Boundary feedback stabilization of distributed parameter systems

An application of pseudo-differential boundary operators

Pedersen, Michael

Published in:

Proceedings of the 27th IEEE Conference on Decision and Control

Publication date:

1988

Document Version

Publisher's PDF, also known as Version of record

[Link back to DTU Orbit](#)

Citation (APA):

Pedersen, M. (1988). Boundary feedback stabilization of distributed parameter systems: An application of pseudo-differential boundary operators. In *Proceedings of the 27th IEEE Conference on Decision and Control* (pp. 366-368). IEEE.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

BOUNDARY FEEDBACK STABILIZATION OF DISTRIBUTED
PARAMETER SYSTEMS: An Application of Pseudo-
Differential Boundary Operators.

Michael Pedersen

Institute of Mathematics and Physics
Roskilde University Centre
4000 Roskilde DENMARK

ABSTRACT

The theory of pseudo-differential boundary operators proves to be a fruitful approach to problems arising in control and stabilization theory of distributed parameter systems. By use of the basic pseudo-differential calculus we can in a direct and simple way obtain existence and stability theorems for boundary feedback semigroups.

I. INTRODUCTION

In this paper we present a brief introduction to the method of pseudo-differential stabilization as developed in [9], and based on the fundamentals from refs. [3] and [4].

Let A be a formally selfadjoint, uniformly strongly elliptic differential operator of order $2m$, with smooth coefficients on $\bar{\Omega}$, where Ω is an open, bounded set in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary Γ . The Dirichlet realization A_Y of A is then the operator acting like A in $L^2(\Omega)$, and with domain

$$D(A_Y) = \{u \in H^{2m}(\Omega) : \gamma u = 0\} \\ = H^{2m}(\Omega) \cap H_0^m(\Omega). \quad (1)$$

Here γ is the Dirichlet trace operator

$$\gamma u = (u|_{\Gamma}, (\partial/\partial n)u|_{\Gamma}, \dots, (\partial/\partial n)^{m-1}u|_{\Gamma})^T \quad (2)$$

$(\partial/\partial n)$ is the normal derivative, and $H^{2m}(\Omega)$ is the usual Sobolev space of order $2m$, consisting of L^2 -functions with L^2 -derivatives up to order $2m$.

The realization A_Y is associated with the parabolic evolution equation:

$$\frac{d}{dt}u(x,t) + Au(x,t) = 0 \text{ for } x \in \Omega \text{ and } t > 0, \\ \gamma u(x,t) = 0 \text{ for } x \in \Omega \text{ and } t > 0, \quad (3) \\ u(x,0) = u_0(x) \text{ for } x \in \Omega;$$

and it is well known that A_Y is the infinitesimal generator of an analytic semigroup, $\exp(-A_Y t)$, $t \geq 0$, on $L^2(\Omega)$, giving the solution to (3) as

$$u(x,t) = \exp(-A_Y t)u_0(x), \quad (4)$$

for $u_0 \in L^2(\Omega)$, $x \in \Omega$ and $t \geq 0$.

Since A_Y has a compact resolvent, the spectrum of A_Y consists of a sequence of real eigenvalues, converging to infinity. There are only finitely many negative eigenvalues, so we write them as a nondecreasing sequence

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_{K-1} \leq 0 < \lambda_K \leq \dots \quad (5)$$

where λ_K is the first positive eigenvalue. Moreover, for simplicity assume that all the negative eigenvalues are simple. Because of the negative eigenvalues of A_Y

there are initial data u_0 for which the corresponding solution to (3) blows up (in L^2 -norm) as t tends to infinity. This is easily observed from the spectral representation of the solution

$$u(x,t) = \sum_{j \geq 1} \exp(-\lambda_j t) (u_0, \varphi_j) \varphi_j(x), \quad (6)$$

where the φ_j , $j = 1, 2, \dots$ is the set of eigenfunctions and $(\cdot, \cdot)$ is the usual $L^2(\Omega)$ -inner product. The boundary stabilization problem is to design a boundary feedback mechanism $T'u$, such that if the boundary condition $\gamma u = 0$ in (3) is replaced by a new boundary condition $\gamma u = T'u$, the resulting boundary feedback system is stable, in the sense that for any initial data $u_0 \in L^2(\Omega)$, the L^2 -norm of the corresponding solution goes to zero as t tends to infinity. Moreover, the feedback mechanisms we consider are of the form:

$$T'u = (u, w)g, \quad (7)$$

where $w \in C^\infty(\bar{\Omega})$ and $g \in C^\infty(\Gamma)$ are functions to be determined. (For certain choices of Ω or if some of the negative eigenvalues have multiplicities > 1 , the feedback must consist of a sum of terms like (7); these technical details are discussed in [5] and [9]).

II. THE FEEDBACK SYSTEM AND
THE PSEUDO-DIFFERENTIAL TRANSFORMATION

The boundary feedback stabilization problem can be stated as:

Can we determine functions $w \in C^\infty(\bar{\Omega})$, $g \in C^\infty(\Gamma)$, such that the boundary feedback system

$$\frac{d}{dt}u(x,t) + Au(x,t) = 0 \text{ for } x \in \Omega \text{ and } t > 0, \\ \gamma u(x,t) = (u, w)g(x) \text{ for } x \in \Omega \text{ and for } t > 0, \quad (8) \\ u(x,0) = u_0(x) \text{ for } x \in \Omega,$$

is stable in the sense that the L^2 -norm of a solution $u(x,t)$ is exponentially decreasing as t tends to infinity, for any initial data $u_0 \in L^2(\Omega)$?

The answer to the above problem is affirmative if we assume that:

- The negative eigenvalues are simple (9)
and
the Neumann traces (i.e. the normal boundary derivatives of order $\geq m$)

$$\left(\frac{\partial}{\partial n}\right)^k \varphi_j|_{\Gamma}, \quad k = m, m+1, \dots, 2m-1, \\ j = 1, 2, \dots, K-1, \quad (10)$$

of the eigenfunctions φ_j , $j = 1, 2, \dots, K-1$ are linearly independent.

(When the assumptions (9)-(10) do not hold, the situation is more complicated and, in general, more terms in the feedback are required; for details, see [5],[6] [7] and [9].)

The treatment of the system (8) is complicated by the fact that the associated realization A_1 of the operator A has the domain

$$D(A_1) = \{u \in H^{2m}(\Omega) \mid \gamma u = (u, w)g\}, \quad (11)$$

which in contrast to the domain for A_Y is given by a variable, non-local boundary condition. Consider now the solution operator K_Y to the stationary Dirichlet problem for A , i.e. K_Y maps φ into the solution u of

$$\begin{aligned} Au &= 0 \quad \text{in } \Omega \\ \gamma u &= \varphi \quad \text{on } \Gamma \end{aligned} \quad (12)$$

K_Y is a standard type of Poisson Operator, as defined in the pseudo-differential boundary operator calculus, (see [3],[4]). Moreover, the operator T' (7) is a standard type Trace Operator in this theory. However, the most important property with respect to the problem at hand is that the composition $K_Y T'$ is also a standard operator of the class called Singular Green Operators, (introduced in [2]). The properties of Singular Green Operators is thoroughly discussed in refs. [3] and [4]. In the present case we need only the fact that it is possible to choose T' of the form (7), such that the operator $1 - K_Y T'$ defines a homeomorphism and an isomorphism in $H^{2m}(\Omega)$, such that

$$1 - K_Y T' : D(A_1) \xrightarrow{\sim} D(A_Y). \quad (13)$$

Then, if $u \in D(A_1)$, $v = (1 - K_Y T')u$ belongs to $D(A_Y)$ and $Au = Av$. This establishes in a precise manner the factorization

$$A_1 = A_Y (1 - K_Y T') \quad (14)$$

which can now be used in the discussion of (8). The evolution problem

$$(d/dt)u + Au = 0, \quad u \in D(A_1) \quad (15)$$

transforms by (13) and (14) into

$$(d/dt)(1 - K_Y T')^{-1}v + Av = 0, \quad v \in D(A_Y) \quad (16)$$

or alternatively

$$(d/dt)v + (1 - K_Y T')Av = 0, \quad v \in D(A_Y). \quad (17)$$

Since A is a differential operator with smooth coefficients, the operator $G = -K_Y T'$ is also a Singular Green Operator (of finite rank), so we observe that our feedback problem (8) (by the transformation (15)-(17)) is in fact nothing but a finite dimensional perturbation:

$$(d/dt)v + Av + Gv = 0, \quad v \in D(A_Y) \quad (18)$$

of the Dirichlet evolution problem (3):

$$(d/dt)v + Av = 0, \quad v \in D(A_Y) \quad (19)$$

As shown in refs. [9], [10] and [11], the stabilization of the system (18) is straightforward, as finite dimensional pole placement techniques can be employed, (cf. [12]). The result is that under the assumptions (9)-(10), the operator T' (7) can be chosen such that

$1 - K_Y T'$ has the abovementioned properties, and such that the operator $A + G$ with domain $D(A_Y) = H^{2m}(\Omega) \cap H_0^m(\Omega)$, is the infinitesimal generator of an analytic semigroup, $\exp(-(A+G)t)$, $t \geq 0$, on L^2 , giving the solution to (18) as:

$$v(x, t) = \exp(-(A+G)t)v_0(x) \quad (20)$$

where $x \in \Omega$, $t \geq 0$, for initial data $v_0 \in L^2(\Omega)$.

Also (what is the key point):

$$\|v(\cdot, t)\| \leq M \exp(-(\lambda_K + \varepsilon)t) \|v_0\|, \quad (21)$$

with $M > 1, \varepsilon > 0$.

As shown in [9], the operator A_1 , with domain $D(A_1)$, is then also the infinitesimal generator of an analytic semigroup, $\exp(-A_1 t)$, $t \geq 0$, on $L^2(\Omega)$, which is the transform of the semigroup $\exp(-(A+G)t)$ under $(1 - K_Y T')$:

$$\exp(-A_1 t) = (1 - K_Y T')^{-1} \exp(-(A+G)t) (1 - K_Y T') \quad (22)$$

for which we have the estimate

$$\|u(\cdot, t)\| \leq M \exp(-(\lambda_K + \varepsilon)t) \|u_0\|, \quad (23)$$

for the solution $u(x, t)$ of (8).

The formula (22) shows that when we impose a boundary feedback on the originally "free" system (3), we are performing a pseudo-differential "change of coordinates" in the space $H^{2m}(\Omega)$. The pseudo-differential approach allows us to obtain stabilization results on the system (8), together with other perturbations of the free system (3), in a unified setting. Moreover, we can consider hyperbolic problems as well as parabolic problems, as described in ref. [9].

References

- [1] Balakrishnan, A.V., "Boundary Control of Parabolic equations: L-Q-R-Theory." Proc. Conf. on Theory of Nonlinear Equations, Sept. 1977. Akademie Verlag. Berlin 1978.
- [2] Boutet de Monvel, L., "Boundary Problems for Pseudo-Differential Operators." Acta Math. 126, 1971, pp. 11-51.
- [3] Grubb, G., "Functional Calculus of Pseudo-Differential Boundary Problems." Birkhäuser, Progress in Math. Vol. 65, Boston 1986.
- [4] Grubb, G., "Singular Green Operators and Their Spectral Asymptotics." Duke Math. J. Vol. 51, No. 3, Sept. 1984, pp. 477-528.
- [5] Lasiecka, I., R. Triggiani, "Stabilization and Structural Assignment of Dirichlet Boundary Feedback Parabolic Equations." Siam J. Control and Opt. Vol. 21, No. 5, Sept. 1983, pp. 766-802.
- [6] Lasiecka, I., R. Triggiani, "Feedback Semigroups and Cosine Operators for Boundary Feedback Parabolic and Hyperbolic Equations." J. Diff. Eq. 47, 1983, pp. 245-272.
- [7] Lasiecka, I., R. Triggiani, "Hyperbolic Equations with Dirichlet Boundary Feedback via Position Vector: Regularity and Almost Periodic Stabilization I." Appl. Math. Opt. 8, 1981, pp. 1-37.

- [8] Nambu,T."Feedback Stabilization for Distributed Parameter Systems of Parabolic Type." J.Diff.Eq.33, 1979, pp.167-188.
- [9] Pedersen,M."Pseudo-Differential Perturbations of Distributed Parameter Systems: Dirichlet Feedback control problems." Preprint: IMFUFA Tekst No.161, 1988, Roskilde University Centre.
- [10] Triggiani,R.," On Nambu's Problem for Diffusion Processes." J.Diff.Eq.33,1979, pp.189-200.
- [11] Triggiani,R.,"Boundary Feedback Stabilizability of Parabolic Equations." Appl.Math.Opt.6, 1980, pp.201-220.
- [12] Wonham,W.M.,"On Pole Assignment in Multi-Input Controllable Linear Systems." IEEE.Trans.Automat. Control AC-12,1967, pp.660-665.