On the Minimum Number of Spanning Trees in k-Edge-Connected Graphs

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ON THE MINIMUM NUMBER OF SPANNING TREES IN $k$-EDGE-CONNECTED GRAPHS

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Abstract. We show that a $k$-edge-connected graph on $n$ vertices has at least $n(k/2)^{n-1}$ spanning trees. This bound is tight if $k$ is even and the extremal graph is the $n$-cycle with edge-multiplicities $k/2$. For $k$ odd, however, there is a lower bound $c_k^{n-1}$ where $c_k > k/2$. Specifically, $c_3 > 1.77$ and $c_5 > 2.75$. Not surprisingly, $c_3$ is smaller than the corresponding number for 4-edge-connected graphs. Examples show that $c_4 \leq \sqrt{2} + \sqrt{3} \approx 1.93$.

However, we have no examples of 5-edge-connected graphs with fewer spanning trees than the $n$-cycle with all edge-multiplicities (except one) equal to 3, which is almost 6-regular. We have no examples of 5-regular 5-edge-connected graphs with fewer than $3.09n^{-1}$ spanning trees which is more than the corresponding number for 6-regular 6-edge-connected graphs. The analogous surprising phenomenon occurs for each higher odd edge-connectivity and regularity.

1. Introduction

Every connected graph has a spanning tree, that is, a connected subgraph with no cycles containing all vertices of the graph. The number of spanning trees, denoted $\tau(G)$, is of importance in electrical networks, in particular, for expressing driving point resistances (effective resistances); see e.g. [9]. Kostochka [4] showed that, if $G$ is a connected $k$-regular simple graph, then $k(1-O(\log k/k)) \leq \tau(G)^{1/n} \leq k$. But if we allow multiple edges, there are graphs with far less spanning trees. In this paper, we investigate the minimum number of spanning trees in $k$-edge-connected graphs with multiple edges. Since a loop is never contained in a spanning tree, we consider only graphs without loops.

In Section 2 we investigate how $\tau(G)$ changes when we replace a certain subgraph of $G$ by another graph. In Section 3 we derive the lower bounds stated in the abstract. Since this bound is not tight for any odd edge-connectivity, we show in Section 4 that $\tau(G) \geq 1.774^{n-1}$ for every 3-edge-connected graph $G$ on $n$ vertices. The proof involves a new recursive description of the 3-connected cubic graphs; they can all be obtained from $K_4$ or $K_{3,3}$ by

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successively adding vertices or blowing vertices up to triangles. In Section 5, we consider the class of 5-regular 5-edge-connected graphs. Section 6 presents a class of \( k \)-regular \( k \)-edge-connected graphs which suggests that for odd \( k > 3 \), the minimum number of spanning trees might be obtained by an almost \( (k + 1) \)-regular graph. Even more surprisingly, all examples of 5-regular, 5-edge-connected graphs with \( n \) vertices known to us have more than \( 3.09^{n-1} \) spanning trees while there are 6-regular, 6-edge-connected graphs with only \( n^{3n-1} \) spanning trees.

We adopt the notation and terminology of Diestel [3]. We repeat a few important definitions. A **bridge** is an edge whose removal disconnects the graph. A graph is **\( k \)-edge-connected** if we need to remove at least \( k \) edges to disconnect the graph. A graph is **\( k \)-regular** if each vertex has \( k \) incident edges. A 3-regular graph is also called **cubic**. If \( e \) is an edge in a graph \( G \), then \( G/e \) is the graph obtained by contracting \( e \).

2. **Lifting pairs of edges**

Let \( G, H_1, H_2 \) be connected graphs, and \( X \subseteq V(G) \), \( X_i \subseteq V(H_i) \) for \( i = 1, 2 \) such that \( |X| = |X_1| = |X_2| \). For \( i = 1, 2 \), let \( G_i \) be the graph obtained from \( G \cup H_i \) by identifying \( X_i \) with \( X \). We are interested in \( \tau(G_1)/\tau(G_2) \). Let \( T \) be a spanning tree of \( G_1 \) or \( G_2 \). Then \( T \cap G \) is a spanning forest of \( G \). By comparing the number of ways of extending \( T \cap G \) into a spanning tree of \( G_i \) using \( H_i \), and taking the minimum ratio over all possible such forests, we can find a lower bound for \( \tau(G_1)/\tau(G_2) \). Note that the number of ways of extending \( T \cap G \) in \( G_i \) using \( H_i \) is exactly the number of spanning trees of the graph obtained from \( H_i \) by contracting each component of \( T \cap G \) into a single vertex. This is made more precise in the following observation.

**Observation 1.** Let \( G \) be a graph, and let \( X \subseteq V(G) \) be a set of vertices. Suppose that \( G \) has two connected subgraphs \( G_0, G_1 \) such that \( G_0 \cup G_1 = G \), \( V(G_0 \cap G_1) = X \) and \( E(G_0 \cap G_1) = \emptyset \). Let \( T \) be a spanning forest of \( G_0 \) such that each component contains at least one vertex in \( X \). Then the number of ways of extending \( T \) to a spanning tree of \( G \) using edges in \( G_1 \) is \( \tau(S_0) \), where \( S_0 \) is the graph obtained from \( G_1 \cup T \) by contracting each component of \( T \) into a single vertex.

Let \( e = vu, f = vw \) be two adjacent edges of a graph. **Lifting** \( e, f \) means that we replace \( e, f \) by an edge \( uw \) if \( u \neq w \). If \( u = w \) we remove both edges \( e, f \) as we do not allow loops. By **lifting at** \( v \) we mean that we lift a pair of edges incident with \( v \). A **complete lifting** at a vertex \( v \) with even degree is a sequence of liftings at \( v \) until no edges are left at \( v \). Then we remove \( v \).
For the following lemma, we define a constant $c_d$ depending on a positive integer $d$:

$$c_d = \min_{d_1, d_2, \ldots, d_k} \min_H \frac{\prod_{i=1}^k d_i}{\tau(H)},$$

where the minimum is taken over all sequences of positive integers $d_1, d_2, \ldots, d_k$ with varying length $k$ such that $\sum_{i=1}^k d_i = 2d$, and over all connected graphs $H$ on $k$ vertices with degree sequence $d'_1, d'_2, \ldots, d'_k$ such that $d'_i \leq d_i$ for each $i$.

In the above definition of $c_d$, $H$ has at most $d$ edges, so $c_1 = 1$. Furthermore, $c_2 = 2$, $c_3 = 8/3$ and $c_4 = 18/5 = 3.6$, which are attained by a 2-cycle, a 3-cycle, and a 3-cycle plus an edge, respectively.

**Lemma 1.** Let $G$ be a graph with a vertex $v$ of degree $2d$. Let $G'$ be a graph obtained from $G$ by a complete lifting at $v$. Then $\tau(G) \geq c_d \tau(G')$, where $c_d$ is defined as above.

**Proof:** Denote $G_0 = G - v$ and the neighbors of $v$ in $G$ by $v_1, v_2, \ldots, v_{2d}$, which are not necessarily distinct. We may assume that for each $i$, $v_{2i-1}v_{2i} \in E(G') \setminus E(G)$ resulting from lifting $vv_{2i-1}$ and $vv_{2i}$ unless $v_{2i-1} = v_{2i}$.

We consider a spanning forest, say $T_0$, of $G_0$ in which each component contains at least one of the neighbors of $v$. We shall estimate the number of ways of extending $T_0$ to a spanning tree using only edges not in $G_0$. The forest $T_0$ partitions the neighbors of $v$, say into $P_1, P_2, \ldots, P_k$ with sizes $|P_i| = d_i$, $\sum_{i=1}^k d_i = 2d$. By Observation 1, the number of ways of extending $T_0$ to a spanning tree of $G$ (using no other edge of $G_0$) is precisely $\tau(S_0)$, where $S_0$ is the star graph at $v$ with edge-multiplicities $d_1, d_2, \ldots, d_k$. Thus $\tau(S_0) = \prod_{i=1}^k d_i$. Likewise, the number of ways of extending $T_0$ to a spanning tree of $G'$ is $\tau(S'_0)$ where $S'_0$ is the graph obtained from $G'$ by contracting each component of $T_0$ into a single vertex, and then remove the remaining edges of $G_0$, if any. Let $p_i$ be the vertex of $S'_0$ corresponding to $P_i$. Then $\deg(p_i) \leq d_i$, since each $v_j \in P_i$ provides $p_i$ with at most one edge from $E(G') \setminus E(G_0)$. Therefore, the number of extensions of $T_0$ into spanning trees of $G$ divided by the number of extensions to $G'$ is at least $\min_H \prod_{i=1}^k d_i / \tau(H)$, where $H$ is as described in the definition of $c_d$. Now we consider all possibilities for $T_0$ and get the inequality. \hfill \Box

**Lemma 2.** Let $G$ be a graph with a vertex $v$ of degree $d \geq 3$. Let $G'$ be a graph resulting from lifting edges $vu, vw$ in $G$. Then $\tau(G) \geq (1 + \frac{1}{d-1}) \tau(G')$.

**Proof:** We consider a spanning forest, say $T_0$, of $G - v$ in which each component contains at least one of the neighbors of $v$. Then $T_0$ partitions the neighbors of $v$, say into $P_1, P_2, \ldots, P_k$ with sizes $|P_i| = d_i$, $\sum_{i=1}^k d_i = d$. By Observation 1, the number of ways to extend $T_0$ to a spanning tree of $G$ is $\tau(S_0) = \prod_{i=1}^k d_i$, where $S_0$ is the star graph at $v$ with edge-multiplicities
Thus, \( S_0' \) are a single vertex and then remove the remaining edges of \( G - v \), if any. By Observation 1, there are \( \tau(S_0') \) ways of extending \( T \) to a spanning tree of \( G' \). If some \( P_j \) contains both \( u, w \), then 
\[
\tau(S_0') = (d_j - 2) \prod_{i \neq j} d_i, \text{ so that either } \tau(S_0') = 0 \text{ or } \tau(S_0')/\tau(S_0') = d_j/(d_j - 2) > 1 + 4/(d^2 - 4).
\]

If \( u, w \) are contained in two different parts, say \( P_i, P_j \) respectively, then \( S_0' \) is obtained from \( S_0 \) by lifting two edges connecting \( v \) to the two vertices corresponding to \( P_i \) and \( P_j \). Thus, 
\[
\frac{\tau(S_0)}{\tau(S_0')} = \frac{d_id_j}{d_id_j - 1} \geq 1 + \frac{d^2 - 4}{4},
\]
since \( d_i + d_j \leq d \) which implies \( d_id_j \leq [(d_i + d_j)/2]^2 \leq d^2/4 \).

By considering all possible such forests \( T_0 \), we get the inequality. \( \square \)

3. \( k \)-edge-connected graphs

Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Consider the pairs \( (e, T) \) where \( e \in E(G) \) and \( T \) a spanning tree of \( G \) containing \( e \). For each \( e \in E(G) \) we have \( \tau(G/e) \) such pairs and for each \( T \), we have \( n - 1 \) such pairs. Therefore 
\[
(n - 1)\tau(G) = \sum_{e \in E(G)} \tau(G/e).
\]
Hence, \( G \) has an edge \( e \) such that \( \tau(G/e)/\tau(G) \leq (n - 1)/m \). We restate this conclusion as the following observation.

**Observation 2.** Let \( G \) be a connected graph with \( n > 1 \) vertices and \( m \) edges. Then \( G \) has an edge \( e \) such that \( \tau(G) \geq \frac{m}{n-1}\tau(G/e) \).

Now we prove the first lower bound stated in the abstract.

**Theorem 1.** Let \( G \) be a \( k \)-edge-connected graph on \( n \) vertices. Then \( G \) has at least \( n(k/2)^{n-1} \) spanning trees. Moreover, \( G \) has more than \( n(k/2)^{n-1} \) spanning trees unless \( k \) is even and \( G \) is a cycle whose edge-multiplicities are all \( k/2 \).

**Proof:** We shall use induction on \( n \). Since \( G \) is \( k \)-edge-connected, the minimum degree of \( G \) is at least \( k \) and thus \( m \geq kn/2 \). By Observation 2, \( G \) has an edge \( e \) such that 
\[
\tau(G) \geq \frac{m}{n-1}\tau(G/e) \geq \frac{kn}{2(n-1)}\tau(G/e).
\]
By the induction hypothesis, \( \tau(G/e) \geq (n - 1)(k/2)^{n-2} \) so that \( \tau(G) \geq n(k/2)^{n-1} \). If equality holds, then \( k \) is even, \( m = kn/2 \), and \( G/e \) is a cycle where all edge-multiplicities are \( k/2 \). Moreover, any edge can play the role of \( e \). This implies that all edge-multiplicities in \( G \) are \( k/2 \). If \( H \) denotes the subgraph of \( G \) obtained by replacing every multiple edge by a single edge, then \( H \) has the property that the contraction of any edge results in a cycle. Then also \( H \) is a cycle. \( \square \)
For $k$ even Theorem 1 is tight. However, for $k$ odd we shall present a lower bound for the number of spanning trees in a $k$-edge-connected graph of the form $c_k^{n-1}$ with $c_k > k/2$. For that, we shall use the following Theorem by Mader [6].

**Theorem 2.** Let $G$ be a connected graph on a vertex set $V \cup \{s\}$. If $\deg(s) \neq 3$ and $s$ is not incident with bridges, then $G$ has a lifting at $s$ such that for each pair $u, v$ of vertices in $V$, the maximum number of edge-disjoint paths between $u, v$ does not decrease after the lifting.

By Theorem 2 and Menger’s Theorem, given a $k$-edge-connected graph and a vertex of degree $\geq k + 2$, we can find a lifting without decreasing the edge-connectivity. Thus by Lemma 2, the minimum number of spanning trees of a $k$-edge-connected graph on $n$ vertices must be obtained by a graph whose degrees are only $k$ or $k+1$. We state this as an observation for later use.

**Observation 3.** If $G$ is a $k$-edge-connected graph on $n$ vertices with minimum $\tau(G)$, then each vertex of $G$ has either $k$ or $k+1$ incident edges.

Now we prove the following lower bound for odd edge-connectivity.

**Theorem 3.** Let $k > 1$ be an odd number and let $G$ be a $k$-edge-connected graph on $n$ vertices. Then $\tau(G) \geq (kc_k/2)^{n-1}$, where $c_k = \sqrt{1 + \frac{4}{(k+3)^2 - 4}} > 1$.

**Proof:** Let $e$ be an edge for which $\tau(G)/\tau(G/e)$ is maximum. By Observation 2 we know $\tau(G)/\tau(G/e) \geq k/2$. If the vertex of $G/e$ resulting from the contraction of $e$, say $v$, has degree bigger than $k + 1$, then using Theorem 2 we can lift some pair of edges at $v$ such that $G/e$ after the lifting is still $k$-edge-connected. We do the lifting at $v$ until the degree of $v$ is at most $k + 1$. Let $H$ be the resulting graph. If $\tau(G)/\tau(H) \geq kc_k^2/2$ then we call $e$ a good edge. Note that, if $H \neq G/e$, then by applying Lemma 2 at the last lifting, we see that $e$ is good. Also, if $e$ has multiplicity at least $(k+1)/2$, then $\tau(G)/\tau(H) \geq \tau(G)/\tau(G/e) \geq (k+1)/2 > kc_k^2/2$ so that $e$ is good. If one of the ends of $e$ has degree at least $k+1$, then either $e$ has multiplicity at least $(k+1)/2$, or the vertex obtained by the contraction of $e$ has degree at least $k + 2$, so that $e$ is good. Thus $e$ is not good only if the ends of $e$ both have degree precisely $k$. In particular, both ends of $e$ have odd degree.

Now we repeat the contractions of an edge with maximum $\tau(G)/\tau(G/e)$, followed by liftings whenever possible, until only two vertices are left. Because of parity, among the $n - 2$ contractions, at most $[(n - 2)/2]$ of them are edges whose ends both have odd degree. Thus at least $[(n - 2)/2]$ times we get an additional factor of $c_k^2$, so $\tau(G) \geq k \cdot (k/2)^{n-2} \cdot c_k^{2[(n-2)/2]} > (kc_k/2)^{n-1}$. \(\square\)
By Theorem 3, Theorem 1 is not tight for any odd edge-connectivity, although it is tight for all even edge-connectivity. In the following we focus on $k$-edge-connected graphs where $k = 3, 5$.

4. 3-EDGE-CONNECTED GRAPHS

Let $G$ be a 3-edge-connected graph on $n$ vertices. By Theorem 3, the lower bound $\tau(G) \geq n(3/2)^{n-1}$ is not tight. Kostochka [4] showed that a cubic simple 2-connected graph on $n$ vertices has at least $8n/4 \approx 1.68n$ spanning trees. This result is essentially best possible because of the cubic 2-connected graphs obtained by a collection of $K_4$’s minus an edge by adding a matching. In this section, we prove the following theorem.

**Theorem 4.** Let $G$ be a 3-edge-connected graph on $n$ vertices. Then $\tau(G) > 1.774^{n-1}$.

Kreweras [5] showed that the prism graph on $n$ vertices has approximately $1.93^n$ spanning trees; see Section 6. By Observation 3, a 3-edge-connected graph on $n$ vertices with minimum number of spanning trees has vertex degrees only 3 and 4. Thus by Lemma 1, the following is enough to prove Theorem 4. Note that a cubic graph with more than two vertices has the same connectivity and edge-connectivity.

**Theorem 5.** Let $G$ be a 3-connected cubic graph on $n$ vertices. Then $\tau(G) > 1.774^{n-1}$.

An often used operation to construct a 3-connected cubic graph is to join two edges, i.e. for non-parallel edges $e, f$, we replace each edge by a path of length 2 and connect the two new vertices of degree 2 by an edge. Note that joining two non-parallel edges in a 3-connected cubic graph results in another 3-connected cubic graph. The following lemma explains how the number of spanning trees changes after joining.

**Lemma 3.** Let $G$ be a graph with two non-parallel edges $e$ and $f$. Let $G'$ be the graph obtained from $G$ by joining $e$ and $f$. Then $\tau(G') \geq (4 - r)\tau(G)$, where $r = \tau(G/e/f)/\tau(G) \leq 1$.

**Proof:** We shall use Observation 1. We only consider the case when $e, f$ are not adjacent, but the other case can be done likewise. Let $e = ab$ and $f = cd$. Let $T$ be a spanning tree of $G$. Then $T - e - f$ is a spanning forest of $G$ in which each component contains at least one of $a, b, c$ and $d$. We shall consider how many ways $T - e - f$ can be extended to a spanning tree in $G$ and $G'$ respectively. For example, if $T - e - f$ has two components such that one of them contains $a, c$ and the other contains $b, d$, then we can extend $T - e - f$ in two ways to a spanning tree of $G$, whereas there are eight ways for $G'$. In fact, there are at least four times as many extensions in $G'$ as extensions in $G$, unless $T$ contains both $e$ and $f$, in which case we have a factor 3. Thus, $\tau(G') \geq 4(\tau(G) - \tau(G/e/f)) + 3\tau(G/e/f) = (4-r)\tau(G)$. □
To prove Theorem 5, we shall consider the following two operations to construct 3-connected cubic graphs.

1. Let \( v \) be a vertex \( v \) in a graph such that \( \deg(v) = 3 \) and all three neighbors of \( v \) are distinct. Then the **blow-up** of \( v \) is obtained by joining two of the incident edges of \( v \).

2. Select three edges, which may not be pairwise distinct, but not all the same, and subdivide each of them so that we have three new vertices of degree 2. Add a new vertex \( v \) and an edge from \( v \) to each of the three vertices of degree 2. We call this a **vertex-addition**.

Since a blow-up is a join of two non-parallel edges, we get the following observation by Lemma 3.

**Observation 4.** Let \( G \) be a graph with a vertex \( v \) of degree 3 whose neighbors are all distinct. Let \( G' \) be the graph obtained from \( G \) by a blow-up of \( v \). Then \( \tau(G') \geq 3\tau(G) \).

Barnette and Grünbaum [1] and independently Titov [10] gave a characterization of 3-connected graphs which implies that every 3-connected cubic graph can be obtained from \( K_4 \) by successively joining edges. We shall here prove a stronger result for cubic graphs.

**Theorem 6.** Let \( G \) be a 3-connected cubic graph with more than two vertices. Then \( G \) can be constructed from \( K_4 \) or \( K_3,3 \) by blow-ups and vertex-additions, such that blow-ups are never used consecutively.

**Proof:** Our proof consists of two parts. We show that if \( G \) has no induced subgraph which is a subdivision of another 3-connected graph, then \( G \) is one of \( K_4 \), \( K_{3,3} \) or the prism on 6 vertices defined in Section 6. Then we assume that \( G \) has a maximal induced subgraph, say \( H \), which is a subdivision of another 3-connected graph \( H^* \), and we show that \( G \) can be obtained from \( H^* \) by a vertex addition, possibly followed by a blow-up.

Suppose that \( G \) has no proper induced subgraph which is a subdivision of a 3-connected cubic graph. Let \( C \) be a cycle in \( G \) of minimum length so that \( C \) has no chord. Let \( v \) be a vertex in \( G - V(C) \). Since \( G \) is 3-connected, Menger’s Theorem implies that \( G \) has three paths \( P_1, P_2, P_3 \) where \( P_i = vu_1^i u_2^i \) \( \ldots \) \( u_k^i u_i \), \( C \cap P_i = \{ u_i \} \) for each \( i \) and the paths \( P_1, P_2, P_3 \) share only \( v \). Let \( v \) be such a vertex with \( k_1 + k_2 + k_3 \) being smallest. Note that some \( k_i \) may be 0, implying that \( P_i \) is an edge. If \( G \) has an edge between the non-endvertices of two \( P_i \)'s, say \( u_1^i u_2^j \), then by taking \( v = u_1^i \) instead and using \( P_1 \cup P_3 \) and \( u_1^i u_2^j u_{j+1}^2 \ldots u_{k_2}^2 \), we get a smaller sum of the lengths of the paths unless \( u_2^2 \) is the neighbor of \( v \) in \( P_2 \). Similarly, we deduce that \( u_1^1 \) is also the neighbor of \( v \) in \( P_1 \). In this case, \( vu_1^1 u_2^2 \) is a triangle and hence \( C \)
must also be a triangle, so that the vertex set of \( C \cup P_1 \cup P_2 \cup P_3 \), say \( V \), induces a subgraph of \( G \) which is a subdivision of the prism graph. Thus by the assumption, \( G \) itself is the prism graph.

Hence we may assume that \( G \) has no edge between the non-endvertices of \( P_i \)'s. Denote by \( G[V] \) the subgraph of \( G \) induced by \( V \). Suppose \( k_1 \geq 1 \) and some \( u_i \) has a neighbor on \( C \) different from \( u_1 \). Because of the minimality of \( k_1 + k_2 + k_3 \), we have \( i = k_1 \) and by taking \( v = u_{k_1} \) and using its two neighbors on \( C \), we see \( k_2 = k_3 = 0 \). Therefore \( G[V] \) is a subgraph of either the prism graph or \( K_{3,3} \), so that again \( G \) itself is either the prism graph or \( K_{3,3} \). The remaining case leaves no other edge in \( G[V] \) than \( C \cup P_1 \cup P_2 \cup P_3 \), which is a subdivision of \( K_4 \). Thus in this case \( G \) itself is \( K_4 \). This completes the first part.

Now we assume that \( G \) has an induced proper subgraph which is a subdivision of a 3-connected cubic graph. Let \( H \) be a maximal such subgraph. Let us call a path in \( H \) suspended if its ends both have degree 3 in \( H \) and all other vertices in the path have degree 2 in \( H \). Suspended paths intersect only at their ends. By replacing each suspended path of \( H \) by an edge between its ends, we get a 3-connected cubic graph, which we denote \( H^* \). Since \( G \) is 3-connected, \( H \) has at least two suspended paths. If \( G \) has a vertex, say \( v \), outside \( H \) which has neighbors in at least two distinct suspended paths of \( H \), then the subgraph of \( G \) induced by \( V(H) \cup \{v\} \) is a subdivision of a 3-connected graph, which must be \( G \) because of the maximality of \( H \). Then \( G \) can be obtained from \( H^* \) by the vertex-addition of \( v \). Thus we may assume that for each vertex in \( V(G) \setminus V(H) \), its neighbors in \( H \), if any, are in a single suspended path of \( H \). Also, we may assume that \( |V(G) \setminus V(H)| > 1 \). If \( V(G) \setminus V(H) = \{u, v\} \), then \( u \) and \( v \) are adjacent, and they have neighbors in distinct suspended paths. Thus we can obtain \( G \) from \( H^* \) by first vertex-adding \( u \) and then a blow-up to make \( v \). Therefore, we assume that \( |V(G) \setminus V(H)| > 2 \).

Since \( G \) is 3-connected, at least one component of \( G - V(H) \) has edges to two distinct suspended paths of \( H \). Thus \( G \) has a path of length \( > 1 \) between distinct suspended paths of \( H \) which intersects \( H \) at only its ends. Let \( P = v_0v_1 \ldots v_k \) be such a path with smallest length. Since \( P \) has no chord, the subgraph of \( G \) induced by \( H \cup P \) is a subdivision of a 3-connected graph, so that \( V(H) \cup V(P) = V(G) \), implying \( k \geq 4 \). By assumption, the neighbors of \( v_1 \) and \( v_{k-1} \), respectively, are in different suspended paths of \( H \). Let \( v \) be the neighbor of \( v_2 \) in \( H \). Then either \( v_0v_1v_2v \) or \( vv_2v_3 \ldots v_k \) contradicts the minimality of \( P \), a contradiction which completes the proof. \( \square \)
Let $c$ be the positive real solution of the equation $x^4 - 3x^2 - 1 = 0$ which is approximately $c \approx 1.8174$. Note that a vertex-addition is equivalent to a joining of two edges and then joining the new edge with an edge.

**Lemma 4.** Let $G_0$ be a 3-connected graph and let $G$ be a graph obtained from $G_0$ by joining two non-parallel edges of $G_0$, where $e$ denotes the joining edge. Let $G'$ be a graph obtained from $G$ by joining $e$ with another edge $f$. Then either $\tau(G') \geq c^2 \tau(G)$ or $\tau(G') \geq c^4 \tau(G_0)$.

**Proof:** Let $r = \tau(G/e/f)/\tau(G)$ be as in Lemma 3. Let $r' = \tau(G/e)/\tau(G)$ so that $\tau(G)/\tau(G-e) = 1/(1-r')$. Since $r' \geq r$, Lemma 3 implies $\tau(G') \geq (4-r')\tau(G) \geq (4-r')\tau(G)$. If $4-r' \geq c^2$ then we are done. Thus we may assume that $4-r' < c^2$, equivalently $1-r' < c^2-3$

By modifying the equation for $c$, we get $1 + 3/(c^2-3) = c^4$, so that

$$\tau(G') \geq (4-r')\tau(G) = \frac{(4-r')\tau(G)}{\tau(G_0)} \geq \frac{(4-r')\tau(G)}{\tau(G-e)} \geq \frac{4-r'}{1-r'}\tau(G_0) = 4 - r'

= \left(1 + \frac{3}{1-r'}\right)\tau(G_0) > \left(1 + \frac{3}{c^2-3}\right)\tau(G_0) = c^4\tau(G_0).

\hfill \square

**Proof of Theorem 5:** We shall prove $\tau(G) \geq (3c^2)^{(n-1)/4}$ by induction on $n = |V(G)|$, where $c$ is the constant used in Lemma 4. We may assume that $n \geq 8$ because $K_4$, $K_{3,3}$ and the prism on 6 vertices have 16, 81 and 75 spanning trees, respectively. By Theorem 6, $G$ can be obtained from $K_4$ or $K_{3,3}$ by repeatedly applying vertex-additions and blow-ups. If the last operation is a vertex-addition, then by Lemma 4, $\tau(G) \geq c^2\tau(G')$ or $\tau(G) \geq c^4\tau(G'')$ for some 3-connected cubic graph $G'$ with $n-2$ vertices or $G''$ with $n-4$ vertices, so we are done. Otherwise, $G$ can be obtained from a 3-connected cubic graph using a vertex-addition and then a blow-up. By Observation 4, a blow-up multiplies the number of spanning trees by at least 3, so that using Lemma 4, $\tau(G) \geq 3c^2\tau(G')$ or $\tau(G) \geq 3c^4\tau(G'')$ for some 3-edge-connected cubic graph $G'$ with $n-4$ vertices or $G''$ with $n-6$ vertices. By the induction hypothesis, $\tau(G) \geq (3c^2)^{(n-1)/4} > 1.774^{n-1}$. \hfill \square

5. 5-regular 5-edge-connected graphs

Let $G$ be a 5-regular 5-edge-connected graph. A **5-cut** is a set of edges $E$ with $|E| = 5$ such that $G - E$ is disconnected. If one of the components of $G - E$ is a single vertex, then we call $E$ trivial. Otherwise we call $E$ nontrivial. A **5-side** is a set $X \subseteq V(G)$ such that $\delta(X)$ (that is, the set of edges with precisely one end in $X$) is a nontrivial 5-cut. If a 5-side $X$ has the property that no nontrivial 5-cut contains an edge with both ends in $X$, then $X$ is called minimal.
Lemma 5. Let $G$ be a 5-regular 5-edge-connected graph. If $G$ has a nontrivial 5-cut, then $G$ has a minimal 5-side.

Proof: Let $A$ be a 5-side which is not minimal. Then some nontrivial 5-cut $S = \delta(B)$ contains an edge $uv$ with $u \in A \cap B$ and $v \in A \cap B^c$. Let $T = \delta(A)$. One of the sets $A \cap B$, $A \cap B^c$, $A^c \cap B$ or $A^c \cap B^c$ is empty because $G$ is 5-edge-connected, $S, T$ are 5-cuts and 5 is odd. Since $u \in A \cap B$ and $v \in A \cap B^c$, either $A^c \cap B$ or $A^c \cap B^c$ is empty, so that either $A \cap B$ or $A \cap B^c$ is a 5-side strictly smaller than $A$. If it is not minimal, then we repeat the argument until we eventually find a minimal 5-side. □

Lemma 6. Let $G$ be a connected graph with a connected subgraph $H$. If $G'$ is the graph obtained by contracting $H$ into a single vertex, then $\tau(G) \geq \tau(H) \tau(G')$.

Proof: For each pair $S, T$ of spanning trees of $H, G'$, we can expand the contracted vertex of $G'$ using $S$ to get a spanning tree of $G$. □

Theorem 7. Let $G$ be a 5-regular 5-edge-connected graph on $n$ vertices. Then $\tau(G) \geq 7.6^{(n-1)/2} \approx 2.7568^{n-1}$.

Proof: We shall use induction on $n$. Being 5-regular and 5-edge-connected, $G$ has no edge of multiplicity at least 3. If $G$ has a nontrivial 5-cut, then by Lemma 5, we can find a minimal 5-side, and we let $e = uv$ be an edge inside that minimal side. Otherwise let $e = uv$ be an arbitrary edge.

Suppose first $e$ has multiplicity 1. $G/e$ has a vertex of degree 8, which we can completely lift using Theorem 2. Denote the resulting 5-regular 5-edge-connected graph by $G'$. By Lemma 1, $\tau(G/e) \geq 3.6\tau(G')$. Now we consider $G - e$. Since $e$ is not contained in any nontrivial 5-cut, $G - e$ has at least 5 edge-disjoint paths between any pair of vertices distinct from the ends of $e$. Thus by Theorem 2, we can completely lift $u, v$ in $G - e$ so that the resulting graph, say $G''$, is 5-edge-connected and 5-regular. By Lemma 1, $\tau(G - e) \geq 4\tau(G'')$ and by the induction hypothesis,

$$\tau(G) = \tau(G/e) + \tau(G - e) \geq 3.6\tau(G') + 4\tau(G'') \geq 7.6^{(n-1)/2}.$$

Now we may assume that every edge of $G$ with multiplicity 1 is contained in a nontrivial 5-cut. Let $X$ be a minimal 5-side. Since the edges inside $X$ are not contained in any nontrivial 5-cut, every edge inside $X$ must be a double edge. Hence every vertex in $X$ is incident with $\delta(X)$, so that $X$ is the 5-double-cycle which has 80 spanning trees. By Lemma 6, $\tau(G) \geq 80\tau(G/X)$, and by the induction hypothesis, $\tau(G) \geq 7.6^{(n-1)/2}$. □
6. EXAMPLES OF $k$-REGULAR $k$-EDGE-CONNECTED GRAPHS WITH FEW SPANNING TREES

In this section, we describe some $k$-regular $k$-edge-connected graphs for odd $k$, leading to a conjecture that the minimum number of spanning trees of a $k$-edge-connected graph is obtained by a nearly $(k + 1)$-regular graph if $k$ is odd. See Open Problems 2, 3 in Section 7.

Let $kC_n$ be the cycle of length $n$ whose edge multiplicities are all $k$. By Theorem 1, when $k$ is even, $\frac{k}{2}C_n$ has the minimum number of spanning trees among all $k$-edge-connected graphs on $n$ vertices. If $k$ is odd, $\frac{k+1}{2}C_n$ minus an edge, say $\frac{k+1}{2}C_n - e$, gives an upper bound on the minimum number of spanning trees of a $k$-edge-connected graph on $n$ vertices. The spanning trees of $\frac{k+1}{2}C_n - e$ belong to either the unique path with uniform edge-multiplicity $\frac{k+1}{2}$ or the $(n - 1)$ paths in which the edge-multiplicities are $\frac{k+1}{2}$ except an edge with one less multiplicity. Thus, the number of spanning trees of $\frac{k+1}{2}C_n - e$ is

$$
\left(\frac{k + 1}{2}\right)^{n-1} + (n - 1)\left(\frac{k + 1}{2}\right)^{n-2} \frac{k - 1}{2} = \left(1 + (n - 1)\frac{k - 1}{k + 1}\right)\left(\frac{k + 1}{2}\right)^{n-1}.
$$

We conjecture that this number is the minimum number of spanning trees of a $k$-edge-connected graph on $n$ vertices when $k$ is an odd number bigger than 3, and $\frac{k+1}{2}C_n - e$ is the unique extremal graph realizing the number.

We do not know any $k$-regular $k$-edge-connected graphs with that few spanning trees. Instead, there are $k$-regular $k$-edge-connected graphs with $(\frac{k+2}{2} + O(\frac{1}{k}))^{n-1}$ spanning trees, namely multiprisms defined below.

The prism $P_{2n}$ is the Cartesian product of $C_n$ and $K_2$. If $n > 2$ is a natural number and $k$ is odd then the multiprism $MP_{2n}(k)$ is defined as follows:

1. Let $v_0, v_1, \ldots, v_{2n-1}$ be the vertices of $\frac{k+1}{2}C_{2n}$, where $v_i$ and $v_{i+1}$ are adjacent for all $i$.
2. Add edges $v_0v_3, v_2v_5, \ldots, v_{2n-4}v_{2n-1}$ and $v_{2n-2}v_1$. 

![Figure 1. Two different drawings of $MP_{12}(5)$](image-url)
If $n$ is even, $MP_{2n}(k)$ can also be obtained by choosing a Hamilton cycle of $P_{2n}$ and replace its edges by $(k - 1)/2$-multiple edges. See Figure 1.

Kreweras [5] determined the exact number of spanning trees in the prisms. Rubey [8, p. 40] showed another method, which can be used to give the exact formula for $\tau(MP_{2n}(k))$; c.f. [7]. Let $k = 2s + 1$. Then

$$\tau(MP_{2n}(2s + 1)) = \frac{sn}{A - B} A^n \left[ 1 + 2 \frac{s^2 A^{n-2} - s^n}{A^n - s^2 A^{n-2}} + 1 + \frac{s^2}{A} \frac{A^n - s^n}{A^n - s^2 A^{n-2}} \right] - B^n \left[ 1 + 2 \frac{s^2 B^{n-2} - s^n}{B^n - s^2 B^{n-2}} + 1 + \frac{s^2}{B} \frac{B^n - s^n}{B^n - s^2 B^{n-2}} \right],$$

where $A = \frac{s}{2} \left( s + 3 + \sqrt{s^2 + 6s + 5} \right)$ and $B = \frac{s}{2} \left( s + 3 - \sqrt{s^2 + 6s + 5} \right)$.

Thus $\lim_{n \to \infty} \tau(MP_{2n}(k))^{1/2n} = A^{1/2} = s + \frac{3}{2} + O\left(\frac{1}{s}\right) = \frac{k + 2}{2} + O\left(\frac{1}{k}\right)$.

In particular, $\tau(MP_n(5)) > 3.09^n$ for large even $n$.

Note again that the number of spanning trees of $MP_{2n}(k)$, which is $k$-regular $k$-edge-connected, is asymptotically $\left(\frac{k+2}{2}\right)^{2n}$. As we have a $(k + 1)$-regular $(k + 1)$-edge-connected graph, namely $\frac{k+1}{2}C_{2n}$, with asymptotically less spanning trees, we suspect that the minimum number of spanning trees of a $k$-edge-connected graph, when $k$ is odd, may be achieved by an almost $(k + 1)$-regular graph. Specifically, we believe that for every odd $k \geq 5$, $\frac{k+1}{2}C_n$ minus an edge has the fewest spanning trees among all $k$-edge-connected graphs on $n$ vertices.

7. Open Problems

For $\mathcal{C}$ an infinite class of finite graphs, define $c(\mathcal{C}) = \lim inf \{\tau(G)^{1/n} : G \in \mathcal{C}, n = |V(G)|\}$. Let $\mathcal{C}_k$ be the class of $k$-edge-connected graphs. Let $\mathcal{C}'_k$ be the class of $k$-regular $k$-edge-connected graphs. We have proved that $c(\mathcal{C}_k) = c(\mathcal{C}'_k) = k/2$ for $k$ even and that $k/2 < c(\mathcal{C}_k) \leq c(\mathcal{C}'_k)$ for $k$ odd. Moreover $1.774 < c(\mathcal{C}_3) = c(\mathcal{C}'_3) \leq 1.932$, $2.75 < c(\mathcal{C}_5) \leq 3$ and $c(\mathcal{C}_5) \leq c(\mathcal{C}'_5) < 3.1$.

Open Problem 1. Is $c(\mathcal{C}_3) = \sqrt{2 + \sqrt{3}} \approx 1.93$, which is obtained by the prisms?

Open Problem 2. Is $c(\mathcal{C}_k) = c(\mathcal{C}'_{k+1}) = \frac{k+1}{2}$ for $k$ odd, $k \geq 5$?

Open Problem 3. Is $c(\mathcal{C}'_k) = k/2 + 1 + O(1/k)$ for $k$ odd?

Open Problem 4. Is $c(\mathcal{C}'_k) = \sqrt{5 + \sqrt{21}} \approx 3.0956$, which is obtained by the multiprisms $MP_n(5)$?
Even if Problems 2 and 3 both have negative answers, we may still ask if \( c(C'_k) > c(C_{k+1}) \) for each odd \( k \geq 5 \).

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