Decomposing series-parallel graphs into paths of length 3 and triangles

Merker, Martin

Published in:
Electronic Notes in Discrete Mathematics

Link to article, DOI:
[10.1016/j.endm.2015.06.051](https://doi.org/10.1016/j.endm.2015.06.051)

Publication date:
2015

Document Version
Peer reviewed version

[Link back to DTU Orbit](#)

Citation (APA):
Merker, M. (2015). Decomposing series-parallel graphs into paths of length 3 and triangles. *Electronic Notes in Discrete Mathematics*, 49(November 2015), 367-370. <https://doi.org/10.1016/j.endm.2015.06.051>

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Decomposing series-parallel graphs into paths of length 3 and triangles

Martin Merker
Technical University of Denmark
marmer@dtu.dk

March 14, 2015

Abstract

An old conjecture by Jünger, Reinelt and Pulleyblank states that every 2-edge-connected planar graph can be decomposed into paths of length 3 and triangles, provided its size is divisible by 3. We prove the conjecture for a class of planar graphs including all 2-edge-connected series-parallel graphs. We also present a 2-edge-connected non-planar graph that can be embedded on the torus and admits no decomposition into paths of length 3 and triangles.

If G is a graph and $\mathcal{H} = \{H_1, \dots, H_k\}$ is a collection of graphs, then an $\mathcal{H}$ -decomposition of G is a collection of graphs in G such that each of them is isomorphic to some H_i and every edge of G is contained in exactly one of them. Let P_n and C_n denote the path and cycle on n vertices respectively. An old conjecture by Jünger, Reinelt and Pulleyblank in [2] states that every 2-edge-connected planar graph of size divisible by 3 admits a $\{P_4, C_3\}$ -decomposition. We shall give further evidence for this conjecture by proving it for a class of graphs including all 2-edge-connected series-parallel graphs as well as for all subdivisions of prisms.

It is easy to see that all 2-edge-connected cubic graphs can be decomposed into paths of length 3, see for example [2] and [1]. By considering the dual graph it was shown by Häggkvist and Johansson in [1] that planar triangulations can be decomposed into paths of length 3. They also verified the conjecture for outerplanar graphs. Thomassen [4] showed that in general every 171-edge-connected graph can be decomposed into paths of length 3. This bound on the edge-connectivity has been pushed down to 63 in [3], and it is open whether 3-edge-connectivity is sufficient.

A graph is called *series-parallel* if it contains no K_4 -minor. Every connected series-parallel graph can be constructed starting from a single edge using the following operations: subdividing an edge, replacing an edge by a pair of parallel edges, or adding a new vertex and an edge connecting it to the graph. The class of outerplanar graphs is a proper subclass of the class of series-parallel graphs. We give a short proof that 2-edge-connected series-parallel graphs have a $\{P_4, C_3\}$ -decomposition, thus generalizing the main theorem in [1].

Theorem 1. *Every 2-edge-connected series-parallel graph of size divisible by 3 has a $\{P_4, C_3\}$ -decomposition.*

In [1] a slightly stronger statement was proved where prepaths at two different vertices were allowed under certain constraints, which also simplified the induction. While Conjecture 2 below might seem like a strengthening of the conjecture by Jünger et al. on the first sight, it is easy to see that they are equivalent.

Figure 1: A 2-edge-connected non-planar graph with no $\{P_4, C_3\}$ -decomposition

Conjecture 2. *If u and v are two vertices on the same face of a 2-edge-connected planar graph G , and we attach a path to u and a path to v such that the resulting graph G' has size divisible by 3, then G' has a $\{P_4, C_3\}$ -decomposition.*

Given a counterexample H to Conjecture 2, we can construct a counterexample to the original conjecture by taking three copies of H and identifying the three vertices of degree 1 on the paths attached to u , respectively v .

We prove the following strengthening of Theorem 1 and thus verify Conjecture 2 for series-parallel graphs. Notice that this also implies the result for outerplanar graphs in [1] in its full strength.

Theorem 3. *Let G be a 2-edge-connected series-parallel graph with m edges and let k and l be natural numbers such that $m + k + l$ is divisible by 3. Then for any vertices x and y in G , the graph we get by attaching a path of length k at vertex x and a path of length l at vertex y has a $\{P_4, C_3\}$ -decomposition.*

Notice that it is necessary in Conjecture 2 that the paths are attached to vertices on the same face. Starting with a subdivision of K_4 , we can attach two edges to two vertices not sharing a face such that the resulting graph has no $\{P_4, C_3\}$ -decomposition. Hence, if the graph is not series-parallel, then it is not always possible to choose x and y arbitrarily, so Theorem 3 is in some sense best possible. It is also not possible to allow three exterior paths since a triangle with an edge attached to each vertex admits no $\{P_4, C_3\}$ -decomposition. Finally, notice that it is also easy to construct a 2-edge-connected series-parallel graph of size divisible by 3 that cannot be decomposed into paths of length 3, so it is necessary to allow triangles in our decomposition.

Using the subdivision of K_4 with two additional edges as a building stone, we construct a 2-edge-connected graph that admits no $\{P_4, C_3\}$ -decomposition, see Figure 1. This example is smaller than the one given in [2] and can also be embedded on the torus, unlike the previous example.

We also use Theorem 3 to construct a larger class of graphs with $\{P_4, C_3\}$ -decomposition. Let e be an edge of a graph G and let H be a 2-edge-connected series-parallel graph with two specified vertices x and y . We say that G' is constructed from G by *replacing e by H* if G' is formed from the disjoint union of $G - e$ and H by identifying x and y with distinct endpoints of e . We call such an operation an *edge-replacement*. Let $\mathcal{G}_0$ denote the class of all 2-edge-connected series-parallel graphs with up to two paths attached to it. Note that a graph in $\mathcal{G}_0$ need not be series-parallel.

Corollary 4. *If G can be constructed from a graph in $\mathcal{G}_0$ by finitely many edge-replacements, and G has size divisible by 3, then G has a $\{P_4, C_3\}$ -decomposition.*

The conjecture by Jünger et al. can be reduced to planar subcubic graphs by splitting vertices of degree larger than 3 such that the resulting graph is still planar and 2-edge-connected. Thus it is of interest to check whether all subdivisions of a given cubic graph admit a $\{P_4, C_3\}$ -decomposition. The k -prism is defined as the cartesian product of C_k and P_2 . As an application of Theorem 3 we get the following corollary.

Corollary 5. *Every subdivision of a k -prism ($k \geq 3$) can be decomposed into paths of length 3, provided the size is divisible by 3.*

References

- [1] R. Häggkvist, R. Johansson, A note on edge-decompositions of planar graphs. *Discrete Mathematics* 283 (2004) 263–266.
- [2] M. Jünger, G. Reinelt, W.R. Pulleyblank, On Partitioning the Edges of Graphs into Connected Subgraphs, *J. Graph Theory* 9 (1985) 539–549.
- [3] M. Merker, Decomposition of graphs, PhD thesis, in preparation.
- [4] C. Thomassen, Decompositions of highly connected graphs into paths of length 3, *J. Graph Theory* 58 (2008) 286–292.