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Published in:
Journal of Sound and Vibration

Link to article, DOI:
[10.1016/j.jsv.2017.10.028](https://doi.org/10.1016/j.jsv.2017.10.028)

Publication date:
2018

Document Version
Peer reviewed version

[Link back to DTU Orbit](#)

Citation (APA):
de Souza Reboucas, G. F., Santos, I. F., & Thomsen, J. J. (2018). Validation of Vibro-Impact Force Models by Numerical Simulation, Perturbation Methods and Experiments. *Journal of Sound and Vibration*, 413, 291–307. <https://doi.org/10.1016/j.jsv.2017.10.028>

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Validation of Vibro-Impact Force Models by Numerical Simulation, Perturbation Methods and Experiments

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Abstract

The frequency response of a single degree of freedom vibro-impact oscillator is analyzed using Harmonic Linearization, Averaging and Numeric Simulation, considering three different impact force models: one given by a piecewise-linear function (Kelvin-Voigt model), another by a high-order power function, and a third one combining the advantages of the other two. Experimental validation is carried out using control-based continuation to obtain the experimental frequency response, including its unstable branch.

Keywords: vibro-impact dynamics ; impact force models ; Kelvin-Voigt impact model ; Power-law impact model ; vibro-impacting beam ; lumped mass beam ; experimental validation ; control-based continuation ;

1. Introduction

Generally speaking, the modeling of vibro-impact systems considers two distinct situations: with and without contact. Hence, one can solve each case separately, connecting them using the contact condition [1]. In [2, 3], the impact condition of piecewise linear systems is used to obtain discrete maps, enabling investigations of bifurcations and stability of periodic motions. However, while it is straightforward to attach solutions of linear systems, the same cannot be said when additional nonlinearities are also present between impacts.

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9 The contact between two bodies can be modeled as a temporary association of the
10 stiffness and damping properties of the colliding bodies, each one modeled as a linear
11 system. Despite its simplicity, this approach has some limitations, mainly the non-
12 zero values of the impact force on the initial and final parts of the contact phase due
13 to dissipative forces. This motivated the development of other contact models such
14 as the one suggested by Hunt and Crossley [4], who derived the damping coefficient
15 as a power function of the impact deformation. This solved the physical inaccuracy
16 of the linear model and lead to an ongoing discussion about the appropriate form of
17 the nonlinear dissipative term [5, 6]. Most of the expressions for the dissipative term
18 combine the velocity immediately before impact and the coefficient of restitution with
19 a rational power of the contact deformation, leading to complicated expressions whose
20 analysis is possible only through numerical simulation.

21 One of the oldest techniques to model impacting systems is to use a coefficient of
22 restitution (CoR) to relate the velocities before and after impact. This classic approach
23 has been used in many applications [7–11]. For instance, Bishop et al. [7] used a single
24 degree of freedom (SDOF) oscillator with a CoR rule to reproduce the experimental
25 behavior of an impacting cantilever beam around its first natural frequency. Also, the
26 coefficient of restitution, together with a power-law elastic force, can be used to obtain
27 various nonlinear contact force models, such as the one mentioned previously by Hunt
28 and Crossley [4]. Despite its popularity, it should be pointed out that the CoR is not an
29 intrinsic property of the material, depending on the impact velocity [12]. Also, using
30 a simple CoR kinematic rule does not give any direct information about contact forces
31 or stresses.

32 From the numerical point of view, one can replace discontinuities by smooth equiv-
33 alent functions and use standard techniques to solve the smoothed model as done by
34 Savi et al. in [13], where the impact condition is smoothed, or by Elmegård et al. [14],
35 who applied numeric continuation to a smoothed model of a lumped-mass impact-
36 ing beam. In these cases, while it is safer to use standard ODE solvers provided by
37 accredited sources, it is also necessary to properly tune smooth approximations of dis-
38 continuous functions; see [15] for a discussion on the effect of smoothing functions on
39 the frequency response of oscillators with clearance. There are also several numerical

40 integration algorithms designed for non-smooth systems [16–18], but their comparison
41 is out of the scope of this work.

42 Vibro-impact oscillators can also be analysed using common perturbation tech-
43 niques, such as Harmonic Linearization [19, 20], Averaging [21–23], and the Lindstedt-
44 Poincaré method [24]. Besides their widespread use with nonlinear problems, these
45 methods assume weak and smooth nonlinearities, which are not reasonable assump-
46 tions for general impacting systems. This limitation motivated the development of
47 non-smooth transformations with respect to state [25, 26] and time [27] variables. By
48 these, one can remove non-smooth terms from a certain model, or make them small,
49 allowing the usage of common perturbation methods. An example of such combina-
50 tion can be found in [8] where non-smooth transformations were used to weaken a
51 near-elastic kinematic impact condition, enabling subsequent application of extended
52 averaging.

53 In some cases, the applicability of the theoretical/numerical tools mentioned above
54 is accompanied by physical experiments. In [28] the experimental chaotic behavior of
55 a base-excited cantilever beam with one-sided stop is qualitatively compared with re-
56 sults from numeric simulations of a piecewise linear oscillator. In [7] the experimental
57 frequency response of a forced cantilever beam with a unilateral constraint is compared
58 with the one obtained by numeric simulation of a SDOF oscillator with a coefficient
59 of restitution, showing reasonable accuracy. The Averaging method has been exten-
60 sively used to obtain analytical frequency-amplitude expressions for piecewise linear
61 oscillators with one [21, 22] and two [23] degrees of freedom.

62 In [13] the nonlinear dynamics of a mass-spring system with discontinuous stiffness
63 and damping was analyzed experimentally and numerically by smoothing the impact
64 condition. A similar numerical-experimental analysis was performed by Aguiar and
65 Weber [29], focusing on the behavior of the impact force. Using control-based con-
66 tinuation Bureau et al. [30] obtained experimental frequency responses of a cantilever
67 beam with lumped mass and bilateral constraints. A single-DOF numerical model for
68 this system was proposed by Elmegård et al. [14], who used the experimental data
69 from [30] to validate the model and predict the existence of an isola, which was later
70 identified experimentally by Bureau et al. [31].

71 From the overview presented above, one can realize that besides the match be-
72 tween results from particular numerical/analytical techniques with experimental find-
73 ings there are little efforts on comparing the different paradigms used to model and
74 analyze vibro-impact systems.

75 The objective and main originality of the present work is to compare different
76 impact force models using analytical, numerical and experimental techniques. The
77 frequency response of a SDOF vibro-impact oscillator is analyzed using Harmonic
78 Linearization and Averaging, considering three different impact force models: one
79 given by the Kelvin-Voigt model (piecewise-linear function), another using a power-
80 law function and a third one combining the strengths of the first two. Experimentally,
81 control-based continuation [30, 31] is used to obtain frequency responses of an impact-
82 ing beam, including its unstable branch. Numerical simulations are used to validate the
83 simple analytic approximations obtained by perturbation methods.

84 Despite the ability of the mentioned models to produce different qualitative be-
85 haviors such as quasiperiodicity and chaos, the analysis presented here is restricted to
86 single-periodic oscillations only.

87 As the main contribution of this manuscript is to compare different impact force
88 formulations, only the most common models are considered. That is the case of the
89 Kelvin-Voigt model, which is widely used despite its inaccuracies [2, 5, 23]. The
90 Power-law model can be viewed as a generic version of the compliant force models
91 first derived by Hunt and Crossley [4] and further developed by others [5, 32].

92 **2. Experimental setup and procedure**

93 The experimental setup has been described previously in [30, 31] and is shown
94 in Fig. 1. In Fig. 1a,b an electrodynamic shaker (1) [B&K® 4808] is used to apply a
95 harmonic excitation to a platform (2), containing a cantilever beam and a pair of sym-
96 metrically located stops (5) to restrain the lateral movement of the beam. The shaker is
97 driven by a power amplifier [B&K® 2712] and is connected to the platform by a stinger.
98 The impacting beam can be seen in detail in Fig. 1c,d, where two DC holding electro-
99 magnetic actuators (6) [Magnet-Schultz® G MH X 030] are placed on each side of the

100 lumped mass (4) to execute control-based continuation. The displacement of both plat-
101 form and lumped mass are measured by two laser sensors (3) [OMRON® ZX-LD40].
102 Due to the electromagnetic actuators around the lumped mass, its displacement is mea-
103 sured below the mass location, Fig. 1d. A dSPACE® DS1104 R&D controller board
104 is used to perform data acquisition and control-based continuation of the experimental
105 setup.

106 The experimental frequency response of the impacting beam, shown in Fig. 2a,
107 was obtained by the authors, who repeated some of the experiments done by [30, 31]
108 using control-based continuation. In this model-free approach, the equilibrium states
109 are found by a predictor-corrector algorithm. A non-invasive proportional-derivative
110 controller is used to stabilize the system through its unstable branch (lower part of the
111 bent peak) using the electromagnetic actuators (6).

112 Figure 2b illustrates the harmonic decomposition of the experimental frequency re-
113 sponse from its static component, represented by $0\tilde{\Omega}$, up to five times the excitation
114 frequency $\tilde{\Omega}$. In this figure one can see that the impacting regime increases the ampli-
115 tude of higher harmonics but they are still negligible if compared with the fundamental
116 harmonic, $1\tilde{\Omega}$.

117 In this work, the experimental observations are restricted to the neighborhood of
118 the system's fundamental linear natural frequency, measured as $f_n = 7.6$ Hz, which is
119 much smaller than its second linear natural frequency, which is around 200 Hz. Also,
120 the lumped mass ($m = 0.2$ kg) is approximately 8 times heavier than the beam's mass,
121 being the dominant inertial element. In addition to that, the lumped mass is located at a
122 reasonable distance from the stops, which can be considered as rigid supports, causing
123 inelastic impacts.

124 Based on the above, one can simplify the mathematical modeling of the impacting
125 beam shown in Fig. 1 in the neighborhood of its fundamental linear natural frequency,
126 by using a SDOF model, where the lumped mass dynamics dominate the oscillations
127 in comparison to the flexible beam alone.

(b)

(d)

Figure 1: Experimental setup (a,c) and its schematic representation (b,d).

Figure 2: (a) Experimental frequency response and its (b) harmonic decomposition

3. Equations of motion

As the measurement and impact locations L_S and L_C are not coincident and due to beam deflection, the measured grazing amplitude Δ_G will differ from the gap width Δ_C . This difference can be seen in Fig. 3 for a cantilever beam oscillating at its first mode. The deflection is caused by the inertial force $F = -m\ddot{U}(L_m, \tilde{t})$. The measured grazing amplitude will be used to normalize the equations of motion later on and to define impact force models in Section 4.

The electrodynamic shaker is not feedback controlled, being not able to provide constant excitation amplitude on the investigated frequency range. Also, the displacement of the sub-system shaker-platform around resonance is affected by a nonlinear coupling with the impacting beam as mentioned in [30]. Alternatively, one could consider the single harmonic electric signal fed into the shaker's power amplifier as the system's external excitation, since it has constant amplitude (in volts). This electric signal excites the shaker-platform subsystem, which in turn excites the impacting beam.

So, the base excitation can be written as:

$$\tilde{b}(\tilde{t}) = \tilde{\gamma}\tilde{V}\sin(\tilde{\Omega}\tilde{t}), \quad (1)$$

where $\tilde{V}$ is the electric amplitude in V and $\tilde{\gamma}$ is the unit conversion gain in m/V. In the schematic representation of the forced impacting beam in Fig. 1d, L_C , L_m , L_S and L represent the axial location of the stops, lumped mass, displacement sensor and beam length, respectively, Δ_C represents the gap width. Using Bernoulli-Euler beam theory, one can write its equation of motion for transverse displacement $U(X, \tilde{t})$ as [14]:

$$(\rho A + m\delta(X - L_m))\ddot{U} + DEI\dot{U}'''' + EIU'''' + \tilde{f}_C(U, \dot{U})\delta(X - L_C) = (\rho A + m\delta(X - L_m))\tilde{\Omega}^2\tilde{\gamma}\tilde{V}\sin(\tilde{\Omega}\tilde{t}), \quad (2)$$

$$U(0, \tilde{t}) = U'(0, \tilde{t}) = U''(L, \tilde{t}) = U'''(L, \tilde{t}) = 0, \quad (3)$$

where $(\dot{\cdot}) \equiv \partial(\cdot)/\partial\tilde{t}$, $(\cdot)' \equiv \partial(\cdot)/\partial X$, ρ , E , A and I represents the beam's density, elasticity modulus, cross-sectional area and cross-section area moment of inertia, respectively, m is the lumped mass and $\delta(\cdot)$ is Dirac's delta function. The function $\tilde{f}_C(U, \dot{U})$ represents the transverse impact force, resulting from the contact between beam and

Figure 3: Deflection of a cantilever beam.

151 stops. This function can be modeled by several means from which three will be ad-
 152 dressed in Section 4.

153 The present model is similar to the one proposed by Elmegård et al. [14], which
 154 did not define an impact force explicitly but used a piecewise relation for the spatial
 155 discretization of the equations of motion.

156 One way of inserting the stiffness proportional damping coefficient D in the Bernoulli-
 157 Euler model is to assume the beam's material to have a viscoelastic behavior, described
 158 by Kelvin and Voigt's (KV) model. Assuming that lumped mass is much bigger than
 159 the beam's mass, i.e. $m \gg \rho AL$, and defining $U = u\Delta_G$ and $X = xL$ as normalized
 160 transverse displacement and axial position, one arrives at:

$$\begin{aligned} & \delta(x - x_m)\ddot{u} + (D\dot{u}'''' + u'''')EI/(mL^3) + \\ & + \tilde{f}_C(u\Delta_G, \dot{u}\Delta_G)\delta(x - x_C)/(m\Delta_G) = \delta(x - x_m)\tilde{V}(\tilde{\gamma}/\Delta_G)\tilde{\Omega}^2 \sin(\tilde{\Omega}\tilde{t}), \end{aligned} \quad (4)$$

161 where now $(\cdot)' \equiv \partial(\cdot)/\partial x$, $x_C = L_C/L$, and $x_m = L_m/L$. This system can be spatially
 162 discretized by defining:

$$u(x, \tilde{t}) = \sum_{i=1}^N \phi_i(x)q_i(\tilde{t}), \quad (5)$$

163 with the mode shapes $\phi_i(x)$ satisfying all essential boundary conditions and $q_i(\tilde{t})$ as the
 164 modal coordinates. When the discretized version of Eq. (4) is driven around its first

165 resonance the high order modes ($i \geq 2$) are expected to have low influence because
 166 they are increasingly damped by the stiffness proportional damping. The weak signif-
 167 icance of higher modes for the experimental setup was already illustrated by Fig. 2b.
 168 So, a single-DOF discretization of Eq. (4) appears to be an appropriate model for the
 169 experimental setup around its first resonance. Using standard Galerkin approximation
 170 this model is obtained as:

$$\ddot{q} + (D\dot{q} + q)\omega_n^2 + \bar{f}_C(q, \dot{q}) = \tilde{\Omega}^2 \gamma \tilde{V} \sin(\tilde{\Omega} \tilde{t}), \quad (6)$$

171 where:

$$\omega_n^2 = K_S / M_S, \quad M_S = m\phi(x_m)^2, \quad (7)$$

$$K_S = \frac{EI}{L^3} \int_0^1 (\phi''(x))^2 dx, \quad \gamma = \frac{\tilde{\gamma}}{\phi(x_m)\Delta_G}, \quad (8)$$

$$\bar{f}_C(q, \dot{q}) = \frac{\phi(x_C)}{M_S \Delta_G} \tilde{f}_C(\phi(x_C)q\Delta_G, \phi(x_C)\dot{q}\Delta_G), \quad (9)$$

172 are the squared fundamental linear natural frequency, equivalent structural mass and
 173 stiffness coefficient, and normalized unit conversion gain and impact force, respec-
 174 tively. Additionally, $\phi(\cdot)$ is the static deformation pattern of a cantilever beam loaded at
 175 $x = x_m$. Defining the structural damping ratio as $\beta_S = \frac{1}{2}D\omega_n$, substituting it into Eq. (4)
 176 and normalizing time $\tilde{t}$ using the system's fundamental linear natural frequency ω_n , one
 177 arrives at:

$$\ddot{q} + 2\beta_S \dot{q} + q + f_C(q, \dot{q}) = \Omega^2 \gamma \tilde{V} \sin(\Omega t), \quad (10)$$

178 where:

$$t = \omega_n \tilde{t}, \quad \Omega = \tilde{\Omega} / \omega_n \quad \text{and} \quad f_C(q, \dot{q}) = \bar{f}_C(q, \omega_n \dot{q}) / \omega_n^2 \quad (11)$$

179 are the normalized time, normalized forcing frequency, and nondimensional impact
 180 force, respectively. Also, from now on $(\dot{\cdot}) \equiv \partial(\cdot) / \partial t$.

181 The vibro-impact oscillations measured from the experimental setup can be related
 182 to Eq. (10) by multiplying the model solution by the measured grazing amplitude and
 183 by the first mode shape at the measurement location, $U(L_S, \tilde{t}) = q(\omega_n \tilde{t}) \Delta_G \phi(L_S / L)$.

184 This relation can be even simpler if the first mode shape is normalized with respect to
 185 the measurement location, L_S . By doing so, the measured vibrations become propor-
 186 tional to the grazing amplitude, i.e., $U(L_S, \tilde{t}) = q(\omega_n \tilde{t}) \Delta_G$.

187 **4. Impact force models**

188 As mentioned in Section 1, there are several ways of modeling vibro-impact forces.
 189 In this work three impact force models will be considered. The piecewise-linear Kelvin-
 190 Voigt model, the Power-law function impact force and a combination of the two called
 191 the modified Kelvin-Voigt model. This section describes these models, showing their
 192 physical basis, advantages and limitations.

193 *4.1. Kelvin-Voigt model*

194 The mass and stops are mounted in different locations of the beam, being not coin-
 195 cident ($L_m \neq L_C$), see Fig. 1c,d. Thus the vibrations of the lumped mass are not directly
 196 constrained by the stops, allowing points of the beam to suffer further deformations be-
 197 yond those necessary to reach the stops. With first-mode beam oscillations, in a contact
 198 configuration only a segment of the beam, from L_C to L_m , suffers further deformation.
 199 This segment can be considered as a lumped mass beam in a pinned-free configuration,
 200 with its own dynamic properties, such as natural frequency and damping.

201 So one can model the contact configuration applying Kelvin-Voigt's approach to the
 202 beam segment as well. This secondary system is inactive by default, being active only
 203 if the measured transverse mass displacement exceeds the measured grazing amplitude,
 204 Δ_G . For symmetric bilateral impacts:

$$\tilde{f}_C(U, \dot{U}) = \begin{cases} 0, & \text{if } |U_C| \leq \Delta_C, \\ (K_C - K_S)(U_C - \Delta_C) + D_C \dot{U}_C, & \text{if } U_C \geq \Delta_C, \\ (K_C - K_S)(U_C + \Delta_C) + D_C \dot{U}_C, & \text{if } U_C \leq -\Delta_C, \end{cases} \quad (12)$$

205 where K_C and D_C are the stiffness and damping coefficients for the beam in the contact
 206 configuration and $U_C = U(L_C, \tilde{t})$ because the force it is being applied at L_C . Its non-
 207 dimensional version can be obtained by applying the same steps taken from Eq. (2)
 208 to Eq. (10), giving:

$$f_C(q, \dot{q}) = \begin{cases} 0, & \text{if } |q| \leq 1, \\ \phi_C^2((\omega_R^2 - 1)(q - 1) + 2\beta_C\omega_R\dot{q}), & \text{if } q \geq 1, \\ \phi_C^2((\omega_R^2 - 1)(q + 1) + 2\beta_C\omega_R\dot{q}), & \text{if } q \leq -1, \end{cases} \quad (13)$$

where:

$$\beta_C = \frac{D_C}{2M_S\omega_C}, \quad \omega_C^2 = K_C/M_S, \quad \omega_R = \omega_C/\omega_n, \quad \phi_C = \phi(x_C) = \frac{\Delta_C}{\Delta_G} \quad (14)$$

are the contact damping ratio, the contact natural frequency, the natural frequency ratio and the cantilever beam mode shape evaluated at x_C , which is equal to Δ_C/Δ_G .

The factor ϕ_C is purely geometric, accounting for the distance between impact and measurement points. For the experimental setup in Fig. 1: $\phi_C = 0.241$. Comparing the locations of measurement and impact on the beam, see Figs. 1 and 3, one can see that the contact point is closer to the beam's clamped end than the measurement point, so for first mode oscillations, $\phi_C < 1$ attenuating the impact force. On the other hand, swapping impact and measurement positions would strengthen the impact force.

Despite its straightforward physical interpretation, Fig. 4a illustrates that KV's model dissipative term is discontinuous with respect to velocity. Also, the impact force changes its sign at the end of the contact, indicating that the impacting element is being pulled towards the stop again, a situation which does not make sense from the physical point of view.

Besides the drawbacks mentioned previously, the impact condition enables one to estimate the impact duration. From the contact intervals (where $|q| \geq 1$) shown in Fig. 4b, one can define the normalized contact duration for a full oscillation as:

$$\tau = 2(\pi - 2\varphi_0)/\Omega, \quad (15)$$

where $\varphi_0 = \arcsin(1/Q)$ is the grazing angle and Q is the first-order modal amplitude, which heavily depends on the model parameters, ω_R and β_C . When using this model, one can estimate the normalized contact duration for any pair (Ω, Q) as shown in Fig. 4c, where curves of constant τ are presented with normalized experimental data (crosses). The experimental data was taken from Fig. 1c and normalized using the fundamental linear natural frequency and the measured grazing amplitude, given by $f_n \approx$

Figure 4: (a): Elastic and dissipative components of the piecewise linear impact force model ($\omega_R = 7.278, \beta_C = 0.258$ and $\phi_C = 0.241$). (b): Time series of elastic Kelvin-Voigt impact force together with $q = Q \sin(\varphi)$. (c): Normalized experimental data (red x's) and curves of constant normalized contact duration (blue lines), the numbers on each curve represent the contact duration.

232 7.6 Hz and $\Delta_G \approx 1.6$ mm, respectively. As expected, the contact duration grows for
 233 higher amplitudes Q .

234 4.2. Power-law impact force model

235 The drawbacks of Kelvin and Voigt's model can be overcome by approximating the
 236 impact force by a power function. However, instead of considering powers of impact
 237 deformation ($U_C \pm \Delta_C$) as done by Hunt and Crossley [4], one can consider powers of
 238 the ratio between displacement and the gap width, U_C/Δ_C . With this approach, one can
 239 define a smooth force impact which is weak in the region $|U_C| < \Delta_C$ but grows very
 240 fast around the gap width, $\pm\Delta_C$. Also, the participation of the Power-law impact force
 241 on the non-contact regime is not necessarily a problem; a similar effect can actually
 242 occur due to a small measurement error on the gap width, shifting the ratio U_C/Δ_C a
 243 little bit away from 1, which is the value where the power function starts to grow faster.
 244 Also, the participation in the non-contact phase can be increased/decreased according
 245 to model parameters. For symmetric bilateral impacts, the elastic force should be an
 246 odd exponent power, while the dissipative force should have an even exponent term
 247 multiplying the velocity. So:

$$\tilde{f}_C(U, \dot{U}) = F \left(\frac{U_C}{\Delta_C} \right)^{2n-1} + D_C \left(\frac{U_C}{\Delta_C} \right)^{2p} \dot{U}_C, \quad (16)$$

248 where $n \geq 1$ and $p \geq 0$ are integer exponents for the restoring and dissipative terms,
 249 respectively. As U_S/Δ_G is non dimensional, D_C has the same physical dimension of a
 250 linear viscous damping coefficient, while F has dimensions of force. So, considering
 251 $F = K_C \Delta_G$ and applying the same steps as from Eq. (2) to Eq. (10), one can obtain the
 252 non-dimensional Power-law impact force as:

$$f_C(q, \dot{q}) = \phi_C \omega_R^2 q^{2n-1} + 2\beta_C \omega_R q^{2p} \dot{q} \phi_C^2, \quad (17)$$

253 where β_C , ϕ_C and ω_R were defined previously for the KV force model. However, it
 254 should be mentioned that despite having dynamic parameters with the same name and
 255 definition, the models are not equivalent. With the exception of ϕ_C , one should not ex-
 256 pect their parameters to have similar numeric values or physical meaning. For instance,

Figure 5: Elastic and dissipative components of Power-law impact force model for $q = Q \sin(\varphi)$, $\omega_R = 0.715$, $\beta_C = 0.659$, $n = 4$, $p = 3$ and $\phi_C = 0.241$.

the terms ω_R and β_C have no physical meaning for the Power-law model. Also, notice that ϕ_C affects the restoring and dissipative parts of the impact force differently.

Comparing Fig. 4a and Fig. 5, the smoothness of the Power-law model is clearly seen. Instead of jumping to a nonzero value, the dissipative Power-law force grows smoothly. However, nonphysical sign changes similar to KV model are possible if the model is purely dissipative, $\omega_R \equiv 0$.

Finally, it is important to remember that numeric simulations involving the Power-law model become demanding for high n and p , due to the stiffening of the ODE, losing the advantage of being easier to simulate than the KV model, which is given by a piecewise linear function.

4.3. Modified Kelvin-Voigt model

While the Power-law model solves the problems presented by Kelvin and Voigt's model, it also removes most of the physical meaning of the parameters involved in the impact phenomenon. In order to obtain the expected behavior for the impact force while keeping part of the physical insight given by the KV approach, a combination of both models is presented.

273 As the dissipative term seems to be the problematic part of the KV model, one
 274 can multiply it by a power of the displacement. However, instead of using U_C/Δ_C as
 275 power term, one can add ± 1 to it, obtaining $(U_C/\Delta_C \pm 1)$. In this way, it is assured that
 276 the damping term starts to grow smoothly from zero, without any discontinuity. Also,
 277 this term can be rewritten as $(U_C \pm \Delta_C)/\Delta_C$ which can be interpreted as a mechanical
 278 impact strain. Another option would be using the impact deformation $(U_C \pm \Delta_C)$ as
 279 power term, but this term is highly dependent on the gap width, which is very small
 280 and would demand an extremely high damping coefficient to compensate for it. So, the
 281 combined model is given by:

$$\tilde{f}_C(U, \dot{U}) = \begin{cases} 0, & \text{if } |U_C| \leq \Delta_C, \\ (K_C - K_S)(U_C - \Delta_C) + D_C \dot{U}_C \left(\frac{U_C}{\Delta_C} - 1\right)^{2p}, & \text{if } U_C \geq \Delta_C \\ (K_C - K_S)(U_C + \Delta_C) + D_C \dot{U}_C \left(\frac{U_C}{\Delta_C} + 1\right)^{2p}, & \text{if } U_C \leq -\Delta_C, \end{cases} \quad (18)$$

282 which leads to the original KV model if $p = 0$. The nondimensionalization of this
 283 expression gives:

$$f_C(q, \dot{q}) = \begin{cases} 0, & \text{if } |q| \leq 1, \\ \phi_C^2 \left((\omega_R^2 - 1)(q - 1) + 2\beta_C \omega_R \dot{q} (q - 1)^{2p} \right), & \text{if } q \geq 1, \\ \phi_C^2 \left((\omega_R^2 - 1)(q + 1) + 2\beta_C \omega_R \dot{q} (q + 1)^{2p} \right), & \text{if } q \leq -1, \end{cases} \quad (19)$$

284 with β_C and ω_R being the same as before and ω_R keeping its straightforward physical
 285 meaning. This model can be seen as a particular case of Hunt and Crossley's model [4],
 286 which originally assumed the elastic term to be a power function as well.

287 The advantages of mixing both models can be seen by comparing Fig. 6 with Fig. 4a
 288 and Fig. 5. While the original KV model changes its sign during impact, leading to non-
 289 physical situations, and the Power-law model is active also in the non-contact phase,
 290 the combined model has none of this undesired behaviors. Also, it is possible to obtain
 291 the normalized contact duration for a full oscillation using this model, shown in Fig. 4c
 292 and given by Eq. (15). However, all the three models considered in this work present
 293 nonphysical sign changes, when elastic forces are neglected.

Figure 6: Elastic and dissipative components of modified Kelvin-Voigt impact force model for $q = Q \sin(\varphi)$, $\omega_R = 7.306$, $\beta_C = 4.636$, $p = 1$ and $\phi_C = 0.241$.

5. Approximate analysis of time periodic response

After the presentation of three different models for the impact force, it is necessary to solve the equation of motion, Eq. (10), for each of them. That is the objective of this section, which employs two analytical techniques commonly used to obtain approximated solutions of nonlinear systems.

5.1. Harmonic Linearization

The method of Harmonic Linearization assumes that the steady-state response of a nonlinear system, like Eq. (10), can be well approximated by a mono-frequency solution, neglecting terms with higher harmonics [33]. Then one can approximate model nonlinearities, like $f_C(q, \dot{q})$ in Eq. (10), using a single harmonic Fourier series:

$$f_C(q(t), \dot{q}(t)) \approx a_1 \sin(\Omega t) + b_1 \cos(\Omega t), \quad (20)$$

where:

$$a_1 = \frac{2}{T} \int_0^T f_C(q(t), \dot{q}(t)) \sin(\Omega t) dt, \quad b_1 = \frac{2}{T} \int_0^T f_C(q(t), \dot{q}(t)) \cos(\Omega t) dt \quad (21)$$

are Fourier coefficients, Ω is the forcing frequency and T is the forcing period. To obtain some physical insight, this approximation can be rearranged as a spring-dashpot system, $f_C(q, \dot{q}) \approx \kappa q + 2\sigma \dot{q}$, whose coefficients can be found by comparing the responses of its mechanical and Fourier approximations to a solution $q(t) = Q \sin(\Omega t + \theta)$, as follows:

$$f_C(q(t), \dot{q}(t)) \approx \kappa Q \sin(\Omega t + \theta) + 2\sigma \Omega Q \cos(\Omega t + \theta) \approx a_1 \sin(\Omega t) + b_1 \cos(\Omega t) \quad (22)$$

Defining $\varphi = \Omega t + \theta$, the harmonic linearization coefficients κ and σ can be defined as:

$$\kappa = \frac{1}{\pi Q} \int_0^{2\pi} f_C(q(\varphi), \dot{q}(\varphi)) \sin \varphi \, d\varphi, \quad (23)$$

$$\sigma = \frac{1}{2\pi Q \Omega} \int_0^{2\pi} f_C(q(\varphi), \dot{q}(\varphi)) \cos \varphi \, d\varphi. \quad (24)$$

Substituting the spring-dashpot approximation into Eq. (10) leads to:

$$\ddot{q} + 2(\beta_S + \sigma)\dot{q} + (1 + \kappa)q = \Omega^2 \gamma \tilde{V} \sin(\Omega t), \quad (25)$$

which corresponds to a standard mechanical oscillator with a pair of springs and dampers in parallel. Its stationary frequency response is given by $q(t) = Q \sin(\Omega t + \theta)$, where:

$$Q = \frac{\Omega^2 \gamma \tilde{V}}{\sqrt{(\Omega^2 - 1 - \kappa)^2 + 4\Omega^2(\beta_S + \sigma)^2}}, \quad (26)$$

or:

$$Q^2 \left((\Omega^2 - 1 - \kappa)^2 + 4\Omega^2 (\beta_S + \sigma)^2 \right) = (\Omega^2 \gamma \tilde{V})^2. \quad (27)$$

The difference between the current case and a standard mechanical oscillator is that the harmonic linearization coefficients κ and σ depend of the response amplitude, Q . Both coefficients will be defined in Section 5.3 for different models of the impact force.

319 5.2. Averaging

320 Standard first-order averaging [34] implicitly uses the same assumption made for
 321 harmonic linearization, that higher harmonics can be neglected and the solution can be
 322 approximated by a single harmonic expression, $q(t) \approx Q \sin \varphi$, $\varphi = \Omega t + \theta$. However
 323 standard averaging is restricted to weakly nonlinear systems only [35], i.e., systems
 324 with small forcing amplitude $\tilde{V}$, damping β_S and nonlinearities $f_C(q, \dot{q})$. The equation
 325 of motion can be written as a weakly nonlinear system in the following way:

$$\ddot{q} + q = \epsilon(\Omega^2 \gamma \tilde{V} \sin(\Omega t) - 2\beta_S \dot{q} - f_C(q, \dot{q})) \quad (28)$$

326 where the parameter $\epsilon \ll 1$ indicates which terms are small in comparison with linear
 327 terms. Another feature of the standard averaging is that it allows both amplitude and
 328 phase to vary in time, i.e. $Q = Q(t)$ and $\theta = \theta(t)$. Defining the time-derivative of
 329 the solution as $\dot{q}(t) \approx \Omega Q \cos \varphi$ implies that $\dot{\theta} Q \cos \varphi = -\dot{Q} \sin \varphi$. Substituting these
 330 relations into Eq. (28) results in a system of differential equations in terms of amplitude
 331 and phase:

$$\begin{aligned} \Omega \dot{Q} = & \frac{1}{2} Q(\Omega^2 - 1) \sin 2\varphi + \epsilon(\cos \varphi(\gamma \tilde{V} \Omega^2 \sin(\varphi - \theta) + \\ & - f_C(q, \dot{q})) - 2\Omega \beta_S Q \cos^2 \varphi) \end{aligned} \quad (29)$$

$$\begin{aligned} \Omega Q \dot{\theta} = & -Q(\Omega^2 - 1) \sin^2 \varphi + \epsilon(\sin \varphi(f_C(q, \dot{q}) + \\ & - \gamma \tilde{V} \Omega^2 \sin(\varphi - \theta)) + \beta_S \Omega Q \sin 2\varphi) \end{aligned} \quad (30)$$

332 Almost all terms on the right-hand sides of these equations are multiplying ϵ and there-
 333 fore are small. Around the primary external resonance the term $\Omega^2 - 1$ is also small
 334 and so the whole right-hand side of these differential equations is small, which means
 335 that both Q and θ vary slowly with time. In this case one can approximate Eqs. (29)
 336 and (30) by their average over one period of oscillations, obtaining:

$$2\pi \Omega \dot{Q} = -\epsilon(2\pi \Omega \beta_S Q + \pi \gamma \tilde{V} \Omega^2 \sin \theta + \int_0^{2\pi} f_C(q(\varphi), \dot{q}(\varphi)) \cos \varphi d\varphi), \quad (31)$$

$$2\pi \Omega Q \dot{\theta} = -\epsilon \pi Q(\Omega^2 - 1) + \epsilon \left(\int_0^{2\pi} f_C(q(\varphi), \dot{q}(\varphi)) \sin \varphi d\varphi - \pi \gamma \tilde{V} \Omega^2 \cos \theta \right), \quad (32)$$

where ϵ is multiplying the term $(\Omega^2 - 1)$ to explicitly indicate that this term is small. The integrals of the impact force can be replaced by the harmonic linearization coefficients κ , σ given by Eqs. (23) and (24). The equilibria of the resulting system correspond to the steady-state response of Eq. (10), so:

$$\Omega^2 \gamma \tilde{V} \sin \theta + 2\Omega Q(\beta_S + \sigma) = 0, \quad (33)$$

$$Q(1 + \kappa - \Omega^2) - \Omega^2 \gamma \tilde{V} \cos \theta = 0. \quad (34)$$

Eliminating θ from the last expressions gives:

$$(\gamma \tilde{V} \Omega^2)^2 = Q^2((\Omega^2 - 1 - \kappa)^2 + 4\Omega^2(\beta_S + \sigma)^2), \quad (35)$$

which is identical to Eq. (27). This same relation could be obtained using Multiple Scales or modified Lindstedt-Poincaré [24] methods, but Averaging was chosen due to its well developed mathematical basis and effectiveness for other vibro-impacting systems [21, 22]. In this context, despite its loose mathematical basis, Harmonic Linearization can be seen as a shortcut method, giving the same results for stationary solutions as Averaging, but bypassing its intermediate steps such as defining small terms, obtaining a system of differential equations for amplitude and phase and finding its equilibrium state.

5.3. Application to specific impact force models

5.3.1. Kelvin-Voigt impact force

As mentioned previously, Kelvin-Voigt's impact force is not active during the whole period of oscillation. So, the limits of integration in Eqs. (23) and (24) can be restricted to the contact period, i.e. for $\varphi \in [\varphi_0; \pi - \varphi_0]$ for $q \geq 1$ and $\varphi \in [\pi + \varphi_0; 2\pi - \varphi_0]$ for $q \leq -1$, as shown in Fig. 4b. Considering $q \approx Q \sin \varphi$ and substituting Eq. (13) into Eqs. (23) and (24) leads to:

$$\kappa = \phi_C^2(\omega_R^2 - 1)W(Q), \quad (36)$$

$$\sigma = \phi_C^2 \beta_C \omega_R W(Q), \quad (37)$$

where:

$$W(Q) = 1 - \frac{2}{\pi}(\varphi_0 + \frac{1}{2} \sin(2\varphi_0)), \quad \varphi_0 = \arcsin(1/Q), \quad (38)$$

358 is a function of the forcing amplitude Q . Despite its complicated formula, $W(Q)$ does
 359 not grow unbounded, being limited between 0 and 1 for Q between 1 and infinity.
 360 Substituting κ and σ in the frequency response, Eq. (27) leads to:

$$Q^2((\Omega^2 - 1 - \phi_C^2(\omega_R^2 - 1)W(Q))^2 + 4\Omega^2(\beta_S + \phi_C^2\beta_C\omega_R W(Q))^2) - (\Omega^2\gamma\tilde{V})^2 = 0. \quad (39)$$

361 As $W(Q)$ varies between 0 and 1, it works as a tuning parameter between two linear
 362 oscillators, one given by Eq. (10) and $f_C \equiv 0$, and another given by the same equation
 363 and $f_C \equiv 0$, but with natural frequency ω_R instead of unity and damping $\beta_S + \beta_C\omega_R$
 364 instead of β_S . The frequency response equations for these oscillators are given by
 365 respectively:

$$Q^2((\Omega^2 - 1)^2 + 4\Omega^2\beta_S^2) - (\Omega^2\gamma\tilde{V})^2 = 0, \quad (40)$$

$$Q^2((\Omega^2 - \phi_C^2\omega_R^2)^2 + 4\Omega^2(\beta_S + \phi_C^2\beta_C\omega_R)^2) - (\Omega^2\gamma\tilde{V})^2 = 0. \quad (41)$$

366 5.3.2. Power-law impact force

367 Substituting Eq. (17) into Eqs. (23) and (24) leads to:

$$\kappa = 2\phi_C G_n \frac{\omega_R^2}{Q^2}, \quad (42)$$

$$\sigma = \phi_C^2\beta_C\omega_R \frac{G_p}{p+1}, \quad (43)$$

368 where:

$$G_j = \frac{Q^{2j}}{\sqrt{\pi}} \frac{\Gamma(j+1/2)}{\Gamma(j+1)}, \quad j = n, p \quad (44)$$

369 is a function of the exponent n or p and the steady-state amplitude Q and Γ denotes
 370 the Gamma function. As expected for a power-function term, G_j grows unbounded for
 371 $j \rightarrow \infty$ and $|Q| > 1$. Substituting κ and σ in the frequency response equation, Eq. (27)
 372 gives:

$$Q^2((\Omega^2 - 1 - 2\phi_C G_n \omega_R^2 / Q^2)^2 + 4\Omega^2(\beta_S + \phi_C^2\beta_C\omega_R G_p / (p+1))^2) - (\Omega^2\gamma\tilde{V})^2 = 0. \quad (45)$$

373 5.3.3. Modified Kelvin-Voigt impact model

374 The equivalent stiffness κ is given by Eq. (36) and the equivalent damping is given
375 by:

$$\begin{aligned} \pi\sigma = & \beta_C \omega_R \phi_C^2 \int_{\varphi_0}^{\pi-\varphi_0} \cos^2 \varphi (Q \sin \varphi - 1)^{2p} d\varphi + \\ & + \beta_C \omega_R \phi_C^2 \int_{\pi+\varphi_0}^{2\pi-\varphi_0} \cos^2 \varphi (Q \sin \varphi + 1)^{2p} d\varphi, \end{aligned} \quad (46)$$

376 whose integration limits are the same as the original Kelvin-Voigt model. When $p = 0$,
377 σ is given by Eq. (37), but there is no general formula for this integral as a function
378 of Q and p . As an example of its complexity, the formula for σ when $p = 1$ is shown
379 below:

$$\sigma = \frac{\beta_C \omega_R \phi_C^2}{4\pi} \left((Q^2 + 4)(\pi - 2\varphi_0) - \frac{2}{3} \cos \varphi_0 (13Q + 2 \sin \varphi_0) \right) \quad (47)$$

380 6. Model tuning and analysis

381 The frequency response for each impact model can be obtained by substituting the
382 harmonic linearization coefficients found in Section 5.3 into the generic amplitude-
383 frequency relationship, Eq. (27). Then one can use a nonlinear least squares solver to
384 fit the frequency response equations of each model to experimental data, such as the
385 one from Fig. 2a. Again, the experimental data is normalized using the fundamental
386 linear natural frequency, $f_n \approx 7.6$ Hz, and grazing amplitude $\Delta_G \approx 1.6$ mm.

387 6.1. Linear range

388 Before defining appropriate values for the impact force model parameters, one
389 should deal with the conversion gain γ and structural damping ratio β_S , which are
390 necessary to model both impacting and non-impacting vibrations. Fig. 7 illustrates the
391 normalized linear experimental frequency response together with its model approxima-
392 tions for $\gamma = 0.195 \text{ V}^{-1}$ and $\beta_S = 14 \times 10^{-3}$. Despite been able to reproduce the reso-
393 nance peak accurately, model predictions deteriorate outside the peak neighbourhood.
394 That can be explained by the unmodeled dynamics of the subsystem shaker-platform.

Figure 7: Linear frequency response, $\tilde{V} = 0.125$ V.

For vibro-impact oscillations the excitation amplitude was chosen as $\tilde{V} = 0.6$ V in order to maximize the nonlinear frequency range without saturating the displacement sensors.

6.2. Power function exponents

The fitting of models with power function was made considering pairs of $1 \leq n \leq 10$ and $0 \leq p \leq 10$ as constant while varying ω_R and β_C . This revealed some insights into the role of the power exponents on the frequency response, see Fig. 8. For instance, by increasing p the Power-law model response gets closer to the experimental upper folding point (Fig. 8b). In this case, the fitted ω_R keeps almost constant, while β_C decreases. By increasing n , as shown in Fig. 8a, the model response gets closer to both the upper and lower folding points, going further away from the upper one if n grows too much. Also, by increasing n the model response approaches the curvature of the bent peak. In this case, the fitted β_C goes up, while ω_R goes down. This approach to the lower folding point and peak curvature for growing exponents can be explained by the fact that for small n and p the Power-law force is not negligible in the non-impacting region. For the MKV model, Fig. 8c shows that there are no significant changes for $1 \leq p \leq 5$, with the parameter ω_R remaining constant while β_C grows.

Figure 8: Comparison of power exponents on frequency response of the Power-law model (a) varying n and $p = 3$, (b) varying p and $n = 4$, and (c) varying p for the Modified Kelvin-Voigt model. $\tilde{V} = 0.6$ V. The arrows indicate how the frequency response changes as the value of each parameter increases.

Based on this analysis, the model parameters are chosen as the ones which led their models' frequency response closer to experimental data with the lower exponent. For the MKV model, the exponent $p = 1$ is chosen as the one whose response is closer to

Table 1: Fitted model parameters.

Model	Parameter			
	ω_R	β_C	n	p
Kelvin-Voigt	7.278	0.258	—	—
Power-law	0.715	0.659	4	3
Mod. Kelvin-Voigt	7.306	4.636	—	1

415 experimental data. The fitted model parameters are shown in Table 1 were obtained
 416 by applying the same reasoning to all models. In this table one can see that the MKV
 417 model has lower exponent coefficient than the Power-law model. That can be explained
 418 by the fact that both KV models are only defined on contact areas, while the power
 419 function is defined also on the non-contact region and demands high order exponents
 420 to be negligible in this area.

421 6.3. Model validation

422 To check the validity of the assumptions made, the fitted values shown in Table 1
 423 are used to obtain the frequency response of the system by numeric simulation of the
 424 single-DOF model from Eq. (10) together with the force models from Eqs. (13), (17)
 425 and (19) using a standard MATLAB[®] ODE solver `ode45()` together with the option
 426 ‘Events’ to handle the transition between impacting and non-impacting regimes
 427 when necessary.

428 Looking at the results from the perturbation methods, numeric simulations and
 429 experiments shown in Fig. 9 one can see that all of the mentioned mathematical mod-
 430 els give results reasonably close to experimental observations. Thus, the assumptions
 431 made appear adequate. Also, one can see that the modified Kelvin-Voigt model is
 432 able to predict the experimental behavior more precisely, capturing both fold points,
 433 while Kelvin-Voigt’s classic model is further away from the upper folding point and
 434 the Power-law model is further away from both folding points.

Figure 9: Comparison of frequency responses obtained using (a) the Kelvin-Voigt model (b) the Power-law model and (c) the Modified Kelvin-Voigt model. $\tilde{V} = 0.6$ V.

6.4. Parameter analysis

After finding appropriate parameter values for the impact force models, one can analyze the individual effects of each parameter on the frequency response. That can expand one's knowledge about the different models, being used to design other vibro-impacting devices, which enhance or mitigate certain effects presented here such as the maximum response amplitude or the length and curvature of the bent peak.

Starting with Kelvin-Voigt's model, shown in Fig. 10, one can see that as expected, the peak curvature is very sensitive to ω_R , with the peak length (hysteresis region) growing as ω_R increases. Also, by increasing the contact damping one can decrease the oscillation amplitude.

Similarly, for the Power-law impact model, shown in Fig. 11, one can see that the parameters related to the elastic force control the peak curvature and length, while the ones related to dissipative forces control the response amplitude. Increasing β_C and p causes the response amplitude to decrease. While increasing ω_R moves both folding

Figure 10: Parameter analysis of Kelvin-Voigt's model varying (a) ω_R and (b) β_C . The arrows indicate how the frequency response changes as the value of each parameter increases.

Figure 11: Parameter analysis of Power-law model varying (a) ω_R , (b) β_C , (c) n and (d) p . The arrows indicate how the frequency response changes as the value of each parameter increases.

points to the right, increasing n moves the upper folding point to the right and the lower one to the left. Also, only the restoring force parameters have an influence on the non-impacting region of the frequency response.

The influence of the natural frequency ratio and contact damping ratio are the same for both versions of Kelvin-Voigt's model, see Figs. 10 and 12a,b. However, different

Figure 12: Parameter analysis of Modified Kelvin-Voigt model varying (a) ω_R , (b) β_C and (c) p . The arrows indicate how the frequency response changes as the value of each parameter increase.

from the Power-law model, the amplitude increases for higher values of p , see Fig. 12c. This happens because the dissipative force is proportional to $(q \pm 1)^{2p}$, which, for lower exponents, is stronger in the neighborhood of impact, $Q \approx 1$. As p increases the dissipative term becomes negligible for low impacting amplitudes. Also, it is worth to point out that despite the modified Kelvin-Voigt model being equivalent to its standard version for $p = 0$, the frequency response on Fig. 12c is not equivalent to the one in Fig. 9a for the standard Kelvin-Voigt model because the contact damping has different numerical values.

7. Conclusions

The experimental behavior of an impacting forced lumped mass cantilever beam around its first resonance was modeled as a single degree of freedom oscillator. Harmonic decomposition of the frequency response of the experimental setup showed that higher harmonic components were negligible if compared with the fundamental one.

467 The SDOF model was analyzed using Averaging and Harmonic Linearization to obtain
468 frequency-amplitude relations. Numeric values for model parameters were obtained by
469 fitting the nonlinear frequency response relationship to the experimental frequency re-
470 sponse obtained using control-based continuation. After choosing the most appropriate
471 model parameters, their role on the frequency response was also analyzed.

472 For all the impact models discussed in the present work, it was found that length and
473 curvature of the bent peak are governed by the parameter ω_R , defined as the natural fre-
474 quency ratio, while the contact damping ratio controls the amplitude of vibration. For
475 the Power-law model, while the exponent of the elastic term has an influence on both
476 impacting and non-impacting regions of the frequency response, the damping exponent
477 affects only the amplitude of oscillation of the impacting region. Also, the exponent of
478 the dissipative term has opposite effects on MKV and power function models.

479 Experimental observations could be reproduced by all of the impact force mod-
480 els under analysis, with the modified Kelvin-Voigt model describing the experimental
481 frequency response more accurately, predicting both fold points, while the other mod-
482 els fail to predict the upper fold point. The reason for that relies on the modified KV
483 model being a combination of the other models, mixing their positive characteristics
484 to obtain a physically more accurate model. Nevertheless, numeric simulations of the
485 MKV model are more demanding than the continuous Power-law model, due to its
486 non-smoothness with respect to displacement and velocity. Also, the lack of a general
487 compact formula for the equivalent damping term, which has to be obtained for every
488 power exponent, gives rise to complicated expressions.

489 The experimental setup could be modified in order to analyze systems whose mass
490 displacement is directly limited by the stops, becoming a test bed for experimental val-
491 idation of kinematic impact models together with nonsmooth transformations [8, 36],
492 thus, making it possible to compare also to kinematically based vibro-impact models
493 in terms of ease of use, applicability and reliability.

494 The discussion presented in this work can be applied to the analysis of other me-
495 chanical systems with non-rigid constraints and whose steady-state oscillations are
496 single-periodic. However, for different parameter configurations, the considered mod-
497 els can produce other qualitative behaviors, such as quasiperiodicity and chaos, not

discussed here.

Acknowledgments

The authors express their acknowledgment to CAPES through the Science Without Borders Program with the process number: 99999.013736/2013-02, which sponsored the elaboration of this paper.

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