



Quantum optical effective-medium theory and transformation quantum optics for metamaterials

Wubs, Martijn; Amooghorban, Ehsan; Zhang, Jingjing; Mortensen, N. Asger

Published in:
Proceedings of SPIE

Link to article, DOI:
[10.1117/12.2239328](https://doi.org/10.1117/12.2239328)

Publication date:
2016

Document Version
Publisher's PDF, also known as Version of record

[Link back to DTU Orbit](#)

Citation (APA):
Wubs, M., Amooghorban, E., Zhang, J., & Mortensen, N. A. (2016). Quantum optical effective-medium theory and transformation quantum optics for metamaterials. In *Proceedings of SPIE* (Vol. 9918). Article 99180P SPIE - International Society for Optical Engineering. <https://doi.org/10.1117/12.2239328>

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Quantum optical effective-medium theory and transformation quantum optics for metamaterials

Martijn Wubs^{a,b}, Ehsan Amooghorban^{c,d}, Jingjing Zhang^{e,a,f}, and N. Asger Mortensen^{a,b}

^aDepartment of Photonics Engineering, Technical University of Denmark, DK-2800 Kgs. Lyngby, Denmark

^bCenter for Nanostructured Graphene, Technical University of Denmark, DK-2800 Kgs. Lyngby, Denmark

^cDepartment of Physics, Faculty of Basic Sciences, Shahrekord University, P.O. Box 115, Shahrekord 88186-34141, Iran

^dPhotonics Research Group, Shahrekord University, P.O. Box 115, Shahrekord 88186-34141, Iran

^eDepartment of Physics, Kings College London, Strand, London WC2R 2LS, United Kingdom

^fSchool of Electrical and Electronic Engineering, Nanyang Technological University, Nanyang Avenue 639798, Singapore

ABSTRACT

While typically designed to manipulate classical light, metamaterials have many potential applications for quantum optics as well. We argue why a quantum optical effective-medium theory is needed. We present such a theory for layered metamaterials that is valid for light propagation in all spatial directions, thereby generalizing earlier work for one-dimensional propagation. In contrast to classical effective-medium theory there is an additional effective parameter that describes quantum noise. Our results for metamaterials are based on a rather general Lagrangian theory for the quantum electrodynamics of media with both loss and gain. In the second part of this paper, we present a new application of transformation optics whereby local spontaneous-emission rates of quantum emitters can be designed. This follows from an analysis how electromagnetic Green functions transform under coordinate transformations. Spontaneous-emission rates can be either enhanced or suppressed using invisibility cloaks or gradient index lenses. Furthermore, the anisotropic material profile of the cloak enables the directional control of spontaneous emission.

Keywords: Metamaterials, quantum electrodynamics, effective-medium theory, quantum noise, transformation optics, Green tensor, spontaneous emission

1. INTRODUCTION

Metamaterials are engineered on the sub-wavelength scale to produce desired effective optical parameters, which often have no counterpart in naturally occurring materials. Important examples are negative-index materials that enable a new flat type of lenses, and hyperbolic metamaterials which are effectively highly anisotropic media with negative-valued effective dielectric function in some directions and positive values in other directions. Metamaterials are typically finite periodic structures with unit cells smaller than the operating wavelength. Homogenization theory describes and compares methods to infer the effective optical parameters of metamaterials.^{1,2} Besides three-dimensional metamaterials, metasurfaces have also started to attract wide interest, because they enable new ways to manipulate the reflection and refraction of light by nanostructuring a surface.

Further author information: (Send correspondence to M. W.)

M.W.: E-mail: mwubs@fotonik.dtu.dk, Telephone: +45 45 25 63 74

E.A.: E-mail: ehsan.amooghorban@sci.sku.ac.ir

J.Z.: E-mail: zhangjj@ntu.edu.sg

N.A.M.: E-mail: asger@mailaps.org

Most though not all research in metamaterials belongs to classical electrodynamics, where light is described as a classical wave and the material by its (effective or not) dielectric function. Here instead we are interested how metamaterials can be used in quantum optics. Now many people expect the next great technological revolution to be a quantum revolution. And many quantum information processing proposals involve combinations of stationary quantum bits and flying quantum bits, where often photons are used for the latter. Thus there would be an important role to play for quantum optics. Important topics in quantum optics are the design and control of spontaneous emission, protecting and preserving quantum states of light, and mapping quantum states of light to the states of quantum bits, to name a few. Now photons are quanta of the electromagnetic field which in turn depends on the dielectric environment. So here the question arises how the engineered environments called metamaterials can be of help in quantum optics.

Before discussing our recent work, let us give some examples of other interesting research in the quantum optics of metamaterials: hyperbolic metamaterials have a strong broadband enhancement of the local optical density of states leading to fast spontaneous-emission rates of embedded quantum emitters.³⁻⁶ Quantum states of light can survive the propagation through metamaterials.⁷ Metamaterials can be used for entanglement distillation of photon states,^{8,9} for the efficient creation of maximally entangled two-photon states.¹⁰ Metasurfaces can break the symmetry of quantum vacuum fluctuations and create a strong anisotropic quantum vacuum as experienced by a distant quantum emitter.¹¹ Vice versa, metamaterials can be characterized by probing them with quantum states of light. For example, dispersion and absorption parameters of a metamaterials can be determined using entangled photons.¹² Moreover, squeezed states of light can tell apart lossless media from effectively lossless ones as obtained by compensating loss by gain.¹³ More about the latter application will follow below.

In this paper we address two of our recent contributions to the quantum optics of metamaterials. In the first place we will discuss in Section 2 effective-medium theories for quantum optics, mainly based on our Refs. 13 and 14. We will argue that in quantum optics the effective index of a metamaterial is the same as in classical optics, but that in quantum optics often an additional polarization-dependent effective parameter is needed to faithfully describe the quantum state of light that has propagated through the metamaterial. In Sec. 3 we present a new way to engineer spontaneous-emission rates using transformation optics, basically because we show how the Green function transforms under the coordinate transformation of transformation optics. Here we highlight some results of Ref. 15 where further details can be found. Controlling spontaneous-emission rates in this new way is both fundamentally interesting and potentially useful for devices including lasers, LEDs, and solar cells.

2. QUANTUM OPTICAL EFFECTIVE-MEDIUM THEORY

Here we first give an idea of the quantum electrodynamic theory upon which we base our quantum optical effective-medium theory for metamaterials. We then argue that in quantum optics, effective-medium theories should also describe quantum noise correctly. We show that in some situations the effective quantum noise of a layered metamaterial is well described by the effective index, but not in all cases. By identifying an additional effective parameter for metamaterials in quantum optics, we find a quantum optical effective-medium theory both for TE- and TM-polarized light, and for all angles of incidence. We give some numerical examples to test and illustrate that our QOEM theory is very accurate.

2.1 From a Lagrangian to a Green-function theory for layered media with loss and gain

All media except vacuum are lossy and dispersive, and these two properties are connected as described by the Kramers-Kronig relations. Often in optics one works with media where loss can be neglected at the operating frequency, while in other situations loss is reluctantly accepted as a disadvantage because other properties of the medium are advantageous. As an example, metal-based metamaterials are lossy at optical frequencies. But instead of accepting some loss, it is actually possible to compensate loss in metallic components of metamaterials by also incorporating gain media into the unit cell.¹⁶ Now in quantum mechanics in general and in quantum optics in particular, one cannot describe loss or gain simply by adding a damping or gain term to the equation of motion: loss or gain of a quantum system arise from its coupling to a typically much larger system, and this coupling is also a source of quantum noise. In a quantum optical description of metamaterials, we therefore need to take this quantum noise into account.

Lossy dielectric media are described by a complex-valued dielectric function with positive-valued imaginary part $\varepsilon_I(\omega)$, while gain media are characterized by $\varepsilon_I(\omega) < 0$ at the operating frequency. In quantum mechanics, loss of a quantum system is often described by the coupling of the system to a bath of harmonic oscillators. It is less well known that gain can likewise be described by coupling the quantum system to a bath of inverted harmonic oscillators.¹⁷ If we now wish to describe a dielectric medium that is lossy^{18–21} at some places and amplifying^{22,23} at other locations, then one can do this by writing a Lagrangian for the free electromagnetic field that has a frequency- and position-dependent coupling to matter, including a bath of oscillators at the lossy locations and of inverted oscillators where there is gain. For simplicity we neglect nonlocal response, which becomes important in plasmonic metamaterials with sub-10-nm features within unit cells.^{5,24}

The result of all this^{14,25} is that the material polarization field (a quantum operator) is not described by $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$, but rather by

$$\mathbf{P}(\mathbf{x}, t) = \varepsilon_0 \int_0^\infty dt' \chi(\mathbf{x}, t - t') \mathbf{E}(\mathbf{x}, t') + \mathbf{P}^N(\mathbf{x}, t), \quad (1)$$

where the novelty is the electric polarization noise density $\mathbf{P}^N(\mathbf{x}, t)$. The Cartesian components (labeled by subscript i) of this vector field are quantum operators

$$P_i^{N(+)}(\mathbf{x}, t) = i \int_0^\infty d\omega \sqrt{\frac{\hbar \varepsilon_0 |\varepsilon_I(\mathbf{x}, \omega)|}{\pi}} f_i(\mathbf{x}, \omega) e^{-i\omega t}, \quad (2)$$

in terms of the more elementary operators $f_i(\mathbf{x}, \omega)$ of the form $d_i(\mathbf{x}, \omega, 0) \Theta[\varepsilon_I(\mathbf{x}, \omega)] + d_i^\dagger(\mathbf{x}, \omega, 0) \Theta[-\varepsilon_I(\mathbf{x}, \omega)]$, where the $d_i(\mathbf{x}, \omega, 0)$ are standard bosonic ladder operators with commutation relations $[d_j(\mathbf{x}, \omega, t), d_{j'}^\dagger(\mathbf{x}', \omega', t)] = \delta_{jj'} \delta(\omega - \omega') \delta^3(\mathbf{x} - \mathbf{x}')$ and Θ is the step function. A brief inspection shows that $f_i(\mathbf{x}, \omega)$ is an annihilation operator for a frequency- and position combination with loss, and a creation operator otherwise, which explains the name “inverted oscillator” somewhat and allows a compact description of our theory for media with both loss and gain.¹⁴ For example, the (positive-frequency component of the) vector potential satisfies $\nabla \times \nabla \times \mathbf{A}^{(+)} - \frac{\omega^2}{c^2} \varepsilon \mathbf{A}^{(+)} = -i\mu_0 \omega \mathbf{P}^{N(+)}$, the solution of which is given by

$$\mathbf{A}^{(+)}(\mathbf{x}, t) = \frac{-i\mu_0}{\sqrt{2\pi}} \int_0^\infty d\omega \omega \int d^3 \mathbf{x}' \mathbf{G}(\mathbf{x}, \mathbf{x}', \omega) \cdot \mathbf{P}^{N(+)}(\mathbf{x}', \omega) e^{-i\omega t}, \quad (3)$$

in terms of the classical electromagnetic Green function $\mathbf{G}(\mathbf{x}, \mathbf{x}', \omega)$, which has the defining equation

$$\left[\nabla \times \nabla \times - \frac{\omega^2}{c^2} \varepsilon(\mathbf{x}, \omega) \right] \mathbf{G}(\mathbf{x}, \mathbf{x}', \omega) = \delta^3(\mathbf{x} - \mathbf{x}') \mathbf{1}_3. \quad (4)$$

So from a Lagrangian theory an effective Green-function theory emerges for the quantum electrodynamics of media with both loss and gain: given the dielectric function $\varepsilon(\mathbf{x}, \omega)$ and the corresponding classical Green function, the vector potential and all the Maxwell fields can be determined. This is a quite general theory, and as a special application it can be used to describe the quantum electrodynamics of metamaterials.

In the following we consider layered media. The classical Green function of layered media is known,²⁶ and we follow the approach of Ref. 27 regarding some causality subtleties for active (i.e. gain) media. Starting from the expression for the electric field operators in terms of the Green function, we derived a recursive expression for the electric field output of layered media given the input states of light and the quantum noise sources inside the metamaterial.¹⁴ The difference with classical electrodynamics is that not only light sources are to be considered as input states, but also unused input ports with vacuum states need to be taken into account as noise sources, just like for beam splitters in quantum optics, and moreover the quantum noise within lossy or amplifying layers should also be considered as a source. So if there are many layers, then there are also many sources that together recursively determine the electric-field operator corresponding to outgoing light. The many additional noise sources constitute the added complexity of quantum optical input-output theory for multilayer media as compared to the corresponding classical input-output theory. So if simple and accurate effective descriptions of multilayer media exists in quantum optics, then they are also highly desirable, and even more so than in classical optics.

2.2 Quantum optical effective-index (QOEI) and effective-medium (QOEM) theory

Unit cells of metamaterials are typically much smaller than the wavelength of the light. The light does not probe the inner structure of the unit cell but rather feels its average effect as described by the effective index (which is a scalar for isotropic and a tensor for anisotropic metamaterials). The same holds true in quantum optics. The “only” novel question is what average quantum noise the light experiences from a unit cell. If the effective index describes this effective amount of quantum noise, then no new effective-medium parameters are needed for metamaterials in quantum optics, even when properly taking quantum noise into account. On the other hand, if the effective index is not sufficient to describe the average quantum noise, then the new questions arise whether in quantum optics an effective-medium theory exists at all, and if so what are its effective parameters and how to determine them.

In Ref. 13 we studied these questions for the first time, limiting our studies to light propagation normal to the layers. We found that for passive multilayers systems as well as for binary multilayer metamaterials with gain in both types of layers, that the effective index actually describes the average amount of quantum noise well. So if we describe the metamaterial as an effectively homogeneous medium with the same effective index as in classical optics but additionally with effective quantum noise (which has no classical counterpart) also described by the effective index, then this together constitutes a Quantum Optical Effective-Index (QOEI) theory for metamaterials. For example, in QOEI theory, the effective noise current polarization of the metamaterial is given by Eq. (2), but with $|\varepsilon_I(\mathbf{x}, \omega)|$ replaced by the absolute value of the imaginary part of the *effective* index, i.e. by $|\varepsilon_{\text{eff},I}(\omega)|$. Without the quantum noise, QOEI theory reduces to the usual classical effective-medium theory.

However, already for normal propagation we found in Ref. 13 that the QOEI theory breaks down for the important class of loss-compensated metamaterials, because it underestimates the effective amount of quantum noise. This has observable consequences. For example, when studying whether quadrature-squeezed states of light (with variance below unity²⁸) are still squeezed after propagation through a loss-compensated metamaterial, QOEI theory may predict this to be the case whereas the correct answer is negative (because the variance actually exceeds unity). Both theories have measurably different predictions, for example for balanced homodyne detection experiments.

QOEI theory breaks down for loss-compensated metamaterials, because not only does it compensate loss by gain, but also quantum noise due to loss by quantum noise due to gain. But the exact multilayer theory illustrates that these independent noise sources do not cancel. Our Quantum Optical Effective-Medium (QOEM) theory of Ref. 13 repairs this by not only determining an effective index but also an effective noise photon distribution $N_{\text{eff},\sigma}(\mathbf{k}, \omega, T)$, depending on the polarization state σ (s -polarized (TE) or p -polarized (TM)), on the wavevector parallel to the layers \mathbf{k} , and on temperature T . For loss-compensated metamaterials the effective noise photon distribution is given by¹⁴

$$N_{\text{eff},\sigma} = -\frac{1}{2} + \frac{1}{2} \sum_{j=1,\text{g}} \eta_{j,\sigma} [2N_{\text{th}}(\omega, |T_j|) + 1], \quad \text{with} \quad \eta_{j,\sigma}(\theta) = p_j \frac{\mathcal{K}_{j,\sigma}(\theta)}{\mathcal{K}_{\text{eff},\sigma}(\theta)} \left| \frac{\varepsilon_{j,I}(\omega)}{\varepsilon_{\text{eff},I}(\omega)} \right|, \quad (5)$$

where the p_j are the volume fractions of the lossy and gain layers, $\mathcal{K}_{j,s}(\theta) = 1$ and $\mathcal{K}_{j,p}(\theta) = (\sin^2 \theta + |\varepsilon_j|^2 \cos^2 \theta) / |\varepsilon_j|^2$, and $\mathcal{K}_{\text{eff},\sigma}(\theta)$ equals $\mathcal{K}_{j,\sigma}(\theta)$ with ε_j replaced by ε_{eff} . For loss-loss or gain-gain metamaterials, the expressions for $N_{\text{eff},\sigma}$ differ slightly from Eq. (5).¹⁴ It follows from the above expressions that the effective noise photon distribution increases when more loss is compensated by more gain, and even diverges in the limit of exact loss compensation ($\varepsilon_{j,I}(\omega) = 0$). For normal incidence, the \mathcal{K} 's reduce to unity and the simpler expression for normal incidence of Ref. 13 is recovered as it should.

In the absence of loss compensation and for normal incidence, the $N_{\text{eff},\sigma}$ reduces to a thermal distribution.¹³ We find that for s -polarized light incident on metamaterials without loss compensation, the effective noise photon density is a thermal distribution for *all* angles of incidence. This is a direct generalization of the result we found earlier for normally incident light.¹³ However, for p -polarized light we do find differences between the thermal and the effective noise-photon distributions, also in the absence of loss compensation.¹⁴ This is one of the surprises of our study, because we anticipated that non-thermal effective noise-photon distributions would only occur in loss-compensated media.

2.3 Numerical tests based on power spectra and homodyne signals

Let us now briefly present a few numerical tests of QOEI and QOEM theory, both compared to exact multilayer calculations. In Figure 1 we relate power spectra of loss-compensated metamaterials to the net effective gain or loss in the metamaterials and we present the corresponding effective noise photon distributions, for an angle of incidence of 45 degrees, and for both polarizations. While QOEI theory would predict that no light would be emitted at all by effectively lossy metamaterials at zero temperature, QOEM theory and the exact multilayer theory agree that the metamaterial is in fact “glowing” because of spontaneous emission of noise photons in the amplifying layers, resulting in non-vanishing power spectra. It can be seen that QOEM theory is a very accurate effective-medium theory. Notice also that while the effective noise photon distribution diverges at exact loss compensation, the resulting predicted power spectra are continuous.

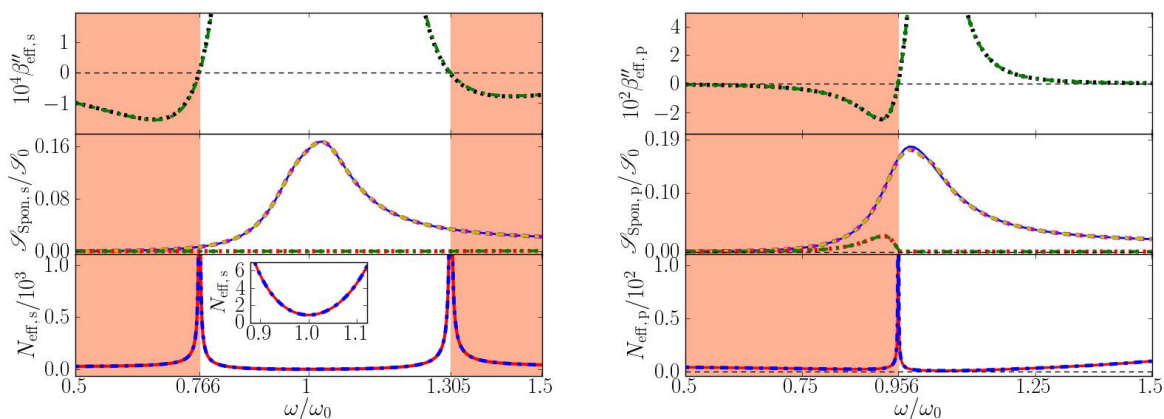


Figure 1. Left panels correspond to TE-polarized light and right panels to TM. Upper panels: net effective gain or loss of the metamaterial, with negative values of β_{eff} corresponding gain. Almost overlapping black and green lines correspond to two different methods of retrieval of the effective index. Middle panels: power spectra. Almost overlapping red and green curves both correspond to effective-index theory, but based on different parameter retrieval methods for the effective index. Blue curve is the exact power spectrum for the five unit cell thick multilayer structure, yellow dashes and magenta dash-dots are QOEM theory based on two different ways of obtaining the effective index. Lower panels: effective noise current densities both according to QOEM theory, with two colors corresponding to two different ways of obtaining the effective index. (Calculations by E.A.)

In Figure 2 we consider the output variances as measurable with balanced homodyne detection, when squeezed light impinges on both sides of the same metamaterial as in the previous figures. The angle of incidence is again 45 degrees (as in Fig. 1). Also in this case we can see that QOEM theory accurately follows the exact multilayer theory in describing the output variance, while QOEI theory is far off. This illustrates that QOEM theory is not only designed to be accurate for just one type of experiment (only to describe power spectra for example), but instead has a more general use.

3. TRANSFORMATION QUANTUM OPTICS

We finally present a recent result in transformation optics that could become important for quantum optics, because it allows the design of metamaterials with specific spontaneous-emission rates.¹⁵ Transformation optics (TO)^{29,30} has become most famous because it allows the design of optical invisibility cloaks, and it is a design tool also for other applications, in plasmonics for example.^{31,32} Examples of electric-field distributions in and near perfect and near-perfect cylindrical invisibility cloaks are given in Figure 3. TO is based on the fact that Maxwell’s equations are invariant under a coordinate transformation, at least when the dielectric function $\epsilon(\mathbf{x})$ and the magnetic permeability $\mu(\mathbf{x})$ are transformed accordingly. But when this is taken care of (and no need to say that this can be challenging in practice), then solutions of Maxwell’s equations in the original coordinates will then also be solutions in the transformed coordinates.³³ So if we started with an electromagnetic problem

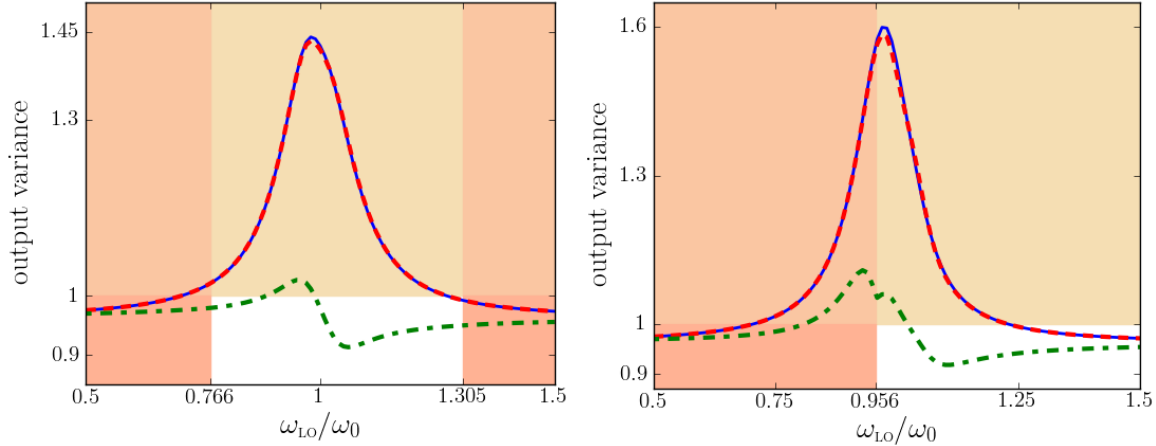


Figure 2. Output variance of a loss-compensated metamaterial when squeezed light (with quadrature variance smaller than unity) impinges on both sides of the metamaterial. Green dash-dotted curves correspond to effective-index theory, the blue curve is the exact multilayer theory, and the red dashed curve is based on QOEM theory. Left panel is TE-, right panel is TM-polarized light. Colored fields mark the regions of effective gain and effective loss. (Calculations by E.A.)

for which a solution is known, perhaps even analytically, then a solution for the otherwise apparently much more complex problem in the transformed basis is then also solved. Thus TO can provide insight into otherwise analytically impenetrable problems, see the recent review in Ref. 34 and an application to nonlocal response in plasmonics.³⁵

As stated in the Introduction, controlling spontaneous-emission rates is both of fundamental and practical interest. Even though spontaneous emission is a purely quantum mechanical process, spontaneous-emission rates in a medium can be determined already from the *classical* Green function [as defined in Eq. (4)] for that medium. At first sight it may seem nearly impossible to find the Green function of TO media, because TO media such as in Fig. 3 are typically graded-index media, and Green functions for graded-index media are generally hard to find. Nevertheless spontaneous-emission rates near invisibility cloaks have been studied before.^{36–38}

What to our knowledge had not been considered before is whether spontaneous-emission rates in the transformed medium can be related to those in the original medium. A central question in Transformation Quantum Optics (TQO) is thus whether the classical Green function in the transformed medium is related to the Green function in the original medium, and if so how.

The standard expression (see for example Ref. 39) for the spontaneous-emission rate Γ of an atomic dipole with dipole moment $\boldsymbol{\mu}$ at position \mathbf{x}_0 in a medium described by a Green function \mathbf{G} is

$$\Gamma = \frac{2\omega_0^2}{\hbar\epsilon_0 c^2} \boldsymbol{\mu} \cdot [\text{Im}\mathbf{G}(\mathbf{x}_0, \mathbf{x}_0, \omega_0)] \cdot \boldsymbol{\mu}. \quad (6)$$

This also holds for transformation media, so the challenge is rather to find the Green function after a coordinate transformation. For an example of a simple but useful coordinate transformation, see Fig. 3. For a general coordinate transformation that maps the space (x, y, z) onto (x', y', z') , we have $d\mathbf{r}' = \mathbf{A} \cdot d\mathbf{r}$, where $\mathbf{A} = \partial(x', y', z')/\partial(x, y, z)$ is the Jacobian matrix. It actually turns out that the transformed Green function in the transformed coordinates has a rather simple relation to the original Green function in the initial coordinates, so that the Green function is known if the original one was known. Here we merely state the result that was derived in Ref. 15, namely

$$\mathbf{G}'(\mathbf{x}', \mathbf{x}'_0) = \left(\mathbf{A}^T\right)^{-1} \cdot \mathbf{G}(\mathbf{x}, \mathbf{x}_0) \cdot \mathbf{A}^{-1}, \quad (7)$$

where the superscript T denotes the matrix transpose. For an atomic dipole with dipole moment $\boldsymbol{\mu}'$ and located

on position \mathbf{x}'_0 in the transformed medium, it follows that the spontaneous-emission rate is given by

$$\Gamma' = \frac{2\omega_0^2}{\hbar\epsilon_0 c^2} \boldsymbol{\mu}' \cdot [\text{Im}\mathbf{G}'(\mathbf{x}'_0, \mathbf{x}'_0, \omega_0)] \cdot \boldsymbol{\mu}' = \frac{2\omega_0^2}{\hbar\epsilon_0 c^2} (\mathbf{A}^{-1} \cdot \boldsymbol{\mu}') \cdot [\text{Im}\mathbf{G}(\mathbf{x}_0, \mathbf{x}_0, \omega_0)] \cdot (\mathbf{A}^{-1} \boldsymbol{\mu}'). \quad (8)$$

Comparing this with Eq. (6) we find the simple result that the spontaneous-emission rate of the atomic dipole with dipole moment $\boldsymbol{\mu}'$ on position \mathbf{x}'_0 in the transformed medium is equal to the spontaneous-emission rate of a fictitious atomic dipole with dipole moment $\boldsymbol{\mu} \equiv \mathbf{A}^{-1} \cdot \boldsymbol{\mu}'$ on position \mathbf{x}_0 in the original medium. Although our compact notation does not show it explicitly, it should be realized that the fictitious dipole $\boldsymbol{\mu}$ in general has a position-dependent magnitude and orientation. If the original medium is homogeneous, then $\text{Im}\mathbf{G}(\mathbf{x}_0, \mathbf{x}_0, \omega_0)$ actually does not depend on \mathbf{x}_0 . The position dependence of Γ' then fully originates from the position dependence of $\boldsymbol{\mu}$ via the Jacobian matrix \mathbf{A} . Since the transformation is quite general, one can design transformations either to enhance or to suppress spontaneous-emission rates. Green functions also contain all information about preferred emission directions, so by Eq. (7), TO can also be used to engineer directional emission.

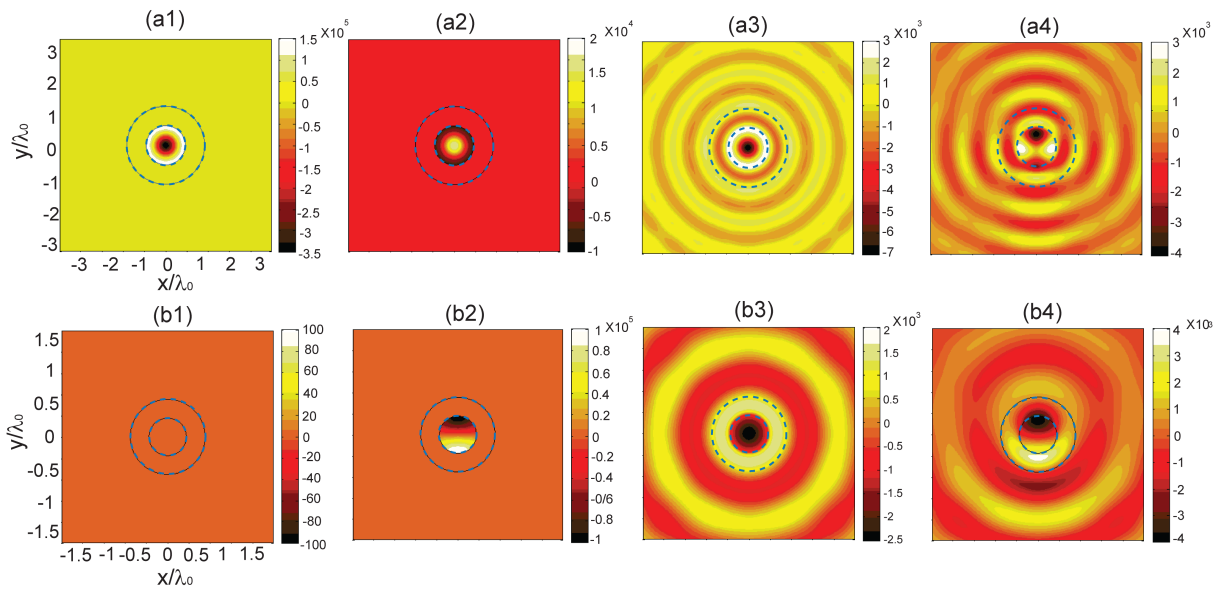


Figure 3. Electric-field distributions for two modes of cylindrical cloaks,⁴⁰ obtained from a transformation of only the radial coordinate as $r' = R_2 \sqrt{(r^2 - R_1^2)/(R_2^2 - R_1^2)}$. (a1-a4) Electric-field distributions for mode $n = 0$ ($k_0 R_1 = 3.8317$, $R_2 = 2R_1$). In (a1) and (a3), the dipole is located at $(0, 0)$. In (a2) and (a4), the dipole is located at $(0, 0.655R_1)$. (b1-b4) Electric-field distributions for mode $n = 1$ ($k_0 R_1 = 1.8412$, $R_2 = 2R_1$). In (b1) and (b3), the dipole is located at $(0, 0)$. In (b2) and (b4), the dipole is located at $(0, 0.682R_1)$. In the left two columns [(a1), (a2), (b1), (b2)], a dipole source is placed inside the perfect cloak shell. In the right two columns [(a3), (a4), (b3), (b4)], a dipole source is placed inside a cloak where a perturbation is applied on the permittivity of the inner boundary, making the cloak no longer perfect. The blue dashed lines indicate the contours of the cloak. (Calculations by J.Z.)

4. CONCLUSIONS

There is an increasing number of applications of metamaterials in quantum optics. Just like in classical optics, effective descriptions of metamaterials are called for in quantum optics. We found that established effective-medium descriptions of metamaterials do not always work in quantum optics, even for strongly sub-wavelength unit cells, because quantum noise due to loss and/or gain are not always negligible.^{13,14} Therefore effective-medium theories in quantum optics should describe the effective amount of quantum noise well. We described two theories. The simplest one is a Quantum Optical Effective-Index (QOEI) theory where the effective amount of quantum noise is given by the usual effective index. It describes the propagation of s -polarized light through passive metamaterials well, but for example active metamaterials where loss is compensated by gain require

instead a Quantum Optical Effective-Medium (QOEM) theory that with the help of an additional effective parameter very accurately describes the effective amount of quantum noise produced by lossy and amplifying constituents of the unit cell. We recently generalized both theories such that they now describe three-dimensional propagation of quantum states of light in layered metamaterials.¹⁴

Moreover, we presented a new way to calculate the Green function of transformation media,¹⁵ which enables the calculation and engineering of position-dependent spontaneous-emission rates using transformation optics.

ACKNOWLEDGEMENTS

M.W. and J.Z. thank P. Ginzburg, G. Wurtz, and A.V. Zayats for the collaboration leading to Ref. 15. N.A.M. and M.W. acknowledge support from the Danish Council for Independent Research (FNU 1323-00087). M.W. acknowledges support from the Villum Foundation via the VKR Centre of Excellence NATEC-II, and from the Otto Mønsted Foundation. The Center for Nanostructured Graphene is sponsored by the Danish National Research Foundation, Project DNRF103. E.A. wishes to thank the Shahrekord University for support. J.Z. acknowledges support from the Royal Society Newton International Fellowship, from the Danish Research Council for Technology and Production Sciences (FTP grants 11-104559), and from the Singapore Ministry of Education (MOE) AcRF TIER 1 under Grant No RG72/15.

REFERENCES

- [1] Bergman, D. J., "The dielectric constant of a composite material – a problem in classical physics," *Phys. Rep.* **43**, 377 (1978).
- [2] Menzel, C., Rockstuhl, C., Paul, T., Lederer, F., and Pertsch, T., "Retrieving effective parameters for metamaterials at oblique incidence," *Phys. Rev. B* **77**, 195328 (2008).
- [3] Noginov, M. A., Li, H., Barnakov, Y. A., Dryden, D., Nataraj, G., Zhu, G., Bonner, C. E., Mayy, M., Jacob, Z., and Narimanov, E. E., "Controlling spontaneous emission with metamaterials," *Opt. Lett.* **35**, 1863 (2010).
- [4] Cortes, C. L., Newman, W., Molesky, S., and Jacob, Z., "Quantum nanophotonics using hyperbolic metamaterials," *J. Opt.* **14**, 063001 (2012).
- [5] Yan, W., Wubs, M., and Mortensen, N. A., "Hyperbolic metamaterials: nonlocal response regularizes broadband supersingularity," *Phys. Rev. B* **86**, 205429 (2012).
- [6] Lu, D., Kan, J. J., Fullerton, E. E., and Liu, Z., "Enhancing spontaneous emission rates of molecules using nanopatterned multilayer hyperbolic metamaterials," *Nature Nanotech.* **9**, 48 (2014).
- [7] Wang, S. M., Mu, S. Y., Zhu, C., Gong, Y. X., Xu, P., Liu, H., Li, T., Zhu, S. N., and Zhang, X., "Hong-Ou-Mandel interference mediated by the magnetic plasmon waves in a three-dimensional optical metamaterial," *Opt. Express* **20**, 5213 (2012).
- [8] al Farooqui, M. A., Breeland, J., Aslam, M. I., Sadatgol, M., Şahin K. Özdemir, Tame, M., Yang, L., and Güney, D. O., "Quantum entanglement distillation with metamaterials," *Opt. Express* **23**, 17941 (2015).
- [9] Asano, M., Bechu, M., Tame, M., Şahin Kaya Özdemir, Ikuta, R., Güney, D. O., Yamamoto, T., Yang, L., Wegener, M., and Imoto, N., "Distillation of photon entanglement using a plasmonic metamaterial," *Sci. Rep.* **5**, 18313 (2015).
- [10] Siomau, M., Kamli, A., Moiseev, S. A., and Sanders, B. C., "Entanglement creation with negative index metamaterials," *Phys. Rev. A* **85**, 050303 (2012).
- [11] Jha, P. K., Ni, X., Wu, C., Wang, Y., and Zhang, X., "Metasurface enabled remote quantum interference," *Phys. Rev. Lett.* **115**, 025501 (2015).
- [12] Zhou, Z.-Y., Ding, D.-S., Shi, B.-S., Zou, X.-B., and Guo, G. C., "Characterizing dispersion and absorption parameters of metamaterial using entangled photons," *Phys. Rev. A* **85**, 023841 (2012).
- [13] Amooghorban, E., Mortensen, N. A., and Wubs, M., "Quantum optical effective-medium theory for loss-compensated metamaterials," *Phys. Rev. Lett.* **110**, 153602 (2013).
- [14] Amooghorban, E. and Wubs, M., "Quantum optical effective-medium theory for layered metamaterials." Preprint ArXiv:1606.07912v1 (2016).

- [15] Zhang, J., Wubs, M., Ginzburg, P., Wurtz, G., and Zayats, A. V., “Transmission quantum optics: designing spontaneous emission using coordinate transforms,” *J. Opt.* **18**, 044029 (2016).
- [16] Xiao, S., Drachev, V. P., Kildishev, A. V., Ni, X., Chettiar, U. K., Yuan, H.-K., and Shalaev, V. M., “Loss-free and active optical negative-index metamaterials,” *Nature* **466**, 735 (2010).
- [17] Glauber, R. J., [*Amplifiers, attenuators, and the quantum theory of measurement*], Hilger, Bristol (1986).
- [18] Huttner, B. and Barnett, S. M., “Quantization of the electromagnetic field in dielectrics,” *Phys. Rev. A* **46**, 4306 (1992).
- [19] Gruner, T. and Welsch, D.-G., “Quantum-optical input-output relations for dispersive and lossy multilayer dielectric plates,” *Phys. Rev. A* **54**, 1661–1677 (1996).
- [20] Wubs, M. and Suttorp, L. G., “Transient QED effects in absorbing dielectrics,” *Phys. Rev. A* **63**, 043809 (2001).
- [21] Suttorp, L. G. and Wubs, M., “Field quantization in inhomogeneous absorptive dielectrics,” *Phys. Rev. A* **70**, 013816 (2004).
- [22] Matloob, R., Loudon, R., Artoni, M., Barnett, S. M., and Jeffers, J., “Electromagnetic field quantization in amplifying dielectrics,” *Phys. Rev. A* **55**, 1623 (1997).
- [23] Scheel, S., Knöll, L., and Welsch, D.-G., “QED commutation relations for inhomogeneous Kramers-Kronig dielectrics,” *Phys. Rev. A* **58**, 700 (1998).
- [24] Yan, W., Mortensen, N. A., and Wubs, M., “Hyperbolic metamaterial lens with hydrodynamic nonlocal response,” *Opt. Express* **21**, 15026 (2013).
- [25] Amooghorban, E., Wubs, M., Mortensen, N. A., and Kheirandish, F., “Casimir forces in multilayer magnetodielectrics with both gain and loss,” *Phys. Rev. A* **84**, 013806 (2011).
- [26] Tomaš, M. S., “Green function for multilayers: light scattering in planar cavities,” *Phys. Rev. A* **51**, 2545 (1995).
- [27] Nistad, B. and Skaar, J., “Causality and electromagnetic properties of active media,” *Phys. Rev. E* **78**, 036603 (2008).
- [28] Vasylyev, D. Y., Vogel, W., Manzke, G., Henneberger, K., and Welsch, D.-G., “Nonclassicality of radiation fields propagating in complex material systems,” *Phys. Status Solidi B* **246**, 293 (2009).
- [29] Leonhardt, U., “Optical conformal mapping,” *Science* **312**, 1777 (2006).
- [30] Pendry, J. B., Schurig, D., and Smith, D. R., “Controlling electromagnetic fields,” *Science* **312**, 1780 (2006).
- [31] Zhang, J., Xiao, S., Wubs, M., and Mortensen, N. A., “Surface plasmon wave adapter designed with transformation optics,” *ACS Nano* **5**, 4359 (2011).
- [32] Kraft, M., Luo, Y., Maier, S. A., and Pendry, J. B., “Designing plasmonic gratings with transformation optics,” *Phys. Rev. X* **5**, 031029 (2015).
- [33] Ward, A. J. and Pendry, J. B., “Refraction and geometry in Maxwell’s equations,” *J. Mod. Opt.* **43**, 773 (1996).
- [34] Pendry, J. B., Luo, Y., and Zhao, R., “Transforming the optical landscape,” *Science* **348**, 521 (2015).
- [35] Fernández-Domínguez, A. I., Wiener, A., García-Vidal, F. J., Maier, S. A., and Pendry, J. B., “Transformation-optics description of nonlocal effects in plasmonic nanostructures,” *Phys. Rev. Lett.* **108**, 106802 (2012).
- [36] Kort-Kamp, W. J. M., Rosa, F. S. S., Pinheiro, F. A., and Farina, C., “Spontaneous emission in the presence of a spherical plasmonic metamaterial,” *Phys. Rev. A* **87**, 023837 (2013).
- [37] Rahmani, A., Steel, M. J., and Chaumet, P. C., “Invisibility and supervisibility: Radiation dynamics in a discrete electromagnetic cloak,” *Phys. Rev. B* **87**, 045430 (2013).
- [38] Morshed Behbahani, M., Amooghorban, E., and Mahdifar, A., “Spontaneous emission and the operation of invisibility cloaks,” *Phys. Rev. A* **94**, 013854 (2016).
- [39] Novotny, L. and Hecht, B., [*Principles of Nano-Optics*], Cambridge University Press, 2nd ed. (2012).
- [40] Luo, Y., Zhang, J., Chen, H., Xi, S., and Wu, B.-I., “Cylindrical cloak with axial permittivity/permeability spatially invariant,” *Appl. Phys. Lett.* **93**, 033504 (2008).