + 2s^2 \frac{\lambda_+^{n-1} - \lambda_+ s^{n-3}}{\lambda_+^2 - s^2} + (1+s^2) \frac{\lambda_+^n - s^n}{\lambda_+^2 - s^2} \\ &-\lambda_-^{n-1} - 2s^2 \frac{\lambda_-^{n-1} - \lambda_- s^{n-3}}{\lambda_-^2 - s^2} - (1+s^2) \frac{\lambda_-^n - s^n}{\lambda_-^2 - s^2} \end{aligned} \right].$$

Acyclic and Totally Cyclic Orientations

4.1 Acyclic and totally cyclic orientations

In [MW99], Merino and Welsh posed the following conjecture which is still open. The numbers $t(G)$, $a(G)$ and $c(G)$ are respectively the number of spanning trees, acyclic orientations, and totally cyclic orientations of G .

Conjecture 4.1.1 ([MW99]). *Let G be a bridgeless loopless graph. Then*

$$t(G) \leq \max\{a(G), c(G)\}. \quad (4.1)$$

Thomassen showed that Inequality (4.1) is true for graphs with n vertices and m edges where $m \leq \frac{16}{15}n$ or $m \geq 4n - 4$. His strategy is to find bounds for $a(G)$, $c(G)$, and $t(G)$ separately in terms of the number of vertices and edges.

The purpose of this section is to improve these results as follows:

Theorem (4.1.9). *Let G be a loopless graph with n vertices and m edges. If $m \leq 1.29(n - 1)$, then $t(G) \leq a(G)$.*

Theorem (4.1.15). *Let G be a 3-edge-connected graph with n vertices and m edges. If $m \geq 3.58(n - 1)$, then $t(G) \leq c(G)$.*

We begin with recalling some definitions. Given a graph G , an **orientation** π (or $\pi(G)$) on G is a directed graph on G . For $e \in E(G)$ with tail u and head v , we write $\pi(e) = (u, v)$ and call it the **orientation of e** (with respect to π). A loop is considered to have two orientations by convention.

The following is an upper bound for $t(G)$ used in [Tho10].

Theorem 4.1.2 ([Tho10]). *Let G be a graph on n vertices with degrees d_1, d_2, \dots, d_n . Then*

$$t(G) \leq d_1 d_2 \dots d_{n-1}$$

with the equality if and only if $d_i = 0$ for some $i < n$ or the vertex of degree d_n is incident with all edges.

Proof. The theorem is clearly true for graphs on two vertices. We may also assume that G has no loops.

We shall prove the theorem by induction on n . If $d_i = 0$ for some $i < n$ or the vertex of degree d_n , say v_n , is incident with all edges then $t(G) = d_1 d_2 \dots d_{n-1}$. So assume that G has an edge e not incident with v_n . We may assume that e connects two vertices of degrees d_1 and d_2 . Then

$$\begin{aligned} t(G) &= t(G - e) + t(G/e) \\ &\leq (d_1 - 1)(d_2 - 1)d_3 d_4 \dots d_{n-1} + (d_1 - 1 + d_2 - 1)d_3 d_4 \dots d_{n-1} \\ &= (d_1 d_2 - 1)d_3 d_4 \dots d_{n-1} \\ &< d_1 d_2 d_3 \dots d_{n-1} \end{aligned}$$

□

Corollary 4.1.3 ([Tho10]). *Let G be a graph on n vertices and m edges. Then*

$$t(G) < \left(\frac{2m}{n} \right)^{n-1}.$$

Proof. Let d_n be the maximum degree of G . The sum of degrees is $2m$, and the product of other degrees is maximized when $2m - d_n$ is evenly distributed. \square

Also we have simple bounds for $a(G)$ and $c(G)$.

Lemma 4.1.4 ([Tho10]). *Let G be a loopless connected graph on n vertices. Then*

$$a(G) \geq 2^{n-1}.$$

Proof. A spanning tree of G has precisely 2^{n-1} acyclic orientations. It is enough to show that any acyclic orientation can be extended to another acyclic orientation when a non-loop edge is added.

Let D be an acyclic digraph and let u, v be two distinct vertices of D . If D has a directed path from u to v and also from v to u , then the union of these two paths is a closed walk (with possibly repeating directed edges) so that it has a directed cycle inside, contradiction to the assumption that D is acyclic. Thus we may assume that D has no directed path from u to v , and an edge uv can be added to D , oriented from v to u so that the augmented orientation is still acyclic. \square

Lemma 4.1.5 ([Tho10]). *Let G be a bridgeless connected graph on n vertices. Then*

$$c(G) \geq 2^{m-n+1}.$$

Proof. A bridgeless connected graph can be constructed from a cycle by recursively adding paths which intersect the previous graph only at its ends. We consider such a construction of G . The starting cycle has two totally cyclic orientations. Whenever we add a new path, we may orient the path into two different directed paths (or cycles) and still get a totally cyclic orientation. Since adding a new path always increases $m - n$ by 1, we have $c(G) \geq 2^{m-n+1}$. \square

Using Corollary 4.1.3 and Lemma 4.1.5, Thomassen proved the following.

Theorem 4.1.6 ([Tho10]). *Let G be a bridgeless graph with n vertices and m edges. If $m \geq 4n - 4$, then*

$$t(G) \leq c(G).$$

Also, using an inductive argument, he proved the following.

Theorem 4.1.7 ([Tho10]). *Let G be a loopless graph with n vertices and m edges. If $m \leq \frac{16}{15}n$, then*

$$t(G) \leq a(G).$$

For graphs with small average degree, the following simple bound is stronger than Corollary 4.1.3.

Observation 4.1.8. *Let G be a graph with n vertices and m edges. Then*

$$t(G) \leq \binom{m}{n-1}.$$

Using Observation 4.1.8, we can improve the constant $\frac{16}{15} \approx 1.067$ in Theorem 4.1.7.

Theorem 4.1.9. *Let G be a loopless bridgeless graph with n vertices and m edges. If $m \leq 1.29(n-1)$, then $t(G) \leq a(G)$.*

Proof. We shall use the Stirling's approximation:

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \leq n! \leq e\sqrt{n} \left(\frac{n}{e}\right)^n.$$

By Observation 4.1.8, the number $t(G)$ is at most

$$\begin{aligned} \binom{m}{n-1} &= \frac{m!}{(n-1)!(m-n+1)!} \\ &\leq \frac{e}{2\pi} \left(\frac{m}{(n-1)(m-n+1)} \right)^{1/2} \frac{m^m}{(n-1)^{n-1}(m-n+1)^{m-n+1}} \end{aligned}$$

Dividing the numerators and denominators by $n-1$, and setting $\alpha = \frac{m}{n-1}$, the last formula becomes

$$\frac{e}{2\pi\sqrt{n-1}} \left(\frac{\alpha}{\alpha-1} \right)^{1/2} \left(\frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}} \right)^{n-1} \quad (4.2)$$

By Theorem 4.1.7, we may assume that $m \geq \frac{16}{15}(n-1)$, so that $\alpha \geq \frac{16}{15}$ and $\frac{\alpha}{\alpha-1} \leq 16$. Therefore, it is enough to show that

$$\frac{2e}{\pi\sqrt{n-1}} \left(\frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}} \right)^{n-1} \leq 2^{n-1}.$$

We have $\frac{2e}{\pi\sqrt{n-1}} < 1$ for $n > 3$ and the function $f(x) = \frac{x^x}{(x-1)^{x-1}}$ is increasing for $x > 1$ and $f(x) = 2$ at $x = 1.2938\dots$. Since the statement is easy to check for $n \leq 3$, we conclude that if $\alpha \leq 1.29$, or equivalently $m \leq 1.29(n-1)$, then $t(G) \leq a(G)$. \square

Note that when $m = 4n$, the number $\binom{m}{n}$ is approximately 9.48^n whereas Thomassen's bound is $\left(\frac{2m}{n}\right)^n = 8^n$. Thus for $c(G)$, this approach does not improve Theorem 4.1.6. Instead, we shall improve the multiplicative constant 4 in Theorem 4.1.6 by imposing an additional condition on the graphs. For that we shall use the notion of flippable edges defined below.

Let π be an acyclic (totally cyclic) orientation on a graph G . Let e be an edge. If the orientation on G obtained from π by reversing the orientation of e is still acyclic (totally cyclic), then e is a **flippable edge** of π . Recall that we write $\pi(e) = (x, y)$ when $e = xy$ and the orientation of e in π is from x to y . If $X \subseteq V$ is a vertex subset of a graph $G = (V, E)$, then $\delta(X)$ is the set of edges having one end in X and the other end not in X . In a directed graph, $\delta^+(X)$ is the set of directed edges with tail in X and head not in X . We define $\delta^-(X) = \delta^+(V \setminus X)$.

The following observations follow immediately.

Observation 4.1.10. *Let G be a graph with an acyclic orientation π . A directed edge $\pi(e) = (v, w)$ is not flippable if and only if $\pi(G) - e$ has a directed path from v to w .*

Observation 4.1.11. *Let G be a graph with a totally cyclic orientation π . A directed edge $\pi(e) = (v, w)$ is not flippable if and only if G has a vertex set X_e such that $\delta^+(X_e) = \{e\}$.*

The concept of flippable edges in acyclic and totally cyclic orientations were considered in [FPS01, CF], as a connection between the so-called hyperplane arrangement and the orientations of graphs.

Proposition 4.1.12 ([CF]). *Let G be a connected simple graph with n vertices and m edges. Then each acyclic orientation on G has at least $n - 1$ flippable edges.*

Proof. Let π be an acyclic orientation on G , and let v, w be two distinct vertices of G connected by a directed path from v to w . Let P be a longest directed path in $\pi(G)$ from v to w . Suppose that a directed edge $\pi(e) = (x, y)$ in P is not flippable. Then $\pi(G) - \pi(e)$ has a directed path from x to y , which must not intersect with P because of acyclicity. With this path and $P - \pi(e)$, $\pi(G)$ has a path from v to w longer than P which is a contradiction to the choice of P . Thus every edge of P is flippable, and the set of flippable edges of π form a connected subgraph of G , which means π has at least $n - 1$ flippable edges. \square

Note that we need G to be simple in Proposition 4.1.12. A multigraph in which each edge has another parallel edge does not have an acyclic orientation with flippable edges.

We shall need the following counterpart to Proposition 4.1.12. Note that for planar graphs, Propositions 4.1.12, 4.1.13 are equivalent by duality.

Proposition 4.1.13. *Let G be a 3-edge-connected graph with n vertices and m edges. Then each totally cyclic orientation on G has at least $m - n + 1$ flippable edges.*

Proof. We prove that the set of non-flippable edges has no cycle in the underlying graph G . We shall use induction on the number of vertices. We may assume that G has at least three vertices. For contradiction, suppose that G is a smallest counterexample and π be a totally cyclic orientation on G in which some of its non-flippable edges form a cycle C . Each non-flippable edge, say e_0 , is associated with a vertex set, denoted by X_{e_0} , such that $\delta^+(X_{e_0}) = \{e_0\}$. Let e be an edge of C . Because C is a cycle, π has another non-flippable edge $f \in \delta(X_e)$. Let π' be the orientation on G/f induced by π . Note that π' is also totally cyclic. We claim that e is nonflippable in π' .

Let us consider the four subsets of $V(G)$ divided by the cuts $\delta(X_e)$ and $\delta(X_f)$. From $f \in \delta(X_e)$ and $\delta_\pi^+(X_e) = \{e\}$, we see that $\pi(f)$ is from $X_e^c \cap X_f$ to $X_e \cap X_f^c$. There are three cases for e :

1. If $\pi(e)$ is from $X_e \cap X_f^c$ to $X_e^c \cap X_f$, then $\delta_\pi^+(X_e \cap X_f) = \delta_\pi^-(X_e^c \cap X_f^c) = \emptyset$ since $\delta^+(X_e) = \{e\}$ and $\delta^+(X_f) = \{f\}$. It implies that $X_e \cap X_f = X_e^c \cap X_f^c = \emptyset$ because $\pi(G)$ is strongly connected. Thus e and f form a 2-edge-cut, a contradiction to G being 3-edge-connected.
2. Suppose that $\pi(e)$ is from $X_e \cap X_f$ to $X_e^c \cap X_f$. Then $\delta_\pi^-(X_e^c \cap X_f^c) = \emptyset$, implying $X_e^c \cap X_f^c = \emptyset$. Note that $\delta_\pi^+(X_e \cap X_f) = \{\pi(e)\}$, which is an edge cut in G/f making e non-flippable in π' .
3. The remaining case is that $\pi(e)$ is from $X_e \cap X_f^c$ to $X_e^c \cap X_f^c$. Similarly to (2), we see $X_e \cap X_f = \emptyset$ and $\delta_\pi^-(X_e^c \cap X_f^c) = \{\pi(e)\}$.

Suppose that C has an edge e' different from e and f . If $f \notin \delta(X_{e'})$ then e' remains non-flippable in π' . If $f \in \delta(X_{e'})$ then by the claim above e' is non-flippable in π' . Thus every edge of C remains non-flippable in π' , contradiction to the induction hypothesis.

Therefore, the set of non-flippable edges form a forest, so that the number of flippable edges is at least $m - n + 1$. \square

Using Proposition 4.1.13, we can find a lower bound on $c(G)$ which is stronger than Lemma 4.1.5.

Proposition 4.1.14. *Let G be a 3-edge-connected graph. If G has n edges and m vertices, then $c(G) \geq \left(\frac{6}{5}\right)^{n-2} 6 \cdot 2^{m-n-1}$*

Proof. We use induction on n . We may assume that G has no loop. Since it is easy to check the inequality when $n = 2$, we also assume that $n > 2$.

By Proposition 4.1.13, each totally cyclic orientation on G has at least $m - n + 1$ flippable edges. Hence the sum of the numbers of flippable edges in all totally cyclic orientations on G is at least $(m - n + 1)c(G)$, so that we can choose an edge e such that e is flippable in at least $\frac{m - n + 1}{m}c(G)$ totally cyclic orientations.

Let $2D$ be the number of totally cyclic orientations on G in which e is flippable. Let $S = c(G) - 2D$. Since a totally cyclic orientation π on G has e as its flippable edge if and only if $\pi(G - e)$ is totally cyclic on $G - e$, we have $c(G - e) = D$ and

$c(G/e) = D + S$. Because of the choice of e ,

$$S \leq \frac{n-1}{m} c(G) \leq \frac{2(n-1)}{m-n+1} \frac{m-n+1}{2m} c(G) \leq \frac{2(n-1)}{m-n+1} D.$$

Thus, $\frac{c(G)}{c(G/e)} = \frac{2D+S}{D+S} \geq \frac{2m}{m+n-1} \geq \frac{6}{5}$. The last inequality comes from $2m \geq 3n$, because G is 3-edge-connected. Note that G/e is again 3-edge-connected with $n-1$ vertices and $m-1$ edges. By the induction hypothesis,

$$c(G) \geq \frac{6}{5} c(G/e) \geq \left(\frac{6}{5}\right)^{n-2} 6 \cdot 2^{m-n-1}.$$

□

Using Proposition 4.1.14 with Lemma 4.1.5, we can improve Theorem 4.1.6 for 3-edge-connected graphs.

Theorem 4.1.15. *Let G be a 3-edge-connected graph with n vertices and m edges. If $m \geq 3.58(n-1)$, then $t(G) \leq c(G)$.*

Proof. Let $\alpha = \frac{m}{n-1}$. Lemma 4.1.5 says that

$$t(G) < \left(\frac{2m}{n}\right)^{n-1} < (2\alpha)^{n-1}.$$

And Proposition 4.1.14 says that

$$c(G) \geq \left(\frac{6}{5}\right)^{n-2} 6 \cdot 2^{m-n+1} > \left(\frac{6}{5}\right)^{n-1} 2^{(\alpha-1)(n-1)}.$$

Thus it is enough to have that

$$2\alpha \leq \frac{6}{5} \cdot 2^{\alpha-1},$$

which is true for $\alpha \geq 3.5748\dots$.

□

On the other hand, Proposition 4.1.12 cannot give a useful lower bound for $a(G)$ because of the following example. See Figure 4.1.

Let us fix a constant $c > 0$. For sufficiently large t and $n = \lfloor ct^2 \rfloor$, we consider the graph G obtained from the complete graph K_t by adding a path of length $n-t+1$

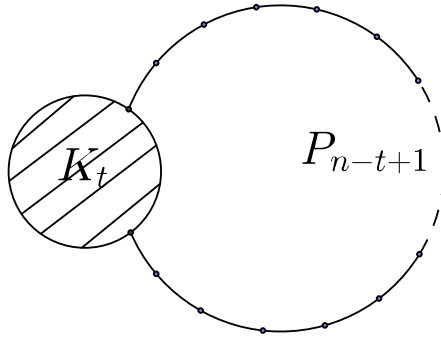


Figure 4.1: Graphs showing that it is impossible to get an exponential lower bound $> 2^n$ for $a(G)$ even when G has high average degree

between two vertices of K_t so that G has n vertices. The average degree of G is asymptotically

$$\frac{t^2 - t + 2(n - t + 1)}{n} = 2 + \frac{1}{c} + o(1),$$

and we have an upper bound for the acyclic orientation such that

$$a(G) \leq t! \cdot 2^{n-t+1} < t^t 2^{n-t+1} = (2 + o(1))^n,$$

so that a lower bound of type $a(G) > K^n$ for a constant $K > 2$ is impossible even if we assume that G is 2-connected, simple and has high average degree.

In [Tho10], Thomassen proved the following theorem.

Theorem 4.1.16. *If G is a loopless graph of maximum degree at most 3, then $t(G) \leq a(G)$.*

Thomassen used the following reduction lemma but the proof in [Tho10] missed a little detail. Here we prove the lemma again to complete Theorem 4.1.16.

Lemma 4.1.17. *A counterexample to Theorem 4.1.16 with minimum number of edges must be a simple bridgeless cubic graph.*

Proof. Assume that G is a counterexample to Theorem 4.1.16 with the minimum number of edges. If G has a bridge then we remove it and by applying the minimality of G to the resulting two components, we get $t(G) \leq a(G)$ which is impossible. If $v \in V(G)$ is incident with a double edge but no other edge, then

$t(G) = 2t(G-v) \leq 2a(G-v) = a(G)$, hence G has no such vertex. If $v \in V(G)$ has only two incident edges vv_1, vv_2 where $v_1 \neq v_2$, then let $H = G - v + v_1v_2$. By the assumption on G we have $t(H) \leq a(H)$ and it is easy to check that $t(G) \leq 2t(H)$ and $a(G) \geq 3a(H)$, so that $t(G) \leq a(G)$ which is a contradiction. Thus we may assume that G is a cubic bridgless connected graph.

Suppose that G has a double edge between two vertices y, z and also $xy, zu \in E(G)$ where $\{x, u\} \cap \{y, z\} = \emptyset$. We may assume that $x \neq u$. Let $H = G - y - z + xu$. Then $t(H) \leq a(H)$ by the minimality of G . It is easy to see that $a(G) \geq 4a(H)$. We shall show that $t(G) \leq 3.5t(H)$, so that $t(G) \leq a(G)$ which is a contradiction. By the Rayleigh's monotonicity (Theorem 3.2.19), the effective resistance $r_H(x, u) = \frac{t(H/xu)}{t(H)}$ is at least the effective resistance between x and u after identifying all other vertices in H into one, which is $1/2$. Thus $2t(H/xu) \geq t(H)$ or equivalently $t(H/xu) \geq t(H - xu)$, and

$$t(G) = 5 t(H - xu) + 2 t(H/xu) \leq 3.5 t(H - xu) + 3.5 t(H/xu) = 3.5 t(H).$$

□

Thomassen asked the following questions in [Tho10].

1. Is $t(G) \leq a(G)$ when $m \leq 2n - 2$?
2. Is $t(G) \leq c(G)$ when $m \geq 2n - 2$?

But both were answered negatively by Noble and Royle [NR14]. They provided an example with $2n - 2$ edges, see Figure 4.2 for $t(G) > a(G)$ and consider its dual for $t(G) > c(G)$.

Noble and Royle stated that they do not know of a graph with $m < 2n - 2$ and $t(G) < a(G)$, but by slightly modifying their graph, we can find such an example.

Let G be a cycle of length n where $n - k$ of its edges are replaced by two parallel edges. It is easy to see that

$$\begin{aligned} t(G) &= (n + k)2^{n-k-1}, \\ a(G) &= 2^n - 2. \end{aligned}$$

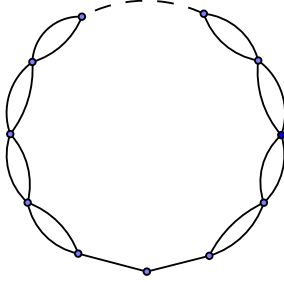


Figure 4.2: Noble and Royle's example

Hence, if $n > 2^{k+2}$ then $t(G) > a(G)$. By taking the dual of G we obtain a graph with n' vertices and $2n' + k - 4$ edges such that $t(G) > c(G)$ where $n' = n - k + 2$.

Therefore, we replace Thomassen's questions by the following weaker questions:

Open Problem 1. *Let G be a graph with n vertices and m edges. Is there a constant $c > 0$ such that the following hold?*

1. *If G is loopless and $m \leq 2n - c \cdot \log n$ edges, then $t(G) \leq a(G)$.*
2. *If G is bridgeless and $m \geq 2n + c \cdot \log n$ edges, then $t(G) \leq c(G)$.*

The above example suggests that the constant c , if it exists, must be at least 1 when the base of the logarithm is 2.

If Open Problem 1 has a negative answer perhaps the following holds:

Open Problem 2. *Let G be a loopless bridgeless graph with n vertices and m edges. Are the following true for each $\epsilon > 0$ and sufficiently large n ?*

1. *If G is loopless and $m \leq (2 - \epsilon)n$, then $t(G) \leq a(G)$.*
2. *If G is bridgeless and $m \geq (2 + \epsilon)n$, then $t(G) \leq c(G)$.*

The Merino-Welsh Conjecture for Bounded Treewidth

5.1 Introduction

In chapters 3, 4 we considered the Merino-Welsh conjecture. The multiplicative version of the Merino-Welsh conjecture is stated below for convenience.

Conjecture 5.1.1. *Let G be a loopless bridgeless graph. Then*

$$T(G; 1, 1)^2 \leq T(G; 2, 0)T(G; 0, 2).$$

In 2014, Noble and Royle proved Conjecture 5.1.1 for series-parallel graphs.

Theorem 5.1.2 ([NR14]). *Let G be a 2-connected series-parallel graphs. Then*

$$T(G; 1, 1)^2 \leq T(G; 2, 0)T(G; 0, 2).$$

Noble and Royle’s proof is based on an algorithm which resulted in a ‘test-space’ of 18 graphs such that, the statement holds for all series-parallel graphs if it holds for the test-space.

I extended their method using my version of the splitting formula, Theorem 2.2.6, so that it is applicable to the class of graphs with bounded treewidth or pathwidth. Let \mathcal{C}_k be the class of loopless bridgeless graphs with treewidth (or pathwidth) at most k . If my algorithm halts in finite time for \mathcal{C}_k , then the algorithm produces a test-space of k -fragments such that the Conjecture 5.1.1 holds for \mathcal{C}_k if it holds for the test-space. I applied the algorithm for graphs with pathwidth at most 3, and it resulted in a test-space of 5242 2- and 3-fragments, and it showed the following.

Theorem 5.1.3. *Let G be a loopless bridgeless graph of pathwidth at most 3. Then*

$$T(G; 1, 1)^2 \leq T(G; 2, 0)T(G; 0, 2).$$

In Section 5.2 we define the pathwidth and present well-known basic properties of pathwidth. Then we present a natural construction of all 2-connected graphs with pathwidth at most 3 using simple local operations, mostly adding one vertex at a time. The detailed algorithm using the splitting formula shall be given in Section 5.3.

5.2 Treewidth and pathwidth

This section is to present the concept of pathwidth for completeness and provide a list of simple local operations on 2- and 3-fragments which construct all 2-connected graphs of pathwidth at most 3, starting from a single edge.

Given a graph G , a **path decomposition** of G is a pair $(P, \{B_x\}_{x \in V(P)})$ where P is a path and $B_x \subseteq V(G)$ for each $x \in V(P)$ such that the following conditions hold.

- For each edge e of G , there is an $x \in V(P)$ such that B_x contains both ends of e .

- For each $u \in V(G)$, the set $\{x \in V(P) : u \in B_x\}$ induces a subpath of P , namely P_u .

The sets B_x are called the **bags**. The **width** of a path decomposition $(P, \{B_x\}_{x \in V(P)})$ is defined by $\max_{x \in V(P)} |B_x| - 1$ and the **pathwidth** of G , denoted by $\text{pw}(G)$, is the minimum width over all tree decompositions of G . For surveys on width parameters, see [Bod93, Lov05].

Let $(P, \{B_x\}_{x \in V(P)})$ be a path decomposition of G . We may define an assignment $\phi : E(G) \rightarrow \{B_x\}_{x \in V(P)}$ such that for each $e \in E(G)$, the bag $\phi(e)$ contains both ends of e . Throughout this chapter, we shall assume that every path decomposition comes with such an assignment, normally assumed implicitly.

Let $S \subseteq V(P)$ where $(P, \{B_x\}_{x \in V(P)})$ is a path decomposition of G . The subgraph of G with vertex set $\bigcup_{x \in S} B_x$ and edge set $\bigcup_{x \in S} \phi^{-1}(B_x)$ is called the subgraph of G **induced by** S and denoted by $G[S]$. If P' is a subpath of P , then $G[V(P')]$ is also denoted by $G[P']$.

We shall add the following two restrictions on path decompositions, which do not change the pathwidth.

1. For each edge xy of P , each component of $P - xy$ contains at least one P_u for some $u \in V(G)$.
2. $B_x \not\subseteq B_y$ for any distinct $x, y \in V(P)$.

Note that $u, w \in V(G)$ are adjacent only if P_u, P_w intersect in P . Let xy be an edge of P . By the condition 1, $B_x \cap B_y$ is a vertex cut of $V(G)$ if G is connected. By the condition 2, this vertex cut has size at most $\text{pw}(G)$. Let C_x and C_y be the components of $P - xy$ containing x and y respectively, where xy is an edge of P . The graphs $G[C_x], G[C_y]$ are two subgraphs of G , meeting at $B_x \cap B_y$ and partitioning $E(G)$. We shall consider these subgraphs as $|B_x \cap B_y|$ -fragments.

Let G be a graph with a path decomposition $(P, \{B_x\}_{x \in V(P)})$. Let the vertices of P be p_0, p_1, \dots, p_r following the order on P . Theorem 5.2.2 below is about what kind of operations are needed to obtain $G[p_0 p_1 \dots p_{i+1}]$ from $G[p_0 p_1 \dots p_i]$.

We begin with the following lemma.

Lemma 5.2.1. *Let G be a loopless 2-connected graph. If $\text{pw}(G) \leq 3$, then G has a path decomposition $(P, \{B_x\}_{x \in V(P)})$ of width $\text{pw}(G)$ such that for each edge e of P , the two components of $P - e$ both induce connected subgraphs of G .*

Proof. It is trivial if $|V(G)| \leq 4$. We shall assume $|V(G)| > 4$.

Let $(P, \{B_x\}_{x \in V(P)})$ be a path decomposition of G with width $\text{pw}(G)$ such that $\sum_{x \in V(P)} |B_x|$ is the smallest. If a bag B_x contains another bag B_y then each of the bags between x and y in P contains B_y , so that we may contract the first edge of P on the path from y to x and get a smaller $\sum_{x \in V(P)} |B_x|$. Thus no bag can contain another bag.

Suppose that P has an edge e such that among the two components of $P - e$, namely P_1 and P_2 , one of them, say P_1 , induces a disconnected subgraph of G . Let $e = xy$ such that $x \in V(P_1)$. Let C be one of the components of $G[P_1]$. If $V(C) = \{v\}$ is a singleton, then we may delete v from all the bags in P_1 and still get a path decomposition of G with the same width, contradicting to the minimality of $\sum_{x \in V(P)} |B_x|$. Thus we may assume that each component of $G[P_1]$ contains at least two vertices. Since G is connected, the set $V(C) \cap B_x$ separates C from the rest of G . But $|B_x \cap B_y| \leq 3$, so that one of the components of $G[P_1]$ is separated from the rest of G by one vertex, contradicting to the assumption that G is 2-connected. Thus $G[P_1]$ is connected, resulting in a contradiction to the existence of e . \square

To illustrate the operations in the following theorem, we shall use the notation (H, C) for a k -fragment H whose labelled vertices are the elements of the set C .

Theorem 5.2.2. *Let G be a loopless 2-connected graph. If $\text{pw}(G) \leq 3$ and $|V(G)| > 4$, then G can be constructed from the smallest connected 2-fragment $(K_2, \{u, v\})$ by applying one of the following six types of operations repeatedly. Moreover, we may assume that the last operation to obtain G is to add an edge between the labelled vertices of a 2-fragment.*

1. An **expansion** of a 2-fragment $(H, \{u, v\})$ is the 3-fragment $(H', \{u, v, v'\})$ where H' is the graph obtained from H by adding a new vertex v' and an edge vv' .

2. A **restriction** of a 3-fragment $(H, \{u, v, w\})$ is the 2-fragment $(H, \{u, v\})$.
3. **Adding an edge to a fragment** (H, C) is to make a fragment (H', C) where H' is obtained from H by adding an edge connecting two distinct vertices in C .
4. **Adding a wedge to a fragment** (H, C) is to make a fragment (H', C) where H' is the graph obtained from H by adding a new vertex v being adjacent to two distinct vertices in C . The edges incident with v have multiplicities at least 1.
5. **Adding a claw to a 3-fragment** (H, C) is to make a 3-fragment (H', C) where H' is the graph obtained from H by adding a new vertex v being adjacent to three distinct vertices in C . The new edges have multiplicities at least 1.
6. A **transition** of a 3-fragment $(H, \{u, v, w\})$ is the 3-fragment $(H', \{u, v, w'\})$ where H' is the graph obtained from H by adding a new vertex w' which is joined to w with at least one edge.

Proof. By Lemma 5.2.1, we can find a path decomposition $(P, \{B_x\}_{x \in V(P)})$ of G such that for each edge e of P , the two components of $P - e$ both induce connected subgraphs of G . Let $P = x_0 e_0 x_1 \dots e_{r-1} x_r$ where $x_i \in V(P)$ and $e_j \in E(P)$ for all i, j . For $i = 0, 1, \dots, r - 1$, let $G_i = G[\{x_0, x_1, \dots, x_i\}]$ and let $C_i = B_{x_i} \cap B_{x_{i+1}}$. We shall consider G_i as a $|C_i|$ -fragment where C_i is the set of labelled vertices. If G_0 is not K_2 , then we can choose two vertices $u, v \in B_{x_0}$ such that u, v are adjacent in G since G_0 is connected and $|G_0| > 1$. By adding a bag $B_{x_{-1}} = \{u, v\}$ to B_{x_0} in P and assign one of the edges connecting u, v to $B_{x_{-1}}$, we may assume that G_0 is K_2 and $|C_0| = 2$. Likewise, we shall add $B_{x_{r+1}}$ to B_{x_r} in P where $G[x_{r+1}]$ is K_2 , so that G_r is a 2-fragment from which G is obtained by adding an edge between the two labelled vertices.

The proof is done once we show that we may obtain G_{i+1} from G_i in all cases using the operations in the statement. Note that $|C_i| = |B_{x_i} \cap B_{x_{i+1}}|$ so that $2 \leq |C_i| \leq 3$. We may assume that no bag is contained in another bag except B_{x_0} and $B_{x_{r+1}}$. Each of the following cases are reduced using appropriate relabelling of the labelled vertices.

Case 1. $|B_{x_{i+1}}| = 3$. Let $B_{x_{i+1}} = \{a, b, c\}$. We may assume that $C_i = \{a, b\}$ since no other bag contains $B_{x_{i+1}}$. There are two possibilities for C_{i+1} , either $\{a, b\}$ or $\{a, c\}$. If $C_{i+1} = \{a, b\}$, then the vertex c is adjacent to only a, b in G . Since G

is 2-connected, both a, b are neighbors of c so that (G_{i+1}, C_{i+1}) can be obtained from (G_i, C_i) by adding a wedge for c , possibly with additional applications of the operation (3) if some edges between a, b are assigned to $B_{x_{i+1}}$.

Suppose $C_{i+1} = \{a, c\}$. If b and c are not adjacent in G , then a is a cutvertex of G , which is impossible. Thus b and c are adjacent, and we expand (G_i, C_i) using an edge between b and c . Afterwards, we may add edges between a, b, c and then we restrict the set of labelled vertices to $\{a, c\}$, obtaining (G_{i+1}, C_{i+1}) .

Case 2. $|B_{x_{i+1}}| = 4$. Let $B_{x_{i+1}} = \{a, b, c, d\}$. We divide the case up to the sizes of C_i and C_{i+1} . The addition of edges between the vertices a, b, c and d shall be done implicitly except those edges which are added explicitly by other operations.

Case 2-1. $|C_i| = 2$ and $|C_{i+1}| = 2$. Let $C_i = \{a, b\}$. There are three possibilities for C_{i+1} , which are $\{a, b\}$, $\{a, c\}$ and $\{c, d\}$. Suppose $C_{i+1} = \{a, b\}$. If c and d are not adjacent in G then we add two wedges to (G_i, C_i) . Otherwise, we may assume that c is adjacent to either a or b and we firstly expand C_i into $\{a, b, c\}$. And then we add either a wedge or a claw for d , and finish by restricting $\{a, b, c\}$ back to $\{a, b\}$ to obtain (G_{i+1}, C_{i+1}) .

Suppose $C_{i+1} = \{a, c\}$. If d is not adjacent to c then we add a wedge for d to (G_i, C_i) . After that we expand $\{a, b\}$ to $\{a, b, c\}$ and finish by restricting it to $\{a, c\}$. If d is adjacent to c , the vertex d is adjacent to at least one of a, b , so we expand $\{a, b\}$ to $\{a, b, d\}$ then transit to $\{a, b, c\}$. After that we restrict to $\{a, c\}$ and obtain (G_{i+1}, C_{i+1}) .

If $C_{i+1} = \{c, d\}$, then the edges between $\{a, b\}$ and $\{c, d\}$ contains two independent edges; otherwise G has a cutvertex. We may assume that a, c are adjacent and so do b, d . We can expand $\{a, b\}$ to $\{a, b, c\}$ then transit to $\{a, c, d\}$. We finish by restricting $\{a, c, d\}$ to $\{c, d\}$.

Case 2-2. $|C_i| = 2$ and $|C_{i+1}| = 3$. Let $C_i = \{a, b\}$. The set C_{i+1} can be either $\{a, b, c\}$ or $\{a, c, d\}$. Suppose $C_{i+1} = \{a, b, c\}$. If c is not adjacent to d then both a and b are neighbors of d . Thus we add a wedge to (G_i, C_i) and then expand C_i to $\{a, b, c\}$. If c and d are adjacent, we firstly expand C_i to $\{a, b, d\}$ and then transit to $\{a, b, c\}$.

Suppose $C_{i+1} = \{a, c, d\}$. We may assume that b, d are adjacent in G , because if

both c, d are not adjacent to b then a is a cutvertex of G . Now we expand $\{a, b\}$ to $\{a, b, c\}$ then transit to $\{a, c, d\}$.

Case 2-3. $|C_i| = 3$ and $|C_{i+1}| = 2$. Let $C_i = \{a, b, c\}$. The set C_{i+1} can be either $\{a, b\}$ or $\{a, d\}$. If $C_{i+1} = \{a, b\}$ then we add a wedge or a claw for d and then restrict $\{a, b, c\}$ to $\{a, b\}$. Suppose $C_{i+1} = \{a, d\}$. The vertex d must be adjacent in G to at least one of b, c . We may assume that b, d are adjacent. Now we transit from $\{a, b, c\}$ to $\{a, c, d\}$ then restrict it to $\{a, d\}$.

Case 2-4. $|C_i| = 3$ and $|C_{i+1}| = 3$. Let $C_i = \{a, b, c\}$. The set C_{i+1} can be either $\{a, b, c\}$ or $\{a, b, d\}$. If $C_{i+1} = \{a, b, c\}$ then we add a wedge or a claw for d . Suppose $C_{i+1} = \{a, b, d\}$. If c, d are adjacent in G then we transit from $\{a, b, c\}$ to $\{a, b, d\}$. Otherwise we restrict $\{a, b, c\}$ to $\{a, b\}$ and then expand it to $\{a, b, d\}$.

We covered all possible changes from G_i to G_{i+1} using the given operations. Thus G_r can be obtained from the 2-fragment K_2 , and at last, G can be obtained from G_r by adding an edge. \square

We remark that for each of the operations (1) – (6) in Theorem 5.2.2, say O_i , there is a fragment K_{O_i} such that applying O_i to a fragment H is equivalent to take $H \oplus K_{O_i}$, possibly followed by relabelling.

5.3 An algorithm to test the Merino-Welsh conjecture

We present an algorithm to test the multiplicative Merino-Welsh conjecture (Conjecture 5.1.1; **mMW-conjecture** for short), for the class of graphs with treewidth (or pathwidth) at most k . It is a generalization of Noble and Royle’s idea [NR14] which was used to series-parallel graphs, or in other words, the graphs of treewidth at most 2. My version of the splitting formula (Theorem 2.2.6) is essential in my generalization of the algorithm. Noble and Royle did not know whether the algorithm would finish in finite time before actually running it on computer, and neither do I know whether my algorithm finishes in finite time. But for pathwidth 3, my algorithm produced a list of 5242 fragments which confirmed that the mMW-conjecture holds for graphs of pathwidth at most 3.

The crucial idea is the following concept of **replacement** between two fragments with respect to the mMW-conjecture.

Definition 5.3.1. *Let G and H be connected k -fragments. We say G is **replaceable by H** if for each k -fragment K , $T(H \oplus K; 1, 1)^2 \leq T(H \oplus K; 2, 0)T(H \oplus K; 0, 2)$ implies $T(G \oplus K; 1, 1)^2 \leq T(G \oplus K; 2, 0)T(G \oplus K; 0, 2)$.*

In other words, if the mMW-conjecture has a counterexample which is $G \oplus K$ for some k -fragment K , then $H \oplus K$ is also a counterexample to the mMW-conjecture. Hence when G is replaceable by H , we may ‘ignore’ G from consideration regarding the mMW-conjecture.

To use the concept of replacement in an algorithm, we present a sufficient condition below that a fragment is replaceable by another. Recall that $\Gamma(k)$ is the set of partitions of $[k] = \{1, 2, \dots, k\}$. The Tutte vector $T_v(G)$ and the graph G_P for a k -fragment G and $P \in \Gamma(k)$ are defined in Section 2.2, before Theorem 2.2.1.

Theorem 5.3.2. *Let G and H be connected k -fragments. Let the three numbers $a_{G/H}$, $c_{G/H}$, and $t_{G/H}$ be defined as the following:*

- $a_{G/H} = \min\{a(G_P)/a(H_P) : P \in \Gamma(k), a(H_P) \neq 0\}$.
- $c_{G/H} = \min\{c(G_P)/c(H_P) : P \in \Gamma(k), c(H_P) \neq 0\}$.
- $t_{G/H} = \infty$ if for some $P \in \Gamma(k)$, we have $t(H_P) = 0$ but $t(G_P) \neq 0$.
Otherwise $t_{G/H} = \max\{t(G_P)/t(H_P) : P \in \Gamma(k), t(H_P) \neq 0\}$.

If $a_{G/H}c_{G/H} \geq t_{G/H}^2$, then G is replaceable by H .

Proof. Let $a_v(G)$, $c_v(G)$, and $t_v(G)$ be the evaluations of the Tutte vector $T_v(G)$ at $(2, 0)$, $(0, 2)$, and $(1, 1)$ respectively, and similarly define $a_v(H)$, $c_v(H)$, and $t_v(H)$ for H . Let us fix a k -fragment K . By Theorem 2.2.6, there is a vector $T_f(K)$ whose entries are polynomials in x, y with positive integer coefficients such that $T(G \oplus K) = T_v(G) \cdot T_f(K)$ and $T(H \oplus K) = T_v(H) \cdot T_f(K)$. By evaluating both equations at $(2, 0)$, we have a vector $a_f(K)$ whose entries are nonnegative integers such that $a(G \oplus K) = a_v(G) \cdot a_f(K)$ and $a(H \oplus K) = a_v(H) \cdot a_f(K)$. Using the definition of $a_{G/H}$, we get

$$a_{G/H}a(H \oplus K) = \sum_{P \in \Gamma(k)} (a_{G/H}a(H_P)) [a_f(K)]_P \leq \sum_{P \in \Gamma(k)} a(G_P)[a_f(K)]_P = a(G \oplus K),$$

where $[a_f(K)]_P$ denotes the P -entry of the vector $a_f(K)$.

Similarly, we can show $c_{G/H}c(H \oplus K) \leq c(G \oplus K)$ and $t_{G/H}t(H \oplus K) \geq t(G \oplus K)$. Therefore, if the multiplicative Merino-Welsh conjecture holds for $H \oplus K$, i.e., $t(H \oplus K)^2 \leq a(H \oplus K)c(H \oplus K)$, then by the assumption $t_{G/H}^2 \leq a_{G/H}c_{G/H}$, we get

$$t(G \oplus K)^2 \leq t_{G/H}^2 t(H \oplus K)^2 \leq a_{G/H}c_{G/H} a(H \oplus K)c(H \oplus K) \leq a(G \oplus K)c(G \oplus K),$$

and hence G is replaceable by H . \square

My algorithm recursively extends a list of 2-fragments and 3-fragments starting from $\mathcal{L} = \{K_2\}$, where K_2 is the 2-fragment whose base graph is the complete graph on 2 vertices. At each step, we apply the operations (1) – (6) in Theorem 5.2.2 to the fragments in \mathcal{L} , and if the resulting fragment is not replaceable by any fragment in \mathcal{L} then we add the new fragment into \mathcal{L} . If the new fragment is replaceable then we discard it. The algorithm stops when no more irreplaceable fragment is found. In the algorithm, we only need the vectors $a_v(H)$, $c_v(H)$ and $t_v(H)$ for each fragment H in \mathcal{L} instead of the whole graph structure of H .

Theorem 5.3.3 below explains that if my algorithm stops, then the mMW-conjecture holds for graphs with pathwidth 3 if and only if it holds for my list \mathcal{L} . However, it should be noted that the operations in Theorem 5.2.2 may produce graphs with bridges. For example, a path can be constructed from the 2-fragment K_2 by repeatedly applying the expansion (operation (1)) and restriction (operation (2)). The mMW-conjecture does not hold for graphs with bridges, so we need to exclude graphs with bridges. Thus we are interested in only the fragments arising from 2-edge-connected graphs. We need to exclude such a k -fragment H that even if we add a complete graph on its labelled vertices, H still has a bridge. Such a case happens if and only if the number of totally cyclic orientation of H_P , or $c(H_P)$, is zero where P is the partition with a single block containing all elements. Therefore, in the algorithm, we discard the new fragments whenever its number $c(H_P)$ is zero. This exclusion of bridges is assumed implicitly in the proof of Theorem 5.3.3.

Now we explain why my list \mathcal{L} is a ‘test-space’ for the mMW-conjecture if the algorithm stops.

Theorem 5.3.3. *Suppose that we have a finite list \mathcal{L} of 2-fragments and 3-fragments such that*

1. \mathcal{L} contains the smallest connected 2-fragment K_2 and
2. For each fragment $F \in \mathcal{L}$, the result of applying any of the operations (1)–(6) in Theorem 5.2.2 to F is replaceable by a fragment $F' \in \mathcal{L}$.

If Conjecture 5.1.1 holds for every graph obtained from a 2-fragment in \mathcal{L} by adding an edge between its labelled vertices, then the conjecture holds for all loopless bridgeless graphs with pathwidth at most three.

Proof. Let G be a loopless bridgeless graph of pathwidth three. The mMW-conjecture holds for G if it holds for each block of G , so we may assume that G is 2-connected. Note that the operations in Theorem 5.2.2 always produce connected fragments when applied to connected fragments, so that we can talk about the replaceability.

We shall write the operations in Theorem 5.2.2 on the right side of the fragment to which they are applied, so that HO_1O_2 , or $(HO_1)O_2$ denotes the fragment obtained from a fragment H by applying O_1 and then O_2 , where O_i is one of the operations in Theorem 5.2.2 for each i .

By Theorem 5.2.2, there is a sequence of operations O_1, O_2, \dots, O_N such that $K_2O_1O_2 \cdots O_N$ is the graph G . Moreover, the statement of Theorem 5.2.2 also allows us to assume that $K_2O_1O_2 \cdots O_{N-1}$ is a 2-fragment, and O_N is the addition of an edge between the labelled vertices of a 2-fragment. We prove that the mMW-conjecture holds for all graphs obtained by this sequence of operations applied to the 2-fragment K_2 , which include all 2-connected loopless graphs of pathwidth at most three.

Let O_1, O_2, \dots, O_N be a sequence of operations in Theorem 5.2.2 such that $K_2O_1O_2 \cdots O_N$ is well-defined and O_N is the addition of an edge between the labelled vertices of a 2-fragment $K_2O_1O_2 \cdots O_{N-1}$. Because of the property (2) of our list \mathcal{L} , the fragment K_2O_1 is replaceable by a fragment $F_1 \in \mathcal{L}$. Note that for each k , applying the sequence $O_kO_{k+1} \cdots O_N$ to a fragment, say H , is equivalent to take $H \oplus K$ for a fragment K depending on the sequence of operations. By the definition of replaceability (Definition 5.3.1), the mMW-conjecture holds for $K_2O_1O_2 \cdots O_N$ if

it holds for $F_1O_2O_3 \cdots O_N$. Likewise, for $i = 2, 3, \dots, N - 1$, we can find a fragment $F_i \in \mathcal{L}$ such that $F_{i-1}O_i$ is replaceable by F_i . At the end, the condition we assumed in the statement implies that the mMw-conjecture holds for $F_{N-1}O_N$, and by the definition of the replaceability, the conjecture holds for $F_{N-2}O_{N-1}O_N$, $F_{N-3}O_{N-2}O_{N-1}O_N$, and so on. Eventually we have that the conjecture holds for $K_2O_1O_2 \cdots O_N$, thereby completing the proof. \square

We used a computer program to find such a finite list \mathcal{L} as stated in Theorem 5.3.3. The result is a sequence of 5242 fragments, say $F_1, F_2, \dots, F_{5242}$. The sequence has the property that for each F_i and for each operation O_j , the fragment F_iO_j is either equal to $F_{i'}$ for some $i' > i$ or replaceable by $F_{i''}$ for some $i'' < i$. A fragment F_i is never replaceable by another fragment $F_{i'}$ with $i' < i$, but we cannot guarantee that F_i is also not replaceable by a fragment $F_{i''}$ with $i'' > i$. In fact, 469 fragments out of those 5242 are replaceable by another fragment included later in the list. Thus we may say that we have a list of 4773 mutually irreplaceable fragments which satisfy the assumptions of Theorem 5.3.3. For the actual code and the complete list, see <http://www2.compute.dtu.dk/~seok/>.

An algorithm for graphs with treewidth (or pathwidth) at most k for $k \geq 3$ can be obtained by simply changing the operation set (Theorem 5.2.2). A tree decomposition or a path decomposition of bounded width naturally provides a finite list of simple operations to construct all of such graphs, and the replacement criterion (Theorem 5.3.2) can be used regardless of the widths. But the list for treewidth 3, if finite, seems to contain more than 10,000 fragments, which was beyond the computational power I used.

Convexity of the Tutte Polynomial

6.1 Convexity of the Tutte polynomials on the line segments with slope -1

Recall that the Merino-Welsh conjecture holds for any graph G for which the Tutte polynomial $T(G)$ is convex on the line segment between $(0, 2)$ and $(2, 0)$.

In this section we prove that the Tutte polynomial $T(G)$ is convex on the line segment from $(0, 2)$ to $(2, 0)$ if G is a minimally 2-edge-connected graph. The proof is in the context of matroids and therefore we introduce some notions from matroid theory.

Two distinct elements e, f in a matroid are called **parallel** if $\{e, f\}$ is a circuit. The relation $e \sim f$ if $e = f$ or $\{e, f\}$ are parallel defines an equivalence relation on the ground set of a matroid. We call a corresponding equivalence class a **parallel class**. It is easy to check the following.

(*) If e, f are parallel in a matroid M and C is a parallel class of M not containing e , then e, f are parallel in both M/C and $M - C$.

The following two properties of the Tutte polynomial can be obtained directly from the definition (2.1) of the Tutte polynomial. A **coloop** is an element which is contained in every basis.

1. $T(M; x, y) = T(M - e; x, y) + T(M/e; x, y)$ if e is neither a loop nor a coloop.
2. $T(M; x, y) = yT(M - e; x, y) = yT(M/e; x, y)$ if e is a loop.

The formula (1) is called the deletion-contraction formula. From (1) and (2), it is easy to prove the following by induction on k .

Observation 6.1.1. *Let M be a matroid and let C be a parallel class of M . Let $|C| = k$. If C itself is a component of M , then*

$$T(M; x, y) = (y^{k-1} + y^{k-2} + \cdots + y + x)T(M/C; x, y).$$

Otherwise,

$$T(M; x, y) = (y^{k-1} + y^{k-2} + \cdots + 1)T(M/C; x, y) + T(M - C; x, y).$$

Now we prove the following.

Theorem 6.1.2. *Let M be a matroid in which each element has another element parallel to it. Then the polynomial $T(M; x, y)$ is a sum of the terms $y^a(x + y)^b$ with nonnegative integer coefficients. Therefore for each real number $c > 0$, the polynomial $T(M; c - y, y)$ is convex in the interval $[0, c]$.*

Proof. We prove by induction on the number of parallel classes of M . If M has only one parallel class and it has size k , then

$$T(M; x, y) = y^{k-1} + y^{k-2} + \cdots + y + x$$

so the statement holds since $k \geq 2$.

Suppose that M has a parallel class C and $C \subsetneq M$. By (*) we can apply the induction hypothesis to both M/C and $M - C$. From Observation 6.1.1, if C is

itself a component of M then

$$T(M; x, y) = (y^{k-1} + y^{k-2} + \cdots + y + x)T(M/C; x, y)$$

and hence $T(M; x, y)$ has the desired property since $k \geq 2$. If C is not a component of M , then

$$T(M; x, y) = (y^{k-1} + y^{k-2} + \cdots + 1)T(M/C; x, y) + T(M - C; x, y)$$

and again by the induction hypothesis the polynomial $T(M; x, y)$ is a sum of the terms $y^a(x + y)^b$ with nonnegative integer coefficients. \square

We remark that we get the same conclusion in Theorem 6.1.2 when M has loops. By restricting the class to the cycle matroids of graphs we get the following.

Corollary. *Let G be a graph in which each edge is parallel to another edge. Then the polynomial $T(G; 2 - y, y)$ is convex in $[0, 2]$.*

Since $T(M; x, y) = T(M^*; y, x)$ for each matroid M , the dual statement of Theorem 6.1.2 is the following.

Corollary 6.1.3. *Let M be a matroid in which each edge is in a cocircuit of size 2. Then $T(M; x, c - x)$ is convex in $[0, c]$ for each $c > 0$.*

Corollary 6.1.4. *Let G be a minimally 2-edge-connected graph. Then the polynomial $T(G; x, 2 - x)$ is convex in $[0, 2]$.*

Corollary 6.1.3 also provides a shorter proof of the following theorem of Chávez-Lomelí et al. [CLMNRI11]. A matroid is called **paving** if the minimum size of a circuit is at least the rank of the matroid.

Theorem 6.1.5 (Chávez-Lomelí et al.[CLMNRI11]). *Let M be a coloopless paving matroid. For each real number $c > 0$, the Tutte polynomial $T(M; x, y)$ is convex on the line segment between $(0, c)$ and $(c, 0)$.*

Proof. If M has a loop then either every element of M is a loop or M has rank 1 and it is the cycle matroid of a graph on two vertices possibly with parallel edges and loops. Thus the Tutte polynomial $T(M; x, y)$ is a sum of the terms y^a and $y^b(x + y)$ so that the statement holds.

If each element of M is contained in a cocircuit of size 2 then we apply Corollary 6.1.3.

Thus we may assume that M has no loops and has an element e not in a cocircuit of size 2. Since every minor of a paving matroid is again paving, we use induction on $M - e$ and M/e so that the equation $T(M; x, y) = T(M - e; x, y) + T(M/e; x, y)$ completes the proof. \square

6.2 Convexity of the Tutte polynomial in the first quadrant

This section is about almost all matroids. We prove, among other things, that the Tutte polynomial of a sparse paving matroid is asymptotically almost surely convex in the first quadrant $\{(x, y) : x, y \geq 0\}$. To be precise, we consider the set \mathbb{M}_n of all matroids on the set $[n] = \{1, 2, \dots, n\}$ and by writing **almost all matroids with property P have property Q** we mean

$$\lim_{n \rightarrow \infty} \frac{|\{M \in \mathbb{M}_n : M \text{ has both } P \text{ and } Q\}|}{|\{M \in \mathbb{M}_n : M \text{ has } P\}|} = 1.$$

Recall that a matroid is **paving** if the minimum size of a circuit is at least the rank of the matroid. A **sparse paving matroid** is a paving matroid whose dual matroid is also paving. We prove the following.

Theorem 6.2.1. *Almost all sparse paving matroids have the property that their Tutte polynomials are convex in $\{(x, y) : x, y \geq 0\}$.*

We shall use the following characterization of the sparse paving matroids. An r -set is a subset of $[n]$ of size r .

Lemma 6.2.2 (Knuth [Knu74]). *Let \mathcal{C} be a collection of r -sets. There is a sparse paving matroid on $[n]$ with rank r whose set of circuits of size r is \mathcal{C} if and only if for all distinct $C_1, C_2 \in \mathcal{C}$, $|C_1 \triangle C_2| > 2$.*

Let M be a sparse paving matroid on $[n]$ with rank r and λ circuits of size r . By Lemma 6.2.2, each $(r + 1)$ -set contains a basis of M . Hence from the definition

(2.1) of the Tutte polynomial,

$$T(M; x, y) = \sum_{i=0}^{r-1} \binom{n}{i} (x-1)^{r-i} + \binom{n}{r} + \lambda(xy - x - y) + \sum_{i=r+1}^n \binom{n}{i} (y-1)^{i-r}. \quad (6.1)$$

Although Equation 6.1 provides a simple formula for the Tutte polynomial of a sparse paving matroid, what we need in this section is the coefficients of the monomials $x^i y^j$, instead of $(x-1)^i (y-1)^j$. I found the following simple expression of the coefficients.

Theorem 6.2.3. *Let M be a sparse paving matroid on n elements with rank r and λ circuits of size r . If $1 \leq r \leq n-1$, then the Tutte polynomial of M is*

$$T(M; x, y) = \sum_{i=0}^r \binom{n-i-1}{r-i} x^i + \binom{n}{r} + \lambda(xy - x - y) + \sum_{i=0}^{n-r} \binom{n-i-1}{n-r-i} y^i.$$

Proof. Let

$$L(n, r) = \sum_{i=0}^r \binom{n}{r-i} (x-1)^i \quad \text{for } n \geq r \geq 0,$$

$$R(n, r) = \sum_{i=0}^r \binom{n-i-1}{r-i} x^i \quad \text{for } n > r \geq 0$$

and $R(n, n) = x^n$ for each $n \geq 0$.

We shall prove $L(n, r) = R(n, r)$ for $n \geq r \geq 0$ by induction.

The following are immediate:

$$L(n, 0) = \sum_{i=0}^0 \binom{n}{0-i} (x-1)^i = 1$$

$$L(n, n) = \sum_{i=0}^n \binom{n}{n-i} (x-1)^i = ((x-1) + 1)^n = x^n$$

$$R(n, 0) = \sum_{i=0}^0 \binom{n-i-1}{0-i} x^i = 1$$

$$R(n, n) = x^n \quad \text{by definition.}$$

Using the binomial equation $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for $n-1 \geq r \geq 1$, we also have

$$L(n, r) = L(n-1, r-1) + L(n-1, r) \quad \text{for } n-1 \geq r \geq 1 \quad \text{and}$$

$$R(n, r) = R(n-1, r-1) + R(n-1, r) \quad \text{for } n-1 > r \geq 1$$

By directly calculating we have $R(n, n-1) = R(n-1, n-2) + R(n-1, n-1)$ as well.

Therefore, using induction on n , it is true that $L(n, r) = R(n, r)$ for all $n \geq r \geq 0$. \square

The following lemma decides whether the Tutte polynomial of a sparse paving matroid is convex in the first quadrant.

Lemma 6.2.4. *Let M be a sparse paving matroid on $[n]$ with rank r , $2 \leq r \leq n-2$. Let λ be the number of circuits of size r in M . The Tutte polynomial $T(M; x, y)$ is convex in $\{(x, y) : x, y \geq 0\}$ if and only if*

$$\lambda^2 \leq 4 \binom{n-3}{r-2} \binom{n-3}{n-r-2}.$$

Proof. As a two-variable polynomial, $T = T(M; x, y)$ is convex in $\{(x, y) : x, y \geq 0\}$ if and only if $T_{xy}^2 \leq T_{xx}T_{yy}$ in $\{(x, y) : x, y \geq 0\}$ where the subscripts denote the partial derivatives. From Equation 6.1 we know that $T_{xy} = \lambda$ is a constant. Since every coefficient of the Tutte polynomial is a positive integer, so are the coefficients of the polynomials T_{xx} and T_{yy} and hence the Tutte polynomial $T(M; x, y)$ is convex in $\{(x, y) : x, y \geq 0\}$ if and only if

$$\lambda^2 \leq T_{xx}(0, 0)T_{yy}(0, 0).$$

By Theorem 6.2.3,

$$T_{xx}(0, 0)T_{yy}(0, 0) = 4 \binom{n-3}{r-2} \binom{n-3}{n-r-2}$$

and the proof is complete. \square

There are precisely two sparse paving matroids on $[n]$ with rank 1, namely the uniform matroid and the cycle matroid of the graph with $n - 1$ parallel edges and a loop. The Tutte polynomial of a uniform matroid is convex in the first quadrant whereas the Tutte polynomial of the latter is not. The sparse paving matroids on $[n]$ with rank $n - 1$ are precisely the dual matroids of those two and the dual operation preserves the convexity of the Tutte polynomial in the first quadrant.

Let $J(n, r)$ be the Johnson graph whose vertices are r -sets in $[n]$ and two vertices A, B are adjacent if and only if $|A \triangle B| = 2$. By Lemma 6.2.2, a set \mathcal{C} of vertices in $J(n, r)$ is the set of circuits of size r in a sparse paving matroid of rank r if and only if \mathcal{C} is independent in $J(n, r)$. Let \mathcal{C} be an independent set of $J(n, r)$. Since each $(r - 1)$ -set is contained in at most one element of \mathcal{C} and each r -set contains r subsets of size $r - 1$, we have

$$|\mathcal{C}| \leq \frac{1}{r} \binom{n}{r-1} = \frac{1}{n-r} \binom{n}{r}.$$

Also, the set of complements $\mathcal{C}^c = \{C^c : C \in \mathcal{C}\}$ satisfies the same condition $|C_1 \triangle C_2| > 2$ for each pair of distinct $C_1, C_2 \in \mathcal{C}^c$ and hence

$$|\mathcal{C}| = |\mathcal{C}^c| \leq \frac{1}{r} \binom{n}{n-r}.$$

On the other hand, $J(n, r)$ has a proper \mathbb{Z}_n -coloring such that a set A is assigned the color $\sum_{i \in A} i \pmod{n}$. Therefore, $J(n, r)$ has an independent set of size at least $\frac{1}{n} \binom{n}{r}$. We summarize the above as the following observation.

Observation 6.2.5. *A sparse paving matroid on $[n]$ with rank r has at most*

$$\min \left\{ \frac{1}{n-r} \binom{n}{r}, \frac{1}{r} \binom{n}{r} \right\}$$

circuits of size r . There is a sparse paving matroid on $[n]$ with rank r with at least $\frac{1}{n} \binom{n}{r}$ circuits of size r .

We remark that the independence number of $J(n, r)$ was also considered in the context of constant weight codes; c.f. [EB96].

Theorem 6.2.6. *Let M be a sparse paving matroid on n elements with rank r . If $2 \leq r \leq n - 2$, then the Tutte polynomial $T(M; x, y)$ is convex in $\{(x, y) : x, y \geq 1\}$.*

Proof. Let $T = T(M; x, y)$ and let λ be the number of circuits of M of size r . As in Lemma 6.2.4, the two-variable polynomial T is convex in $\{(x, y) : x, y \geq 1\}$ if and only if $\lambda^2 \leq T_{xx}(1, 1)T_{yy}(1, 1)$ where the subscripts denote the partial derivatives. From Equation 6.1 we have $T_{xx}(1, 1) = 2\binom{n}{r-2}$ and $T_{yy}(1, 1) = 2\binom{n}{r+2}$. By

Observation 6.2.5, $\lambda^2 \leq \frac{1}{r} \frac{1}{n-r} \binom{n}{r}^2$ and it is easy to check

$$\frac{1}{r} \frac{1}{n-r} \binom{n}{r}^2 \leq 4 \binom{n}{r-2} \binom{n}{r+2}$$

when $2 \leq r \leq n-2$ which completes the proof. \square

Let $T = T(M; x, y)$ be the Tutte polynomial of a sparse paving matroid on $[n]$ with rank r , $2 \leq r \leq n-2$. By Theorem 6.2.6, T is convex in $\{(x, y) : x, y \geq 1\}$. By Theorem 6.1.5, T is convex on the line segment between $(0, c)$ and $(c, 0)$ for each positive real number c , since a sparse paving matroid with a coloop has corank 1. But it is not true in general that T is convex in $\{(x, y) : x, y \geq 0\}$. One such example is the sparse paving matroid on 14 elements with rank 2 and 7 disjoint circuits of size 2; see Lemma 6.2.4. There are infinitely many sparse paving matroids whose Tutte polynomials are not convex in $\{(x, y) : x, y \geq 0\}$, in contrast to Theorem 6.2.6.

Proposition 6.2.7. *There are infinitely many sparse paving matroids whose Tutte polynomials are not convex in $\{(x, y) : x, y \geq 0\}$.*

Proof. Let r be a fixed number and let n be large enough. By Observation 6.2.5 we can find a sparse paving matroid on $[n]$ with rank r and $\Theta(n^{r-1})$ circuits of size r . By Lemma 6.2.4, the Tutte polynomial of such a matroid is not convex in the first quadrant if the square of the number of circuits of size r is larger than

$$4 \binom{n-3}{r-2} \binom{n-3}{n-r-2} = \Theta(n^{2r-3}),$$

which is true and thus completes the proof. \square

On the other hand, we shall now prove that almost all sparse paving matroids on $[n]$ have the property that their Tutte polynomials are convex in $\{(x, y) : x, y \geq 0\}$. The proof uses the fact that almost all matroids and almost all sparse paving matroids have rank close to $n/2$; c.f. [MNWW11, LOSW13, PvdP14]. We prove the following weaker version here for completeness.

Lemma 6.2.8. *Almost all sparse paving matroids on $[n]$ have rank between $n/3$ and $2n/3$.*

Proof. A matroid is completely determined by its set of bases. Thus, the number of matroids on $[n]$ with rank r is less than $2^{\binom{n}{r}}$ and the number of matroids on $[n]$ with rank at most $n/3$ is less than

$$\sum_{i=0}^{\lfloor n/3 \rfloor} 2^{\binom{n}{i}} < n 2^{\binom{n}{\lfloor n/3 \rfloor}}.$$

Using Stirling's formula, we get $\binom{n}{\lfloor n/3 \rfloor} < \left(\frac{3^3}{2^2}\right)^{n/3} < 1.89^n$ for sufficiently large n . The number of matroids on $[n]$ with rank bigger than $2n/3$ is also less than $n 2^{1.89^n}$ by duality.

On the other hand, by Observation 6.2.5 the graph $J(n, \lfloor n/2 \rfloor)$ has an independent set of size $\frac{1}{n} \binom{n}{\lfloor n/2 \rfloor} = O\left(\frac{2^n}{n^{3/2}}\right)$. Using all of its subsets, we can find more than $2^{1.99^n}$ distinct sparse paving matroids on $[n]$ with rank $\lfloor n/2 \rfloor$ when n is sufficiently large. Therefore, the number of matroids on $[n]$ with rank either less than $n/3$ or bigger than $2n/3$ is vanishing compared to the number of sparse paving matroids with rank $\lfloor n/2 \rfloor$. \square

Theorem 6.2.9. *Almost all sparse paving matroids have the property that their Tutte polynomials are convex in $\{(x, y) : x, y \geq 0\}$.*

Proof. By Lemma 6.2.4 and Observation 6.2.5, it is enough to show that

$$\frac{1}{r^2} \binom{n}{r}^2 \leq 4 \binom{n-3}{r-2} \binom{n-3}{n-r-2} \quad (6.2)$$

for almost all sparse paving matroids. By Lemma 6.2.8 almost all sparse paving matroids have rank between $n/3$ and $2n/3$, and it is routine to check Equation (6.2) when $n/3 \leq r \leq 2n/3$ and n is sufficiently large, by expanding the factorials. \square

Based on the proof of Proposition 6.2.7, I conjecture that if we fix the rank, then the Tutte polynomial is almost never convex in the first quadrant, in contrast to Theorem 6.2.9.

Conjecture 6.2.10. *Let r be a fixed positive integer. The proportion of the sparse paving matroids on $[n]$ with rank r , whose Tutte polynomials are convex in $\{(x, y) : x, y \geq 0\}$, tends to 0 as n tends to infinity.*

By Proposition 6.2.7 and the discussion above Observation 6.2.5, if we can show that most of the independent sets of the Johnson graph $J(n, r)$ has relatively large size then Conjecture 6.2.10 is true. For example, we may ask whether the following property **P** holds for an infinite class of graphs.

Property P: We say that an infinite class \mathcal{G} of graphs has property **P** if there exists a constant $c > 0$ such that

$$\lim_{n \rightarrow \infty} \min_{\substack{G \in \mathcal{G} \\ |V(G)|=n}} \frac{\# \text{ of independent sets of } G \text{ with size at least } c\alpha(G)}{\# \text{ of independent sets of } G} = 1$$

where $\alpha(G)$ is the maximum size of an independent set of G .

In other words, a class \mathcal{G} has property **P** if for each graph in \mathcal{G} , almost all independent sets have relatively large size.

Suppose that for a fixed positive integer r , the class $\mathcal{J}_r = \{J(n, r) : n > r\}$ of the Johnson graphs has **P**. Then Observation 6.2.5 implies that almost all independent sets of $J(n, r)$ has size at least $\frac{c}{n} \binom{n}{r}$ and in turn, it implies that almost all sparse paving matroids on $[n]$ with rank r has at least $\frac{c}{n} \binom{n}{r}$ circuits of size r . Since

$$\frac{c^2}{n^2} \binom{n}{r} > 4 \binom{n-3}{r-2} \binom{n-3}{n-r-2} = \Theta(n^{2r-3})$$

for fixed r and large n , Lemma 6.2.4 shows that Conjecture 6.2.10 is true.

So, if the class \mathcal{J}_r of the Johnson graphs $J(n, r)$ with fixed r has the property **P**, then Conjecture 6.2.10 holds.

In the rest of this section we make some general comments on the property **P** for various graph classes. For example, the graphs with a fixed upper bound on the chromatic number has the property **P** whereas we loose this property if we only assume that the chromatic number is bounded by a function of n , the number of vertices, which tends to ∞ as $n \rightarrow \infty$.

Let us consider the class $\mathcal{C}_k = \{G : G \text{ a graph with chromatic number } \chi(G) \leq k\}$.

If $G \in \mathcal{C}_k$ has n vertices, then the independence number $\alpha(G)$ is at least $\frac{n}{k}$. Let $\alpha(G) = dn$ where $d \geq \frac{1}{k}$. Let $c \ll \frac{1}{k}$ be a small positive number. Let I be an independent set of size dn in G .

By choosing c small enough compared to $\frac{1}{k}$, we may assume that almost all subsets of I have size greater than cn . These sets are all independent so that G has at least $\frac{2^{dn}}{2}$ independent of size greater than cn . On the other hand, the number of subsets of $V(G)$ of size at most cn is

$$\sum_{i=0}^{cn} \binom{n}{i} < cn \binom{n}{cn}$$

and by Stirling's formula,

$$\binom{n}{cn} = O\left(\frac{1}{\sqrt{n}} \left(\frac{1}{c^c(1-c)^{1-c}}\right)^n\right).$$

Since the fraction $\frac{1}{c^c(1-c)^{1-c}}$, as a function of c in the interval $(0,1)$, has a maximum at $c = 1/2$ and tends to 0 as c tends to either end, by choosing small enough c we can ensure that the number of subsets of $V(G)$ of size at most cn is much less than 2^{dn} . That is, almost all independent sets of G have size at least cn and thus the class \mathcal{C}_k has the property **P**.

On the other hand, if we consider the class

$$\mathcal{C}_f = \{G : G \text{ a graph with chromatic number } \chi(G) \leq f(|V(G)|)\}$$

where f is any increasing function which tends to infinity, for example $f(n) = \log n$, then \mathcal{C}_f does not have **P**. We give a proof below.

Let b be a large integer and let d be so large that $f(d) > b + 1$. We may assume that d is much larger than b . Let $a = \lceil \frac{\log 4}{\log b} d \rceil$. Note that

$$b^a \geq b^{\frac{\log 4}{\log b} d} = 4^d.$$

We construct a graph G from a disjoint copies of K_b and adding d new mutually independent vertices all adjacent to each of the ab vertices of the disjoint complete graphs.

The independent sets of G are either a subset of the d pairwise independent vertices or a set obtained by choosing at most one vertex from each of the copies of K_b .

Thus the independence number of G is $\alpha(G) = d$. Since G has K_{b+1} as a subgraph the chromatic number $\chi(G) \geq b + 1$ and it is easy to color G with $b + 1$ colors. Thus $\chi(G) = b + 1$.

Since f is an increasing function, $f(|V(G)|) = f(ab + d) \geq f(d) > b + 1 = \chi(G)$. Thus G is in the class \mathcal{C}_f . Suppose that \mathcal{C}_f has the property **P** with a constant $c > 0$. The number of independent sets in G of size larger than a is definitely less than 2^d . On the other hand, the number of independent sets of size $\leq a$ is at least $b^a \geq 4^d \gg 2^d$. We may take b sufficiently large such that $a = \lceil \frac{\log 4}{\log b} d \rceil < cd$. In that case, almost all independent sets of G has smaller size than cd which implies that the class \mathcal{C}_f does not have the property **P**.

Let us consider again the motivation for property **P**, namely Conjecture 6.2.10. It suffices to prove that the class \mathcal{J}_r of the Johnson graphs $J(n, r)$ with fixed r has the property **P**, but the chromatic number of the Johnson graph $J(n, r)$ is at least $n - r + 1$ since $J(n, r)$ has a clique of size $n - r + 1$ consisting of all r -subsets of $[n]$ containing a fixed $(r - 1)$ -set. So our discussion above on the property **P** cannot be used to prove Conjecture 6.2.10.

Instead, we may consider the symmetry of the Johnson graph. I conjecture the following.

Conjecture 6.2.11. *There is a constant $c > 0$ such that if G is a vertex-transitive graph, then the proportion of the independent sets of size at least $c\alpha(G)$ in the set of all independent sets of G tends to 1 as $|V(G)|$ tends to infinity.*

Or we may consider just the regularity, and ask the following question.

Question 1. *Is there a constant $c > 0$ such that if G is regular, then almost all independent sets of G has size at least $c\alpha(G)$?*

Fractions involving the Tutte polynomials

7.1 Introduction

This section is an outline of Section 7.2. We conclude this section with a possible consequence to physics.

Thomassen's partial proof [Tho10] of the Merino-Welsh conjecture [MW99] is based on the following idea.

- In sparse graphs, the acyclic orientations dominate the spanning trees.
- In dense graphs, the totally cyclic orientations dominate the spanning trees.

In Section 7.2, I consider Conjecture 7.2.1 below as an attempt to strengthen Thomassen's idea. Recall that the numbers $a(G)$, $c(G)$ and $t(G)$ respectively denote the number of acyclic orientations, totally cyclic orientations and spanning trees of a graph G . The parentheses around labels of conjectures and theorems indicate that they will be introduced formally later.

Conjecture (7.2.1). *Let G be a connected loopless graph. If $e \in E(G)$ is not a bridge, then*

$$\frac{a(G)}{t(G)} \leq \frac{a(G-e)}{t(G-e)}$$

Conjecture 7.2.1 claims that the sparser the graph is, the more dominant acyclic orientations are over spanning trees. The conditions on G and e are simply to ensure that all numbers in the inequality are positive. Conjecture 7.2.1 and its counterpart for totally cyclic orientations lead to the following.

Conjecture (7.2.4). *Let H and G be connected, loopless and bridgeless graphs in the following statements.*

- *If $a(G) \geq t(G)$, then $a(H) \geq t(H)$ for any spanning subgraph H of G .*
- *If $c(G) \geq t(G)$, then $c(H) \geq t(H)$ for any supergraph H of G on the same vertex set.*
- *If H is a subgraph of G such that*

$$t(G) > \max\{a(G), c(G)\} \quad \text{and} \quad t(H) > \max\{a(H), c(H)\},$$

then any subgraph K of G containing H also satisfies

$$t(K) > \max\{a(K), c(K)\}.$$

The last statement is a type of interpolation theorem for the counterexamples of the Merino-Welsh conjecture.

I shall explain in Section 7.2 that Conjecture 7.2.4 follows from Conjecture 7.2.3 below. The subscript x denotes the partial derivative with respect to the variable x .

Conjecture (7.2.3). *Let M be a matroid and let e be an element of M . At each point on the line segment between $(0, 2)$ and $(2, 0)$, we have*

$$T^{M/e} T_x^{M-e} - T_x^{M/e} T^{M-e} \geq 0. \tag{7.1}$$

The line segment between $(0, 2)$ and $(2, 0)$ is drawn as B in Figure 7.1. The regions A, B and C in Figure 7.1 are respectively:

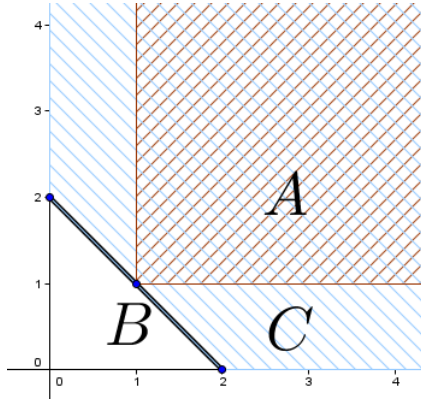


Figure 7.1: The regions related to Conjecture 7.2.3

- $A = \{(x, y) : x, y \geq 1\}$,
- $B = \{(x, y) : x, y \geq 0, \ x + y \leq 2\}$,
- $C = \{(x, y) : x, y \geq 0, \ x + y \geq 2\}$.

If Inequality (7.1) holds on B for every matroid M and every edge e then Conjecture 7.2.4 is true. As a supporting evidence, I prove that (7.1) holds in A when M is the cycle matroid of a series-parallel graph. I expect that (7.1) holds in a larger region, namely C in Figure 7.1, for every matroid M . However, we cannot extend the region to entire first quadrant since there are matroids M for which (7.1) fails in a small region close to the origin. The cycle matroids of the wheel graphs provide such examples. If M is the Tutte polynomial of the wheel graph on 11 vertices, then $T^{M/e}T_x^{M-e} - T_x^{M/e}T^{M-e}$ is negative when x, y are positive numbers smaller than 0.3.

My proof that Inequality (7.1) holds in the region A for series-parallel graphs uses a stronger statement, namely Theorem 7.2.9 below.

Theorem (7.2.9). *If $M = (E, r)$ is the cycle matroid of a series-parallel graph, then for each positive integer k ,*

$$\sum_{\substack{e \in A \subseteq E \\ r(A) + r(A^c) = k}} r(A) - r(A^c) \geq 0 \quad (7.2)$$

We will see in the next section that if (7.2) holds for every matroid M in a minor-closed class of matroids, say \mathcal{C} , then (7.1) holds for every $M \in \mathcal{C}$ in the region

$A = \{(x, y) : x, y \geq 1\}$. Thus, it follows from Theorem 7.2.9 that the inequality (7.1) in Conjecture 7.2.3 holds for series-parallel graphs in the region $A = \{(x, y) : x, y \geq 1\}$.

We finish this introductory section with a possible consequence of Theorem 7.2.9 to physics.

Sokal explained in [Sok05] that the Tutte polynomial of a graph G on the curve $(x-1)(y-1) = q$ is the partition function, say $Z^{Potts}(G)$, of the q -state **Potts model** where the so-called coupling constant (in this case $y-1$) is the same for all edges. In particular, in the region $\{(x, y) : x, y > 1\}$ the model is called **ferromagnetic**. Conjecture 7.2.6 below claims that Theorem 7.2.9 holds for all matroids. If true, it implies that the gradient vector field of the ratio $T(G/e)/T(G-e)$ in the region $\{(x, y) : x, y > 1\}$ always points toward upper-left for any graph (or matroid) G . Therefore, it follows that the ratio $Z^{Potts}(G/e)/Z^{Potts}(G-e)$ is an increasing function of the coupling constant $y-1$, regardless of the base graph G and the number of states q .

7.2 Fractions involving the Tutte polynomial and the Merino-Welsh conjecture

In this section we formulate some conjectures involving fractions of the Tutte polynomial. We verify one of those conjectures for series-parallel graphs. For an overview of the relationship between the conjectures, see Section 7.1 first.

Intuitively, my idea here is that maybe the sparser the graph is, the more dominant the acyclic orientations are over the spanning trees. To be precise, I expect the following to hold.

Conjecture 7.2.1. *Let G be a connected loopless graph and let e be an edge, not a bridge. Then*

$$\frac{a(G)}{t(G)} \leq \frac{a(G-e)}{t(G-e)}.$$

By interchanging terms, we get the following inequality about a fraction of the

Tutte polynomial, as follows.

$$\frac{T(G-e; 1, 1)}{T(G; 1, 1)} \leq \frac{T(G-e; 2, 0)}{T(G; 2, 0)}$$

Thus, I conjecture the following.

Conjecture 7.2.2. *For each loopless connected graph G and an edge e which is not a bridge, the fraction $\frac{T(G-e)}{T(G)}$ is an increasing function of x on the line segment $\{(x, 2-x) : 0 \leq x \leq 2\}$.*

For notational convenience, I shall write as T^G for the Tutte polynomial of G instead of $T(G)$ throughout this chapter.

Let L be the line segment between the points $(0, 2)$ and $(2, 0)$. By the assumptions of Conjecture 7.2.2, e is neither a bridge nor a loop. Using the deletion-contraction formula, we have

$$\frac{T^{G-e}}{T^G} = \frac{T^{G-e}}{T^{G-e} + T^{G/e}} = \frac{1}{1 + T^{G/e}/T^{G-e}}$$

and the question is whether $\frac{T^{G/e}}{T^{G-e}}$ is a decreasing function of x on L .

One way to show that $\frac{T^{G/e}}{T^{G-e}}$ is indeed decreasing on L is to consider the gradient.

If the gradient of $\frac{T^{G/e}}{T^{G-e}}$, as a function on the xy -plane, points toward the left-upper side on L then the fraction is a decreasing function of x on L . To be precise, what I want to prove is the following where the subscripts x and y denotes the partial derivatives with respect to the variables x and y .

$$\left[\frac{T^{G/e}}{T^{G-e}} \right]_x \leq 0 \quad \text{and} \quad \left[\frac{T^{G/e}}{T^{G-e}} \right]_y \geq 0 \quad \text{on } L. \quad (7.3)$$

To prove the two inequalities in (7.3), it is enough to prove the inequality in the following conjecture.

Conjecture 7.2.3. *Let M be a matroid and let e be an element of M . At each point on the line segment between $(0, 2)$ and $(2, 0)$, we have*

$$T^{M/e} T_x^{M-e} - T_x^{M/e} T^{M-e} \geq 0.$$

Now we explain why Conjecture 7.2.3 implies (7.3).

The partial derivatives in (7.3) are

$$\begin{aligned} \left[\frac{T^{G/e}}{T^{G-e}} \right]_x &= \frac{T_x^{G/e} T^{G-e} - T^{G/e} T_x^{G-e}}{(T^{G-e})^2}, \text{ and} \\ \left[\frac{T^{G/e}}{T^{G-e}} \right]_y &= \frac{T_y^{G/e} T^{G-e} - T^{G/e} T_y^{G-e}}{(T^{G-e})^2}. \end{aligned}$$

If we consider G as a matroid instead of a graph and take its dual G^* , then since $T^G(x, y) = T^{G^*}(y, x)$, we have $T_y^G(a, b) = T_x^{G^*}(b, a)$ and

$$\left[T_y^{G/e} T^{G-e} - T^{G/e} T_y^{G-e} \right]_{\text{at } (a,b)} = \left[T_x^{G^*-e} T^{G^*/e} - T^{G^*-e} T_x^{G^*/e} \right]_{\text{at } (b,a)},$$

which means that the numerators of the two gradients are in some sense dual to each other. Therefore, Conjecture 7.2.3 implies (7.3).

Suppose that Conjecture 7.2.3 is true. Then by the previous discussion,

$$\left[\frac{T^{M/e}}{T^{M-e}} \right]_{\text{at } (x,2-x)}$$

is decreasing in $[0, 2]$, and hence

$$\left[\frac{T^{M-e}}{T^M} \right]_{\text{at } (x,2-x)}$$

is increasing in $[0, 2]$, so that

$$\frac{t(G-e)}{t(G)} = \frac{T^{G-e}(1,1)}{T^G(1,1)} \leq \frac{T^{G-e}(2,0)}{T^G(2,0)} = \frac{a(G-e)}{a(G)}$$

whenever all four numbers $t(G-e), t(G), a(G-e)$ and $a(G)$ are positive.

The last inequality implies

$$\frac{a(G)}{t(G)} \leq \frac{a(G-e)}{t(G-e)}$$

so that if $a(G) \geq t(G)$ for some connected graph G then $a(H) \geq t(H)$ for any connected spanning subgraph H of G .

On the other hand, by applying Conjecture 7.2.3 between the points $(1, 1)$ and $(0, 2)$, it implies that if $c(G) \geq t(G)$ for some connected graph G then for any graph H obtained from G by adding edges between vertices of G , we have $c(H) \geq t(H)$.

We may combine last two paragraphs to conjecture the following.

Conjecture 7.2.4. *In this conjecture, we consider only the connected, loopless and bridgeless graphs.*

- *If $a(G) \geq t(G)$, then $a(H) \geq t(H)$ for any spanning subgraph H of G .*
- *If $c(G) \geq t(G)$, then $c(H) \geq t(H)$ for any supergraph H of G on the same vertex set.*
- *If H is a subgraph of G such that*

$$t(G) > \max\{a(G), c(G)\} \quad \text{and} \quad t(H) > \max\{a(H), c(H)\},$$

then any subgraph K of G containing H also satisfies

$$t(K) > \max\{a(K), c(K)\}.$$

Conjecture 7.2.4 is a type of interpolation theorem for the counterexamples of the Merino-Welsh conjecture. Recall that Conjecture 7.2.3 is stronger than Conjecture 7.2.4.

If each coefficient of the polynomial $T^{M/e}T_x^{M-e} - T_x^{M/e}T^{M-e}$ is a positive integer then Conjecture 7.2.3 is obviously true. But it is not the case. If we choose M to be the cycle matroid of the wheel graph on 6 vertices and e to be an edge incident with the center, then the low-order terms of $T^{M/e}T_x^{M-e} - T_x^{M/e}T^{M-e}$ have negative coefficients, for example $-4xy$. But all the higher order terms have positive coefficients and Conjecture 7.2.3 holds for this M .

As a generalization of Conjecture 7.2.3, I suspect that the polynomial $T^{M/e}T_x^{M-e} - T_x^{M/e}T^{M-e}$ is nonnegative on the region $\{(x, y) : x, y \geq 0, x+y \geq 2\}$. As a starting approach, I considered the following.

Conjecture 7.2.5. *Let M be a matroid and let e be an element of M . If we express*

$$T^{M/e}T_x^{M-e} - T_x^{M/e}T^{M-e}$$

as a polynomial in $x-1$ and $y-1$ instead of x and y , then every coefficient is a positive integer.

If true, Conjecture 7.2.5 implies that $T^{M/e}T_x^{M-e} - T_x^{M/e}T^{M-e}$ is nonnegative in the region $\{(x, y) : x, y \geq 1\}$.

We now consider a possible approach to Conjecture 7.2.5. We focus on the polynomial $T^{M/e}T_x^{M-e} - T_x^{M/e}T^{M-e}$.

When e is a loop or a coloop, $M/e = M - e$ and $T^{M/e}T_x^{M-e} - T_x^{M/e}T^{M-e} = 0$, so Conjecture 7.2.5 holds trivially. Suppose that M is a direct sum of two matroids, say N and K (meaning that every circuit of M is either in N or in K). We may assume that $e \in N$. Then

$$T^{M/e} = T^{N/e}T^K \quad \text{and} \quad T^{M-e} = T^{N-e}T^K,$$

so that

$$\begin{aligned} T^{M/e}T_x^{M-e} - T_x^{M/e}T^{M-e} &= T^{N/e}T^K (T_x^{N-e}T^K + T^{N-e}T_x^K) \\ &\quad - T^{N-e}T^K (T_x^{N/e}T^K + T^{N/e}T_x^K) \\ &= (T^{N/e}T_x^{N-e} - T_x^{N/e}T^{N-e}) (T^K)^2, \end{aligned}$$

which has the same sign as $T^{N/e}T_x^{N-e} - T_x^{N/e}T^{N-e}$. Thus we shall consider only the connected, loopless and coloopless matroids.

Now we find a way to express the coefficient of $(x-1)^i(y-1)^j$ in the polynomial $T^{M/e}T_x^{M-e} - T_x^{M/e}T^{M-e}$. For convenience, we write $E-e$ for $E \setminus \{e\}$ when E is the ground set of M and $e \in E$. I shall use the following rank-generating formulation of the Tutte polynomial, where r is the rank function of M .

$$T^M(x, y) = \sum_{A \subseteq E} (x-1)^{r(M)-r(A)} (y-1)^{|A|-r(A)}$$

We shall use also $r_{M/e}(A) = r(A+e) - 1$ and $r_{M-e}(A) = r(A)$ for every $A \subseteq E-e$. Putting them into the above equation, we get (for $T^{M/e}$ and T^{M-e} instead of T^M)

$$\begin{aligned} T^{M/e}(x, y) &= \sum_{A \subseteq E-e} (x-1)^{r(M/e)-r_{M/e}(A)} (y-1)^{|A|-r_{M/e}(A)} \\ &= \sum_{A \subseteq E-e} (x-1)^{r(M)-r(A+e)} (y-1)^{|A|+1-r(A+e)} \end{aligned}$$

and

$$\begin{aligned} T^{M-e}(x, y) &= \sum_{B \subseteq E-e} (x-1)^{r(M-e)-r_{M-e}(B)} (y-1)^{|B|-r_{M-e}(B)} \\ &= \sum_{B \subseteq E-e} (x-1)^{r(M)-r(B)} (y-1)^{|B|-r(B)}. \end{aligned}$$

Hence,

$$\begin{aligned} T^{M/e}T_x^{M-e} - T_x^{M/e}T^{M-e} &= \sum_{A \subseteq E-e} \sum_{B \subseteq E-e} (r(M) - r(B))(x-1)^{X(A,B)} (y-1)^{Y(A,B)} \\ &\quad - \sum_{A \subseteq E-e} \sum_{B \subseteq E-e} (r(M) - r(A+e))(x-1)^{X(A,B)} (y-1)^{Y(A,B)} \\ &= \sum_{A \subseteq E-e} \sum_{B \subseteq E-e} (r(A+e) - r(B))(x-1)^{X(A,B)} (y-1)^{Y(A,B)}, \end{aligned}$$

where $X(A, B) = 2r(M) - r(A + e) - r(B) - 1$ and $Y(A, B) = |A| + |B| + 1 - r(A + e) - r(B)$.

Therefore, the coefficient of $(x - 1)^i(y - 1)^j$ in $T^{M/e}T_x^{M-e} - T_x^{M/e}T^{M-e}$ is

$$\sum_{\substack{A, B \subseteq E - e \\ X(A, B) = i \\ Y(A, B) = j}} r(A + e) - r(B).$$

If this number is nonnegative for all i, j and all matroids $M = (E, r)$ and $e \in E$, then Conjecture 7.2.5 follows.

Fixing $X(A, B)$ is equivalent to fixing $r(A + e) + r(B)$, and fixing $Y(A, B)$ additionally is in effect fixing also $|A| + |B|$. Thus, I would like to show that

$$\sum_{\substack{A, B \subseteq E - e \\ |A| + |B| = i \\ r(A + e) + r(B) = j}} r(A + e) - r(B) \geq 0$$

regardless of the matroid M , element e , and numbers i and j .

The summation can be refined by fixing the union of A and B , as a multiset, instead of $|A| + |B|$. Let us write the multiset formed by unifying two sets A and B as $A \uplus B$. In order to prove Conjecture 7.2.5, it is sufficient to show that

$$\sum_{\substack{A, B \subseteq E - e \\ A \uplus B = C \\ r(A + e) + r(B) = k}} r(A + e) - r(B) \geq 0. \quad (7.4)$$

In other words, in this summation we fix k and C , and the summation is taken over all pairs A, B of subsets of $E - e$ satisfying those two conditions. If C contains two copies of an element f , then each pair (A, B) counted in the summation must have f in both A and B . Let C' be the subset of C consisting of the elements counted twice. Then

$$\sum_{\substack{A, B \subseteq E - e \\ A \uplus B = C \\ r(A + e) + r(B) = k}} r(A + e) - r(B) = \sum_{\substack{A, B \subseteq E - e \\ A' \uplus B' = C \setminus C' \\ r(A' + e + C') + r(B' + C') = k}} r(A' + e + C') - r(B' + C').$$

Since

$$\begin{aligned} r(A' + e + C') - r(B' + C') &= (r(A' + e + C') - r(C')) - (r(B' + C') - r(C')) \\ &= r_{M/C'}(A' + e) - r_{M/C'}(B'), \end{aligned}$$

if we can show that

$$\sum_{\substack{A \cup B = E - e, A \cap B = \emptyset \\ r(A+e) + r(B) = k}} r(A+e) - r(B) = \sum_{\substack{e \in A \subseteq E \\ r(A) + r(A^c) = k}} r(A) - r(A^c) \geq 0$$

for all k , all matroids M and every element e , then (7.4) holds and hence Conjecture 7.2.5 follows.

These considerations show that Conjecture 7.2.5 follows from Conjecture 7.2.6 below.

Conjecture 7.2.6. *Let $M = (E, r)$. For all $e \in E$ and all $k \geq 1$,*

$$\sum_{\substack{e \in A \subseteq E \\ r(A) + r(A^c) = k}} r(A) - r(A^c) \geq 0.$$

We now prove Conjecture 7.2.6 for series-parallel graphs. For that we need the following two lemmas.

Lemma 7.2.7. *Let $M = (E, r)$ be a matroid with a cocircuit of size 2, say $\{e, f\}$. Let $M_1 = M/f$ and say $M_1 = (E_1, r_1)$. If Conjecture 7.2.6 holds for M_1 , then it holds for M .*

Lemma 7.2.8. *Let $M = (E, r)$ be a matroid with a circuit of size 2, say $\{e, f\}$. Let $M_1 = M - f$ and say $M_1 = (E_1, r_1)$. If Conjecture 7.2.6 holds for M_1 , then it holds for M .*

Note that, although Lemmas 7.2.7 and 7.2.8 are similar, one does not follow from the other by taking dual matroids.

Proof of Lemma 7.2.7. I shall divide both summations into cases up to $r(A+f) - r(A)$ and $r(A^c + e + f) - r(A^c)$ then compare them individually to prove the lemma.

By the definition of M_1 , we have $E_1 = E - f$, $r_1(A) = r(A+f) - 1$. Assuming Conjecture 7.2.6 for M_1 we have the following for each k .

$$\sum_{\substack{e \in A \subseteq E_1 \\ r_1(A) + r_1(E_1 - A) = k}} r_1(A) - r_1(E_1 - A) \geq 0. \quad (7.5)$$

I shall partition the summation in (7.5) into two parts according to $r_1(A) - r_1(A - e)$ (which is the same as $r(A + f) - r(A - e) - 1$, and either 0 or 1) as follows. The last terms in the following equations, $I(k)$ and $J(k)$, are brief notations of the summations for later use.

$$\sum_{\substack{e \in A \subseteq E_1 \\ r_1(A) + r_1(E_1 - A) = k \\ r_1(A) = r_1(A - e) + 1}} r_1(A) - r_1(E_1 - A) = \sum_{\substack{e \in A \subseteq E_1 \\ r_1(A - e) + r_1(E_1 - A) = k - 1 \\ r_1(A) = r_1(A - e) + 1}} r_1(A - e) + 1 - r_1(E_1 - A) = I(k),$$

$$\sum_{\substack{e \in A \subseteq E_1 \\ r_1(A) + r_1(E_1 - A) = k \\ r_1(A) = r_1(A - e)}} r_1(A) - r_1(E_1 - A) = \sum_{\substack{e \in A \subseteq E_1 \\ r_1(A - e) + r_1(E_1 - A) = k \\ r_1(A) = r_1(A - e)}} r_1(A - e) - r_1(E_1 - A) = J(k)$$

The inequality (7.5) is equivalent to $I(k) + J(k) \geq 0$ for each k . Note that if $e \in A \subseteq E_1$, then $r_1(A - e) = r(A - e)$ and $r_1(E_1 - A) = r(E_1 - A)$. Thus

$$I(k) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A - e) + r(E_1 - A) = k - 1 \\ r(A + f) = r(A - e) + 2}} r(A - e) + 1 - r(E_1 - A) \quad \text{and}$$

$$J(k) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A - e) + r(E_1 - A) = k \\ r(A + f) = r(A - e) + 1}} r(A - e) - r(E_1 - A).$$

We want to show the following inequality for each $k \geq 1$. Again, $F(k)$ is a notation for later use.

$$F(k) = \sum_{\substack{e \in A \subseteq E \\ r(A) + r(E - A) = k}} r(A) - r(E - A) \geq 0.$$

The summation is on the pairs $(A, E - A)$. I shall partition the pairs depending on $f \in A$ or $f \notin A$. I shall also divide the cases up to $r(A + f) - r(A - e - f)$, which is either 1 or 2 and equal to $r_1(A \cap E_1) - r_1(A \cap E_1 - e) + 1$ when $e \in A \subseteq E$. There are four cases.

Case 1. $f \in A$, and $r(A + f) = r(A - e - f) + 2$.

The part of the summation $F(k)$ corresponding to this case is the following.

$$\sum_{\substack{e, f \in A \subseteq E \\ r(A) + r(E - A) = k \\ r(A) = r(A - e - f) + 2}} r(A) - r(E - A)$$

Let us consider the summation as $A_1 = A - f$ ranging over the subsets of $E_1 = E - f$ containing e instead of A ranging over $A \subseteq E$ containing both e and f . Since we assumed $f \in A$ and $r(A) = r(A - e - f) + 2$, we have

$$k = r(A) + r(E - A) = r(A_1 - e) + 2 + r(E_1 - A_1),$$

and

$$r(A) - r(E - A) = r(A_1 - e) + 2 - r(E_1 - A_1)$$

so that this partial summation equals to

$$\sum_{\substack{e \in A_1 \subseteq E_1 \\ r(A_1 - e) + r(E_1 - A_1) = k - 2 \\ r(A_1 + f) = r(A_1 - e) + 2}} r(A_1 - e) + 2 - r(E_1 - A_1).$$

By renaming the dummy variable A_1 to A again and putting a short notation $F_1(k)$, we get the partial summation of $F(k)$ for this case as

$$F_1(k) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A - e) + r(E_1 - A) = k - 2 \\ r(A + f) = r(A - e) + 2}} r(A - e) + 2 - r(E_1 - A).$$

The remaining cases use exactly same process, hence we only write the results.

Case 2. $f \in A$, and $r(A + f) = r(A - e - f) + 1$.

$$\sum_{\substack{e, f \in A \subseteq E \\ r(A) + r(E - A) = k \\ r(A) = r(A - e - f) + 1}} r(A) - r(E - A) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A - e) + r(E_1 - A) = k - 1 \\ r(A + f) = r(A - e) + 1}} r(A - e) + 1 - r(E_1 - A) = F_2(k).$$

Case 3. $f \notin A$, and $r(A + f) = r(A - e - f) + 2$.

$$\sum_{\substack{e \in A \subseteq E, f \notin A \\ r(A) + r(E - A) = k \\ r(A + f) = r(A - e) + 2}} r(A) - r(E - A) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A - e) + r(E_1 - A) = k - 2 \\ r(A + f) = r(A - e) + 2}} r(A - e) - r(E_1 - A) = F_3(k).$$

Case 4. $f \notin A$, and $r(A + f) = r(A - e - f) + 1$.

$$\sum_{\substack{e \in A \subseteq E, f \notin A \\ r(A) + r(E - A) = k \\ r(A + f) = r(A - e) + 1}} r(A) - r(E - A) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A - e) + r(E_1 - A) = k - 2 \\ r(A + f) = r(A - e) + 1}} r(A - e) - r(E_1 - A) = F_4(k).$$

Note that $F_1(k) + F_3(k) = 2I(k-1)$ and $F_2(k) \geq J(k-1)$. Also,

$$\begin{aligned}
I(k-1) + F_4(k) &= \sum_{\substack{e \in A \subseteq E_1 \\ r(A-e) + r(E_1-A) = k-2 \\ r(A+f) = r(A-e) + 2}} (r(A-e) + 1 - r(E_1 - A)) \\
&+ \sum_{\substack{e \in A \subseteq E_1 \\ r(A-e) + r(E_1-A) = k-2 \\ r(A+f) = r(A-e) + 1}} (r(A-e) - r(E_1 - A)) \\
&\geq \sum_{\substack{e \in A \subseteq E_1 \\ r(A-e) + r(E_1-A) = k-2 \\ r(A+f) = r(A-e) + 2}} (r(A-e) - r(E_1 - A)) \\
&+ \sum_{\substack{e \in A \subseteq E_1 \\ r(A-e) + r(E_1-A) = k-2 \\ r(A+f) = r(A-e) + 1}} (r(A-e) - r(E_1 - A)) \\
&= \sum_{\substack{e \in A \subseteq E_1 \\ r(A-e) + r(E_1-A) = k-2}} (r(A-e) - r(E_1 - A)) \\
&= 0,
\end{aligned}$$

since in the last summation a set A with $e \in A \subseteq E_1$ satisfies $r(A-e) + r(E_1 - A) = k - 2$ if and only if $r((E_1 - A + e) - e) + r(E_1 - (E_1 - A + e)) = k - 2$ so that their corresponding summands cancel out. Therefore,

$$F(k) = F_1(k) + F_2(k) + F_3(k) + F_4(k) \geq 2I(k-1) + J(k-1) - I(k-1) = I(k-1) + J(k-1)$$

is nonnegative for each k by the assumption. \square

Proof of Lemma 7.2.8. The proof idea of Lemma 7.2.8 is same as Lemma 7.2.7 but the calculations are little different. Recall that M is a matroid (E, r) with a circuit of size 2, namely $\{e, f\}$, and $M_1 = M - f$. We write $M_1 = (E_1, r_1)$, $E_1 = E - f$ and $r_1(A) = r(A)$ for all $A \subseteq E_1$. We assumed Conjecture 7.2.6 for M_1 , that is,

$$\sum_{\substack{e \in A \subseteq E_1 \\ r_1(A) + r_1(E_1 - A) = k}} r_1(A) - r_1(E_1 - A) = \sum_{\substack{e \in A \subseteq E \\ r(A) + r(E_1 - A) = k}} r(A) - r(E_1 - A) \geq 0 \quad (7.6)$$

for each k . I shall partition the summation in (7.6) into two parts upto $r(A) - r(A - e)$ as follows. The last terms in the following equations, $I(k)$ and $J(k)$, are

brief notations for later use.

$$\sum_{\substack{e \in A \subseteq E_1 \\ r(A) + r(E_1 - A) = k \\ r(A) = r(A - e) + 1}} r(A) - r(E_1 - A) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A) + r(E_1 - A) = k \\ r(A) = r(A - e) + 1}} r(A - e) + 1 - r(E_1 - A) = I(k),$$

$$\sum_{\substack{e \in A \subseteq E_1 \\ r(A) + r(E_1 - A) = k \\ r(A) = r(A - e)}} r(A) - r(E_1 - A) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A) + r(E_1 - A) = k \\ r(A) = r(A - e)}} r(A - e) - r(E_1 - A) = J(k).$$

The inequality (7.6) is equivalent to $I(k) + J(k) \geq 0$ for each k .

What we want to show is that the following formula

$$\sum_{\substack{e \in A \subseteq E \\ r(A) + r(E - A) = k}} r(A) - r(E - A) = F(k)$$

is nonnegative for each k . I used A' instead of A to indicate that A' is a subset of $M_1 - e$ instead of M . The summation is over the sets $A \subseteq E$ such that $e \in A$. I shall divide the cases depending on (1) $f \in A$ or $f \notin A$, (2) $r(A) - r(A - e - f)$, and (3) $r(E - A + f) - r(E - A - f)$. The values of (2) and (3) are either 0 or 1, but if $f \in A$, we ignore $r(E - A + f) - r(E - A - f)$ so there are six cases in total. We use $r(A + e + f) = r(A + e) = r(A + f)$ for all $A \subseteq E - e - f$ without mentioning.

Case 1. $f \in A$, $r(A) = r(A - e - f) + 1$.

The partial summation of $F(k)$ corresponding to the sets A with these two conditions is the following.

$$\sum_{\substack{e, f \in A \subseteq E \\ r(A) + r(E - A) = k \\ r(A) = r(A - e - f) + 1}} r(A) - r(E - A).$$

Let $A_1 = A - f$ so that $r(A) + r(E - A) = r(A_1 - e) + 1 + r(E_1 - A_1)$ and $r(A) - r(E - A) = r(A_1 - e) + 1 - r(E_1 - A_1)$. Using A_1 instead of A , we may rewrite the above summation as the following.

$$\sum_{\substack{e \in A_1 \subseteq E_1 \\ r(A_1 - e) + r(E_1 - A_1) = k - 1 \\ r(A_1) = r(A_1 - e) + 1}} r(A_1 - e) + 1 - r(E_1 - A_1).$$

By renaming the dummy variable A_1 back to A and putting a brief notation $F_1(k)$, we get the partial summation of $F(k)$ for this case as

$$F_1(k) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A-e)+r(E_1-A)=k-1 \\ r(A)=r(A-e)+1}} r(A-e) + 1 - r(E_1 - A).$$

The remaining cases use exactly same process, and hence we only write the results.

Case 2. $f \in A$, $r(A) = r(A - e - f)$.

$$\sum_{\substack{e, f \in A \subseteq E \\ r(A)+r(E-A)=k \\ r(A)=r(A-e-f)}} r(A) - r(E - A) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A-e)+r(E_1-A)=k \\ r(A)=r(A-e)}} r(A-e) - r(E_1 - A) = F_2(k).$$

Case 3. $f \notin A$, $r(A) = r(A - e - f) + 1$, $r(E - A + f) = r(E - A - f) + 1$

$$\sum_{\substack{e \in A \subseteq E, f \notin A \\ r(A)+r(E-A)=k \\ r(A)=r(A-e)+1 \\ r(E-A+f)=r(E-A-f)+1}} r(A) - r(E - A) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A-e)+r(E_1-A)=k-2 \\ r(A)=r(A-e)+1 \\ r(E_1-A+f)=r(E_1-A-f)+1}} r(A-e) - r(E_1 - A) = F_3(k).$$

Case 4. $f \notin A$, $r(A) = r(A - e - f) + 1$, $r(E - A + f) = r(E - A - f)$

$$\sum_{\substack{e \in A \subseteq E, f \notin A \\ r(A)+r(E-A)=k \\ r(A)=r(A-e)+1 \\ r(E-A+f)=r(E-A-f)}} r(A) - r(E - A) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A-e)+r(E_1-A)=k-1 \\ r(A)=r(A-e)+1 \\ r(E_1-A+f)=r(E_1-A-f)}} r(A-e) + 1 - r(E_1 - A) = F_4(k).$$

Case 5. $f \notin A$, $r(A) = r(A - e - f)$, $r(E - A + f) = r(E - A - f) + 1$

$$\sum_{\substack{e \in A \subseteq E, f \notin A \\ r(A)+r(E-A)=k \\ r(A)=r(A-e) \\ r(E-A+f)=r(E-A-f)+1}} r(A) - r(E - A) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A-e)+r(E_1-A)=k-1 \\ r(A)=r(A-e) \\ r(E_1-A+f)=r(E_1-A-f)+1}} r(A-e) - 1 - r(E_1 - A) = F_5(k).$$

Case 6. $f \notin A$, $r(A) = r(A - e - f)$, $r(E - A + f) = r(E - A - f)$

$$\sum_{\substack{e \in A \subseteq E, f \notin A \\ r(A)+r(E-A)=k \\ r(A)=r(A-e) \\ r(E-A+f)=r(E-A-f)}} r(A) - r(E - A) = \sum_{\substack{e \in A \subseteq E_1 \\ r(A-e)+r(E_1-A)=k \\ r(A)=r(A-e) \\ r(E_1-A+f)=r(E_1-A-f)}} r(A-e) - r(E_1 - A) = F_6(k).$$

Note that $F_6(k) = 0$ since the summands for A cancels out with the summand for $E_1 - A + e$. By the same reason, $F_3(k) = 0$. Also, by setting $B = A - e$ in $F_5(k)$,

we get

$$\begin{aligned}
F_5(k) &= \sum_{\substack{e \in A \subseteq E_1 \\ r(A-e) + r(E_1-A) = k-1 \\ r(A) = r(A-e) \\ r(E_1-A+f) = r(E_1-A-f)+1}} r(A-e) - 1 - r(E_1-A) \\
&= \sum_{\substack{B \subseteq E_1-e \\ r(B) + r(E_1-e-B) = k-1 \\ r(B+e) = r(B) \\ r(E_1-B+e) = r(E_1-B-e)+1}} r(B) - 1 - r(E_1-B-e).
\end{aligned}$$

Let $C = E_1 - B$. Then

$$F_5(k) = \sum_{\substack{e \in C \subseteq E_1 \\ r(E_1-C) + r(C-e) = k-1 \\ r(E_1-C+e) = r(E_1-C) \\ r(C) = r(C-e)+1}} r(E_1-C) - 1 - r(C-e).$$

By renaming the dummy variable C to A , we get $F_5(k) = -F_4(k)$.

Thus $F(k) = \sum_{i=1}^6 F_i(k) = F_1(k) + F_2(k)$. Since $F_1(k) = I(k)$ and $F_2(k) = J(k)$, we have $F(k) = I(k) + J(k) \geq 0$ for each k . \square

Theorem 7.2.9. *Conjecture 7.2.6 is true for all series-parallel graphs. If $M = (E, r)$ is the cycle matroid of a series-parallel graph, then for each integer $k \geq 1$,*

$$\sum_{\substack{e \in A \subseteq E \\ r(A) + r(E-A) = k}} r(A) - r(E-A) \geq 0.$$

Proof. Every series-parallel graph can be constructed from a single edge by serial- and parallel-extensions. Therefore, by Lemmas 7.2.7 and 7.2.8, it is enough to prove Conjecture 7.2.6 for when M is the cycle matroid of the graph with a single edge, which is trivial to check. \square

I believe that a stronger version of Conjecture 7.2.5 is also true, by deleting and contracting a subset instead of an element, as follows.

Conjecture 7.2.10. *Let M be a matroid and S be a proper nonempty subset of M . If we consider*

$$T^{M/S} T_x^{M-S} - T_x^{M/S} T^{M-S}$$

as a polynomial in $x - 1$ and $y - 1$ instead of x and y , then every coefficient is a positive integer.

In the same way as we strengthened Conjecture 7.2.5 to Conjecture 7.2.6, we now strengthen Conjecture 7.2.10 to Conjecture 7.2.11. We give the statement first, and then explain why it is stronger than Conjecture 7.2.10.

Conjecture 7.2.11. *Let $M = (E, r)$ be a matroid and $\emptyset \neq S \subseteq E$. Then*

$$\sum_{\substack{S \subseteq A \subseteq E \\ r(A) + r(M-A) = k}} r(A + S) - r(E - A + S) + r(E - S) - r(E) \geq 0$$

for each nonnegative integer k .

Note that if S consists of a single element e which is not a bridge of $M + e$, then Conjecture 7.2.11 becomes Conjecture 7.2.6.

Proof that Conjecture 7.2.11 implies Conjecture 7.2.10: We start by expressing each term in the formula $T^{M/S}T_x^{M-S} - T_x^{M/S}T^{M-S}$ using the rank-generating formulation of the Tutte polynomial.

$$\begin{aligned} T^{M/S}(x, y) &= \sum_{A \subseteq E-S} (x-1)^{r_{M/S}(E-S) - r_{M/S}(A)} (y-1)^{|A| - r_{M/S}(A)} \\ &= \sum_{A \subseteq E-S} (x-1)^{r(E) - r(A+S)} (y-1)^{|A| + r(S) - r(A+S)} \end{aligned}$$

$$\begin{aligned} T^{M-S}(x, y) &= \sum_{B \subseteq E-S} (x-1)^{r_{M-S}(E-S) - r_{M-S}(B)} (y-1)^{|B| - r_{M-S}(B)} \\ &= \sum_{B \subseteq E-S} (x-1)^{r(E-S) - r(B)} (y-1)^{|B| - r(B)} \end{aligned}$$

Hence, the formula can be expressed as

$$\begin{aligned} T^{M/S}T_x^{M-S} - T_x^{M/S}T^{M-S} &= \sum_{A, B \subseteq E-S} (r(E-S) - r(B))(x-1)^{X(A,B)} (y-1)^{Y(A,B)} \\ &\quad - \sum_{A, B \subseteq E-S} (r(E) - r(A+S))(x-1)^{X(A,B)} (y-1)^{Y(A,B)} \\ &= \sum_{A, B \subseteq E-S} Z(A, B, S)(x-1)^{X(A,B)} (y-1)^{Y(A,B)}, \end{aligned}$$

where

$$\begin{aligned} X(A, B) &= r(E) + r(E - S) - r(A + S) - r(B) - 1, \\ Y(A, B) &= |A| + |B| + r(S) - r(A + S) - r(B), \\ Z(A, B) &= r(A + S) - r(B) + r(E - S) - r(E). \end{aligned}$$

Thus, the coefficient of $(x - 1)^i (y - 1)^j$ in $T^{M/S} T_x^{M-S} - T_x^{M/S} T^{M-S}$ is the sum of $Z(A, B)$ over all pairs (A, B) such that $A, B \subseteq E - S$, $X(A, B) = i$ and $Y(A, B) = j$.

Fixing $X(A, B)$ and $Y(A, B)$ is equivalent to fixing both $r(A + S) + r(B)$ and $|A| + |B|$. Thus, Conjecture 7.2.10 claims that

$$\sum_{\substack{A, B \subseteq E - S \\ |A| + |B| = i \\ r(A + S) + r(B) = j}} (r(A + S) - r(B) + r(E - S) - r(E))$$

is nonnegative for all i, j .

I shall replace the condition $|A| + |B| = i$ in the above summation with $A \uplus B = C$ where C is a multiset consisting of some elements of $E - S$ with multiplicity at most 2, and $A \uplus B$ is the multiset obtained by unifying A and B . This replacement refines the summation and hence strengthens the claim. Thus, if for all matroids M , subsets S, C and all j the following holds:

$$\sum_{\substack{A, B \subseteq E - S \\ A \uplus B = C \\ r(A + S) + r(B) = j}} (r(A + S) - r(B) + r(E - S) - r(E)) \geq 0,$$

then Conjecture 7.2.10 follows. We may simplify the summation by removing the cases where C contains an element of multiplicity 2. Suppose that C' is the set of elements of C with multiplicity 2. If $A \uplus B = C$, then $C' \subseteq A$ and $C' \subseteq B$, and if we denote $A' = A - C'$ and $B' = B - C'$,

$$\begin{aligned} r(A + S) - r(B) &= r(A' + S + C') - r(B' + C') \\ &= (r(A' + S + C) - r(C)) - (r(B' + C) - r(C)) \\ &= r_{M/C}(A' + S) - r_{M/C}(B'), \end{aligned}$$

so that Conjecture 7.2.11 implies Conjecture 7.2.10. \square

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