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Abstract For a probability measure with compact and non-polar support in the complex plane we relate dynamical properties of the associated sequence of orthogonal polynomials $\{P_n\}$ to properties of the support. More precisely we relate the Julia set of P_n to the outer boundary of the support, the filled Julia set to the polynomial convex hull K of the support, and the Green's function associated with P_n to the Green's function for the complement of K.

Keywords Orthogonal polynomials · Julia set · Green's function

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1 Introduction and Main Results

In this paper, we study orthonormal polynomials $\{P_n(z)\} \equiv \{P_n(\mu; z)\}$ given by a Borel probability measure μ on $\mathbb{C}$ with compact and non-polar support $S(\mu)$. We relate the non-escaping set for P_n, the locus of non-normality (the boundary of the non-escaping set) for P_n, and an associated Green's function to the support of the measure, getting a fairly complete picture of the limiting behavior of these objects as $n \to \infty$.

We build on the classical monograph [9] by Stahl and Totik, where the authors relate potential and measure theoretic properties of, e.g., the asymptotic zero distribution for the sequence of orthonormal polynomials defined by μ to the potential and measure theoretic properties of the support of μ. We shall also use [10] as a reference to the basic concepts of potential theory in the complex plane.

Recall that $\{P_n(z)\}$ is the unique orthonormal sequence in $L^2(\mu)$ with

$$P_n(z) = \gamma_n z^n + \text{ lower order terms}, \tag{1}$$

where $\gamma_n > 0$.

Definition 1.1 Let $\mathcal{B}$ denote the set of Borel probability measures on $\mathbb{C}$ with compact, non-polar support. Furthermore, let $\mathcal{B}_+ \subset \mathcal{B}$ be defined as

$$\mathcal{B}_+ := \{\mu \in \mathcal{B} \mid \limsup_{n\to\infty} \gamma_n^{1/n} < \infty\}, \tag{2}$$

where γ_n is given in Eq. 1.

For $\mu \in \mathcal{B}$ we denote by Ω the unbounded connected component of $\mathbb{C} \setminus S(\mu)$ and define

$$K = \mathbb{C} \setminus \Omega, \quad J = \partial K.$$

The set K is the *filled* $S(\mu)$ and $J = \partial\Omega \subset S(\mu)$ is the outer boundary of $S(\mu)$. We shall also say that $S(\mu)$ is *full* if $\mathbb{C} \setminus S(\mu)$ has no bounded connected components.

Furthermore, we define the exceptional subset $E \subset S(\mu)$ by

$$E = \{z \in J \mid z \text{ is } \textit{not} \text{ a Dirichlet regular boundary point}\}. \tag{3}$$

This set is an F_σ polar subset, see [10, Theorem 4.2.5]. We let $g_\Omega : \mathbb{C} \to [0, \infty)$ be the Green's function for Ω with pole at infinity (in short, just the Green's function for Ω). This is the unique non-negative subharmonic function which is harmonic and positive on Ω, zero precisely on $K \setminus E$, (see [10, Theorem 4.4.9]) and which satisfies

$$g_\Omega(z) = \log|z| + O(1) \text{ at infinity}. \tag{4}$$

Finally, we denote by ω_J the equilibrium measure on J, which equals harmonic measure on Ω from ∞ and which is the distributional Laplacian Δg_Ω of the Green's function g_Ω.

We shall also use (see [9, Section 1.2]) the extended notion of the Green's function $g_B : \mathbb{C} \to [0, \infty)$ for an arbitrary connected Borel set $B \subset \mathbb{C}$ with bounded complement L of positive logarithmic capacity, $\mathrm{Cap}(L) > 0$. This is the unique non-negative subharmonic function which is harmonic and positive on the interior of B, satisfies

$$g_B(z) = \log|z| - \log \mathrm{Cap}(L) + o(1) \text{ at infinity}, \tag{5}$$

and equals zero qu. e. on $\mathbb{C} \setminus B$. Here, qu. e. is short for quasi everywhere meaning except on a polar set ([10] uses n. e., nearly everywhere).

Furthermore, for $\mu \in \mathcal{B}$ we denote by $g_\mu : \mathbb{C} \to [0, \infty)$ the minimal-carrier Green's function for μ which is (see [9, Definition 1.1.1]) the pointwise sup of $g_{\mathbb{C}\setminus C}(z)$ over all

bounded carriers C of μ (i.e., Borel sets for which $\mu(\mathbb{C} \setminus C) = 0$). We note that (see [9, Lemma 1.2.4])

$$g_\mu(z) = \log|z| - \log c_\mu + o(1) \text{ at infinity}, \tag{6}$$

where c_μ is the minimal-carrier capacity (i.e., inf of the capacities of bounded carriers of μ). Moreover, we denote by E_μ the exceptional set for g_μ defined by

$$E_\mu = \{z \in S(\mu) \mid g_\mu(z) > 0\}. \tag{7}$$

The following fundamental result concerning the distribution of zeros of the orthogonal polymonials was originally obtained by Fejér in [6]; see also [9, Lemma 1.1.3].

Theorem 1.2 *If $\mu \in \mathcal{B}$, then all zeros of the orthonormal polynomials P_n are contained in the convex hull $\mathrm{Co}(S(\mu))$. Moreover, for any compact subset $V \subset \Omega$ the number of zeros of P_n in V is bounded as $n \to \infty$.*

Our main result, Theorem 1.3, concerns measures in the class $\mathcal{B}_+$ and it is proved in Section 3. The first part of the theorem should be compared with [9, Theorem 1.1.4], while the second part does not have an immediate counterpart in the classical theory of orthogonal polynomials. We remark that $\mathcal{B}_+$ is a large subclass of $\mathcal{B}$ since only measures in $\mathcal{B}$ with zero carrier capacity are left out.

Before stating our main result, some more notation is needed. We denote by Ω_n the attracted basin of ∞ for P_n, by $K_n = \mathbb{C} \setminus \Omega_n$ the filled Julia set, and by $J_n = \partial K_n = \partial \Omega_n$ the Julia set. Theorem 1.3 is motivated by the following questions: What is the relation between K and limits of K_n and, similarly, what is the relation between J and limits of J_n? Inspired by [5], we answer these questions in terms of limits involving the Hausdorff distance on the space of compact subsets of $\mathbb{C}$ (see the beginning of Section 3 for details and the notions of lim inf and lim sup of sequences of compact sets).

Theorem 1.3 *Let $\mu \in \mathcal{B}_+$. Then the following assertions hold.*

(i) *We have*

$$\limsup_{n\to\infty} K_n \subseteq \mathrm{Co}(K). \tag{8}$$

Moreover, for any $\epsilon > 0$ and with $V_\epsilon := \{z \in \mathbb{C} \mid g_\Omega(z) \geq \epsilon\}$,

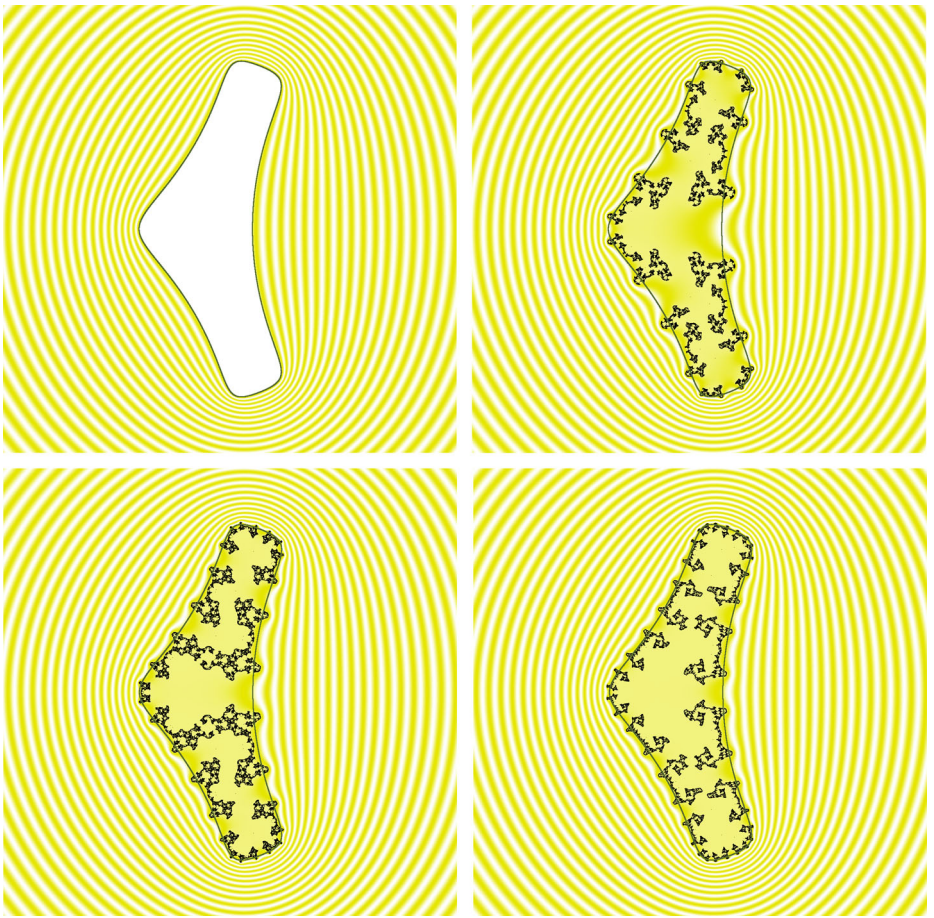
$$\lim_{n\to\infty} \mathrm{Cap}(V_\epsilon \cap K_n) = 0. \tag{9}$$

(ii) *We have*

$$\overline{J \setminus E_\mu} \subseteq \liminf_{n\to\infty} J_n. \tag{10}$$

The figure below illustrates Theorem 1.3 in the case where μ is the equilibrium measure for the boundary of the boomerang-shaped white set K in the top left image. The black fractal sets in the other images are the Julia sets J_{10}, J_{15}, and J_{20} (which in these cases appear to be equal to the filled Julia sets). The Green's functions are visualized by the toning such that level sets have the same tone.

We remark that equilibrium measures belong to a special class of measures, the so-called *regular measures* to be discussed in Sections 4 and 5.



2 Polynomial Dynamics and Technical Preparations

For any polynomial P of degree $d > 1$, there clearly exists $R > 0$ such that $|P(z)| \geq 2|z|$ for all z with $|z| > R$. Thus the orbit, $\{z_n\}$, of such z under iteration by P converges to ∞. The basin of attraction for ∞ for P, denoted Ω_P, may therefore be written as

$$\Omega_P = \{z \in \mathbb{C} \mid P^k(z) \underset{k\to\infty}{\longrightarrow} \infty\} = \bigcup_{k\geq 0} P^{-k}(\mathbb{C} \setminus \overline{\mathbb{D}(0,R)}). \tag{11}$$

Here $P^k = \overbrace{P \circ P \circ \ldots \circ P}^{k \text{ times}}$, whereas P^{-k} denotes inverse image and $\mathbb{D}(0, R)$ is the open ball of radius R centered at 0. It follows immediately that Ω_P is open and completely invariant, that is, $P^{-1}(\Omega_P) = \Omega_P = P(\Omega_P)$. Denote by $K_P = \mathbb{C} \setminus \Omega_P \subseteq \overline{\mathbb{D}(0,R)}$ the filled Julia set for P and by $J_P = \partial\Omega_P = \partial K_P$ the Julia set for P. Then K_P and J_P are compact and also completely invariant. Clearly, any periodic point (i.e., a solution of the equation

$P^k(z) = z$ for some $k \in \mathbb{N}$) belongs to K_P, so that K_P is non-empty. It follows from Eq. 11 that the filled Julia set K_P can also be described as the nested intersection

$$K_P = \bigcap_{k \geq 0} P^{-k}(\overline{\mathbb{D}(0, R)}). \tag{12}$$

To ease notation we denote the Green's function for Ω_P with pole at infinity by g_P (and not by g_{Ω_P}). It follows from Eq. 12 that g_P satisfies

$$g_P(z) = \lim_{k\to\infty} \frac{1}{d^k} \log^+(|P^k(z)|/R) = \lim_{k\to\infty} \frac{1}{d^k} \log^+ |P^k(z)|. \tag{13}$$

Here and elsewhere, $\log^+$ is the positive part of log. Thus g_P vanishes precisely on K_P and hence ([10, Theorem 4.4.9]) every point in J_P is a Dirichlet regular boundary point of Ω_P. Moreover, denoting the leading coefficient of P by γ,

$$g_P(P(z)) = d \cdot g_P(z) \quad \text{and} \quad \mathrm{Cap}(K_P) = \frac{1}{|\gamma|^{\frac{1}{d-1}}}. \tag{14}$$

When $P = P_n$, we thus have in our notation

$$\frac{1}{\gamma_n^{\frac{1}{n-1}}} = \mathrm{Cap}(K_n). \tag{15}$$

As

$$\liminf_{n\to\infty} \gamma_n^{\frac{1}{n-1}} = \liminf_{n\to\infty} \gamma_n^{\frac{1}{n}} \quad \text{and} \quad \limsup_{n\to\infty} \gamma_n^{\frac{1}{n-1}} = \limsup_{n\to\infty} \gamma_n^{\frac{1}{n}}, \tag{16}$$

we immediately obtain, by combining with [9, Cor. 1.1.7, formula (1.13)],

Lemma 2.1 *For $\mu \in \mathcal{B}$ we have*

$$c_\mu \leq \liminf_{n\to\infty} \mathrm{Cap}(K_n) \leq \limsup_{n\to\infty} \mathrm{Cap}(K_n) \leq \mathrm{Cap}(K), \tag{17}$$

where c_μ is the minimal-carrier capacity.

The examples in [9, Section 1.5] show that all the inequalities in Eq. 17 can be strict. However, in this paper we only need $\liminf_{n\to\infty} \mathrm{Cap}(K_n) > 0$, which is implied for $\mu \in \mathcal{B}_+$.

Furthermore, we have

Lemma 2.2 *Let $\mu \in \mathcal{B}$ and choose $R > 0$ so that $K \subset \mathbb{D}(0, R)$. Then there exists N such that for all $n \geq N$:*

$$K_n \subset P_n^{-1}(\overline{\mathbb{D}(0, R)}) \subset \mathbb{D}(0, R). \tag{18}$$

Proof By [9, Theorem 1.1.4], we have

$$\liminf_{n\to\infty} \frac{1}{n} \log |P_n(z)| \geq g_\Omega(z)$$

locally uniformly on $\mathbb{C} \setminus \mathrm{Co}(K)$. Taking R such that $K \subset \mathbb{D}(0, R)$ then $\partial\mathbb{D}(0, R)$ is a compact set disjoint from K on which g_Ω is continuous, and hence $\epsilon = \inf\{g_\Omega(z) \,|\, |z| = R\} > 0$. By the above inequality and compactness of $\partial\mathbb{D}(0, R)$, there exists N such that

$$\forall\, n \geq N\ \forall\, z \in \partial\mathbb{D}(0, R): \quad \frac{1}{n} \log |P_n(z)| \geq \epsilon/2.$$

By increasing N if necessary, we can suppose $\log(R) < N\epsilon/2$. Then since the zeros of P_n are contained in $\mathrm{Co}(K) \subset \mathbb{D}(0, R)$ (by Theorem 1.2), the minimal modulus principle implies

$$\forall\, n \geq N: \quad P_n(\mathbb{C} \setminus \mathbb{D}(0, R)) \subset \mathbb{C} \setminus \overline{\mathbb{D}(0, R)}.$$

Thus, by Eq. 12,

$$\forall\, n \geq N: \quad K_n \subset P_n^{-1}(\overline{\mathbb{D}(0, R)}) \subset \mathbb{D}(0, R)$$

and this completes the proof. □

Proposition 2.3 *Let $\mu \in \mathcal{B}_+$. Then there exists $N \in \mathbb{N}$ and $M > 0$ such that*

$$\forall\, n \geq N: \quad \left\| g_n(z) - \frac{1}{n} \log^+ |P_n(z)| \right\|_\infty \leq \frac{M}{n}. \tag{19}$$

Remark 2.4 Proposition 2.3 plays a key role in the proofs of our main results. It links the Green's functions g_n for Ω_n to the potentials $\frac{1}{n} \log |P_n(z)|$ or rather to the Green's functions $\frac{1}{n} \log^+ |P_n(z)|$ of the set $\{z \,|\, |P_n(z)| > 1\}$. The literature on orthogonal polynomials, and [9] in particular, does not seem to study the latter Green's function in connection with orthogonal polynomials, though this restriction of $\frac{1}{n} \log |P_n(z)|$ is quite natural. For instance, the equilibrium measure $\Delta(\frac{1}{n} \log^+ |P_n(z)|)$ on $\{z \,|\, |P_n(z)| = 1\}$ is the balayage in $\{z \,|\, |P_n(z)| < 1\}$ of the purely atomic measure $\Delta(\frac{1}{n} \log |P_n(z)|)$ with an atom of weight $\frac{1}{n}$ at each root of P_n (counting multiplicities).

Proof of Proposition 2.3 By Eqs. 15–16, we have

$$\mathcal{B}_+ = \{\mu \in \mathcal{B} \,|\, \liminf_{n\to\infty} \mathrm{Cap}(K_n) > 0\}. \tag{20}$$

Hence $c := \liminf_{n\to\infty} \mathrm{Cap}(K_n) > 0$ and we can choose $R' > 1$ such that $K \subset \mathbb{D}(0, R')$. Further, let $R = 2R'$, $c' = c/2$ and choose N so that

$$\forall\, n \geq N: \quad K_n \subset P_n^{-1}(\overline{\mathbb{D}(0, R')}) \subset \mathbb{D}(0, R') \quad \text{and} \quad R' \geq \mathrm{Cap}(K_n) > c'.$$

The Green's functions g_n can be written as

$$g_n(z) = \log |z| - \log \mathrm{Cap}(K_n) + \int \log |1 - w/z| \, d\omega_n(w),$$

where ω_n is harmonic measure from ∞. Writing

$$M' = \max\{\log R', -\log c'\} + \log 2 \;\text{ and }\; M = 3M',$$

we find

$$\forall\, n \geq N \;\forall\, z, |z| \geq R: \quad \left| g_n(z) - \log |z| \right| < M'.$$

For each n, denote by A_n the set $\{z \,|\, |P_n(z)| < R\}$. Then for each $n \geq N$ and all $z \in \mathbb{C} \setminus A_n$, we have $|P_n(z)| \geq R$ so that $\log^+ |P_n(z)| = \log |P_n(z)|$ and

$$\left| g_n(z) - \frac{1}{n} \log^+ |P_n(z)| \right| = \left| \frac{1}{n} g_n(P_n(z)) - \frac{1}{n} \log |P_n(z)| \right| \leq \frac{M'}{n}.$$

Moreover, for all $z \in \partial A_n$,

$$\begin{aligned} 0 < g_n(z) &= \frac{1}{n} \log |P_n(z)| + \left(g_n(z) - \frac{1}{n} \log |P_n(z)| \right) \\ &< \left| \frac{1}{n} \log |P_n(z)| \right| + \left| g_n(z) - \frac{1}{n} \log |P_n(z)| \right| < \frac{2M'}{n}. \end{aligned}$$

Hence, by the maximum principle for subharmonic functions, $g_n(z) < 2M'/n$ on all of A_n. Since

$$0 \leq \frac{1}{n} \log^+ |P_n(z)| \leq \frac{M'}{n}$$

on A_n by construction, we have

$$\left| g_n(z) - \frac{1}{n} \log^+ |P_n(z)| \right| < \frac{3M'}{n} = \frac{M}{n}$$

on A_n and thus on all of $\mathbb{C}$. □

Remark 2.5 (i) If $\liminf_{k\to\infty} \operatorname{Cap}(K_{n_k}) > 0$ for some subsequence $\{n_k\}$, then the proof shows that Eq. 19 holds when n is replaced by n_k.

(ii) By Eq. 17, the hypothesis in the proposition is satisfied if the minimal-carrier capacity is strictly positive. However, according to [9, Example 1.5.4], there are measures $\mu \in \mathcal{B}$ for which $0 = c_\mu < \lim_{n\to\infty} \operatorname{Cap}(K_n)$.

Combining Proposition 2.3 with [9, Theorem 1.1.4], we can now prove the proposition below. For easy reference, we first recall the upper and lower bounds given by Stahl and Totik.

Theorem 2.6 (part of [9, Theorem 1.1.4]) *For any $\mu \in \mathcal{B}$ we have*

$$\limsup_{n\to\infty} |P_n(z)|^{\frac{1}{n}} \leq e^{\,g_\mu(z)} \tag{21}$$

locally uniformly in $\mathbb{C}$ and

$$\liminf_{n\to\infty} |P_n(z)|^{\frac{1}{n}} \geq e^{\,g_\Omega(z)} \tag{22}$$

locally uniformly in $\mathbb{C} \setminus \operatorname{Co}(K)$. In $\operatorname{Co}(K) \cap \Omega$, the lower bound (22) holds true only in capacity, that is, for every compact set $V \subseteq \Omega$ and every $\epsilon > 0$ we have

$$\lim_{n\to\infty} \operatorname{Cap}(\{z \in V \mid |P_n(z)|^{\frac{1}{n}} < e^{\,g_\Omega(z)} - \epsilon\}) = 0. \tag{23}$$

Proposition 2.7 *For all $\mu \in \mathcal{B}_+$ we have*

$$\limsup_{n\to\infty} g_n(z) \leq g_\mu(z) \tag{24}$$

locally uniformly in $\mathbb{C}$ and

$$\liminf_{n\to\infty} g_n(z) \geq g_\Omega(z) \tag{25}$$

locally uniformly in $\mathbb{C} \setminus \operatorname{Co}(K)$. In $\operatorname{Co}(K) \cap \Omega$, the lower bound (25) holds true only in capacity, that is, for every compact set $V \subseteq \Omega$ and every $\epsilon > 0$, we have

$$\lim_{n\to\infty} \operatorname{Cap}(\{z \in V \mid g_n(z) < g_\Omega(z) - \epsilon\}) = 0. \tag{26}$$

Remark 2.8 (i) As with Eq. 21, the bound (24) holds for every $\mu \in \mathcal{B}$.

(ii) For a sequence of real valued functions h_n on an open set U and $h : U \to \mathbb{R}$, the relation

$$\limsup_{n\to\infty} h_n(z) \leq h(z) \text{ locally uniformly in } U$$

means that for every $z \in U$ and every sequence $\{z_n\} \subset U$ converging to z, we have $\limsup_{n\to\infty} h_n(z_n) \leq h(z)$. Similar statements hold for lim inf and lim.

Proof of Proposition 2.7 If $c_\mu = 0$, then $g_\mu \equiv \infty$ and Eq. 24 trivially holds. The relations (24) and (25) are straightforward translations of the relations (21) and (22) by using Proposition 2.3 and noting that for any $\epsilon > 0$,

$$\frac{1}{n}\log|P_n(z)| \le g_\mu(z) + \epsilon \implies \frac{1}{n}\log^+|P_n(z)| \le g_\mu(z) + \epsilon.$$

This implication holds by definition of $\log^+$, since $g_\mu(z) \ge 0$.

For Eq. 26, let $\epsilon > 0$ be given and choose according to Proposition 2.3 an N such that

$$\forall n \ge N\ \forall z \in \mathbb{C}: \quad \left| g_n(z) - \frac{1}{n}\log^+|P_n(z)| \right| < \epsilon/2.$$

Then for $n \ge N$, we have

$$g_n(z) + \epsilon \ge \frac{1}{n}\log^+|P_n(z)| + \epsilon/2 \ge \frac{1}{n}\log|P_n(z)| + \epsilon/2$$

so that $g_n(z) < g_\Omega(z) - \epsilon$ implies $\frac{1}{n}\log|P_n(z)| < g_\Omega(z) - \epsilon/2$. Hence,

$$|P_n(z)|^{\frac{1}{n}} \le e^{g_\Omega(z)-\epsilon/2} = e^{g_\Omega(z)} - (1 - e^{-\epsilon/2})e^{g_\Omega(z)} \le e^{g_\Omega(z)} - (1 - e^{-\epsilon/2}),$$

recalling that $g_\Omega(z) \ge 0$. Thus, with $\epsilon' := (1 - e^{-\epsilon/2}) > 0$ and $V \subset \Omega$ a compact subset, we have

$$\{z \in V \mid g_n(z) < g_\Omega(z) - \epsilon\} \subseteq \{z \in V \mid |P_n(z)|^{\frac{1}{n}} < e^{g_\Omega(z)} - \epsilon'\}$$

and Eq. 26 applies. □

3 Relating the Sequences K_n, J_n to K and J

This section contains the proof of Theorem 1.3. We shall equip the space of non-empty compact subsets of $\mathbb{C}$ with the Hausdorff distance, which is the natural choice in dynamical systems (see, e.g., [5]). We begin by briefly recalling the main definitions and then characterize lim inf and lim sup in this setup.

Let $\mathcal{K}$ denote the set of non-empty compact subsets of $\mathbb{C}$. For $L, M \in \mathcal{K}$, we define the Hausdorff semi-distance from L to M by

$$\mathrm{d_H}(L, M) := \sup\{\mathrm{d}(z, M) \mid z \in L\} = \sup_{z\in L} \inf_{w\in M} |z - w| \tag{27}$$

and the Hausdorff distance between the two sets as

$$\mathrm{D_H}(L, M) := \max\{\mathrm{d_H}(L, M), \mathrm{d_H}(M, L)\}. \tag{28}$$

The Hausdorff distance is a metric on the space $\mathcal{K}$ of compact subsets. When $\{K_n\} \subset \mathcal{K}$ is a bounded sequence of compact sets (i.e., a sequence for which there exists $R > 0$ such that $K_n \subset \mathbb{D}(0, R)$ for all n), we define the symbols

$$\liminf_{n\to\infty} K_n := \{z \in \mathbb{C} \mid \exists\{z_n\},\ K_n \ni z_n \underset{n\to\infty}{\longrightarrow} z\}, \tag{29}$$

$$\limsup_{n\to\infty} K_n := \{z \in \mathbb{C} \mid \exists\{n_k\},\ n_k \nearrow \infty \text{ and } \exists\{z_{n_k}\},\ K_{n_k} \ni z_{n_k} \underset{k\to\infty}{\longrightarrow} z\}. \tag{30}$$

Clearly, $\liminf_{n\to\infty} K_n \subseteq \limsup_{n\to\infty} K_n$ and by Lemma 3.1, the sets

$$I = \liminf_{n\to\infty} K_n, \qquad S = \limsup_{n\to\infty} K_n$$

are compact. The set I may be empty whereas S is always non-empty. Moreover, to illustrate that $(\mathcal{K}, \mathrm{D_H})$ is a nice metric space, let us remark that it can be shown that I is either empty or it is the largest compact set for which

$$\lim_{n\to\infty} \mathrm{d_H}(I, K_n) = 0. \tag{31}$$

Likewise, S is the smallest compact set for which

$$\lim_{n\to\infty} \mathrm{d_H}(K_n, S) = 0. \tag{32}$$

Thus, $I = S$ if and only the sequence $\{K_n\}$ is convergent to the common value $I = S$. If the sequence $\{K_n\}$ is Cauchy, then the equality $I = S$ easily follows and this shows that $\mathcal{K}$ is a complete metric space. Also, the above statements serve to explain the names lim inf and lim sup.

Lemma 3.1 *Let $\{K_n\}$ be a bounded sequence from $\mathcal{K}$. The complements of $I = \liminf_{n\to\infty} K_n$ and $S = \limsup_{n\to\infty} K_n$ are open and characterized by*

$$z_0 \in \mathbb{C} \setminus I \iff \exists\, \delta_0 > 0\ \exists\, \{n_k\},\ n_k \nearrow \infty \text{ s.t. } \forall\, k:\ \mathrm{d}(z_0, K_{n_k}) > \delta_0 \tag{33}$$

and

$$z_0 \in \mathbb{C} \setminus S \iff \exists\, \delta_0 > 0\ \exists\, N \text{ s.t. } \forall n \geq N:\ \mathrm{d}(z_0, K_n) > \delta_0. \tag{34}$$

As a consequence, both I and S are compact.

Proof The implication "$\Leftarrow$" in Eq. 33 is trivial. For the reverse implication, let $z_0 \in \mathbb{C}$ and suppose the right hand side is false. Then

$$\forall \delta > 0\ \exists\, N \text{ s.t. } \forall\, n \geq N:\ \mathrm{d}(z_0, K_n) \leq \delta.$$

For each n, let $z_n \in K_n$ be a point with $|z_n - z_0| = \mathrm{d}(z_0, K_n)$. Then $K_n \ni z_n \to z_0$ which shows that $z_0 \in I$.

The implication "$\Leftarrow$" in Eq. 34 is also trivial. For the reverse implication, take an arbitrary $z_0 \in \mathbb{C}$ and assume the right hand side is false. Then for any $\delta > 0$ there are infinitely many values of n for which $\mathrm{d}(z_0, K_n) \leq \delta$. Thus we may construct an increasing sequence $\{n_k\}$ of integers such that $\mathrm{d}(z_0, K_{n_k}) \leq 1/k$, say. Take as above, for each k, a point $z_k \in K_{n_k}$ with $|z_k - z_0| = \mathrm{d}(z_0, K_{n_k}) \leq 1/k$. Hence $z_0 \in S$.

Openness of $\mathbb{C} \setminus I$ and of $\mathbb{C} \setminus S$ follow from the relations (33) and (34). Thus I and S are both closed, and also bounded. □

After these preliminaries we are ready to prove Theorem 1.3.

Proof of Theorem 1.3(i) Since $z \in K_n$ if and only if $g_n(z) = 0$ and since $g_\Omega(z) > 0$ on $\Omega = \mathbb{C} \setminus K$, the inclusion

$$\limsup_{n\to\infty} K_n \subseteq \mathrm{Co}(K)$$

follows immediately from Eq. 25 and Lemma 3.1. Next, choose $R > 0$ so large that $K_n \subset \mathbb{D}(0, R)$ for all $n \geq 2$. For given $\epsilon > 0$ we obtain from Eq. 26 that

$$\lim_{n\to\infty} \mathrm{Cap}(\{z \in V_\epsilon \cap \overline{\mathbb{D}(0, R)} \,|\, g_n(z) < g_\Omega(z) - \epsilon/2\}) = 0,$$

where $V_\epsilon = \{z \in \mathbb{C} \,|\, g_\Omega(z) \geq \epsilon\}$. Since $g_n(z) = 0$ on K_n, we deduce that

$$\lim_{n\to\infty} \mathrm{Cap}(V_\epsilon \cap K_n) = 0$$

and the proof is complete. □

Proof of Theorem 1.3(ii) Since the right hand side of the relation is closed, it suffices to prove that $J \setminus E_\mu \subseteq \liminf_{n\to\infty} J_n$. Suppose to the contrary that there exists $z_0 \in J \setminus E_\mu$ which does *not* belong to $\liminf_{n\to\infty} J_n$. Then $g_\mu(z_0) = 0$ and according to Lemma 3.1,

$$\exists\, \delta > 0\ \exists \{n_k\},\ n_k \nearrow \infty \text{ s.t. } \forall\, k :\ \mathbb{D}(z_0, \delta) \cap J_{n_k} = \emptyset. \tag{35}$$

Since $z_0 \in J$, there exists $w_0 \in \mathbb{D}(z_0, \delta/4) \cap \Omega$. Choose $r \leq \delta/4$ such that $\overline{\mathbb{D}(w_0, r)} \subset \Omega$. Let $2\epsilon = g_\Omega(w_0) > 0$ and define

$$L := \{w \in \overline{\mathbb{D}(w_0, r)} \mid g_\Omega(w) \geq 2\epsilon\}.$$

Let L_0 denote the connected component of L containing w_0. Since g_Ω is subharmonic, it has no local maxima. It follows that $L_0 \subset \Omega \cap \overline{\mathbb{D}(z_0, \delta/2)}$ is a non-trivial compact continuum and hence $\mathrm{Cap}(L_0) > 0$. Thus, by Eq. 26 there exists N such that

$$\forall\, k \geq N :\ \mathrm{Cap}(\{z \in L_0 | g_{n_k}(z) \leq g_\Omega(z) - \epsilon\}) < \mathrm{Cap}(L_0).$$

Since $g_\Omega(z) \geq 2\epsilon$ on L_0, it follows that

$$\forall\, k \geq N\ \exists\, z_k \in L_0 \text{ s.t. } g_{n_k}(z_k) \geq \epsilon.$$

Combining with Eq. 35, we find that $\mathbb{D}(z_0, \delta) \subset \Omega_{n_k}$ for $k \geq N$. By applying Harnack's inequality, we obtain

$$g_{n_k}(z_0) \geq g_{n_k}(z_k)\frac{1 - 1/2}{1 + 1/2} \geq \epsilon/3 > 0.$$

On the other hand, by Eq. 24,

$$\limsup_{k\to\infty} g_{n_k}(z_0) \leq g_\mu(z_0) = 0,$$

which is a contradiction. □

4 Results for n-th Root Regular Measures

In this section, we specialize the general results of the previous sections to the important class of regular measures. According to Stahl and Totik, a measure $\mu \in \mathcal{B}$ is nth-root regular, in short $\mu \in \mathrm{Reg}$, if

$$\lim_{n\to\infty} \tfrac{1}{n} \log |P_n(z)| = g_\Omega(z) \tag{36}$$

locally uniformly for $z \in \mathbb{C} \setminus \mathrm{Co}(K)$. In particular, we see that $\mathrm{Reg} \subset \mathcal{B}_+$. Note that Eq. 36 is equivalent to [9, Theorem 3.2.1, formula (2.1)]

$$\limsup_{n\to\infty} |P_n(z)|^{1/n} \leq e^{g_\Omega(z)} \tag{37}$$

locally uniformly in $\mathbb{C}$.

A prime example of $\mu \in \mathrm{Reg}$ is the equilibrium measure for the boundary J of a full compact non-polar subset K or, equivalently, the harmonic measure on $\mathbb{C} \setminus K$ viewed from infinity. This follows immediately from Erdös-Turán's theorem, see [9, Theorem 4.1.1].

Combining Eqs. 36–37 with Proposition 2.3, we have as an immediate corollary

Corollary 4.1 *The following statements are equivalent:*

(i) $\mu \in \mathrm{Reg}$,
(ii) $\lim_{n\to\infty} g_n(z) = g_\Omega(z)$ *locally uniformly for* $z \in \mathbb{C} \setminus \mathrm{Co}(K)$,
(iii) $\limsup_{n\to\infty} g_n(z) \le g_\Omega(z)$ *locally uniformly on* $\mathbb{C}$,
(iv) $\mu \in \mathcal{B}_+$ *and* $\lim_{n\to\infty} g_n(z) = 0$ *qu. e. on* J.

Proceeding as in the proof of Theorem 1.3(ii), but using Corollary 4.1(iii) instead of Eq. 24, we obtain a stronger result (compare also with Theorem 1.3(i)).

Corollary 4.2 *Suppose* $\mu \in \mathrm{Reg}$. *Then*

$$\overline{J \setminus E} \subseteq \liminf_{n\to\infty} J_n, \tag{38}$$

where E *denotes the* (F_σ *and polar*) *exceptional set for the Green's function* g_Ω. *In particular, if* J *is Dirichlet regular, then*

$$J \subseteq \liminf_{n\to\infty} J_n. \tag{39}$$

In the convex case we note the following proposition.

Proposition 4.3 *If* $\mu \in \mathrm{Reg}$ *and* $K = \mathrm{Co}(K)$, *then*

$$J \subseteq \liminf_{n\to\infty} K_n \subseteq \limsup_{n\to\infty} K_n \subseteq K. \tag{40}$$

Proof For a compact convex set K, every boundary point is Dirichlet regular. Moreover, $J_n \subset K_n$ so that the first inclusion follows from Eq. 39. The latter follows from Theorem 1.3(i). □

Corollary 4.4 *For any compact convex subset* K *and any* $\epsilon > 0$, *there exists a polynomial* P_n *(of high degree* n*) with*

$$\mathrm{d_H}(\partial K, K_n) < \epsilon \quad \text{and} \quad \mathrm{d_H}(K_n, K) < \epsilon. \tag{41}$$

It has recently been shown that a general compact connected subset $K \subset \mathbb{C}$ can be approximated arbitrarily well in the Hausdorff topology by (filled) Julia sets of polynomials, see Lindsay [7] and Bishop–Pilgrim [1]. Theorem 1.3(i), Corollary 4.2, and Proposition 4.3 of this paper deal with approximation of general compact sets $K \subset \mathbb{C}$ by the (filled) Julia sets of orthogonal polynomials for probability measures supported on ∂K. These results can be viewed as a complement to the results of [1, 7] in the connected case and an extension in the general case. At the same time, our results are statements about the dynamical behavior of orthogonal polynomials.

Remark 4.5 We cannot expect that

$$\limsup_{n\to\infty} K_n \subseteq K \tag{42}$$

for general non-convex sets K. To see this, suppose $K \subset \mathbb{C}$ is any full compact subset of $\mathbb{C}$ with $K = -K$ (i.e., $z \mapsto -z$ is an involution of K) and let ω denote the equilibrium measure on $J = \partial K$. Then the corresponding orthonormal polynomials P_n are even for n even and odd for n odd. In particular, 0 is a fixed point of each P_{2n+1}, $n \ge 0$, and so

$0 \in K_{2n+1}$. This implies that $0 \in \limsup_{n\to\infty} K_n$. However, we may choose K as above with $0 \notin K$. Note that K cannot be connected in this case.

5 The Orthogonal Polynomials for the Measure of Maximal Entropy of a Polynomial

Our main results apply to measures $\mu \in \mathcal{B}_+$ or $\mu \in \mathrm{Reg}$. A natural way of generating non-trivial examples of such measures is to take a monic polynomial Q of degree $d \geq 2$ and construct the unique balanced invariant measure ω for Q (see, e.g., [2]). This measure is known to coincide with the (unique) measure of maximal entropy for Q (see [8]) and is in fact the equilibrium measure of J_Q, the Julia set of Q. Note that, with K_Q the filled Julia set of Q, we have

$$\mathrm{Cap}(J_Q) = \mathrm{Cap}(K_Q) = 1.$$

The orthogonal polynomials associated with ω (as above) were studied in a series of papers of Barnsley et al. (see, e.g., [3, 4]). One of their basic results reads:

Theorem 5.1 [3] *Let $Q(z) = z^d + az^{d-1} + \cdots$ be a polynomial of degree ≥ 2 and let ω denote the unique measure of maximal entropy for Q. Then the monic orthogonal polynomials $\{p_n\}$ with respect to ω satisfy*

(i) $p_1(z) = z + a/d$,
(ii) $\forall k \in \mathbb{N} : p_{kd}(z) = p_k(Q(z))$,
(iii) $\forall k \in \mathbb{N} : p_{d^k}(z) = p_1(Q^k(z)) = Q^k(z) + a/d$.

The last part of this theorem in particular shows that if Q is centered (i.e., $a = 0$), then the iterates of Q fit neatly into the sequence of monic orthogonal polynomials. To be specific,

$$Q^k = p_{d^k} \quad \text{for all } k \geq 0.$$

A natural question in this context is: Are the remaining orthogonal polynomials dynamically related to Q? As a corollary of Theorem 1.3 we obtain the following answer to this question:

Corollary 5.2 *In the setting of Theorem 5.1, let J_n and K_n be the Julia set, resp. filled Julia set, of the orthonormal polynomial $P_n = \gamma_n p_n$. Then*

$$J_Q \subseteq \liminf_{n\to\infty} J_n \subseteq \limsup_{n\to\infty} K_n \subseteq \mathrm{Co}(K_Q). \tag{43}$$

Moreover, for any $\epsilon > 0$ and $V_\epsilon := \{z \in \mathbb{C} \mid g_\Omega(z) \geq \epsilon\}$,

$$\lim_{n\to\infty} \mathrm{Cap}(V_\epsilon \cap K_n) = 0. \tag{44}$$

Proof Since $\omega \in \mathrm{Reg}$ and J_Q is Dirichlet regular, this follows from Corollary 4.2 and Theorem 1.3. □

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