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A Cycle of Maximum Order in a Graph of High Minimum Degree has a Chord

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Abstract

A well-known conjecture of Thomassen states that every cycle of maximum order in a 3-connected graph contains a chord. While many partial results towards this conjecture have been obtained, the conjecture itself remains unsolved. In this paper, we prove a stronger result without a connectivity assumption for graphs of high minimum degree, which shows Thomassen's conjecture holds in that case. This result is within a constant factor of best possible. In the process of proving this, we prove a more general result showing that large minimum degree forces a large difference between the order of the largest cycle and the order of the largest chordless cycle.

Keywords: Euclidean Ramsey theory; Plane; Combinatorial Geometry

1 Introduction

Consider the following famous conjecture of Thomassen:

Conjecture 1 (see [1, 3, 9], but originally conjectured in 1976)**.** *Every largest cycle in every* 3*-connected graph contains a chord.*

This is an elegant conjecture, tempting both for the simplicity of its formulation and for how it would illuminate something of the structure of large cycles in 3-connected graphs. While the conjecture currently remains unsolved, many partial results have been discovered. Thomassen [10] showed that Conjecture 1 holds when we also assume that G

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is cubic. (This result is particularly intriguing, since intuitively the cubic result should probably be the hardest case, since extra edges should increase the potential for chords, but this does not seem to be the case.) Considering sparse graphs more generally, Li and Zhang [6] verified the conjecture for graphs embedded in the projective plane with minimum degree at least 4, and Li and Zhang [7] verified the conjecture when G is 4-connected and embedded in a torus or a Klein bottle. The first result regarding planar graphs was due to Zhang [13], who showed the conjecture holds for cubic planar graphs or planar graphs with minimum degree at least four. Subsequently, Kawarabayashi et al. [5] verified the conjecture for locally 4-connected planar graphs, and Birmelé [2] verified the result for every 3-connected graph with no $K_{3,3}$-minor. Wu et al. [12] verified the result for certain classes of graphs that have a bounded number of removable edges.

Most of the previous results involve very sparse classes of graphs; now we shall explicitly consider dense graphs. In highly dense graphs, the result is clear; for example, if an n-vertex graph G has minimum degree $\delta(G) \geqslant \frac{1}{2}n$ then it is Hamiltonian [4] and so (as long as $n \geqslant 5$) any cycle of maximum order contains a chord. Similarly, it is quite easy to prove that if $\delta(G) \geqslant \frac{1}{3}n + 2$, then any largest cycle contains a chord. (Briefly, allow C to be a cycle of maximum order and assume it has no chord. Then for any edge e of C there is a vertex outside of C adjacent to both end vertices of e, hence there is a cycle longer than C, a contradiction.) Note that neither of these results require us to make any assumption about connectivity. Hence, it is worth determining what bound on the minimum degree is required to force any cycle of maximum order to contain a chord (with no connectivity assumption).

Consider the following key example. Start with an r-vertex cycle C, and for each vertex v of C add a unique copy of K_{r-1} together with $r-1$ edges from v to this new clique. This graph contains r^2 vertices and has minimum degree $r-1$. However, the order of a longest cycle in this graph is r, and C itself is an r-vertex cycle with no chord. Hence this graph shows that minimum degree of $\sqrt{n}-1$ in an n-vertex graph is insufficient to force our desired outcome. Note the graph we constructed is not 2-connected (every vertex of C is a 1-cut) and as such Conjecture 1 is unaffected. We make the following conjecture:

Conjecture 2. *If $\delta(G) > \sqrt{n} - 1$ then any cycle of maximum order in G contains a chord.*

This conjecture would be optimal. In this article we prove a slightly weaker form of this conjecture, which is within a constant factor of optimal:

Theorem 3. *If $\delta(G) \geqslant \frac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n} + 1$ then any cycle of maximum order in G contains a chord.*

As part of proving this result, we instead prove a stronger result about the relationship between chordless cycles and cycles more generally. Given a graph G that is not a forest, let $c(G)$ denote the order of the longest cycle in G, and let $c'(G)$ denote the order of the longest chordless cycle in G. Clearly $c'(G) \leqslant c(G)$. Using this, we can re-frame the Conjecture 1 as follows:

Conjecture 4. *If G is 3-connected, then $c'(G) \leqslant c(G) - 1$.*

Given this interpretation, it is reasonable to consider whether increasing the minimum degree will further increase the difference between $c'(G)$ and $c(G)$. Our main result in this article will be proving the following extension:

Theorem 5. *Let $k \in \mathbb{N}$. If $\delta(G) \geqslant \frac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n} + 5k - 4$ then $c'(G) \leqslant c(G) - k$.*

Theorem 3 follows directly from Theorem 5 by setting $k = 1$.

Consider the following graph H: start with an r-vertex cycle C, and for each vertex v of C construct a copy of K_{k+r-2} and add all edges from each v to its copy of K_{k+r-2}. Thus $\delta(H) = k + r - 2$ and H has $r(k + r - 1)$ vertices. Now if we fix r and fix an $\epsilon > 0$, note that $\delta(H) = k + r - 2 = (1 - \epsilon)k + (r - 2 + \epsilon k) > (1 - \epsilon)k + \sqrt{r(k + r - 1)}$ when we take k to be sufficiently large with respect to ϵ and r. However, the largest chordless cycle in H has r vertices, while a cycle of maximum order contains $r + k - 1$. Thus it is not true that if $\delta(G) \geqslant \sqrt{n} + (1 - \epsilon)k$ then $c'(G) \leqslant c(G) - k$, at least when k is large, and so Theorem 5 is within a constant factor of optimal for large k.

Before proving Theorem 5, we recall the following simple yet helpful result.

Lemma 6 (Dirac [4])**.** *A graph H contains a cycle of order at least $\delta(H) + 1$.*

2 Proof of Theorem 5

We suppose, for the sake of a contradiction, that $\delta(G) \geqslant \frac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n} + 5k - 4$ but $c'(G) \geqslant c(G) - k + 1$. Our result is trivial if $k = 0$ so we may assume $k \geqslant 1$. Let C_0 be a chordless cycle of maximum order in G; clearly $|V(C_0)| = c'(G)$. Let C be a cycle in G of maximal order such that $V(C_0) \subseteq V(C)$. It follows clearly that $c'(G) \leqslant |V(C)| \leqslant c(G) \leqslant c'(G) + (k - 1)$. Call the vertices of C_0 *red* vertices and the vertices of $C - C_0$ *blue* vertices; there are between 0 and $k - 1$ blue vertices. By Lemma 6 it follows $|V(C)| \geqslant c'(G) \geqslant c(G) - k + 1 \geqslant \frac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n} + 4k - 2$.

Claim 1. Red vertices of C have at least $\frac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n} + 4k - 5$ neighbours in $V(G - C)$.

Proof. Let v be a red vertex in $V(C)$. Thus v is adjacent to two other red vertices (since C_0 is a chordless cycle) and at most $k - 1$ blue vertices (as that is the maximum number of blue vertices). Hence v has at least $\delta(G) - 2 - (k - 1) \geqslant \frac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n} + 4k - 5$ neighbours in $V(G - C)$. □

Claim 2. There is no vertex $v \in V(G - C)$ such that $|N(v) \cap V(C)| \geqslant \frac{\sqrt{17}-3}{2\sqrt{2}}\sqrt{n} + k$.

Proof. Assume for the sake of a contradiction that such a vertex v exists. Define $X := N(v) \cap C$. Suppose there exist two vertices $y, z \in X$ that are sequential in the cycle. Then we can construct a new cycle C' from C by removing the edge yz and adding the vertex v and the edges yv, vz. But $|V(C')| > |V(C)|$ and $V(C') \supseteq V(C) \supseteq V(C_0)$, which contradicts our choice of C. Thus we let X' denote the set of all vertices that are clockwise adjacent to a vertex of X. Clearly $|X'| = |X| \geqslant \frac{\sqrt{17}-3}{2\sqrt{2}}\sqrt{n} + k$ and by the previous argument $X \cap X' = \emptyset$.

Consider two distinct vertices $y, z \in X$. Let $y', z' \in X'$ be the vertices clockwise adjacent to y, z respectively. Suppose there exists a vertex $w \in V(G - C)$ such that w is adjacent to y' and z'. Then we can construct a new cycle C' from C by removing the edges yy', zz' and adding the vertices v, w together with the edges $yv, vz, y'w, wz'$. The cycle C' is larger than C but $V(C')$ contains $V(C_0)$, contradicting the definition of C. Hence no such w exists, and as such the sets $N(u) \cap V(G - C)$ are vertex disjoint for all $u \in X'$. Let X'' be the subset of X' containing only the vertices of X' coloured red. Thus $|X''| \geqslant \frac{\sqrt{17}-3}{2\sqrt{2}}\sqrt{n}+1$ and by Claim 1, $|N(u)\cap V(G-C)| \geqslant \frac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n}+4k-5 \geqslant \frac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n}-1$ for each $u \in X''$. Hence

$$\begin{aligned}
|V(G - C)| &\geqslant \left| \bigcup_{u \in X''} N(u) \cap V(G - C) \right| \\
&= \sum_{u \in X''} |N(u) \cap V(G - C)| \\
&\geqslant |X''|\left(\tfrac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n} + 4k - 5\right) \\
&\geqslant \left(\tfrac{\sqrt{17}-3}{2\sqrt{2}}\sqrt{n} + 1\right)\left(\tfrac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n} - 1\right) \\
&\geqslant n + \tfrac{3}{\sqrt{2}}\sqrt{n} - 1 \\
&> n.
\end{aligned}$$

This is clearly a contradiction. $\square$

The following claim follows from Claim 2 directly.

Claim 3. $\delta(G - C) \geqslant \frac{3}{\sqrt{2}}\sqrt{n} + 4k - 4$.

Our next goal is to determine some useful results about the structure of the subgraph $G - C$, so we shall focus on that for the time being. Consider the block decomposition of $G - C$. Let $\mathcal{B}$ denote the set of blocks of $G - C$, each of which is either a maximal 2-connected subgraph or a bridge (together with its endvertices). (Note that there are no isolated vertices of $G - C$.) Let A be the set of cut-vertices of $G - C$. Let D be the forest with vertex set $A \cup \mathcal{B}$ and an edge between $a \in A$ and $B \in \mathcal{B}$ when $a \in B$. Recall that every tree of D corresponds to a connected component of $G - C$.

We call a block $B \in \mathcal{B}$ a *good block* if B contains a cycle of order at least $\sqrt{2}\sqrt{n}+4k-4$ which avoids the cut-vertices inside B. Consider a leaf node or an isolated vertex of D; since each cut-vertex must be in at least two blocks, it follows that each leaf or isolated vertex of D corresponds to a block of $\mathcal{B}$.

Claim 4. If $B \in \mathcal{B}$ corresponds to a leaf or isolated vertex of D, then B is a good block.

Proof. Let Z be the set containing all cut-vertices of B. Hence $|Z| = 0$ or 1. Let $H := G[V(B)\backslash Z]$ and apply Lemma 6 to H. Since $\delta(H) \geqslant \delta(G-C)-1$ and $\delta(G-C)-1 \geqslant \frac{3}{\sqrt{2}}\sqrt{n} + 4k - 5$, we obtain a cycle in B of order at least $\sqrt{2}\sqrt{n} + 4k - 4$ which avoids any cut-vertices. $\square$

Claim 5.

1. There are less than $\frac{1}{\sqrt{2}}\sqrt{n}$ good blocks in $\mathcal{B}$.
2. There are less than $\frac{1}{\sqrt{2}}\sqrt{n}$ leaves in D.
3. $\Delta(D) < \frac{1}{\sqrt{2}}\sqrt{n}$.
4. Any block contains less than $\frac{1}{\sqrt{2}}\sqrt{n}$ cut-vertices.

Proof. Because each good block contains a cycle of order $\sqrt{2}\sqrt{n}+4k-4 \geqslant \sqrt{2}\sqrt{n}$ (without any overlap between these cycles), there cannot be $\frac{1}{\sqrt{2}}\sqrt{n}$ or more such good blocks. Hence there are less than $\frac{1}{\sqrt{2}}\sqrt{n}$ leaves of D by Claim 4. There cannot be a vertex of D of degree $\frac{1}{\sqrt{2}}\sqrt{n}$ or larger without creating that many leaves. Finally, any block $B \in \mathcal{B}$ with $\frac{1}{\sqrt{2}}\sqrt{n}$ or more cut-vertices gives rise to a vertex of the same degree in D. $\square$

Claim 6. If $B \in \mathcal{B}$ contains a vertex that is not a cut-vertex then B is a good block.

Proof. Let Z be the set of cut-vertices of B, and set $H := G[B\backslash Z]$. Note that since $Z \neq V(B)$, H is well-defined, and that by Claim 5 that $|Z| < \frac{1}{\sqrt{2}}\sqrt{n}$. Apply Lemma 6 to H, and note that $\delta(H) \geqslant \delta(G-C) - |Z| > \frac{3}{\sqrt{2}}\sqrt{n} + 4k - 4 - \frac{1}{\sqrt{2}}\sqrt{n}$. Hence B contains a cycle of order at least $\delta(H) + 1 \geqslant \sqrt{2}\sqrt{n} + 4k - 3$ which avoids its cut-vertices, as required. $\square$

Claim 7. Every vertex of $G - C$ is in at least one good block.

Proof. Suppose, for the sake of a contradiction, there exists a vertex $v \in V(G-C)$ such that no good block contains v. By Claim 6 it follows that every block $B \in \mathcal{B}$ containing v only contains cut-vertices. Thus v and the vertices of $N(v) \cap V(G-C)$ are all cut-vertices, and therefore are all in A. Let D' be the tree of D containing v, and treat D' as being rooted at v. If $w \in N(v) \cap V(G-C)$ then let B_w denote the block containing v and w. Hence in D' there is a 3-vertex path v, B_w, w and thus every vertex of $N(v) \cap V(G-C)$ is exactly distance 2 from v in D'. Given there is at least one leaf of D' in each subtree of D' rooted by a vertex of $N(v) \cap V(G-C)$, it follows D has at least $\delta(G-C) \geqslant \frac{3}{\sqrt{2}}\sqrt{n}+4k-4$ leaves, contradicting Claim 5. $\square$

This is enough for us to finally prove our result. Let $\mathcal{S} \subset \mathcal{B}$ be the set of all the good blocks of $\mathcal{B}$. By Claim 5, $|\mathcal{S}| < \frac{1}{\sqrt{2}}\sqrt{n}$ and by Claim 7 every vertex of $G - C$ is in at least one of these blocks. Let U be a subpath of C of order $\lceil \frac{1}{\sqrt{2}}\sqrt{n} \rceil + k$. (Recall $|C| \geqslant \frac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n} + 4k - 2$.) Let U' be the subset of U containing the vertices of U that are coloured red; note $|U'| \geqslant \lceil \frac{1}{\sqrt{2}}\sqrt{n} \rceil + 1$. Each $u \in U'$ has at least $\frac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n} + 4k - 5$ neighbours in $G-C$ by Claim 1. As $\frac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n}+4k-5 > \frac{1}{\sqrt{2}}\sqrt{n}$ it follows by the Pigeonhole Principle that for each $u \in U'$ there exists some $S \in \mathcal{S}$ such that $u \in U'$ is adjacent to at least two vertices in S. Since $|U'| \geqslant \lceil \frac{1}{\sqrt{2}}\sqrt{n} \rceil + 1 > \frac{1}{\sqrt{2}}\sqrt{n} > |\mathcal{S}|$ it follows, again by the Pigeonhole Principle, that there exists $u_1, u_2 \in U'$ and $S \in \mathcal{S}$ such that both u_1 and u_2 have two neighbours each in S.

Pick a neighbour of u_1 in S and label it u_1', and pick a neighbour of u_2 in S distinct from u_1' and label it u_2'. Because S is a good block it contains a cycle of order at least $\sqrt{2}\sqrt{n}+4k-4$; choose one such cycle at label it C_g. As S is 2-connected, by Menger's Theorem there exist two vertex-disjoint paths from $\{u_1', u_2'\}$ to C_g in S; by taking the longer path around C_g between those end vertices we construct a path P in S between u_1' and u_2' which contains at least $\frac{1}{2}|C_g|+1$ vertices of C_g. That is, $|P| \geqslant \frac{1}{\sqrt{2}}\sqrt{n}+2k-1$.

We can construct a new cycle C' from C as follows. First remove from C the subpath of U between u_1 and u_2 except for the vertices u_1 and u_2 themselves. This could be at most $|U|-2 = \lceil \frac{1}{\sqrt{2}}\sqrt{n}\rceil+k-2$ vertices. Then add the edges u_1u_1', u_2u_2' and the path P; this adds at least $\frac{1}{\sqrt{2}}\sqrt{n}+2k-1$ vertices. Hence $|V(C')| \geqslant |V(C)|+\frac{1}{\sqrt{2}}\sqrt{n}+2k-1-\lceil\frac{1}{\sqrt{2}}\sqrt{n}\rceil-k+2 > |V(C)|+k$. Now this new cycle may not contain all of $V(C_0)$ so we cannot necessarily replace C with C', however as $|V(C)| \geqslant |V(C_0)| = c'(G)$, it follows $|V(C')| \geqslant c'(G)+k$. By our initial assumption $c'(G) \geqslant c(G)-k+1$, hence $|V(C')| \geqslant c(G)+1$ and C' is larger than the largest cycle, which gives a contradiction.

This proves the theorem.

3 Further Problems

Obviously, Conjecture 1 is still an obvious candidate for further investigation. Similarly, it may be possible to improve the multiplicative constants in Theorem 3 and Theorem 5 or find better graphs to improve the necessary lower bound. However, Theorem 3 is best possible using the techniques in this paper; it appears that any further improvement will require a more developed approach.

Conjecture 1 concerns 3-connected graphs, whereas the results of this paper are still within a constant multiplier of optimal even if we assume our graphs are 1-connected, as the graphs with which we show our necessary lower bound are themselves 1-connected. However, this leaves an open question for 2-connected graphs. Obviously 2-connectivity itself is not enough to force every cycle of maximum order to contain a chord (consider the chordless cycle itself as a graph). Instead, we make the following conjecture about 2-connected graphs:

Conjecture 7. *Let $k \in \mathbb{N}$. If G is a 2-connected graph and $\delta(G) \geqslant \lceil \frac{1}{2}k+2\rceil$, then $c'(G) \leqslant c(G)-k$.*

If $k=1$, this conjecture states that any cycle of maximum order in a 2-connected graph with minimum degree at least 3 will contain a chord. A recent result of Thomassen [8] extends the result of [10] and shows that every longest cycle in a 2-connected cubic graph contains a chord. This provides some evidence for the assertion that, in general, 3-connectivity is not required, and that 2-connectivity together with a minimum degree condition may be sufficient.

The above conjecture would be best possible. Let T be a complete d-ary tree (where $d \geqslant 3$) with ℓ layers and $f := 2^{\ell-1}$ leaves. To construct G, first take two copies of T, labelling them T_1 and T_2; label the leaves of T_1 by $1,\dots,f$ and the leaves of T_2 by

$1', \ldots, f'$. For each $i \in \{1, \ldots, f\}$ add to G a copy of K_{d-1} labelled X_i, and add all the edges from i to X_i and all the edges from i' to X_i. It is clear that G is d-regular and 2-connected. Let P_1 be the unique path between the leaves labelled 1 and f in T_1, and let P_2 be the equivalent path in T_2. Let C be the cycle consisting of P_1 and P_2 together with a path of $d-1$ vertices from both X_1 and X_f. The cycle C is a cycle of maximum order in G. (We omit the proof of this fact. Also note there are many cycles of the same maximum order.) Thus $c(G) = 2(2\ell - 1) + 2(d-1)$. Alternatively, if we modify C to take only a single vertex from each of X_1 and X_f then we construct a chordless cycle of maximum order (again, we omit the proof, and note this chordless cycle is not unique). Thus $c'(G) = 2(2\ell - 1) + 2 = c(G) - 2(d-2)$. Hence if $d = \lceil \frac{1}{2}k + 2 \rceil$ and k is even, then $c'(G) = c(G) - k$, suggesting Conjecture 7 would be optimal. This construction is based on the work of Voss [11], who constructed the above family for the case $d = 3$.

This conjecture does seem very strong; it is possible that we may require a lower bound on $\delta(G)$ which is a function of both n and k even when G is 2-connected. Either way, the techniques used in this paper do not appear to be strong enough by themselves to give a result of this fashion.

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