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ON THE MINIMUM NUMBER OF SPANNING TREES IN k -EDGE-CONNECTED GRAPHS

S. OK AND C. THOMASSEN

ABSTRACT. We show that a k -edge-connected graph on n vertices has at least $n(k/2)^{n-1}$ spanning trees. This bound is tight if k is even and the extremal graph is the n -cycle with edge-multiplicities $k/2$. For k odd, however, there is a lower bound c_k^{n-1} where $c_k > k/2$. Specifically, $c_3 > 1.77$ and $c_5 > 2.75$. Not surprisingly, c_3 is smaller than the corresponding number for 4-edge-connected graphs. Examples show that $c_3 \leq \sqrt{2 + \sqrt{3}} \approx 1.93$.

However, we have no examples of 5-edge-connected graphs with fewer spanning trees than the n -cycle with all edge-multiplicities (except one) equal to 3, which is almost 6-regular. We have no examples of 5-regular 5-edge-connected graphs with fewer than 3.09^{n-1} spanning trees which is more than the corresponding number for 6-regular 6-edge-connected graphs. The analogous surprising phenomenon occurs for each higher odd edge-connectivity and regularity.

1. INTRODUCTION

Every connected graph has a spanning tree, that is, a connected subgraph with no cycles containing all vertices of the graph. The number of spanning trees, denoted $\tau(G)$, is of importance in electrical networks, in particular, for expressing driving point resistances (effective resistances); see e.g. [9]. Kostochka [4] showed that, if G is a connected k -regular simple graph, then $k^{(1-O(\log k/k))} \leq \tau(G)^{1/n} \leq k$. But if we allow multiple edges, there are graphs with far less spanning trees. In this paper, we investigate the minimum number of spanning trees in k -edge-connected graphs with multiple edges. Since a loop is never contained in a spanning tree, we consider only graphs without loops.

In Section 2 we investigate how $\tau(G)$ changes when we replace a certain subgraph of G by another graph. In Section 3 we derive the lower bounds stated in the abstract. Since this bound is not tight for any odd edge-connectivity, we show in Section 4 that $\tau(G) \geq 1.774^{n-1}$ for every 3-edge-connected graph G on n vertices. The proof involves a new recursive description of the 3-connected cubic graphs; they can all be obtained from K_4 or $K_{3,3}$ by

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successively adding vertices or blowing vertices up to triangles. In Section 5, we consider the class of 5-regular 5-edge-connected graphs. Section 6 presents a class of k -regular k -edge-connected graphs which suggests that for odd $k > 3$, the minimum number of spanning trees might be obtained by an almost $(k + 1)$ -regular graph. Even more surprisingly, all examples of 5-regular, 5-edge-connected graphs with n vertices known to us have more than 3.09^{n-1} spanning trees while there are 6-regular, 6-edge-connected graphs with only $n3^{n-1}$ spanning trees.

We adopt the notation and terminology of Diestel [3]. We repeat a few important definitions. A **bridge** is an edge whose removal disconnects the graph. A graph is **k -edge-connected** if we need to remove at least k edges to disconnect the graph. A graph is **k -regular** if each vertex has k incident edges. A 3-regular graph is also called **cubic**. If e is an edge in a graph G , then G/e is the graph obtained by contracting e .

2. LIFTING PAIRS OF EDGES

Let G, H_1, H_2 be connected graphs, and $X \subseteq V(G)$, $X_i \subseteq V(H_i)$ for $i = 1, 2$ such that $|X| = |X_1| = |X_2|$. For $i = 1, 2$, let G_i be the graph obtained from $G \cup H_i$ by identifying X_i with X . We are interested in $\tau(G_1)/\tau(G_2)$. Let T be a spanning tree of G_1 or G_2 . Then $T \cap G$ is a spanning forest of G . By comparing the number of ways of extending $T \cap G$ into a spanning tree of G_i using H_i , and taking the minimum ratio over all possible such forests, we can find a lower bound for $\tau(G_1)/\tau(G_2)$. Note that the number of ways of extending $T \cap G$ in G_i using H_i is exactly the number of spanning trees of the graph obtained from H_i by contracting each component of $T \cap G$ into a single vertex. This is made more precise in the following observation.

Observation 1. *Let G be a graph, and let $X \subseteq V(G)$ be a set of vertices. Suppose that G has two connected subgraphs G_0, G_1 such that $G_0 \cup G_1 = G$, $V(G_0 \cap G_1) = X$ and $E(G_0 \cap G_1) = \emptyset$. Let T be a spanning forest of G_0 such that each component contains at least one vertex in X . Then the number of ways of extending T to a spanning tree of G using edges in G_1 is $\tau(S_0)$, where S_0 is the graph obtained from $G_1 \cup T$ by contracting each component of T into a single vertex.*

Let $e = vu, f = vw$ be two adjacent edges of a graph. **Lifting** e, f means that we replace e, f by an edge uw if $u \neq w$. If $u = w$ we remove both edges e, f as we do not allow loops. By **lifting at v** we mean that we lift a pair of edges incident with v . A **complete lifting** at a vertex v with even degree is a sequence of liftings at v until no edges are left at v . Then we remove v .

For the following lemma, we define a constant c_d depending on a positive integer d :

$$c_d = \min_{d_1, d_2, \dots, d_k} \min_H \frac{\prod_{i=1}^k d_i}{\tau(H)},$$

where the minimum is taken over all sequences of positive integers $d_1, d_2, \dots, d_k$ with varying length k such that $\sum_{i=1}^k d_i = 2d$, and over all connected graphs H on k vertices with degree sequence $d'_1, d'_2, \dots, d'_k$ such that $d'_i \leq d_i$ for each i .

In the above definition of c_d , H has at most d edges, so $c_1 = 1$. Furthermore, $c_2 = 2$, $c_3 = 8/3$ and $c_4 = 18/5 = 3.6$, which are attained by a 2-cycle, a 3-cycle, and a 3-cycle plus an edge, respectively.

Lemma 1. *Let G be a graph with a vertex v of degree $2d$. Let G' be a graph obtained from G by a complete lifting at v . Then $\tau(G) \geq c_d \tau(G')$, where c_d is defined as above.*

Proof: Denote $G_0 = G - v$ and the neighbors of v in G by $v_1, v_2, \dots, v_{2d}$, which are not necessarily distinct. We may assume that for each i , $v_{2i-1}v_{2i} \in E(G') \setminus E(G)$ resulting from lifting vv_{2i-1} and vv_{2i} unless $v_{2i-1} = v_{2i}$.

We consider a spanning forest, say T_0 , of G_0 in which each component contains at least one of the neighbors of v . We shall estimate the number of ways of extending T_0 to a spanning tree using only edges not in G_0 . The forest T_0 partitions the neighbors of v , say into $P_1, P_2, \dots, P_k$ with sizes $|P_i| = d_i$, $\sum_{i=1}^k d_i = 2d$. By Observation 1, the number of ways of extending T_0 to a spanning tree of G (using no other edge of G_0) is precisely $\tau(S_0)$, where S_0 is the star graph at v with edge-multiplicities $d_1, d_2, \dots, d_k$. Thus $\tau(S_0) = \prod_{i=1}^k d_i$. Likewise, the number of ways of extending T_0 to a spanning tree of G' is $\tau(S'_0)$ where S'_0 is the graph obtained from G' by contracting each component of T_0 into a single vertex, and then remove the remaining edges of G_0 , if any. Let p_i be the vertex of S'_0 corresponding to P_i . Then $\deg(p_i) \leq d_i$, since each $v_j \in P_i$ provides p_i with at most one edge from $E(G') \setminus E(G_0)$. Therefore, the number of extensions of T_0 into spanning trees of G divided by the number of extensions to G' is at least $\min_H \prod_{i=1}^k d_i / \tau(H)$, where H is as described in the definition of c_d . Now we consider all possibilities for T_0 and get the inequality. $\square$

Lemma 2. *Let G be a graph with a vertex v of degree $d \geq 3$. Let G' be a graph resulting from lifting edges vu, vw in G . Then $\tau(G) \geq (1 + \frac{4}{d^2-4})\tau(G')$.*

Proof: We consider a spanning forest, say T_0 , of $G - v$ in which each component contains at least one of the neighbors of v . Then T_0 partitions the neighbors of v , say into $P_1, P_2, \dots, P_k$ with sizes $|P_i| = d_i$, $\sum_{i=1}^k d_i = d$. By Observation 1, the number of ways to extend T_0 to a spanning tree of G is $\tau(S_0) = \prod_{i=1}^k d_i$, where S_0 is the star graph at v with edge-multiplicities

$d_1, d_2, \dots, d_k$. Let S'_0 be the graph obtained from G' by contracting each component of T_0 into a single vertex and then remove the remaining edges of $G - v$, if any. By Observation 1, there are $\tau(S'_0)$ ways of extending T to a spanning tree of G' . If some P_j contains both u, w , then $\tau(S'_0) = (d_j - 2) \prod_{i \neq j} d_i$, so that either $\tau(S'_0) = 0$ or $\tau(S_0)/\tau(S'_0) = d_j/(d_j - 2) > 1 + 4/(d^2 - 4)$.

If u, w are contained in two different parts, say P_i, P_j respectively, then S'_0 is obtained from S_0 by lifting two edges connecting v to the two vertices corresponding to P_i and P_j . Thus,

$$\frac{\tau(S_0)}{\tau(S'_0)} = \frac{d_i d_j}{d_i d_j - 1} \geq 1 + \frac{4}{d^2 - 4},$$

since $d_i + d_j \leq d$ which implies $d_i d_j \leq [(d_i + d_j)/2]^2 \leq d^2/4$.

By considering all possible such forests T_0 , we get the inequality. $\square$

3. k -EDGE-CONNECTED GRAPHS

Let G be a connected graph with n vertices and m edges. Consider the pairs (e, T) where $e \in E(G)$ and T a spanning tree of G containing e . For each $e \in E(G)$ we have $\tau(G/e)$ such pairs and for each T , we have $n - 1$ such pairs. Therefore $(n - 1)\tau(G) = \sum_{e \in E(G)} \tau(G/e)$. Hence, G has an edge e such that $\tau(G/e)/\tau(G) \leq (n - 1)/m$. We restate this conclusion as the following observation.

Observation 2. *Let G be a connected graph with $n > 1$ vertices and m edges. Then G has an edge e such that $\tau(G) \geq \frac{m}{n-1}\tau(G/e)$.*

Now we prove the first lower bound stated in the abstract.

Theorem 1. *Let G be a k -edge-connected graph on n vertices. Then G has at least $n(k/2)^{n-1}$ spanning trees. Moreover, G has more than $n(k/2)^{n-1}$ spanning trees unless k is even and G is a cycle whose edge-multiplicities are all $k/2$.*

Proof: We shall use induction on n . Since G is k -edge-connected, the minimum degree of G is at least k and thus $m \geq kn/2$. By Observation 2, G has an edge e such that $\tau(G) \geq \frac{m}{n-1}\tau(G/e) \geq \frac{kn}{2(n-1)}\tau(G/e)$. By the induction hypothesis, $\tau(G/e) \geq (n-1)(k/2)^{n-2}$ so that $\tau(G) \geq n(k/2)^{n-1}$. If equality holds, then k is even, $m = kn/2$, and G/e is a cycle where all edge-multiplicities are $k/2$. Moreover, any edge can play the role of e . This implies that all edge-multiplicities in G are $k/2$. If H denotes the subgraph of G obtained by replacing every multiple edge by a single edge, then H has the property that the contraction of any edge results in a cycle. Then also H is a cycle. $\square$

For k even Theorem 1 is tight. However, for k odd we shall present a lower bound for the number of spanning trees in a k -edge-connected graph of the form c_k^{n-1} with $c_k > k/2$. For that, we shall use the following Theorem by Mader [6].

Theorem 2. *Let G be a connected graph on a vertex set $V \cup \{s\}$. If $\deg(s) \neq 3$ and s is not incident with bridges, then G has a lifting at s such that for each pair u, v of vertices in V , the maximum number of edge-disjoint paths between u, v does not decrease after the lifting.*

By Theorem 2 and Menger's Theorem, given a k -edge-connected graph and a vertex of degree $\geq k + 2$, we can find a lifting without decreasing the edge-connectivity. Thus by Lemma 2, the minimum number of spanning trees of a k -edge-connected graph on n vertices must be obtained by a graph whose degrees are only k or $k + 1$. We state this as an observation for later use.

Observation 3. *If G is a k -edge-connected graph on n vertices with minimum $\tau(G)$, then each vertex of G has either k or $k + 1$ incident edges.*

Now we prove the following lower bound for odd edge-connectivity.

Theorem 3. *Let $k > 1$ be an odd number and let G be a k -edge-connected graph on n vertices. Then $\tau(G) \geq (kc_k/2)^{n-1}$, where $c_k = \sqrt{1 + \frac{4}{(k+3)^2 - 4}} > 1$*

Proof: Let e be an edge for which $\tau(G)/\tau(G/e)$ is maximum. By Observation 2 we know $\tau(G)/\tau(G/e) \geq k/2$. If the vertex of G/e resulting from the contraction of e , say v , has degree bigger than $k + 1$, then using Theorem 2 we can lift some pair of edges at v such that G/e after the lifting is still k -edge-connected. We do the lifting at v until the degree of v is at most $k + 1$. Let H be the resulting graph. If $\tau(G)/\tau(H) \geq kc_k^2/2$ then we call e a *good* edge. Note that, if $H \neq G/e$, then by applying Lemma 2 at the last lifting, we see that e is good. Also, if e has multiplicity at least $(k + 1)/2$, then $\tau(G)/\tau(H) \geq \tau(G)/\tau(G/e) \geq (k + 1)/2 > kc_k^2/2$ so that e is good. If one of the ends of e has degree at least $k + 1$, then either e has multiplicity at least $(k + 1)/2$, or the vertex obtained by the contraction of e has degree at least $k + 2$, so that e is good. Thus e is not good only if the ends of e both have degree precisely k . In particular, both ends of e have odd degree.

Now we repeat the contractions of an edge with maximum $\tau(G)/\tau(G/e)$, followed by liftings whenever possible, until only two vertices are left. Because of parity, among the $n - 2$ contractions, at most $\lceil (n - 2)/2 \rceil$ of them are edges whose ends both have odd degree. Thus at least $\lfloor (n - 2)/2 \rfloor$ times we get an additional factor of c_k^2 , so $\tau(G) \geq k \cdot (k/2)^{n-2} \cdot c_k^{2\lfloor (n-2)/2 \rfloor} > (kc_k/2)^{n-1}$. $\square$

By Theorem 3, Theorem 1 is not tight for any odd edge-connectivity, although it is tight for all even edge-connectivity. In the following we focus on k -edge-connected graphs where $k = 3, 5$.

4. 3-EDGE-CONNECTED GRAPHS

Let G be a 3-edge-connected graph on n vertices. By Theorem 3, the lower bound $\tau(G) \geq n(3/2)^{n-1}$ is not tight. Kostochka [4] showed that a cubic simple 2-connected graph on n vertices has at least $8^{n/4} \approx 1.68^n$ spanning trees. This result is essentially best possible because of the cubic 2-connected graphs obtained by a collection of K_4 's minus an edge by adding a matching. In this section, we prove the following theorem.

Theorem 4. *Let G be a 3-edge-connected graph on n vertices. Then $\tau(G) > 1.774^{n-1}$.*

Kreweras [5] showed that the prism graph on n vertices has approximately 1.93^n spanning trees; see Section 6. By Observation 3, a 3-edge-connected graph on n vertices with minimum number of spanning trees has vertex degrees only 3 and 4. Thus by Lemma 1, the following is enough to prove Theorem 4. Note that a cubic graph with more than two vertices has the same connectivity and edge-connectivity.

Theorem 5. *Let G be a 3-connected cubic graph on n vertices. Then $\tau(G) > 1.774^{n-1}$.*

An often used operation to construct a 3-connected cubic graph is to **join** two edges, i.e. for non-parallel edges e, f , we replace each edge by a path of length 2 and connect the two new vertices of degree 2 by an edge. Note that joining two non-parallel edges in a 3-connected cubic graph results in another 3-connected cubic graph. The following lemma explains how the number of spanning trees changes after joining.

Lemma 3. *Let G be a graph with two non-parallel edges e and f . Let G' be the graph obtained from G by joining e and f . Then $\tau(G') \geq (4 - r)\tau(G)$, where $r = \tau(G/e/f)/\tau(G) \leq 1$.*

Proof: We shall use Observation 1. We only consider the case when e, f are not adjacent, but the other case can be done likewise. Let $e = ab$ and $f = cd$. Let T be a spanning tree of G . Then $T - e - f$ is a spanning forest of G in which each component contains at least one of a, b, c and d . We shall consider how many ways $T - e - f$ can be extended to a spanning tree in G and G' respectively. For example, if $T - e - f$ has two components such that one of them contains a, c and the other contains b, d , then we can extend $T - e - f$ in two ways to a spanning tree of G , whereas there are eight ways for G' . In fact, there are at least four times as many extensions in G' as extensions in G , unless T contains both e and f , in which case we have a factor 3. Thus, $\tau(G') \geq 4(\tau(G) - \tau(G/e/f)) + 3\tau(G/e/f) = (4 - r)\tau(G)$. $\square$

To prove Theorem 5, we shall consider the following two operations to construct 3-connected cubic graphs.

- (1) Let v be a vertex v in a graph such that $\deg(v) = 3$ and all three neighbors of v are distinct. Then the **blow-up** of v is obtained by joining two of the incident edges of v .
- (2) Select three edges, which may not be pairwise distinct, but not all the same, and subdivide each of them so that we have three new vertices of degree 2. Add a new vertex v and an edge from v to each of the three vertices of degree 2. We call this a **vertex-addition**.

Since a blow-up is a join of two non-parallel edges, we get the following observation by Lemma 3.

Observation 4. *Let G be a graph with a vertex v of degree 3 whose neighbors are all distinct. Let G' be the graph obtained from G by a blow-up of v . Then $\tau(G') \geq 3\tau(G)$.*

Barnette and Grünbaum [1] and independently Titov [10] gave a characterization of 3-connected graphs which implies that every 3-connected cubic graph can be obtained from K_4 by successively joining edges. We shall here prove a stronger result for cubic graphs.

Theorem 6. *Let G be a 3-connected cubic graph with more than two vertices. Then G can be constructed from K_4 or $K_{3,3}$ by blow-ups and vertex-additions, such that blow-ups are never used consecutively.*

Proof: Our proof consists of two parts. We show that if G has no induced subgraph which is a subdivision of another 3-connected graph, then G is one of K_4 , $K_{3,3}$ or the prism on 6 vertices defined in Section 6. Then we assume that G has a maximal induced subgraph, say H , which is a subdivision of another 3-connected graph H^* , and we show that G can be obtained from H^* by a vertex addition, possibly followed by a blow-up.

Suppose that G has no proper induced subgraph which is a subdivision of a 3-connected cubic graph. Let C be a cycle in G of minimum length so that C has no chord. Let v be a vertex in $G - V(C)$. Since G is 3-connected, Menger's Theorem implies that G has three paths P_1, P_2, P_3 where $P_i = vu_1^i u_2^i \dots u_{k_i}^i u_i$, $C \cap P_i = \{u_i\}$ for each i and the paths P_1, P_2, P_3 share only v . Let v be such a vertex with $k_1 + k_2 + k_3$ being smallest. Note that some k_i may be 0, implying that P_i is an edge. If G has an edge between the non-endvertices of two P_i 's, say $u_i^1 u_j^2$, then by taking $v = u_i^1$ instead and using $P_1 \cup P_3$ and $u_i^1 u_j^2 u_{j+1}^2 \dots u_{k_2}^2$, we get a smaller sum of the lengths of the paths unless u_j^2 is the neighbor of v in P_2 . Similarly, we deduce that u_i^1 is also the neighbor of v in P_1 . In this case, $vu_1^1 u_1^2$ is a triangle and hence C

must also be a triangle, so that the vertex set of $C \cup P_1 \cup P_2 \cup P_3$, say V , induces a subgraph of G which is a subdivision of the prism graph. Thus by the assumption, G itself is the prism graph.

Hence we may assume that G has no edge between the non-endvertices of P_i 's. Denote by $G[V]$ the subgraph of G induced by V . Suppose $k_1 \geq 1$ and some u_i^1 has a neighbor on C different from u_1 . Because of the minimality of $k_1 + k_2 + k_3$, we have $i = k_1$ and by taking $v = u_{k_1}^1$ and using its two neighbors on C , we see $k_2 = k_3 = 0$. Therefore $G[V]$ is a subdivision of either the prism graph or $K_{3,3}$, so that again G itself is either the prism graph or $K_{3,3}$. The remaining case leaves no other edge in $G[V]$ than $C \cup P_1 \cup P_2 \cup P_3$, which is a subdivision of K_4 . Thus in this case G itself is K_4 . This completes the first part.

Now we assume that G has an induced proper subgraph which is a subdivision of a 3-connected cubic graph. Let H be a maximal such subgraph. Let us call a path in H *suspended* if its ends both have degree 3 in H and all other vertices in the path have degree 2 in H . Suspended paths intersect only at their ends. By replacing each suspended path of H by an edge between its ends, we get a 3-connected cubic graph, which we denote H^* . Since G is 3-connected, H has at least two suspended paths. If G has a vertex, say v , outside H which has neighbors in at least two distinct suspended paths of H , then the subgraph of G induced by $V(H) \cup \{v\}$ is a subdivision of a 3-connected graph, which must be G because of the maximality of H . Then G can be obtained from H^* by the vertex-addition of v . Thus we may assume that for each vertex in $V(G) \setminus V(H)$, its neighbors in H , if any, are in a single suspended path of H . Also, we may assume that $|V(G) \setminus V(H)| > 1$. If $V(G) \setminus V(H) = \{u, v\}$, then u and v are adjacent, and they have neighbors in distinct suspended paths. Thus we can obtain G from H^* by first vertex-adding u and then a blow-up to make v . Therefore, we assume that $|V(G) \setminus V(H)| > 2$.

Since G is 3-connected, at least one component of $G - V(H)$ has edges to two distinct suspended paths of H . Thus G has a path of length > 1 between distinct suspended paths of H which intersects H at only its ends. Let $P = v_0 v_1 \dots v_k$ be such a path with smallest length. Since P has no chord, the subgraph of G induced by $H \cup P$ is a subdivision of a 3-connected graph, so that $V(H) \cup V(P) = V(G)$, implying $k \geq 4$. By assumption, the neighbors of v_1 and v_{k-1} , respectively, are in different suspended paths of H . Let v be the neighbor of v_2 in H . Then either $v_0 v_1 v_2 v$ or $vv_2 v_3 \dots v_k$ contradicts the minimality of P , a contradiction which completes the proof. $\square$

Let c be the positive real solution of the equation $x^4 - 3x^2 - 1 = 0$ which is approximately $c \approx 1.8174$. Note that a vertex-addition is equivalent to a joining of two edges and then joining the new edge with an edge.

Lemma 4. *Let G_0 be a 3-connected graph and let G be a graph obtained from G_0 by joining two non-parallel edges of G_0 , where e denotes the joining edge. Let G' be a graph obtained from G by joining e with another edge f . Then either $\tau(G') \geq c^2\tau(G)$ or $\tau(G') \geq c^4\tau(G_0)$.*

Proof: Let $r = \tau(G/e/f)/\tau(G)$ be as in Lemma 3. Let $r' = \tau(G/e)/\tau(G)$ so that $\tau(G)/\tau(G-e) = 1/(1-r')$. Since $r' \geq r$, Lemma 3 implies $\tau(G') \geq (4-r)\tau(G) \geq (4-r')\tau(G)$. If $4-r' \geq c^2$ then we are done. Thus we may assume that $4-r' < c^2$, equivalently $1-r' < c^2-3$. By modifying the equation for c , we get $1 + 3/(c^2-3) = c^4$, so that

$$\begin{aligned} \tau(G') &\geq (4-r')\tau(G) = \frac{(4-r')\tau(G)}{\tau(G_0)}\tau(G_0) \geq \frac{(4-r')\tau(G)}{\tau(G-e)}\tau(G_0) = \frac{4-r'}{1-r'}\tau(G_0) \\ &= \left(1 + \frac{3}{1-r'}\right)\tau(G_0) > \left(1 + \frac{3}{c^2-3}\right)\tau(G_0) = c^4\tau(G_0). \end{aligned}$$

□

Proof of Theorem 5: We shall prove $\tau(G) \geq (3c^2)^{(n-1)/4}$ by induction on $n = |V(G)|$, where c is the constant used in Lemma 4. We may assume that $n \geq 8$ because K_4 , $K_{3,3}$ and the prism on 6 vertices have 16, 81 and 75 spanning trees, respectively. By Theorem 6, G can be obtained from K_4 or $K_{3,3}$ by repeatedly applying vertex-additions and blow-ups. If the last operation is a vertex-addition, then by Lemma 4, $\tau(G) \geq c^2\tau(G')$ or $\tau(G) \geq c^4\tau(G'')$ for some 3-connected cubic graph G' with $n-2$ vertices or G'' with $n-4$ vertices, so we are done. Otherwise, G can be obtained from a 3-connected cubic graph using a vertex-addition and then a blow-up. By Observation 4, a blow-up multiplies the number of spanning trees by at least 3, so that using Lemma 4, $\tau(G) \geq 3c^2\tau(G')$ or $\tau(G) \geq 3c^4\tau(G'')$ for some 3-edge-connected cubic graph G' with $n-4$ vertices or G'' with $n-6$ vertices. By the induction hypothesis, $\tau(G) \geq (3c^2)^{(n-1)/4} > 1.774^{n-1}$. □

5. 5-REGULAR 5-EDGE-CONNECTED GRAPHS

Let G be a 5-regular 5-edge-connected graph. A **5-cut** is a set of edges E with $|E| = 5$ such that $G - E$ is disconnected. If one of the components of $G - E$ is a single vertex, then we call E **trivial**. Otherwise we call E **nontrivial**. A **5-side** is a set $X \subseteq V(G)$ such that $\delta(X)$ (that is, the set of edges with precisely one end in X) is a nontrivial 5-cut. If a 5-side X has the property that no nontrivial 5-cut contains an edge with both ends in X , then X is called **minimal**.

Lemma 5. *Let G be a 5-regular 5-edge-connected graph. If G has a nontrivial 5-cut, then G has a minimal 5-side.*

Proof: Let A be a 5-side which is not minimal. Then some nontrivial 5-cut $S = \delta(B)$ contains an edge uv with $u \in A \cap B$ and $v \in A \cap B^c$. Let $T = \delta(A)$. One of the sets $A \cap B$, $A \cap B^c$, $A^c \cap B$ or $A^c \cap B^c$ is empty because G is 5-edge-connected, S, T are 5-cuts and 5 is odd. Since $u \in A \cap B$ and $v \in A \cap B^c$, either $A^c \cap B$ or $A^c \cap B^c$ is empty, so that either $A \cap B$ or $A \cap B^c$ is a 5-side strictly smaller than A . If it is not minimal, then we repeat the argument until we eventually find a minimal 5-side. $\square$

Lemma 6. *Let G be a connected graph with a connected subgraph H . If G' is the graph obtained by contracting H into a single vertex, then $\tau(G) \geq \tau(H)\tau(G')$.*

Proof: For each pair S, T of spanning trees of H, G' , we can expand the contracted vertex of G' using S to get a spanning tree of G . $\square$

Theorem 7. *Let G be a 5-regular 5-edge-connected graph on n vertices. Then $\tau(G) \geq 7.6^{(n-1)/2} \approx 2.7568^{n-1}$.*

Proof: We shall use induction on n . Being 5-regular and 5-edge-connected, G has no edge of multiplicity at least 3. If G has a nontrivial 5-cut, then by Lemma 5, we can find a minimal 5-side, and we let $e = uv$ be an edge inside that minimal side. Otherwise let $e = uv$ be an arbitrary edge.

Suppose first e has multiplicity 1. G/e has a vertex of degree 8, which we can completely lift using Theorem 2. Denote the resulting 5-regular 5-edge-connected graph by G' . By Lemma 1, $\tau(G/e) \geq 3.6\tau(G')$. Now we consider $G - e$. Since e is not contained in any nontrivial 5-cut, $G - e$ has at least 5 edge-disjoint paths between any pair of vertices distinct from the ends of e . Thus by Theorem 2, we can completely lift u, v in $G - e$ so that the resulting graph, say G'' , is 5-edge-connected and 5-regular. By Lemma 1, $\tau(G - e) \geq 4\tau(G'')$ and by the induction hypothesis,

$$\tau(G) = \tau(G/e) + \tau(G - e) \geq 3.6\tau(G') + 4\tau(G'') \geq 7.6^{(n-1)/2}.$$

Now we may assume that every edge of G with multiplicity 1 is contained in a nontrivial 5-cut. Let X be a minimal 5-side. Since the edges inside X are not contained in any nontrivial 5-cut, every edge inside X must be a double edge. Hence every vertex in X is incident with $\delta(X)$, so that X is the 5-double-cycle which has 80 spanning trees. By Lemma 6, $\tau(G) \geq 80\tau(G/X)$, and by the induction hypothesis, $\tau(G) \geq 7.6^{(n-1)/2}$. $\square$

FIGURE 1. Two different drawings of $MP_{12}(5)$

6. EXAMPLES OF k -REGULAR k -EDGE-CONNECTED GRAPHS WITH FEW SPANNING TREES

In this section, we describe some k -regular k -edge-connected graphs for odd k , leading to a conjecture that the minimum number of spanning trees of a k -edge-connected graph is obtained by a nearly $(k+1)$ -regular graph if k is odd. See Open Problems 2, 3 in Section 7.

Let kC_n be the cycle of length n whose edge multiplicities are all k . By Theorem 1, when k is even, $\frac{k}{2}C_n$ has the minimum number of spanning trees among all k -edge-connected graphs on n vertices. If k is odd, $\frac{k+1}{2}C_n$ minus an edge, say $\frac{k+1}{2}C_n - e$, gives an upper bound on the minimum number of spanning trees of a k -edge-connected graph on n vertices. The spanning trees of $\frac{k+1}{2}C_n - e$ belong to either the unique path with uniform edge-multiplicity $\frac{k+1}{2}$ or the $(n-1)$ paths in which the edge-multiplicities are $\frac{k+1}{2}$ except an edge with one less multiplicity. Thus, the number of spanning trees of $\frac{k+1}{2}C_n - e$ is

$$\left(\frac{k+1}{2}\right)^{n-1} + (n-1) \left(\frac{k+1}{2}\right)^{n-2} \frac{k-1}{2} = \left(1 + (n-1)\frac{k-1}{k+1}\right) \left(\frac{k+1}{2}\right)^{n-1}.$$

We conjecture that this number is the minimum number of spanning trees of a k -edge-connected graph on n vertices when k is an odd number bigger than 3, and $\frac{k+1}{2}C_n - e$ is the unique extremal graph realizing the number.

We do not know any k -regular k -edge-connected graphs with that few spanning trees. Instead, there are k -regular k -edge-connected graphs with $(\frac{k+2}{2} + O(\frac{1}{k}))^{n-1}$ spanning trees, namely multiprisms defined below.

The **prism** P_{2n} is the Cartesian product of C_n and K_2 . If $n > 2$ is a natural number and k is odd then the **multiprism** $MP_{2n}(k)$ is defined as follows:

- (1) Let $v_0, v_1, \dots, v_{2n-1}$ be the vertices of $\frac{k-1}{2}C_{2n}$, where v_i and v_{i+1} are adjacent for all i .
- (2) Add edges $v_0v_3, v_2v_5, \dots, v_{2n-4}v_{2n-1}$ and $v_{2n-2}v_1$.

If n is even, $MP_{2n}(k)$ can also be obtained by choosing a Hamilton cycle of P_{2n} and replace its edges by $(k-1)/2$ -multiple edges. See Figure 1.

Kreweras [5] determined the exact number of spanning trees in the prisms. Rubey [8, p. 40] showed another method, which can be used to give the exact formula for $\tau(MP_{2n}(k))$; c.f. [7]. Let $k = 2s + 1$. Then

$$\begin{aligned} \tau(MP_{2n}(2s+1)) = & \frac{sn}{A-B} A^n \left[1 + 2 \frac{s^2 A^{n-2} - s^n}{A^n - s^2 A^{n-2}} + \frac{1+s^2}{A} \frac{A^n - s^n}{A^n - s^2 A^{n-2}} \right] \\ & - B^n \left[1 + 2 \frac{s^2 B^{n-2} - s^n}{B^n - s^2 B^{n-2}} + \frac{1+s^2}{B} \frac{B^n - s^n}{B^n - s^2 B^{n-2}} \right], \end{aligned}$$

where $A = \frac{s}{2} \left(s + 3 + \sqrt{s^2 + 6s + 5} \right)$ and $B = \frac{s}{2} \left(s + 3 - \sqrt{s^2 + 6s + 5} \right)$.

Thus $\lim_{n \rightarrow \infty} \tau(MP_{2n}(k))^{1/2n} = A^{1/2} = s + \frac{3}{2} + O\left(\frac{1}{s}\right) = \frac{k+2}{2} + O\left(\frac{1}{k}\right)$.

In particular, $\tau(MP_n(5)) > 3.09^n$ for large even n .

Note again that the number of spanning trees of $MP_{2n}(k)$, which is k -regular k -edge-connected, is asymptotically $\left(\frac{k+2}{2}\right)^{2n}$. As we have a $(k+1)$ -regular $(k+1)$ -edge-connected graph, namely $\frac{k+1}{2}C_{2n}$, with asymptotically less spanning trees, we suspect that the minimum number of spanning trees of a k -edge-connected graph, when k is odd, may be achieved by an almost $(k+1)$ -regular graph. Specifically, we believe that for every odd $k \geq 5$, $\frac{k+1}{2}C_n$ minus an edge has the fewest spanning trees among all k -edge-connected graphs on n vertices.

7. OPEN PROBLEMS

For $\mathcal{C}$ an infinite class of finite graphs, define $c(\mathcal{C}) = \liminf\{\tau(G)^{1/n} : G \in \mathcal{C}, n = |V(G)|\}$. Let $\mathcal{C}_k$ be the class of k -edge-connected graphs. Let $\mathcal{C}'_k$ be the class of k -regular k -edge-connected graphs. We have proved that $c(\mathcal{C}_k) = c(\mathcal{C}'_k) = k/2$ for k even and that $k/2 < c(\mathcal{C}_k) \leq c(\mathcal{C}'_k)$ for k odd. Moreover $1.774 < c(\mathcal{C}_3) = c(\mathcal{C}'_3) \leq 1.932$, $2.75 < c(\mathcal{C}_5) \leq 3$ and $c(\mathcal{C}_5) \leq c(\mathcal{C}'_5) < 3.1$.

Open Problem 1. Is $c(\mathcal{C}_3) = \sqrt{2 + \sqrt{3}} \approx 1.93$, which is obtained by the prisms?

Open Problem 2. Is $c(\mathcal{C}_k) = c(\mathcal{C}_{k+1}) = \frac{k+1}{2}$ for k odd, $k \geq 5$?

Open Problem 3. Is $c(\mathcal{C}'_k) = k/2 + 1 + O(1/k)$ for k odd?

Open Problem 4. Is $c(\mathcal{C}'_5) = \sqrt{5 + \sqrt{21}} \approx 3.0956$, which is obtained by the multiprisms $MP_n(5)$?

Even if Problems 2 and 3 both have negative answers, we may still ask if $c(\mathcal{C}'_k) > c(\mathcal{C}_{k+1})$ for each odd $k \geq 5$.

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