Comment on “Temporal Correlations of the Running Maximum of a Brownian Trajectory”

Mortensen, Kim; Pedersen, Jonas Nyvold

Published in:
Physical Review Letters

Link to article, DOI:
[10.1103/PhysRevLett.117.248901](https://doi.org/10.1103/PhysRevLett.117.248901)

Publication date:
2016

Document Version
Publisher's PDF, also known as Version of record

[Link back to DTU Orbit](#)

Citation (APA):
Mortensen, K., & Pedersen, J. N. (2016). Comment on “Temporal Correlations of the Running Maximum of a Brownian Trajectory”. *Physical Review Letters*, 117(24), [248901].
<https://doi.org/10.1103/PhysRevLett.117.248901>

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Comment on “Temporal Correlations of the Running Maximum of a Brownian Trajectory”

Bénichou *et al.* [1] use the running maximum (RM) position in a single experimental trajectory of a particle exhibiting 1D Brownian motion (BM) to estimate its diffusion coefficient. This is unreliable: While the estimator’s precision (reproducibility) increases with the suggested parameter tuning, so does its *inaccuracy* (bias), as increasing emphasis is put on the RM’s maximum value.

In the mathematical idealization for BM used in Ref. [1], B_t is the position of a particle diffusing with coefficient D . However, $B_t = \sqrt{2D}W_t$, where W_t is the Wiener process. In this model, BM is a scale-free process.

Experimentally, one samples positions $x_{i=1,\dots,N}$ at time points $t_{i=1,\dots,N}$ [1]. Typically, constant time lapse Δt is used, such that $t_i = i\Delta t$ and $T = N\Delta t$. For BM, measured positions relate as $x_{i+1} = x_i + \sqrt{2D}\eta_i$, where $\eta_i = W_{t_{i+1}} - W_{t_i}$ is a Gaussian white noise with $\langle \eta_i \rangle = 0$ and $\langle \eta_i \eta_j \rangle = \Delta t \delta_{i,j}$ for all i, j . Each of the $N - 1$ displacements $\Delta x_i = x_{i+1} - x_i$ contains information about D ; hence, variances of estimators in this discrete case are limited by N , not T , due to the scale invariance of BM.

A reasonable estimator $\hat{D}$ for D should (i) be unbiased, i.e., $\langle \hat{D} \rangle = D$, and (ii) have a variance that decreases as $1/N$, for sufficiently large but practically relevant N . The discretized version $\hat{D}_{\text{msd}}^{(N)}$ of D_{msd} [1] with $\tau = \Delta t$, i.e., $\hat{D}_{\text{msd}}^{(N)} = \sum_{i=1}^{N-1} (\Delta x_i)^2 / [2(N-1)\Delta t]$, complies with (i) and (ii) for $N \geq 2$ in the present case of instantaneous recording of positions and in the absence of measurement noise. It is even optimal: It achieves the Cramér-Rao lower bound [2,3] and thus has the lowest possible variance among unbiased estimators.

With discrete sampling, the RM is $M_i = \max_{j=1,\dots,i} x_j$, and thus the RM-based estimator of Ref. [1] must read $\hat{D}_{\text{es}}^{(N,k)} = [C(k) \sum_{i=1}^N M_i^k]^{2/k}$, with $C(k) \equiv ([\Delta t \sqrt{\pi}(k/2+1)] / \{2^k \Gamma[(k+1)/2] T^{k/2+1}\})$ and $k > 0$. As a function of N , the information available to $\hat{D}_{\text{es}}^{(N,k)}$ increases so slowly that its variance approaches a constant value [1]. This is in conflict with (ii). The variance can be made arbitrarily small, however, by increasing k [1]; thus it is argued that $\hat{D}_{\text{es}}^{(N,k)}$ is superior to $\hat{D}_{\text{msd}}^{(N)}$ for small T [1].

Application of both estimators to Monte Carlo (MC) simulated BM shows, however, that the estimates of $\hat{D}_{\text{msd}}^{(N)}$ scatter with a normal distribution around D , while the estimates of $\hat{D}_{\text{es}}^{(N,k)}$ are skewed [Figs. 1(a) and 1(b)]. This results in a bias, $\langle \hat{D}_{\text{es}}^{(N,k)} \rangle \neq D$, which is in conflict with (i). The bias becomes worse with increasing k [Fig. 1(c)], while

FIG. 1. (a) Histograms of estimates obtained from application of, respectively, $\hat{D}_{\text{msd}}^{(N)}$ (red) and $\hat{D}_{\text{es}}^{(N,k)}$ with $k = 1$ (blue) to 10^5 MC simulated, discretely sampled BM trajectories using $D = 0.25$, $\Delta t = 1$, and $N = 10^4$. (b) The same as (a) for $k = 100$. (c) Mean values of estimates obtained as in (a) for various values of N . Results are shown for $\hat{D}_{\text{msd}}^{(N)}$ (pluses) and $\hat{D}_{\text{es}}^{(N,k)}$ with, respectively, k values of 1 (full circles), 10 (open circles), and 100 (crosses). (d) The same as (c) for the variances of the estimates. The theoretical variance $2D^2/(N-1)$ for $\hat{D}_{\text{msd}}^{(N)}$, the Cramér-Rao lower bound, is indicated (full line).

the variance indeed decreases [Fig. 1(d)]. The bias of $\hat{D}_{\text{es}}^{(N,k)}$ vanishes too slowly with N to ensure any practical relevance of $\hat{D}_{\text{es}}^{(N,k)}$ relative to $\hat{D}_{\text{msd}}^{(N)}$ [Figs. 1(c) and 1(d)].

In summary, the estimator suggested by Bénichou *et al.* [1] unfortunately yields biased values for the diffusion coefficient, while optimal, plug-and-play alternatives already exist [2,3].

The Danish Council for Strategic Research (Grant No. 10-092322/DSF) and the European Union’s Horizon 2020 research and innovation programme (Grant No. 696656) are acknowledged.

Kim I. Mortensen* and Jonas N. Pedersen†
Department of Micro- and Nanotechnology
Technical University of Denmark
DK-2800 Kongens Lyngby, Denmark

Received 25 August 2016; published 6 December 2016
DOI: 10.1103/PhysRevLett.117.248901

*kim.mortensen@nanotech.dtu.dk

†jonas.pedersen@nanotech.dtu.dk

- [1] O. Bénichou, P. L. Krapivsky, C. Mejía-Monasterio, and G. Oshanin, *Phys. Rev. Lett.* **117**, 080601 (2016).
- [2] C. L. Vestergaard, P. C. Blainey, and H. Flyvbjerg, *Phys. Rev. E* **89**, 022726 (2014).
- [3] C. L. Vestergaard, J. N. Pedersen, K. I. Mortensen, and H. Flyvbjerg, *Eur. Phys. J. Spec. Top.* **224**, 1151 (2015).