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The Effect of Autocorrelation on the Hotelling T^2 Control Chart

Erik Vanhatalo[a*†] and Murat Kulahci[a,b]

One of the basic assumptions for traditional univariate and multivariate control charts is that the data are independent in time. For the latter, in many cases, the data are serially dependent (autocorrelated) and cross-correlated because of, for example, frequent sampling and process dynamics. It is well known that the autocorrelation affects the false alarm rate and the shift-detection ability of the traditional univariate control charts. However, how the false alarm rate and the shift-detection ability of the Hotelling T^2 control chart are affected by various autocorrelation and cross-correlation structures for different magnitudes of shifts in the process mean is not fully explored in the literature. In this article, the performance of the Hotelling T^2 control chart for different shift sizes and various autocorrelation and cross-correlation structures are compared based on the average run length using simulated data. Three different approaches in constructing the Hotelling T^2 chart are studied for two different estimates of the covariance matrix: (i) ignoring the autocorrelation and using the raw data with theoretical upper control limits; (ii) ignoring the autocorrelation and using the raw data with adjusted control limits calculated through Monte Carlo simulations; and (iii) constructing the control chart for the residuals from a multivariate time series model fitted to the raw data. To limit the complexity, we use a first-order vector autoregressive process and focus mainly on bivariate data. © 2014 The Authors. *Quality and Reliability Engineering International* published by John Wiley & Sons Ltd.

Keywords: statistical process control (SPC); Hotelling T^2 chart; autocorrelation; multivariate data; time series modeling, simulation

1. Introduction

Statistical process control (SPC) provides an important toolbox for improving the process performance and maintaining an efficient manufacturing process. Shewhart control charts together with cumulative sum and exponentially weighted moving average charts, to a large extent, form the basis of SPC when a single quality characteristic is of interest. However, in many applications of SPC, data are often collected for more than one quality characteristics, and therefore, multiple variables need to be monitored simultaneously. Process industry provides typical examples where processes often are richly instrumented with sensors and/or people routinely collecting measurements on many process variables and finished product characteristics. The multiple measurements are typically cross-correlated because a few underlying events usually drive the process at any given time. Many of the measured variables are therefore just different reflections of the same underlying event; see, for example, Kourti and MacGregor.[1]

Sometimes, univariate control charts provide sufficient information, but when multiple variables require simultaneous monitoring, a univariate approach is normally neither effective nor efficient; see, for example, MacGregor.[2] An important advantage of multivariate control charts is that the performance of a process can be monitored using a single or a few multivariate charts instead of many univariate charts. Comprehensive overviews of the multivariate SPC (MSPC) methods can be found in Bersimis *et al.*[3] and Kourti.[4] The traditional MSPC charts include the Hotelling T^2,[5] multivariate cumulative sum,[6] and multivariate exponentially weighted moving average[7] control charts. Furthermore, applications of the latent variable techniques such as PCA and partial least squares for multivariate monitoring are commonly used in cases where a large number of highly correlated variables are of interest.

2. Motivation

The traditional SPC techniques assume that the data are independent in time. However, because of system dynamics and/or frequent sampling, successive observations will often be correlated; see Montgomery *et al.*[8] and Bisgaard and Kulahci.[9] This is particularly true

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for continuous processes. The issue of autocorrelation when using traditional univariate control charts has been previously discussed by many authors; see Johnson and Bagshaw,[10] Vasilopoulos and Stamboulis,[11] Alwan and Roberts,[12] Montgomery and Mastrangelo,[13] Wardell *et al.*,[14] Zhang[15] among others.

Two different general solutions to the problem emerge in the literature. The first is to adjust the control limits of the traditional charts, for example, by accounting for the autocorrelation in the estimation of the process standard deviation. The second solution is to fit a time series model to the data and then apply the traditional control charts to the residuals from the model—sometimes referred to as the 'Alwan and Roberts method'.[12] Zhang[15] shows that in the univariate case, the Shewhart chart based on residuals does not have the same properties as the individual Shewhart chart for independent data. While the univariate residuals chart has a higher probability in detecting a shift in the process mean in the first plotted point after the shift occurs, the detection ability at future points depends on the autocorrelation structure potentially liable to cause excessive delays in detecting an out-of-control signal.

The concern related to the impact of autocorrelation in the data extends to the multivariate case as well. For example, an important assumption for desired performance of the Hotelling T^2 control chart is that data are independent in time. However, in reality, data collected in time often exhibit various degrees of serial dependency (autocorrelation). It is to be expected that MSPC control charts that have been developed assuming independent observations should be affected by the violation of this assumption.

A detailed literature review of SPC techniques for autocorrelated univariate and multivariate data can be found in Psarakis and Papaleonida.[16] Kalgonda and Kulkarni[17] propose a control chart called the *Z* chart to monitor a process modeled by a first-order vector autoregressive model (VAR(1)). Pan and Jarrett[18–20] illustrate how multivariate Hotelling T^2 charts can be applied to residuals from state space models as well as from vector autoregressive (VAR) models. Essentially, this is an extension of Alwan and Robert's[12] approach to the multivariate case. Furthermore, Pan and Jarrett[21] show that the Hotelling T^2 chart based on residuals from a VAR model cannot distinguish between shifts in the mean and the variability. Instead, they propose using the Hotelling T^2 chart, the *W* chart, and the portmanteau test on residuals from a VAR model to monitor the variability of a multivariate autocorrelated process. Snoussi[22] proposes a technique for monitoring short-run autocorrelated data using a multivariate transformation technique on the residuals from a VAR(1) model.

In this article, our main goal is to provide a more detailed study of how autocorrelation affects the Hotelling T^2 control chart, which is the most widely used MSPC chart. The shift-detection ability of the Hotelling T^2 control chart for simulated data using a VAR(1) model is evaluated for different shifts in the mean vector. For a comparative study, three different approaches are considered: (i) ignoring the autocorrelation and using the raw data with theoretical upper control limits (UCLs); (ii) ignoring the autocorrelation and using the raw data with adjusted control limits calculated through simulations; and (iii) using the residuals from a multivariate time series model fitted to the raw data. We use the average run length (ARL) as the performance measure. Throughout the study, we focus on the Hotelling T^2 chart for individual observations.

3. The Hotelling T^2 control chart

A popular multivariate process monitoring chart for monitoring the mean vector of a process is the Hotelling T^2 control chart. The method assumes that the quality characteristics of interest are distributed according to a multivariate normal distribution. The multivariate normal distribution is an extension of the univariate normal distribution to a situation with multiple (k) variables (Montgomery [23]). The multivariate normal density function is:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{k/2}|\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}, \tag{1}$$

where $\boldsymbol{x} = [x_1, x_2, \ldots, x_k]'$ is a k-dimensional random vector, $\boldsymbol{\mu}$ is a $k \times 1$ vector with the means of the k variables and $\boldsymbol{\Sigma}$ is the $k \times k$ variance-covariance matrix:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{12} & \sigma_{22}^2 & \cdots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1k} & \sigma_{2k} & \cdots & \sigma_{kk}^2 \end{bmatrix} \tag{2}$$

where σ_{ii}^2 is the variance of the ith variable and σ_{ij} is the covariance between ith and jth variables.

There are two basic versions of the Hotelling T^2 chart; one for subgrouped data and one for individual observations; see Montgomery[23] for further details. In this study we are concerned with the T^2 statistic for individual observations which is:

$$T^2 = (\mathbf{x} - \overline{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x} - \overline{\mathbf{x}}) \tag{3}$$

where $\overline{\mathbf{x}}$ and $\boldsymbol{S}$ are the sample mean vector and sample covariance matrix, respectively.

It should be noted that the proper estimation of the covariance matrix is a concern even for independent data. Sullivan and Woodall[24] compare five different estimators. The traditional estimator which they denote as $\mathbf{S}_1$ is the sample covariance matrix.

$$\mathbf{S}_1 = \frac{1}{m-1} = \sum_{i=1}^{m}(x_i - \overline{x}) \times (x_i - \overline{x})' \tag{4}$$

Sullivan and Woodall[24] recommend using $\mathbf{S}_5$ for detecting a step or ramp shift for individual observations, which is based on the first difference of successive pairs of observations $\mathbf{v}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$ for $i = 1, \ldots, m-1$ and

$$\mathbf{S}_5 = \frac{1}{2}\frac{\mathbf{V}'\mathbf{V}}{(m-1)} \tag{5}$$

where $\mathbf{v}_i$ make up the rows of the $\mathbf{V}$ matrix.

However, Kulahci and Bisgaard[25] show that $\mathbf{S}_5$ underestimates the true covariance matrix compared with $\mathbf{S}_1$ for positive autocorrelation. In this study, we use both $\mathbf{S}_1$ and $\mathbf{S}_5$ to compare the results for all proposed approaches.

When using $\mathbf{S}_1$, Tracy *et al.*[26] give the Phase I UCL as

$$\mathrm{UCL}_{\mathbf{S}_1} = \frac{(m-1)^2}{m}\beta_{\alpha,k/2,(m-k-1)/2} \tag{6}$$

where $\beta_{\alpha,k/2,(m-k-1)/2}$ is the upper α percentile of the β distribution with $k/2$ and $(m\text{-}k\text{-}1)/2$ degrees of freedom, k is the number of variables, m is the number of samples (i.e., observations) in Phase I, and α is the acceptable false alarm rate. The Phase II UCL is given as

$$\mathrm{UCL}_{\mathbf{S}_1} = \frac{k(m+1)(m-1)}{m^2 - mk}F_{\alpha,k,m-k} \tag{7}$$

where $F_{\alpha,k,m-k}$ is the upper α percentile of the F distribution with k and $m\text{-}k$ degrees of freedom.

When $\mathbf{S}_5$ is used to estimate the covariance matrix, the approximate UCL for the T^2 statistic is provided by Sullivan and Woodall[24] and Mason and Young[27] as

$$\mathrm{UCL}_{\mathbf{S}_5} = \frac{(f-1)^2}{f}\beta_{\alpha,k/2,(f-k-1)/2} \tag{8}$$

where $f = 2(m-1)^2/(3m-4)$. The lower control limit is 0 in both Phase I and Phase II for both estimators.

4. Simulating autocorrelated multivariate data

To limit complexity, we use the VAR(1) model. Furthermore, we primarily focus on a process with two variables ($k = 2$) in our simulations. In Section 8, we consider a five-variable case for further generalization. The bivariate VAR(1) model with two quality characteristics, x_1 and x_2, can be expressed as

$$x_{1,t} = c_1 + \phi_{11}x_{1,t-1} + \phi_{12}x_{2,t-1} + \varepsilon_{1,t}$$
$$x_{2,t} = c_2 + \phi_{21}x_{1,t-1} + \phi_{22}x_{2,t-1} + \varepsilon_{2,t}$$

or

$$\mathbf{x}_t = \mathbf{c} + \mathbf{\Phi}\mathbf{x}_{t-1} + \boldsymbol{\varepsilon}_t \tag{9}$$

where $\mathbf{x}_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, $\mathbf{\Phi} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$, and $\boldsymbol{\varepsilon}_t = \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$

For the process to be stationary, the eigenvalues of the autocorrelation coefficient matrix $\mathbf{\Phi}$ should be less than one in absolute value; see Reinsel.[28] For a stationary VAR(1) process, the mean vector is

$$E(\mathbf{x}_t) = \boldsymbol{\mu} = (\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{c} \tag{10}$$

where $\boldsymbol{I}$ is the identity matrix. The covariance matrix of the VAR(1) process is then

$$\mathbf{\Gamma}(0) = \mathbf{\Phi}'\mathbf{\Gamma}(0)\mathbf{\Phi} + \mathbf{\Sigma} \tag{11}$$

where $\mathbf{\Gamma}(0)$ is the covariance matrix of the VAR(1) process (or the autocovariance matrix at lag 0) and $\mathbf{\Sigma}$ is the covariance matrix of the errors (Reinsel[28]). The covariance structure of the first-order autoregressive process is hence dependent on both the autocorrelation matrix $\mathbf{\Phi}$ and the covariance matrix $\mathbf{\Sigma}$ of the errors. For example, for

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.95 \end{bmatrix} \text{ and } \mathbf{\Sigma} = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}, \text{ we have}$$
$$\mathbf{\Gamma}(0) = \begin{bmatrix} 10.256 & 9.231 \\ 9.231 & 10.256 \end{bmatrix} \tag{12}$$

In this study, we investigate how changes to the autocorrelation matrix $\mathbf{\Phi}$ and the covariance matrix $\mathbf{\Sigma}$ of the errors affect the ARL of the Hotelling T^2 control chart using the three different methods. We generate different autocorrelation and cross-correlation structures by changing the elements of the $\mathbf{\Phi}$ and $\mathbf{\Sigma}$ matrices. Shifts in the mean vector are generated as multiples of the standard deviations of the corresponding variables.

5. Approaches for constructing Hotelling T^2 control chart

In the following text, we describe the three approaches that we consider in this study in more detail. The performance of the three approaches are based on simulations using $m = 500$ observations and $k = 2$ variables. That is, we assume that the mean vector and covariance matrices ($\mathbf{S}_1$ and $\mathbf{S}_5$) can be estimated from 500 observations from an in-control process in Phase I. These estimates are then used in the online monitoring stage in Phase II.

The in-control ARL (ARL$_0$) and the out-of-control ARL (ARL$_1$) for different shifts in the mean vector are evaluated. The theoretical UCLs are calculated based on a false alarm rate of 0.0027, which corresponds to an in-control ARL of approximately 370. We have also run simulations with $m = 100$, 1000, 5000, and 10000. The nominal value of 370 for ARL$_0$ is achieved for $m \geq 1000$. However, we used $m = 500$ in our simulations because we found that ARL$_0$ is fairly close to 370 for independent data, while 500 observations in Phase I are still feasible from a practical viewpoint. All simulations in this article are performed in *R* statistics software, and the *R* code for the simulations is available upon request.

To limit the number of cases to simulate, we begin by simplifying the bivariate VAR(1) model:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$$

with

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} \phi_{11} & 0 \\ 0 & \phi_{22} \end{bmatrix} \text{ with } \phi_{11}, \phi_{22} = \pm 0.25, \pm 0.5, \pm 0.75, \pm 0.95 \tag{13}$$

Furthermore we consider three covariance matrices for the errors:

1. Uncorrelated $\mathbf{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
2. Moderately correlated $\mathbf{\Sigma} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$
3. Highly correlated $\mathbf{\Sigma} = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$

5.1. *Theoretical upper control limit*

In this first approach, the autocorrelation is ignored, and the theoretical UCLs are calculated. This approach is expected to provide a benchmark to which the other two approaches are compared. For the first approach, we compare the results using $\mathbf{S}_1$ and $\mathbf{S}_5$.

5.2. *Adjusting the upper control limit through simulations*

In this approach, the UCL is adjusted through Monte Carlo simulation to yield the desired in-control ARL of 370, which corresponds to a false alarm rate of 0.0027.

When $m = 500$, not all simulated samples generate an out-of-control signal. To calculate the control limit corresponding to a desired in-control run length of 370, the following procedure is therefore employed. For a given false alarm rate α for each (independent) observation, the probability that there is at least one signal in a sample of m observations is

$$\alpha_{\text{OVERALL}} = 1 - (1 - \alpha)^m \tag{14}$$

Now, let N_S be the number of samples with one or more out-of-control signals among n simulated samples, and N_{NS} be the number of samples with no out-of-control signal such that $N_S + N_{NS} = n$. The overall false alarm rate can now be expressed as $N_S/n = \alpha_{\text{OVERALL}} =$

$1-(1-\alpha)^m$. Hence, $N_S = n(1-(1-\alpha)^m)$. To find the adjusted UCL value that corresponds to the given overall false alarm rate, we calculate the maximum T^2 value in each sample and rank them in descending order. The adjusted UCL is the N_s^{th} (rounded down to the nearest integer) maximum T^2 value in descending order.

It should be noted that the probability calculation in (14) assumes independent observations. For Hotelling T^2 charts, it can be shown that even for independent data, T^2 values are not independent; see Mason and Young.[27] However, for independent data, the dependence among T^2 values in Phase I is shown to be equal to $-1/(m-1)$ and can therefore be considered negligible for large m as in our case; see Mason and Young[27] and Sullivan and Woodall.[24] On the other hand, when the observations are autocorrelated, the dependence among T^2 values clearly cannot be ignored. We present this approach as an alternative to the first approach and assume that the autocorrelation is once again ignored, and as opposed to the first approach for which the theoretical UCL is used, the UCL is instead calculated using Monte Carlo simulation. As stated earlier, our main goal in this study is to present the repercussions of ignoring or simply not being aware of autocorrelation in the raw data when constructing Hotelling T^2 control charts.

Table I shows the adjusted UCLs for various autocorrelation values and covariance structures for the errors. The adjusted UCLs are based on $n = 100{,}000$ simulations of samples of size $m = 500$ and the false alarm rate $\alpha = 0.0027$. The theoretical UCL for independent data is 11.25 and 10.96 using $\mathbf{S}_1$ and $\mathbf{S}_5$, respectively.

From Table I, we can see that to achieve the specified overall false alarm rate, we need to decrease the UCL using $\mathbf{S}_1$ as the autocorrelation increases, both for positive and negative autocorrelation. The largest decrease in the UCL occurs when both variables exhibit a large magnitude of autocorrelation, $|\phi_{11}| = |\phi_{22}| = 0.95$. This suggests that if the autocorrelation in the data is ignored and the theoretical UCL is used, the resulting control chart will have larger than expected in-control ARLs. This may at first be interpreted as welcoming news, but it is expected to have an adverse effect on the shift-detection ability of the control chart because the UCL would be set too high compared to the UCL that will result in the nominal in-control ARL.

Table I also shows that for $\mathbf{S}_5$, the adjusted UCL increases with increasing positive autocorrelation and decreases with increasing negative autocorrelation. This is due to the fact that $\mathbf{S}_5$ is akin to the estimate of standard deviation based on moving ranges in univariate control charts. Successive differences for positive autocorrelation will tend to be small, whereas the situation is reversed for negative autocorrelation. Therefore for the former, the variation will be underestimated using successive differences, and for the latter, it will be overinflated. The changes in the adjusted UCL using $\mathbf{S}_5$ is rather dramatic suggesting that if the autocorrelation in the data is ignored and the theoretical UCL is used, the resulting control chart can have a very small in-control ARL for positive autocorrelation and a very large in-control ARL for negative autocorrelation depending on the magnitude of autocorrelation.

5.3. Monitor the residuals from a vector autoregressive moving average model

The third approach is an extension of Alwan and Robert's[12] method to the multivariate case. Essentially, the approach filters the data through an appropriate time series model and uses the residuals from the model for monitoring. Although the identification of a suitable time series model may be fairly straightforward in the univariate case, it is much more complicated in the multivariate case.

Consider a stationary vector autoregressive moving average model, VARMA (p,q) process for k variables as

$$\mathbf{x}_t = \mathbf{c} + \mathbf{\Phi}_1\mathbf{x}_{t-1} + \ldots + \mathbf{\Phi}_p\mathbf{x}_{t-p} + \boldsymbol{\theta}_1\boldsymbol{\varepsilon}_{t-1} + \ldots + \boldsymbol{\theta}_q\boldsymbol{\varepsilon}_{t-q} + \boldsymbol{\varepsilon}_t \tag{15}$$

where $\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2, \ldots, \boldsymbol{\Phi}_p$ are all $k \times k$ autoregressive parameter matrices, $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_q$ are moving average parameter matrices of order $k \times k$, $\mathbf{c}$ is a $k \times 1$ vector of constants, and ε_t is a $k \times 1$ vector of multivariate normally distributed uncorrelated error terms with mean zero and variance–covariance matrix $\boldsymbol{\Sigma}_{k \times k}$. In matrix notation (15) can be expressed as

$$\begin{aligned}
\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{k,t} \end{bmatrix} &= \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} + \begin{bmatrix} \phi_{11}^1 & \phi_{12}^1 & \cdots & \phi_{1k}^1 \\ \phi_{21}^1 & \phi_{22}^1 & \cdots & \phi_{2k}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k1}^1 & \phi_{k2}^1 & \cdots & \phi_{kk}^1 \end{bmatrix} \times \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \\ \vdots \\ x_{k,t-1} \end{bmatrix} + \ldots + \begin{bmatrix} \phi_{11}^p & \phi_{12}^p & \cdots & \phi_{1k}^p \\ \phi_{21}^p & \phi_{22}^p & \cdots & \phi_{2k}^p \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k1}^p & \phi_{k2}^p & \cdots & \phi_{kk}^p \end{bmatrix} \times \begin{bmatrix} x_{1,t-p} \\ x_{2,t-p} \\ \vdots \\ x_{k,t-p} \end{bmatrix} \ldots \\
&\ldots + \begin{bmatrix} \theta_{11}^1 & \theta_{12}^1 & \cdots & \theta_{1k}^1 \\ \theta_{21}^1 & \theta_{22}^1 & \cdots & \theta_{2k}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{k1}^1 & \theta_{k2}^1 & \cdots & \theta_{kk}^1 \end{bmatrix} \times \begin{bmatrix} \varepsilon_{1,t-1} \\ \varepsilon_{2,t-1} \\ \vdots \\ \varepsilon_{k,t-1} \end{bmatrix} + \ldots + \begin{bmatrix} \theta_{11}^q & \theta_{12}^q & \cdots & \theta_{1k}^q \\ \theta_{21}^q & \theta_{22}^q & \cdots & \theta_{2k}^q \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{k1}^q & \theta_{k2}^q & \cdots & \theta_{kk}^q \end{bmatrix} \times \begin{bmatrix} \varepsilon_{1,t-q} \\ \varepsilon_{2,t-q} \\ \vdots \\ \varepsilon_{k,t-q} \end{bmatrix} \ldots \\
&\ldots + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \vdots \\ \varepsilon_{k,t} \end{bmatrix}.
\end{aligned} \tag{16}$$

It is evident from (16) that the number of parameters to estimate in the VARMA(p,q) model quickly becomes overwhelmingly large with increasing orders of p and q and can cause estimation issues during the model fitting stage. Some form of simplification or approximation is therefore usually necessary. In this study, the ARL performance of Hotelling T^2 charts based on residuals from a VAR(1) model is calculated assuming that a perfect model with the known parameters is available as in the analysis of univariate control charts with autocorrelated data by Zhang.[15] This is expected to provide the 'best case scenario' for this approach.

6. Performance of the Hotelling T^2 control chart for different autocorrelation and cross-correlation structures for two variables

6.1. ARL_0 with autocorrelation and cross-correlation for two variables

We first consider the in-control Phase II performance of the T^2 control chart for the three approaches: the first approach for which the autocorrelation is ignored and the theoretical UCLs are obtained from Equations (7) and (8), the second approach using the adjusted UCLs from Table I, and finally, the residuals-based approach using the theoretical UCLs. Again, it should be noted that for the last approach, even though we only consider two variables in this study to avoid additional complications due to the estimated parameters, we still use the true parameter values to obtain the residuals.

The in-control ARLs in Phase II monitoring for various scenarios for the autocorrelation parameters and error covariance structures are provided in Tables IIA and IIB, where there is no evidence suggesting a systematic effect of the level of cross-correlation between the errors on the ARL_0 values. The first approach applying the Hotelling T^2 chart to raw autocorrelated data, using $\mathbf{S}_1$, and the theoretical UCL in Equation (7) results in substantially higher ARL_0 values than what is to be expected with a UCL obtained for a false alarm rate of 0.0027. For raw data using $\mathbf{S}_5$ and theoretical limits, the ARL_0 values are dramatically decreased with increasing magnitude of positive autocorrelation and dramatically increased with increasing magnitude of negative autocorrelation. Hence, we conclude that $\mathbf{S}_5$ is clearly more sensitive to autocorrelation than $\mathbf{S}_1$ and results in unacceptably many false alarms for positively autocorrelated data and vice versa for negative autocorrelation.

The results in Tables IIA and IIB also show that the second approach of adjusting the UCLs does a fairly good job of adjusting the ARL_0 values closer to the nominal value of 370. As expected, the adjustment is not as effective for high positive autocorrelation, while it performs somewhat better for high negative autocorrelation. The adjustment of the UCL corresponding to $\mathbf{S}_5$ seems to perform clearly worse than for $\mathbf{S}_1$ for positive autocorrelation and highly correlated errors.

Applying the Hotelling T^2 chart on the residuals from the VAR(1) model results in stable ARL_0 values across all autocorrelation cases. The ARL_0 values are fairly close to the nominal value of 370, although for $\mathbf{S}_1$, the average ARL_0 value lies slightly above 370, and for $\mathbf{S}_5$, the average lies somewhat below 370. Therefore, we should expect that the residuals-based approach using $\mathbf{S}_5$ and theoretical UCLs will produce slightly lower ARL_1 values as well.

As discussed in the previous section, high ARL_0 values may not at first be seen as problematic; however, as it will be shown in the next section, it can have dire repercussions in detecting a shift in the mean in due time.

6.2. Detecting shifts in the means of two variables

In this section, we consider the shift-detection ability through the ARL_1 performance of the Hotelling T^2 chart for individual observations for autocorrelated data. Shifts in the mean of the two variables, δ_{x1} and δ_{x2}, are generated as multiples of their standard deviations. Note that the true standard deviations of the variables are dependent on both $\mathbf{\Phi}$ and $\mathbf{\Sigma}$. Tables III–VI present ARL_1 values for different cases. We generate shifts in only one variable, in both variables, and with different autocorrelation structures. The covariance between the error terms is chosen to be 0.9 in all cases.

Tables III and IV show the shift-detection ability when there is a shift in only one variable. Using the first approach and $\mathbf{S}_1$, the ARL_1 values increase with larger magnitude of autocorrelation. For the first approach using $\mathbf{S}_5$, the ARL_1 values are low for positive autocorrelation and high for negative autocorrelation, which is expected from the results in Tables IIA and IIB. The performance of the second approach with adjusted UCLs is better than of the first approach. Overall, the shift-detection ability is slightly better for adjusted UCLs using $\mathbf{S}_1$. The residuals-based approach performs best overall especially for negative autocorrelation. Although the results are comparable for the residuals-based approach for both covariance matrix estimates, using $\mathbf{S}_5$ results in slightly lower ARL_1 values for small shift sizes. This is again expected based on the results for ARL_0 in Tables IIA and IIB.

As positive autocorrelation seems to pose a bigger challenge also for the residuals-based approach, Tables V and VI show the results from further simulations of different shift scenarios for positive autocorrelation only.

Comparing the results in Table V with Table III, it is interesting to note that although the residuals-based approach can be argued to have the best overall performance in Table V, it is not as effective when both variables have equal shift sizes.

From Table VI, where variables have different shift sizes, we note that the second approach with adjusted UCLs performs worse especially using $\mathbf{S}_5$ compared with the results in Tables III–V. Again, the residuals-based approach has the best overall performance. However, we note that for some combinations of the autocorrelation coefficients in $\mathbf{\Phi}$ and for smaller shifts, the ARL_1 values are actually lower for the first approach with theoretical limits.

From Tables III–VI, we conclude that, as expected, the first approach—the Hotelling T^2 chart based on raw autocorrelated data, $\mathbf{S}_1$, and theoretical UCL—performs the worst with substantially higher ARL_1 values than the other two methods. Comparing the results in Tables III and IV, it is also clear that the worst case is for positive autocorrelation, which results in higher ARL_1 values for all methods compared with negative autocorrelation. This is in line with the conclusions made by Zhang[15] for the univariate control charts. The differences among the three approaches are expectedly more significant for small shift sizes. The second approach applying the Hotelling T^2 chart on raw data but with an adjusted UCL performs better than the first approach, especially for cases with high autocorrelation. The Hotelling T^2 chart based on the residuals from the VAR(1) model clearly outperforms the other approaches when there is a shift in only one variable, especially for negative autocorrelation. However, it should be once again noted that the perfect VAR(1) model fit is assumed in obtaining the residuals. The results for the residuals-based approach should be expected to differ when estimated parameters are used.

The results for the cases with equal shifts in both variables given in Table V are more mixed. On average, the Hotelling T^2 chart based on the residuals from the VAR(1) model has the lowest ARL_1 values for all tested shift combinations but not for all cases of the autocorrelation structure. For equal shift sizes and when $\phi_{11} = \phi_{22}$, there is a visible trend that the ARL_1 values increase using the residuals from the VAR(1) model (Table V). In contrast, when the autocorrelation in one of the variables is high and the autocorrelation in the other variable is low, the Hotelling T^2 chart based on the residuals from the VAR(1) model catches the shift substantially faster than the other methods.

The special case for which $\phi_{11} = \phi_{22} = \phi$ presents an interesting pattern. Note that for this case, we have

$$\begin{aligned}\mathbf{\Gamma}(0) &= \mathbf{\Phi}'\mathbf{\Gamma}(0)\mathbf{\Phi}' + \mathbf{\Sigma}\\ &= \phi\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\mathbf{\Gamma}(0)\phi\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \mathbf{\Sigma}\\ &= \phi^2\mathbf{\Gamma}(0) + \mathbf{\Sigma}\\ &\Rightarrow \mathbf{\Gamma}(0) = \left(1 - \phi^2\right)^{-1}\mathbf{\Sigma}\end{aligned} \tag{17}$$

We can see that in this case, the true covariance matrix is simply the error covariance matrix adjusted for the autocorrelation in both variables.

Comparing the results for $\mathbf{S}_1$ and $\mathbf{S}_5$, we conclude that for the first approach using raw autocorrelated data $\mathbf{S}_5$ is clearly an inappropriate estimate. In the second approach, with adjusted UCLs, $\mathbf{S}_5$ cannot be recommended either because it performs in an unpredictable manner suggesting that the adjustment of the UCL works poorly for $\mathbf{S}_5$. However, in the residuals-based approach using $\mathbf{S}_5$ results in slightly faster shift detection albeit also in lower ARL_0 values.

7. Examples with a more complicated $\mathbf{\Phi}$ matrix

The results in Section 6 were based on simulations with a diagonal $\mathbf{\Phi}$ matrix. To explore more complicated $\mathbf{\Phi}$ matrix structures, we test two additional scenarios for the bivariate VAR(1) model. In the simulations, we assume highly correlated errors:

$$\mathbf{\Sigma} = \begin{bmatrix}1 & 0.9\\ 0.9 & 1\end{bmatrix}$$

and two different $\mathbf{\Phi}$ matrices; the first with one off-diagonal element and the second with two off-diagonal elements as:

1. $\mathbf{\Phi} = \begin{bmatrix}0.25 & 0.25\\ 0 & 0.25\end{bmatrix}$
2. $\mathbf{\Phi} = \begin{bmatrix}0.2 & 0.5\\ 0.5 & 0.2\end{bmatrix}$

Here, we choose the $\mathbf{\Phi}$ matrices to have non-zero eigenvalues. Also, all absolute eigenvalues of the autocorrelation coefficient matrices are less than one so that the resulting VAR(1) processes are stationary.

In the second approach, we adjust the UCLs through simulation as described earlier. Table VII presents the ARL_0 and ARL_1 values of the three approaches for different shift combinations in the two variables.

From Table VII, we conclude that the ARL_0 values are fairly close to the nominal value of 370 except for the first approach using $\mathbf{S}_5$, which yields low ARL_0 values. We again note that for the residuals-based approach using $\mathbf{S}_1$, the average ARL_0 values lie above the nominal value, while the opposite is true when using $\mathbf{S}_5$.

The results for the ARL_1 values are more mixed. The second approach using $\mathbf{S}_1$ performs slightly better than the first approach for the tested cases. However, once again, the second approach using $\mathbf{S}_5$ performs in an unpredictable manner, producing lower ARL_1 values for some cases while higher ARL_1 values for most cases compared to the second approach using $\mathbf{S}_1$.

The difference among the methods is most apparent for the second $\mathbf{\Phi}$ matrix and shifts in only one variable. The Hotelling T^2 chart based on the residuals from the VAR(1) model performs slightly better than the second approach when only one of the variables has a shift in the mean. However, we once again observe that when both variables have equal shifts, the residuals-based approach in some cases performs worse than the first approach using $\mathbf{S}_1$. Using $\mathbf{S}_5$, the residuals-based approach results in slightly lower ARL_1 values but then so are the ARL_0 values. Overall, the performance of the residuals-based approach is best except for cases when both variables have equal shifts for which the second approach has the lowest ARL_1 values.

Table I. The adjusted UCL for the different cases of the autocorrelation structure, three different Σ matrices, and the two covariance matrix estimates. The adjusted UCLs are based on 100,000 simulations in each sub-case. For comparison, the theoretical UCL is 11.25 using $\mathbf{S}_1$ and 10.96 using $\mathbf{S}_5$. The results are based on 100,000 simulations for each case

		$\Sigma=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$		$\Sigma=\begin{bmatrix}1 & .5\\ .5 & 1\end{bmatrix}$		$\Sigma=\begin{bmatrix}1 & .9\\ .9 & 1\end{bmatrix}$				$\Sigma=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$		$\Sigma=\begin{bmatrix}1 & .5\\ .5 & 1\end{bmatrix}$		$\Sigma=\begin{bmatrix}1 & .9\\ .9 & 1\end{bmatrix}$	
ϕ_{11}	ϕ_{22}	$\mathbf{S}_1$	$\mathbf{S}_5$	$\mathbf{S}_1$	$\mathbf{S}_5$	$\mathbf{S}_1$	$\mathbf{S}_5$	ϕ_{11}	ϕ_{22}	$\mathbf{S}_1$	$\mathbf{S}_5$	$\mathbf{S}_1$	$\mathbf{S}_5$	$\mathbf{S}_1$	$\mathbf{S}_5$
0	0	11.81	11.85	11.81	11.83	11.80	11.83		0	11.80	11.84	11.81	11.85	11.80	11.83
	0.25	11.79	14.06	11.80	14.54	11.74	17.14		−0.25	11.80	10.79	11.79	11.02	11.73	12.22
	0.5	11.77	19.23	11.73	21.55	11.54	28.61		−0.5	11.76	10.24	11.71	10.70	11.54	11.93
	0.75	11.61	34.52	11.52	41.84	11.30	56.33		−0.75	11.62	9.93	11.50	10.32	11.32	10.85
	0.95	11.11	119.69	11.03	153.50	10.97	204.17		−0.95	11.08	9.75	11.02	9.83	10.96	9.92
0.25	0	11.80	14.06	11.78	14.53	11.75	17.13	−0.25	0	11.80	10.79	11.80	11.02	11.74	12.23
	0.25	11.81	15.77	11.80	15.76	11.80	15.76		−0.25	11.80	9.47	11.80	9.47	11.78	9.46
	0.5	11.75	20.21	11.74	21.47	11.63	27.9		−0.5	11.75	8.73	11.74	8.86	11.64	9.43
	0.75	11.59	35.12	11.51	42.56	11.30	64.45		−0.75	11.59	8.29	11.52	8.49	11.30	8.86
	0.95	11.08	120.16	11.00	163.27	10.91	248.51		−0.95	11.06	8.02	10.99	8.07	10.91	8.13
0.5	0	11.77	19.23	11.74	21.54	11.53	28.63	−0.5	0	11.77	10.23	11.72	10.72	11.53	11.93
	0.25	11.75	20.23	11.73	21.46	11.64	27.96		−0.25	11.75	8.74	11.74	8.86	11.63	9.42
	0.5	11.70	23.35	11.71	23.35	11.70	23.37		−0.5	11.71	7.84	11.69	7.83	11.70	7.84
	0.75	11.51	36.54	11.46	41.45	11.28	65.31		−0.75	11.51	7.23	11.46	7.29	11.28	7.50
	0.95	10.95	121.21	10.87	171.20	10.77	312.34		−0.95	10.93	6.83	10.85	6.86	10.75	6.89
0.75	0	11.61	34.58	11.51	41.79	11.32	56.41	−0.75	0	11.62	9.93	11.51	10.31	11.29	10.83
	0.25	11.60	35.10	11.52	42.54	11.29	64.46		−0.25	11.61	8.29	11.51	8.49	11.30	8.87
	0.5	11.53	36.51	11.48	41.46	11.28	65.38		−0.5	11.52	7.24	11.48	7.30	11.28	7.50
	0.75	11.29	44.65	11.27	44.51	11.29	44.58		−0.75	11.27	6.47	11.27	6.48	11.29	6.48
	0.95	10.59	123.69	10.51	168.38	10.35	376.96		−0.95	10.55	5.85	10.48	5.86	10.34	5.85
0.95	0	11.10	119.81	11.03	153.51	10.96	205.04	−0.95	0	11.08	9.74	11.01	9.82	10.95	9.91
	0.25	11.07	120.30	10.99	163.06	10.92	248.66		−0.25	11.06	8.02	10.98	8.07	10.90	8.12
	0.5	10.96	121.03	10.87	171.33	10.75	312.82		−0.5	10.93	6.82	10.84	6.86	10.74	6.89
	0.75	10.58	123.67	10.49	168.56	10.36	376.63		−0.75	10.55	5.86	10.48	5.86	10.35	5.86
	0.95	9.38	168.36	9.38	168.62	9.40	168.76		−0.95	9.28	4.79	9.28	4.79	9.27	4.78

Table IIA. The in-control ARL (ARL_0) values in Phase II for the three methods and different combinations of the autocorrelation parameters. The ARL_0 values are based on 1000 simulations in each case

		$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$						$\Sigma = \begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix}$						$\Sigma = \begin{bmatrix} 1 & .9 \\ .9 & 1 \end{bmatrix}$					
		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals	
ϕ_{11}	ϕ_{22}	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5
0	0	382	325	354	349	382	325	387	329	352	358	387	329	375	324	343	350	375	324
	0.25	397	144	361	372	389	322	389	130	352	417	404	330	423	86	359	599	432	349
	0.5	391	44	350	388	404	331	420	39	369	566	405	349	425	25	344	626	385	326
	0.75	408	14	339	338	391	340	429	14	336	543	394	331	509	10	361	491	402	328
	0.95	455	5	307	276	399	337	469	5	308	414	386	330	586	4	354	376	384	327
0.25	0	387	133	358	380	394	330	395	128	357	422	392	326	384	89	343	569	388	325
	0.25	380	79	342	357	404	337	384	76	347	350	380	319	392	80	351	361	414	339
	0.5	374	34	333	343	392	329	406	32	347	449	414	344	421	25	351	710	399	342
	0.75	420	13	332	351	394	318	423	12	339	608	415	358	458	9	340	649	394	328
	0.95	488	5	316	295	400	313	513	4	322	504	398	314	562	3	338	405	400	331
0.5	0	413	44	366	367	398	331	405	44	359	502	428	359	452	24	347	671	402	333
	0.25	394	34	352	377	392	336	408	36	363	472	415	338	406	25	338	693	373	310
	0.5	398	22	349	347	419	345	392	22	345	341	394	317	410	21	356	382	402	334
	0.75	414	10	327	343	396	328	424	10	327	508	389	328	492	8	353	911	398	343
	0.95	490	4	315	274	383	321	512	4	323	516	382	307	601	3	324	487	406	338
0.75	0	412	15	352	339	398	325	434	13	341	567	420	355	512	9	352	543	398	331
	0.25	403	14	326	348	413	333	461	12	367	650	406	334	478	9	320	659	397	312
	0.5	405	10	316	339	409	326	427	10	333	521	380	320	494	8	343	898	386	331
	0.75	485	7	337	342	404	331	462	6	341	321	401	339	471	7	328	336	405	336
	0.95	580	3	303	268	399	326	519	3	280	497	402	326	674	2	337	713	394	337
0.95	0	444	5	307	282	382	313	468	5	294	396	408	335	549	4	318	363	381	317
	0.25	478	5	299	272	392	335	489	5	326	466	375	319	532	3	329	415	406	338
	0.5	476	4	289	282	383	305	539	4	303	533	387	327	587	3	334	494	413	331
	0.75	541	3	270	265	407	332	542	3	294	527	415	344	704	3	332	668	406	325
	0.95	688	2	256	251	416	342	638	2	259	251	375	324	635	2	249	258	404	334

Remark: The ARLs are reported in three columns and for both covariance matrix estimators. 'Raw' denotes the ARL values from the first approach with theoretical UCLs for the raw data. 'Adj. UCL' denotes the ARL values from the approach with adjusted UCLs. 'Residuals' reports the ARL values from residuals-based approach. This holds for all tables in this article.

Table IIB. The ARL_0 values are based on 1000 simulations in each case

		$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$						$\Sigma = \begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix}$						$\Sigma = \begin{bmatrix} 1 & .9 \\ .9 & 1 \end{bmatrix}$					
		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals	
ϕ_{11}	ϕ_{22}	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5
0		399	325	357	55	399	325	398	322	363	358	398	322	378	322	343	351	378	322
	−0.25	376	560	343	341	375	309	400	576	361	413	393	334	419	371	357	477	401	318
	−0.5	401	754	359	377	396	338	414	739	362	430	391	330	447	388	347	443	395	328
	−0.75	454	937	376	387	385	322	476	900	368	435	398	338	569	672	402	455	423	344
	−0.95	567	1041	384	367	378	327	605	1031	348	361	422	338	670	1051	411	410	405	345
−0.25	0	391	593	351	368	408	341	397	586	362	416	406	337	411	379	358	483	417	342
	−0.25	395	1488	354	359	400	340	409	1401	369	376	408	333	385	1343	343	348	377	327
	−0.5	405	2694	354	373	389	320	423	2481	369	372	389	315	410	1754	335	429	379	311
	−0.75	457	3444	365	362	380	317	454	3308	357	415	391	339	544	2433	390	419	419	346
	0.95	611	3952	376	366	391	326	672	4464	413	406	399	324	673	4025	377	376	399	334
−0.5	0	410	828	360	369	399	344	406	714	350	431	403	339	451	423	345	488	389	323
	−0.25	401	2630	359	361	398	338	391	2388	344	383	376	303	430	1760	359	411	363	312
	−0.5	438	5660	363	369	418	333	408	5810	353	354	394	327	418	6260	365	370	402	336
	−0.75	449	10720	345	357	413	341	471	9745	345	371	384	330	541	8946	380	412	400	328
	−0.95	658	13413	378	369	404	348	671	12871	361	373	403	326	742	13489	389	420	85	329
−0.75	0	438	1000	364	374	411	325	473	904	374	415	400	326	513	633	368	417	404	337
	−0.25	410	3400	332	332	397	325	468	3326	366	433	404	343	528	2464	368	432	411	338
	−0.5	465	10608	372	372	406	325	496	10036	386	399	393	328	488	8239	344	385	399	331
	−0.75	539	20079	388	395	398	333	488	17759	351	348	385	320	530	18008	381	385	412	329
	−0.95	690	21482	351	352	384	329	688	20625	356	363	401	335	956	23055	414	411	412	333
−0.95	0	592	1030	387	367	405	333	594	956	378	366	392	312	598	995	365	369	40x8	344
	−0.25	572	4354	369	373	407	336	635	4070	362	390	398	322	638	3938	376	387	393	320
	−0.5	625	13216	386	400	395	338	654	12618	389	396	399	329	771	14400	410	450	398	325
	−0.75	702	20461	332	337	379	305	735	21111	372	383	382	315	901	22828	411	448	411	334
	−0.95	1313	21404	419	427	405	333	1238	20790	375	381	390	324	1508	22253	442	448	419	351

Table III. The out-of-control ARL (ARL_1) performance in Phase II for different shifts in x_1 only, with $\mathrm{cov}(\varepsilon_{x_1}, \varepsilon_{x_2}) = 0.9$, and for positive autocorrelation. The ARL_1 values are based on 1000 simulations in each case

		$\delta_{x_1} = 0.5, \delta_{x_2} = 0$						$\delta_{x_1} = 1.0, \delta_{x_2} = 0$						$\delta_{x_1} = 1.5, \delta_{x_2} = 0$						$\delta_{x_1} = 2.0, \delta_{x2} = 0$						$\delta_{x_1} = 3.0, \delta_{x_2} = 0$					
		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals	
ϕ_{11}	ϕ_{22}	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5
0	0	52	45	48	48	52	45	6	6	6	6	6	6	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1
	0.25	76	30	67	158	51	45	9	7	9	23	6	6	3	2	2	5	2	2	1	1	1	2	1	1	1	1	1	1	1	1
	0.5	125	19	104	352	54	47	21	8	19	128	6	6	5	3	5	38	2	2	2	2	2	13	1	1	1	1	1	2	1	1
	0.75	204	9	148	460	52	46	47	7	36	295	6	6	13	4	11	177	2	2	5	3	4	95	1	1	2	1	1	31	1	1
	0.95	243	4	156	342	50	42	73	3	51	349	6	6	25	3	18	299	2	2	10	2	7	291	1	1	2	1	2	239	1	1
0.25	0	73	22	67	86	93	79	10	5	10	12	13	12	3	2	3	3	2	2	1	1	1	1	1	1	1	1	1	1	1	1
	0.25	54	18	50	51	93	81	7	4	7	7	13	11	2	1	2	2	3	2	1	1	1	1	1	1	1	1	1	1	1	1
	0.5	84	14	74	236	94	80	11	4	10	44	13	12	3	2	3	10	2	2	1		1	3	1	1	1	1	1	1	1	1
	0.75	157	8	123	494	92	79	32	4	26	260	13	11	9	3	8	116	3	2	4	2	3	58	1	1	1	1	1	12	1	1
	0.95	250	3	156	390	95	81	77	3	51	397	13	12	24	2	17	349	3	2	10	2	7	308	1	1	2	1	2	228	1	1
0.5	0	143	13	114	154	140	122	29	5	25	32	28	24	8	2	7	9	4	3	3	1	3	3	1	1	1	1	1	1	1	1
	0.25	87	11	77	118	154	130	14	3	13	19	27	23	4	1	3	5	3	3	2	1	1	2	1	1	1	1	1	1	1	1
	0.5	64	8	57	60	152	126	8	2	8	8	27	23	2	1	2	2	4	3	1	1	1	1	1	1	1	1	1	1	1	1
	0.75	114	6	86	455	138	121	21	3	17	137	28	25	5	1	4	45	3	3	2	1	2	14	1	1	1	1	1	2	1	1
	0.95	251	3	158	465	158	133	75	2	48	431	28	25	25	2	17	378	4	3	10	1	7	313	1	1	3	1	2	185	1	1
0.75	0	254	8	190	221	249	199	84	4	63	63	45	38	26	2	21	23	3	2	10	1	9	9	1	1	2	1	2	2	1	1
	0.25	228	7	170	242	239	200	56	3	44	63	44	38	17	2	15	21	3	3	6	1	5	7	1	1	1	1	1	2	1	1
	0.5	152	5	115	214	233	197	29	2	23	43	42	34	8	1	7	11	2	2	3	1	2	4	1	1	1	1	1	1	1	1
	0.75	84	4	66	68	239	198	13	2	11	11	46	38	3	1	3	3	2	2	1	1	1	1	1	1	1	1	1	1	1	1
	0.95	275	2	143	655	238	196	76	2	43	484	50	40	21	1	14	346	2	2	8	1	6	224	1	1	2	1	2	84	1	1
0.95	0	432	3	277	268	144	112	220	2	155	132	1	1	111	2	76	54	1	1	45	1	33	25	1	1	10	1	7	4	1	1
	0.25	394	3	244	272	143	118	210	2	140	134	1	1	90	2	66	49	1	1	42	1	31	23	1	1	9	1	7	5	1	1
	0.5	430	3	242	322	128	103	192	2	129	141	1	1	85	2	59	61	1	1	33	1	24	22	1	1	7	1	5	5	1	1
	0.75	472	2	240	443	156	120	154	2	93	157	1	1	58	1	37	54	1	1	22	1	14	21	1	1	3	1	2	3	1	1
	0.95	428	2	108	103	139	113	38	1	20	20	1	1	7	1	4	3	1	1	2	1	1	1	1	1	1	1	1	1	1	1

Table IV. The out-of-control ARL (ARL_1) performance in Phase II for different shifts in x_1 only, with $cov(\varepsilon_{x_1}, \varepsilon_{x_2}) = 0.9$, and negative autocorrelation. The ARL_1 values are based on 1000 simulations in each case.

		$\delta_{x_1} = 0.5, \delta_{x_2} = 0$						$\delta_{x_1} = 1.0, \delta_{x_2} = 0$						$\delta_{x_1} = 1.5, \delta_{x_2} = 0$						$\delta_{x_1} = 2.0, \delta_{x_2} = 0$						$\delta_{x_1} = 3.0, \delta_{x_2} = 0$					
		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals	
ϕ_{11}	ϕ_{22}	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5
0	0	51	45	47	48	51	45	6	6	6	6	6	6	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1
	−0.25	73	64	65	78	53	47	10	8	9	10	6	6	3	2	2	3	2	2	1	1	1	1	1	1	1	1	1	1	1	1
	−0.5	126	103	100	113	53	48	23	18	20	19	6	6	6	5	5	5	2	2	2	2	2	2	1	1	1	1	1	1	1	1
	−0.75	212	210	156	147	51	45	48	43	38	33	6	5	14	13	12	10	2	2	5	5	5	4	1	1	2	2	1	1	1	1
	−0.95	280	340	182	155	53	46	82	89	56	45	6	6	27	26	19	15	2	2	10	10	8	7	1	1	2	2	2	2	1	1
−0.25	0	73	97	64	119	27	24	10	14	9	17	3	3	2	3	2	4	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	−0.25	54	138	50	53	26	24	6	11	6	6	3	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	−0.5	79	209	68	72	27	23	10	19	9	10	3	3	3	4	2	3	1	1	1	2	1	1	1	1	1	1	1	1	1	1
	−0.75	159	485	119	120	26	24	32	73	25	24	3	3	8	16	7	6	2	1	3	5	3	3	1	1	1	1	1	1	1	1
	−0.95	269	1258	66	156	28	25	76	246	50	42	3	3	24	57	18	15	1	1	8	18	6	5	1	1	2	3	2	2	1	1
−0.5	0	125	190	102	212	12	10	22	45	18	49	2	2	6	12	5	13	1	1	2	4	2	4	1	1	1	1	1	1	1	1
	0.25	81	369	71	113	11	10	11	37	10	16	2	2	2	6	2	3	1	1	1	2	1	1	1	1	1	1	1	1	1	1
	−0.5	51	423	47	46	11	10	6	25	6	6	2	2	2	3	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	−0.75	115	1079	83	82	11	10	17	92	14	13	2	2	4	12	3	3	1	1	2	3	1	1	1	1	1	1	1	1	1	1
	0.95	270	4051	163	156	1	10	69	728	45	41	2	2	21	123	14	13	1	1	7	32	5	5	1	1	2	4	2	1	1	1
−0.75	0	216	428	156	272	3	3	56	159	42	110	1	1	16	57	13	42	1	1	5	21	4	16	1	1	1	3	1	3	1	1
	−0.25	175	1057	131	224	3	3	40	81	32	3	1	1	9	55	8	6	1	1	3	14	3	5	1	1	1	2	1	1	1	1
	−0.5	119	1871	90	120	3	3	19	198	15	21	1	1	4	25	3	4	1	1	1	5	1	2	1	1	1	1	1	1	1	1
	0.75	65	1684	49	51	3	3	7	72	5	6	1	1	2	7	1	2	1	1	1	2	1	1	1	1	1	1	1	1	1	1
	0.95	264	9210	129	130	3	3	58	1277	33	31	1	1	14	179	9	9	1	1	4	40	3	3	1	1	1	3	1	1	1	1
−0.95	0	374	810	242	327	1	1	162	473	106	198	1	1	62	249	40	103	1	1	21	114	15	52	1	1	3	25	3	11	1	1
	−0.25	384	2796	228	275	1	1	172	1476	108	165	1	1	57	578	38	74	1	1	18	204	13	30	1	1	2	28	2	5	1	1
	−0.5	436	8274	244	276	1	1	161	3246	88	129	1	1	48	975	30	45	1	1	15	251	10	17	1	1	2	30	2	2	1	1
	−0.75	382	14273	193	209	1	1	109	4685	62	71	1	1	32	970	17	20	1	1	9	182	5	6	1	1	1	13	1	1	1	1
	−0.95	157	6294	54	56	1	1	15	532	6	6	1	1	2	42	1	1	1	1	1	4	1	1	1	1	1	1	1	1	1	1

Table V. The out-of-control ARL (ARL_1) performance in Phase II for different cases of equal shifts in both variables, with $\text{cov}(\varepsilon_{x_1},, \varepsilon_{x_2}) = 0.9$, and positive autocorrelation. The ARL_1 values are based on 1000 simulations in each case

		$\delta_{x_1}=0.5, \delta_{x_2}=0.5$						$\delta_{x_1}=1.0, \delta_{x_2}=1.0$						$\delta_{x_1}=2.0, \delta_{x_2}=2.0$						$\delta_{x_1}=3.0, \delta_{x_2}=3.0$					
		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals	
ϕ_{11}	ϕ_{22}	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5
0	0	217	183	200	201	217	183	70	59	64	63	70	59	9	8	9	9	9	8	2	2	2	2	2	2
	0.25	228	58	206	326	213	178	71	24	65	107	67	60	9	5	9	15	8	7	2	2	2	3	2	2
	0.5	239	19	197	359	174	149	74	11	61	130	41	36	9	3	8	23	5	5	3	2	2	6	2	2
	0.75	246	8	181	326	117	105	72	6	56	137	22	20	10	2	8	28	2	2	2	1	2	7	1	1
	0.95	260	3	169	269	56	47	75	3	52	155	1	1	8	1	6	38	1	1	2	1	1	9	1	1
0.25	0	213	56	187	321	205	178	74	25	65	109	67	58	9	5	9	13	8	8	2	2	2	4	2	2
	0.25	207	49	188	196	258	217	74	23	69	69	120	103	10	5	9	9	19	17	3	2	3	3	4	4
	0.5	229	19	191	367	261	215	84	11	72	155	112	98	11	3	10	22	16	15	3	1	3	5	3	3
	0.75	271	8	203	459	208	182	83	5	64	174	58	50	11	2	10	34	5	5	3	1	2	9	1	1
	0.95	252	3	159	300	94	79	84	2	55	175	2	2	9	1	7	41	1	1	2	1	2	10	1	1
0.5	0	242	19	195	394	172	142	73	12	63	138	46	40	10	3	9	22	5	5	3	1	2	5	2	2
	0.25	228	19	192	439	266	226	76	11	65	153	104	88	11	3	10	22	17	15	3	1	3	5	3	3
	0.5	213	16	187	193	289	245	81	9	73	74	180	156	12	3	11	11	42	37	3	1	3	3	7	6
	0.75	276	7	202	568	316	263	98	5	74	215	147	122	15	2	12	40	20	18	3	1	3	10	2	2
	0.95	297	3	185	403	182	150	106	2	69	228	4	4	13	1	9	57	1	1	3	1	2	14	1	1
0.75	0	249	9	194	335	117	98	74	6	58	140	23	20	10	2	8	28	2	2	2	1	2	8	1	1
	0.25	254	7	196	424	201	170	91	5	67	192	60	51	11	2	9	36	5	5	3	1	2	9	1	1
	0.5	257	7	190	559	297	248	95	5	71	233	141	120	14	2	12	40	21	18	3	1	3	10	1	1
	0.75	297	6	219	214	369	304	107	4	84	87	257	212	19	2	16	16	75	64	5	1	4	4	6	5
	0.95	402	2	208	562	274	234	153	2	89	315	24	21	22	1	14	82	1	1	4	1	3	21	1	1
0.95	0	252	3	171	270	57	48	69	3	46	147	1	1	8	1	6	38	1	1	2	1	2	8	1	1
	0.25	271	3	165	346	101	86	80	2	55	175	2	2	9	1	7	41	1	1	2	1	2	9	1	1
	0.5	295	3	173	370	184	154	100	2	67	223	4	3	12	1	9	54	1	1	2	1	2	13	1	1
	0.75	410	2	210	556	284	228	143	2	83	321	27	21	22	1	15	87	1	1	4	1	3	19	1	1
	0.95	543	2	200	202	383	323	273	2	119	116	186	155	59	1	31	30	1	1	12	1	6	6	1	1

Table VI. The out-of-control ARL (ARL_1) performance in Phase II for different cases of unequal shifts in both variables, with $\mathrm{cov}(\varepsilon_{x_1}, \varepsilon_{x_2}) = 0.9$, and positive autocorrelation. The ARL_1 values are based on 1000 simulations in each case

		$\delta_{x_1} = 0.5, \delta_{x_2} = 1.0$						$\delta_{x_1} = 0.5, \delta_{x_2} = 2.0$						$\delta_{x_1} = 0.5, \delta_{x_2} = 3.0$						$\delta_{x_1} = 1.0, \delta_{x_2} = 2.0$						$\delta_{x_1} = 1.0, \delta_{x_2} = 3.0$						$\delta_{x_1} = 2.0, \delta_{x_2} = 3.0$					
		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals	
ϕ_{11}	ϕ_{22}	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5
0	0	34	30	31	32	34	30	2	2	2	2	2	2	1	1	1	1	1	1	4	3	3	3	4	3	1	1	1	1	1	1	2	2	2	2	2	2
	0.25	44	12	40	37	87	75	2	1	2	2	3	2	1	1	1	1	1	1	5	2	5	5	10	10	1	1	1	1	1	1	3	1	3	2	4	3
	0.5	77	8	65	67	182	151	6	2	6	5	4	4	1	1	1	1	1	1	12	2	11	10	28	24	2	1	2	2	1	1	4	1	3	4	2	2
	0.75	128	5	97	93	166	138	18	2	14	11	2	1	4	1	3	2	1	1	23	2	18	15	4	3	5	1	4	3	1	1	5	1	4	5	1	1
	0.95	177	3	116	138	1	1	49	1	36	25	1	1	12	1	9	5	1	1	33	1	25	27	1	1	11	1	8	6	1	1	5	1	4	8	1	1
0.25	0	42	29	38	165	24	22	2	2	2	5	2	2	1	1	1	1	1	1	4	4	4	17	3	3	1	1	1	2	1	1	2	2	2	5	2	2
	0.25	35	12	32	33	65	55	2	1	2	2	2	2	1	1	1	1	1	1	4	2	4	4	7	6	1	1	1	1	1	1	2	1	2	2	3	3
	0.5	58	7	51	57	146	124	3	1	3	3	3	3	1	1	1	1	1	1	8	2	7	7	24	21	1	1	1	1	1	1	3	1	3	3	4	3
	0.75	125	5	96	104	215	173	13	1	11	11	1	1	2	1	2	2	1	1	2	2	18	17	12	8	4	1	3	3	1	1	5	1	5	5	1	1
	0.95	205	2	140	148	2	2	57	1	41	28	1	1	12	1	9	4	1	1	39	1	28	32	1	1	11	1	8	5	1	1	5	1	4	8	1	1
0.5	0	1	16	49	466	17	15	4	3	4	57	2	2	1	1	1	5	1	1	7	5	6	178	2	2	2	2	1	14	1	1	3	2	2	27	2	2
	0.25	46	11	41	321	45	39	2	2	2	10	2	2	1	1	1	1	1	1	5	3	5	44	5	4	1	1	1	3	1	1	2	2	2	13	2	2
	0.5	42	6	37	38	121	98	2	1	2	2	3	3	1	1	1	1	1	1	5	2	5	5	14	13	1	1	1	1	1	1	2	1	2	2	4	4
	0.75	98	4	72	95	239	203	6	1	6	7	2	2	1	1	1	1	1	1	15	1	13	13	9	8	2	1	2	2	1	1	5	1	4	5	1	1
	0.95	222	2	143	175	4	1	68	1	44	30	1	1	12	1	9	5	1	1	47	1	35	37	1	1	14	1	10	6	1	1	7	1	5	10	1	1
0.75	0	80	8	60	419	14	12	7	3	6	303	2	2	2	1	2	93	1	1	10	4	8	247	2	2	2	2	2	202	1	1	3	2	2	40	2	2
	0.25	73	6	57	527	31	27	6	2	5	217	2	2	2	1	1	40	1	1	9	3	8	357	4	4	2	1	2	146	1	1	3	1	3	58	2	2
	0.5	62	5	49	648	84	72	4	1	3	65	3	3	1	1	1	6	1	1	7	2	6	278	9	8	1	1	1	20	1	1	3	1	3	71	3	3
	0.75	59	3	47	48	193	159	2	1	2	2	1	1	1	1	1	1	1	1	7	1	6	6	17	15	1	1	1	1	1	1	3	1	3	3	3	3
	0.95	309	2	166	231	3	2	51	1	31	30	1	1	7	1	5	4	1	1	70	1	42	42	1	1	14	1	9	7	1	1	11	1	7	12	1	1
0.95	0	87	3	58	288	8	8	10	2	8	313	2	2	3	1	2	316	1	1	11	2	8	191	1	1	3	1	2	209	2	1	2	1	2	41	1	1
	0.25	89	3	59	339	20	18	11	2	8	350	3	3	3	1	2	346	1	1	11	2	9	230	2	2	3	1	3	265	2	2	3	1	2	56	1	1
	0.5	95	2	61	430	49	43	12	1	9	466	5	4	3	1	2	366	1	1	14	1	10	322	4	3	4	1	3	351	2	2	4	1	3	79	1	1
	0.75	112	2	67	639	133	113	13	1	9	511	6	5	3	1	2	244	1	1	18	1	12	522	17	15	4	1	3	454	1	1	5	1	4	149	1	1
	0.95	168	1	77	74	67	49	6	1	3	3	1	1	1	1	1	1	1	1	22	1	12	11	1	1	1	1	1	1	1	1	8	1	4	4	1	1

Table VII. The ARL_0 and ARL_1 performance in Phase II for different $\mathbf{\Phi}$ matrices and different shifts in the two variables and $\text{cov}(\varepsilon_{x_1}, \varepsilon_{x_2}) = 0.9$ for all cases. The ARL values are based on 1000 simulations in each case

		$\Phi = \begin{bmatrix} .25 & .25 \\ 0 & .25 \end{bmatrix}$						$\Phi = \begin{bmatrix} .2 & .5 \\ .5 & .2 \end{bmatrix}$					
		Raw		Adj. UCL		Residuals		Raw		Adj. UCL		Residuals	
	Shift sizes	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5	S_1	S_5
a)	No shift (ARL_0)	429	46	373	391	410	331	423	21	370	343	391	331
b)	$\delta_{x_1} = 0.5, \delta_{x2} = 0$	66	15	60	79	78	67	24	11	22	199	10	9
c)	$\delta_{x_1} = 1, \delta_{x2} = 0$	9	4	8	11	9	8	2	3	2	47	2	2
d)	$\delta_{x_1} = 2, \delta_{x2} = 0$	1	1	1	1	1	1	1	1	1	2	1	1
e)	$\delta_{x_1} = 0, \delta_{x2} = 0.5$	66	21	60	138	49	44	27	11	24	202	10	10
f)	$\delta_{x_1} = 0, \delta_{x2} = 1$	8	5	7	23	6	6	2	3	2	49	2	2
g)	$\delta_{x_1} = 0, \delta_{x2} = 2$	1	1	1	2	1	1	1	1	1	2	1	1
h)	$\delta_{x_1} = 0.5, \delta_{x2} = 0.5$	234	29	197	202	255	213	260	15	218	192	355	297
i)	$\delta_{x_1} = 1, \delta_{x2} = 1$	81	15	73	69	116	98	97	9	86	62	240	207

8. A five-variable example

Finally, we explore if the results for the three approaches can be extended to more than two variables. Here, we choose five variables because the Hotelling T^2 chart is generally considered most effective for a moderate number of variables. When the number of variables increases dimensionality reduction techniques, such as PCA, are preferred; see Montgomery.[23]

Even with only five variables, the number of possible combinations of covariance structures for the errors and autocorrelation basically becomes unfeasibly large. Therefore, we only consider a model for a given covariance structure for the errors and vary the autocorrelation through different diagonal $\mathbf{\Phi}$ matrices.

We assume that we have a five-variable VAR(1) model with the error covariance matrix:

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & .8 & .3 & 0 & 0 \\ .8 & 1 & .6 & 0 & 0 \\ .3 & .6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & .6 \\ 0 & 0 & 0 & .6 & 1 \end{bmatrix}$$

which essentially means that the variables are correlated through two blocks of correlated errors. The first block contains correlated errors for x_1, x_2, x_3, and the second block contains correlated errors for x_4, x_5. In the simulations, we change the parameters of the diagonal $\mathbf{\Phi}$ matrix:

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{11} & 0 & 0 & 0 & 0 \\ 0 & \phi_{22} & 0 & 0 & 0 \\ 0 & 0 & \phi_{33} & 0 & 0 \\ 0 & 0 & 0 & \phi_{44} & 0 \\ 0 & 0 & 0 & 0 & \phi_{55} \end{bmatrix}$$

where $\phi_{11} = \phi_{22} = \phi_{33} = \phi_{44} = \phi_{55} = \begin{cases} 0.25 & \text{low autocorrelation} \\ 0.5 & \text{moderate autocorrelation} \\ 0.95 & \text{high autocorrelation} \end{cases}$

We also include two additional cases with high negative autocorrelation and with different autocorrelation parameters for all variables as

$$\phi_{11} = 0.95; \phi_{22} = 0.85; \phi_{33} = 0.75; \phi_{44} = 0.65; \phi_{55} = 0.55.$$

In the second approach for each case, we adjust the UCLs through simulation and then proceed to test the shift-detection ability of the three methods. We test a number of scenarios with various shifts. Table VIII shows the ARL_0 and ARL_1 values for the three methods and different shift combinations in the five-variable case.

From Table VIII, it is again evident that the first approach using $\mathbf{S}_5$ is inappropriate because the ARL_0 values are far too low for positive autocorrelation and too high for negative autocorrelation. Another interesting result is that the ARL_1 values for the first approach using $\mathbf{S}_1$ are fairly competitive, especially for small-to-moderate positive autocorrelation. However, the adjustment of the UCLs in the second approach is less effective in the five-variable case because the ARL_0 values are too low, especially for high autocorrelation and using $\mathbf{S}_1$.

We also note that the residuals-based approach does not appear as competitive as in the two-variable cases for positive autocorrelation. However, it is still clearly the best approach for negative autocorrelation. When the autocorrelation parameters in the $\mathbf{\Phi}$ matrix are positive and of small-to-moderate magnitude, the residuals-based approach actually performs the worst among the three approaches. When the autocorrelation increases, the shift-detection ability of the residuals-based approach is clearly improved for larger shifts. For the high positive autocorrelation case in Table VIII, the residuals-based approach catches shifts of one standard deviation and above faster than the other two methods. Using $\mathbf{S}_1$ in the residuals-based approach seems to produce ARL_0 values closest to the nominal value of 370. However, for small shift sizes and high positive autocorrelation, the residuals-based approach performs worse than the first approach using $\mathbf{S}_1$. A possible explanation might be that if a small shift does not signal instantly in the residuals-based approach, the VAR(1) model may actually incorporate and adapt to the shift resulting in higher ARL_1 values. The analogue phenomenon for univariate residual charts is described by Zhang.[15]

9. Conclusions and discussion

In this article, we study the impact of autocorrelation in the raw data on the Hotelling T^2 control chart. We provide simulation results for in-control and out-of-control ARLs for various autocorrelation and error covariance structures and shifts in the mean. To limit the potentially myriad of possibilities, we primarily explore a two-variable case but also provide an example of a five-variable case.

The results clearly show that the first approach of ignoring the autocorrelation and using theoretical UCLs can lead to erroneous conclusions, in-control ARLs (sometimes significantly) different from the nominal, and poor shift-detection ability, particularly with increasing amount of autocorrelation in the data. Moreover, there is the associated problem of the estimation of the covariance matrix, which is also an issue for independent data. In this article, we compare the performance of the 'traditional' estimate $\mathbf{S}_1$ with $\mathbf{S}_5$, which is based on the first difference of successive pairs of observations and has been recommended for the detection of step or ramp shifts in the mean.

As in the case of univariate Shewhart charts, we find that using a naïve approach that completely ignores the autocorrelation leads to an overestimation of the UCL, when using $\mathbf{S}_1$, and increasing in-control ARL (ARL_0) of the Hotelling T^2 chart. As expected, the consequence of fewer false alarms when the process is in control is that the shift-detection ability diminishes substantially. This approach when using $\mathbf{S}_5$ gives even worse performance with too low ARL_0 values for positive autocorrelation and too high ARL_0 values for negative autocorrelation. We therefore conclude that $\mathbf{S}_5$ is not a proper estimator of the covariance matrix to be used in Hotelling T^2 calculation when data are autocorrelated.

We show that it is possible to reduce the effect of autocorrelation by adjusting the UCLs through simulation. The Hotelling T^2 chart with adjusted UCLs has improved shift-detection ability compared to the first approach for the majority of cases we tested. However, the adjustment of the UCLs we used suffers from the fact that it also assumes independent T^2 values, which is clearly violated for autocorrelated raw data.

We found that the Hotelling T^2 chart based on residuals from the VAR(1) model performs best overall, catching the shifts faster on average, and turns out to be especially effective for shifts larger than one standard deviation and for negative autocorrelation. Using $\mathbf{S}_1$ and theoretical UCLs for the residuals-based approach seems to result in ARL_0 values closest to the nominal value of 370. Using $\mathbf{S}_5$ and corresponding theoretical UCLs produces somewhat too low ARL_0 values. However, the residuals-based approach is not as effective in detecting shifts of smaller magnitude and especially when the variables have the same shift size. In fact, for some cases of smaller shifts of equal size and direction, the first approach using $\mathbf{S}_1$ and theoretical UCLs produced lower ARL_1 values than the residuals-based approach. For smaller shifts, there seems to be a risk; given that the residuals-based chart does not signal instantly after the shift, that the VAR(1) model incorporates and adapts to the shift causing longer run lengths.

Applying the Hotelling T^2 chart to the residuals from a multivariate time series model can improve out-of-control run lengths, but there are of course modeling issues to consider. To avoid such complications, we assumed that the true parameter estimates of the VAR(1) model were known and the residuals were calculated accordingly. Therefore, we believe that the results provided in this article constitute the 'best case scenario' for this method and further research is certainly needed to study the impact of estimated parameters on the control chart performance. The residuals-based approach has further drawbacks when the number of variables gets large because fitting an appropriate multivariate time series model then becomes increasingly difficult.

Since the results in this article produces no clear 'best' method in all situations, we believe that a larger study that compares the performance of different approaches to tackle the autocorrelation issue would be of value to the users of Hotelling T^2 charts. Examples of such methods are those illustrated in this article: to adjust the control limits and to use residuals from a multivariate time series model. Other methods of interest to investigate are to use residuals from univariate time series models for each variable and to include lagged variables in the data matrix. How to properly adjust the control limits for autocorrelated data is another important research question.

Table VIII. The ARL_0 and ARL_1 performance for different $\mathbf{\Phi}$ matrices and different shift combinations for the five-variable example. The ARL values are based on 1000 simulations in each case

$$\mathbf{\Phi}=\begin{bmatrix} .25 & 0 & 0 & 0 & 0 \\ 0 & .25 & 0 & 0 & 0 \\ 0 & 0 & .25 & 0 & 0 \\ 0 & 0 & 0 & .25 & 0 \\ 0 & 0 & 0 & 0 & .25 \end{bmatrix}$$

	Raw		Adj. UCL		Residuals	
Shift sizes	S_1	S_5	S_1	S_5	S_1	S_5
a) No shift (ARL_0)	380	46	284	314	369	252
b) $\delta_{x_1}=0.5$	149	24	120	125	211	150
c) $\delta_{x_1}=1$	28	8	23	24	59	44
d) $\delta_{x_4}=0.5$	224	33	176	189	263	186
e) $\delta_{x_4}=1$	79	15	63	65	128	92
f) $\delta_{x_1}=0.5,\delta_{x_4}=0.5$	106	19	84	90	161	114
g) $\delta_{x_1}=1,\delta_{x_4}=1$	16	5	13	14	31	24
h) $\delta_{x_1}=1,\delta_{x_4}=-1$	15	5	13	14	34	27
i) $\delta_{x_1}=2,\delta_{x_4}=2$	2	1	1	1	2	2
j) $\delta_{x_1}=\delta_{x_2}=\delta_{x_3}=1$	76	15	61	64	131	90
k) $\delta_{x_1}=\delta_{x_2}=\delta_{x_4}=\delta_{x_5}=1$	35	8	29	31	74	51
l) $\delta_{x_1}=\delta_{x_4}=1;\delta_{x_2}=\delta_{x_5}=1$	2	1	1	1	2	2

$$\Phi=\begin{bmatrix} .5 & 0 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 & 0 \\ 0 & 0 & .5 & 0 & 0 \\ 0 & 0 & 0 & .5 & 0 \\ 0 & 0 & 0 & 0 & .5 \end{bmatrix}$$

	Raw		Adj. UCL		Residuals	
Shift sizes	S_1	S_5	S_1	S_5	S_1	S_5
a) No shift (ARL_0)	349	10	254	298	382	261
b) $\delta_{x_1}=0.5$	147	7	108	130	266	184
c) $\delta_{x_1}=1$	32	3	26	29	111	74
d) $\delta_{x_4}=0.5$	237	8	176	197	317	220
e) $\delta_{x_4}=1$	77	5	59	69	199	141
f) $\delta_{x_1}=0.5,\delta_{x_4}=0.5$	106	6	82	90	217	147
g) $\delta_{x_1}=1,\delta_{x_4}=1$	18	2	15	15	67	48
h) $\delta_{x_1}=1,\delta_{x_4}=-1$	18	2	15	16	68	49
i) $\delta_{x_1}=2,\delta_{x_4}=2$	2	1	2	2	2	2
j) $\delta_{x_1}=\delta_{x_2}=\delta_{x_3}=1$	81	5	61	70	195	133
k) $\delta_{x_1}=\delta_{x_2}=\delta_{x_4}=\delta_{x_5}=1$	38	4	31	36	127	88
l) $\delta_{x_1}=\delta_{x_4}=1;\delta_{x_2}=\delta_{x_5}=1$	2	1	2	2	2	2

$$\Phi=\begin{bmatrix} .95 & 0 & 0 & 0 & 0 \\ 0 & .95 & 0 & 0 & 0 \\ 0 & 0 & .95 & 0 & 0 \\ 0 & 0 & 0 & .95 & 0 \\ 0 & 0 & 0 & 0 & .95 \end{bmatrix}$$

	Raw		Adj. UCL		Residuals	
Shift sizes	S_1	S_5	S_1	S_5	S_1	S_5
a) No shift (ARL_0)	202	1	89	214	393	259
b) $\delta_{x_1}=0.5$	131	1	62	127	303	191
c) $\delta_{x_1}=1$	56	1	26	51	13	9
d) $\delta_{x_4}=0.5$	158	1	71	163	367	239
e) $\delta_{x_4}=1$	103	1	47	96	156	95
f) $\delta_{x_1}=0.5,\delta_{x_4}=0.5$	122	1	52	161	216	138
g) $\delta_{x_1}=1,\delta_{x_4}=1$	37	1	18	32	1	1
h) $\delta_{x_1}=1,\delta_{x_4}=-1$	38	1	19	32	2	1
i) $\delta_{x_1}=2,\delta_{x_4}=2$	2	1	2	2	1	1
j) $\delta_{x_1}=\delta_{x_2}=\delta_{x_3}=1$	92	1	44	92	155	91
k) $\delta_{x_1}=\delta_{x_2}=\delta_{x_4}=\delta_{x_5}=1$	68	1	33	58	21	12
l) $\delta_{x_1}=\delta_{x_4}=1;\delta_{x_2}=\delta_{x_5}=1$	3	1	2	2	1	1

$$\Phi=\begin{bmatrix} .95 & 0 & 0 & 0 & 0 \\ 0 & .85 & 0 & 0 & 0 \\ 0 & 0 & .75 & 0 & 0 \\ 0 & 0 & 0 & .65 & 0 \\ 0 & 0 & 0 & 0 & .55 \end{bmatrix}$$

	Raw		Adj. UCL		Residuals	
Shift sizes	S_1	S_5	S_1	S_5	S_1	S_5
a) No shift (ARL_0)	302	2	174	280	389	260
b) $\delta_{x_1}=0.5$	244	2	153	181	306	193
c) $\delta_{x_1}=1$	111	1	69	65	9	7
d) $\delta_{x_4}=0.5$	213	2	129	258	340	224
e) $\delta_{x_4}=1$	89	1	60	248	252	164
f) $\delta_{x_1}=0.5,\delta_{x_4}=0.5$	162	1	100	167	264	178
g) $\delta_{x_1}=1,\delta_{x_4}=1$	46	1	32	56	4	3
h) $\delta_{x_1}=1,\delta_{x_4}=-1$	44	1	29	64	3	3
i) $\delta_{x_1}=2,\delta_{x_4}=2$	4	1	3	6	1	1
j) $\delta_{x_1}=\delta_{x_2}=\delta_{x_3}=1$	110	1	70	131	168	111
k) $\delta_{x_1}=\delta_{x_2}=\delta_{x_4}=\delta_{x_5}=1$	52	1	37	146	107	72
l) $\delta_{x_1}=\delta_{x_4}=1;\delta_{x_2}=\delta_{x_5}=1$	4	1	3	14	1	1

$$\Phi=\begin{bmatrix} -.95 & 0 & 0 & 0 & 0 \\ 0 & -.85 & 0 & 0 & 0 \\ 0 & 0 & -.75 & 0 & 0 \\ 0 & 0 & 0 & -.65 & 0 \\ 0 & 0 & 0 & 0 & -.55 \end{bmatrix}$$

	Raw		Adj. UCL		Residuals	
Shift sizes	S_1	S_5	S_1	S_5	S_1	S_5
a) No shift (ARL_0)	307	10901	115	113	402	263
b) $\delta_{x_1}=0.5$	139	4066	54	53	2	2
c) $\delta_{x_1}=1$	40	1248	17	17	1	1
d) $\delta_{x_4}=0.5$	200	7236	75	74	3	3
e) $\delta_{x_4}=1$	90	2853	38	37	1	1
f) $\delta_{x_1}=0.5,\delta_{x_4}=0.5$	105	3929	44	44	2	2
g) $\delta_{x_1}=1,\delta_{x_4}=1$	21	576	9	9	1	1
h) $\delta_{x_1}=1,\delta_{x_4}=-1$	23	583	8	8	1	1
i) $\delta_{x_1}=2,\delta_{x_4}=2$	1	29	1	1	1	1
j) $\delta_{x_1}=\delta_{x_2}=\delta_{x_3}=1$	96	3042	36	36	1	1
k) $\delta_{x_1}=\delta_{x_2}=\delta_{x_4}=\delta_{x_5}=1$	50	1379	18	18	1	1
l) $\delta_{x_1}=\delta_{x_4}=1;\delta_{x_2}=\delta_{x_5}=1$	1	14	1	1	1	1

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