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Design of four-point SENB specimens with stable crack growth

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Abstract

A four-point single-edge-notch-beam (SENB) test specimen loaded in displacement control (fixed grip) is proposed for studying crack deflection at bi-material interfaces. In order to ensure stable crack growth, a novel analytical model of the four-point SENB specimen in fixed grip is derived and compared with numerical models. Model results show that the specimen should be short and thick, and the start-crack length should be deep for the crack to propagate stable towards the bi-material interface. Observations from experimental tests of four-point SENB specimens with different start-crack lengths, confirmed that the crack grows stable if the start-crack length is deep and unstable if not.

Keywords: Stable crack growth, Bonded joints, Brittle fracture, Adhesive, Finite element analysis

Nomenclature

\begin{itemize}
\item \(a\) actual crack length
\item \(a_0\) start-crack length
\item \(A\) area of cracked surface
\item \(b\) adhesive layer thickness/beam thickness
\item \(B\) horizontal distance between load- and support point
\item \(c\) substrate thickness
\item \(C\) compliance of beam
\item \(D\) length of debond crack at interface
\item \(E_1, E_2\) Young’s modulus (substrate, adhesive)
\item \(\bar{E}\) plane strain Young’s modulus
\end{itemize}

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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>$f$</td>
<td>non-dimensional function for interface stress</td>
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<tr>
<td>$F$</td>
<td>Tada F-function</td>
</tr>
<tr>
<td>$F_\delta$</td>
<td>F-function for bi-material (fixed grip)</td>
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<tr>
<td>$G_I$</td>
<td>mode-I energy release rate</td>
</tr>
<tr>
<td>$G_{IC}$</td>
<td>critical energy release rate</td>
</tr>
<tr>
<td>$G_R$</td>
<td>resistance curve (rising critical energy release rate)</td>
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<tr>
<td>$h$</td>
<td>horizontal distance between crack and load point</td>
</tr>
<tr>
<td>$I$</td>
<td>area moment of inertia per unit width</td>
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<tr>
<td>$K_I$</td>
<td>mode-I stress intensity factor</td>
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<tr>
<td>$K_{IC}$</td>
<td>mode-I critical stress intensity factor</td>
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<tr>
<td>$L$</td>
<td>loading parameter</td>
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<td>$M$</td>
<td>bending moment per unit width</td>
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<tr>
<td>$P$</td>
<td>force per width</td>
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<tr>
<td>$r_p$</td>
<td>radius of plastic zone size</td>
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<tr>
<td>$S$</td>
<td>Tada S-function</td>
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<tr>
<td>$t$</td>
<td>time</td>
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<td>$T$</td>
<td>shear force</td>
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<td>$U$</td>
<td>strain energy</td>
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<tr>
<td>$U_1$</td>
<td>strain energy of part 1 of the beam</td>
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<td>strain energy of part 2 of the beam</td>
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<td>$U_P$</td>
<td>strain energy caused by force</td>
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<td>$U_M$</td>
<td>strain energy caused by bending moment</td>
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<td>strain energy caused by shear force</td>
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<td>$U_{\text{crack}}$</td>
<td>strain energy caused by crack</td>
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<tr>
<td>$U_{\text{no,crack}}$</td>
<td>strain energy in beam without crack</td>
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<tr>
<td>$U_t$</td>
<td>total strain energy</td>
</tr>
<tr>
<td>$V$</td>
<td>work done by external loads</td>
</tr>
<tr>
<td>$V_t$</td>
<td>total work done by external loads</td>
</tr>
</tbody>
</table>
$w$  width of beam

$x, y, z$  coordinates

$\alpha$  first Dundurs’ parameter

$\beta$  second Dundurs’ parameter

$\gamma$  shear strain

$\delta$  displacement

$\epsilon$  normal strain

$\theta$  rotation angle

$\theta_{\text{crack}}$  rotation angle caused by introduction of crack

$\theta_{\text{no,crack}}$  rotation angle for beam without crack

$\theta_t$  total rotation angle

$\kappa$  curvature of beam

$\lambda$  first orthotropy parameter

$\mu$  shear modulus

$\nu$  Poisson’s ratio

$\tilde{\nu}$  modified Poisson’s ratio

$\Pi_{PE}$  potential energy

$\rho$  second orthotropy parameter

$\sigma$  normal stress

$(\sigma_{xx})_{\text{max}}$  maximum normal stress

$\sigma_{yy,i}$  normal stress across interface

$\hat{\sigma}_i$  cohesive strength of interface

$\sigma_{YS}$  yield stress

$\tilde{\sigma}_i$  cohesive strength of the interface

$\tau$  shear stress

$\Phi$  strain energy density

$\Omega$  distance to neutral axis from the bottom of the beam

FE  finite element
1. Introduction

Crack deflection at bi-material interfaces is one of the important cracking phenomena for adhesive joints in large composite structures e.g. wind turbine blades. The bond-line contains defects from which cracks can initiate and turn into transverse cracks in the adhesive. Cracks in the trailing-edge joint, including transverse cracks, are observed in full scale blades in operation as demonstrated by Ataya et al. [1]. After a small crack has formed in the adhesive from a pre-existing defect, the typical cracking sequence is as shown in Figure 1.

![Figure 1: Possible cracking sequences for an adhesive joint loaded in tension. The main crack in the adhesive grows ideally orthogonal towards the adhesive-substrate interface (A). The crack might reach the interface (B) or initiate a new crack at the adhesive-substrate interface (C). If the crack reaches the interface it can stop (B), or penetrate into the substrate (D) or deflect along the adhesive-substrate interface (E).](image)

Historically, the cracking mechanisms in Figure 1 have been addressed using various approaches. Models typically describe the problem where the main crack has reached the interface (Figure 1 (B)) and the subsequent crack deflection/penetration process (Figure 1 (D) or Figure 1 (E)). Modelling the deflection of a crack meeting an interface were, at first, based on either stress criteria [2, 3] or energy criteria [4, 5, 6, 7, 8]. The stress criteria and energy based approach can be unified using a cohesive law, which is a traction-separation law that encompasses a peak stress (strength) and work or separation (energy), typically combined with cohesive zone modelling in finite element (FE) simulations [9, 10, 11, 12, 13, 14]. The parameters for the cohesive law can e.g. be measured by the $J$-integral.
approach [15, 16]. However, accurate experimental determination of peak stress, $\hat{\sigma}_i$, (cohesive strength) for bi-material interfaces is challenging, especially for brittle material interfaces with small separations [17], and therefore novel methodologies are desired.

The problem of interface crack initiation in Figure 1 (C) was addressed using modelling based on stress by Cook and Gordon [2]. However, rigorous experimental tests of crack deflection at interfaces, where the crack deflection process is clearly documented are limited [18] since it is difficult to design an experiment where the crack propagates stable towards a bi-material interface. A few have attempted but with limited success [19, 20].

Zhang and Lewandowski [20] tested bi-material four-point single-edge-notch-beam (SENB) specimens with different interfaces. Unfortunately, crack propagation was unstable such that the crack deflection could only be assessed by visual inspection of the cracked specimens after testing i.e. ex-situ. For the cracking mechanism in Figure 1 (C), the bi-material four-point SENB specimen can also be used to measure the cohesive strength of the interface, $\hat{\sigma}_i$, using a recently developed approach [21]. For this purpose and for the study of crack deflection at interfaces, it is advantageous that the crack grows stable in mode-I towards the interface so that the crack deflection process can be observed in-situ e.g. captured on images by a relatively low camera frame rate during loading. Limited dynamic effects are another advantage of stable crack growth. Therefore, in the present paper the bi-material four-point SENB specimen is modeled to design an experimental test with stable crack growth.

The parameters defining the four-point SENB test specimen geometry and boundary conditions are presented in Figure 2 where the nomenclature follows Tada et al. [22]. The start-crack length is denoted $a_0$ whereas $a$ is the actual crack length. $h$ is the distance between the crack and the load point, and $B$ is the distance between the load- and support points. The depth in z-direction is denoted $w$. $P$ is the total applied force and $\delta$ is the load-point displacement. $\theta$ is the resulting rotation for an applied bending moment, $M$. The primary parameters can be written in non-dimensional form as: $a/b$, $h/b$, and $B/b$. The dimensionless substrate thickness for the bi-material specimen in Figure 2 (C) is denoted $c/b$.

In absence of elastic mismatch, the bi-material four-point SENB specimen (Figure 2 (B)) reduces to the homogenous four-point SENB specimen (Figure 2 (A)). The homogenous four-point SENB specimen shown in Figure 2 (A) is a commonly used test specimen during the last five decades e.g. for measuring fracture toughness, but also for other purposes with varying geometry, material and setup [23, 24, 25, 26, 27, 28, 29, 30, 12, 31, 32]. Four-point SENB specimen geometries, found in the literature [15, 28, 33, 20, 30, 12], vary significant from test to test:

- $0.12 < a_0/b < 0.80$, $0.60 < B/b < 6.90$, $0.50 < h/b < 5.10$, $0.22 < w/b < 9.66$

Justification for the choice of $a_0/b$, $B/b$, $h/b$, and $w/b$ are absent in the papers. Crack growth stability
Figure 2: (A) Homogeneous four-point SENB specimen. (B) Bi-material four-point SENB specimen. (C) Homogenous pure bending specimen.

were tested experimentally by Tandon et al. [33] for three different material systems using homogenous four-point SENB specimens. One of the conclusions was that a smaller \( h \) improved the stability of the four-point SENB test since the amount of elastic energy in the beam was reduced [33]. The effect of \( B \) was not investigated.

Generally, stable crack growth is desirable e.g. when measuring \( R \)-curves [34], for tests where a crack initiates from a notch [20, 35] or for tests with sub-critical crack growth [36, 37]. Furthermore, when measuring fracture toughness, stable crack growth enables measuring several values instead of a single one. Often a crack develops from a notch and "pops-in" meaning that initiation of crack growth starts at a higher energy release rate because the starter crack does not have a sufficiently sharp tip leading to an abrupt advance of the crack front and thus an incorrect determination of the critical energy release rate [38]. Furthermore, for the study of crack deflection at bi-material interfaces [20], it should be avoided that the unstable crack growth associated with pop-in extends completely to the interface, which is another reason why stable crack growth is desirable.

A number of models of the four-point SENB specimen loaded with a pure bending moment exist [39, 25, 22]. However, experimental tests are often conducted by applied displacements i.e. displacement control (fixed grip) [20]. Therefore, this paper presents new models (analytical and numerical) of the four-point SENB specimen with applied displacements. More specifically, for displacement control, analytical expressions will be derived for the homogenous specimen (Figure 2 (A)), whereas numerical models will be applied for the bi-material specimen (Figure 2 (B)).

It is the aim to design the specimen geometry including start-crack length dependent on the loading
configuration such that the crack grows stable and can be observed in-situ during loading. To enable observation of the crack deflection mechanism it is necessary that the crack can grow stable towards the bi-material interface. Therefore, novel models are needed in order to explore a relevant parameter space. To summarise, the goals of the present paper is to:

(i) design the homogenous four-point SENB specimen geometry such that crack propagation is stable.
(ii) design the bi-material four-point SENB specimen geometry such that crack propagation towards the bi-material interface is stable.
(iii) determine the normal stress component across the interface of the bi-material four-point SENB specimen.

The latter is needed for the novel approach to determine the cohesive strength of the interface [21]. These points are addressed in the next sections using a model framework, but first the problems and assumptions are specified.

2. Problem definition

The stiffness mismatch for the bi-material specimen in Figure 2 (B) is presented in terms of $E_1/E_2$ since Dundurs’ parameters ($\alpha, \beta$) can only be used when boundary conditions are prescribed as tractions (not displacement boundary conditions) [40]. For simplicity, the materials are assumed isotropic and Poisson’s ratio are set constant ($\nu_1 = \nu_2 = 1/3$) i.e. in terms of Dundurs’ parameters in plane strain, $\beta = \alpha/4$ [40, 41]. For small-scale yielding the energy- and stress intensity approach are related by the Irwin relation [42]:

$$G_I = \frac{K_I^2}{E}$$

where $E = E$ is for plane stress and $\bar{E} = E/(1 - \nu^2)$ is for plane strain.

To enable observations of crack propagation (and the crack deflection mechanism) during loading, the main crack must propagate in a stable manner i.e. in small increments. Dependent on the specimen geometry, material properties and loading configuration, the crack will propagate stable or dynamic once the magnitude of the critical energy release rate, $G_{IC}$, is reached [43]. For a material exhibiting R-curve behavior, i.e. a material with rising fracture resistance, the condition for continued crack extension is $G = G_R(\Delta a)$, where $G$ is the applied energy release rate. $G_R$ versus $\Delta a$ is the resistance curve of a material when the crack has extended an amount $\Delta a$ to the current crack length, $a$, under quasi-static loading. To ensure stable crack propagation (not dynamic), the following generalised condition must be satisfied [43, 22]:

$$\left[ \frac{\partial G}{\partial a} \right]_L < \left[ \frac{dG_R}{d\Delta a} \right]$$

7
where $L$ is the loading parameter that is kept constant while the crack advances a small increment [43]. A mode-I crack in a perfectly brittle material will propagate under constant $G_I$ hence the condition for stable crack growth reduces to [43]:

$$\left[ \frac{\partial G_I}{\partial a} \right]_L < 0$$

(3)

In words, the mode-I energy release rate, $G_I$, must decrease with crack length. This general condition for stable crack growth in a perfectly brittle material can be specified for fixed displacement loading (fixed grip) as:

$$\left[ \frac{\partial G_I}{\partial a} \right]_\delta < 0$$

(4)

or for a fixed loading condition (dead load) as:

$$\left[ \frac{\partial G_I}{\partial a} \right]_P < 0$$

(5)

Note, the energy release rate function can be expressed in terms of load point displacement as $G_I(\delta, a)$ or in terms of applied force as $G_I(P, a)$. Basically this is two representations of the same function.

The specimens in Figure 2 are analysed for load control, displacement control, and test configuration since the first derivative of energy release rate, $\partial G_I/\partial a$, depends on load conditions, geometry, and stiffness properties/mismatch. It is therefore non-trivial to determine the best possible test setup and rigorous models are needed.

The simple pure bending specimen in Figure 2 (A) is analysed in rotation control in Appendix B. The homogenous four-point SENB specimen in Figure 2 (B) is analysed using analytical expressions that are derived and used to determine a specimen design with stable crack growth for displacement control (fixed grip). The assumptions used in the derivations are: (1) It is assumed that Bernoulli-Euler beam theory can be applied such that shear force- and moment distribution are assumed to take the form depicted in Figure C.19 in Appendix C. (2) The specimen is in static equilibrium (no dynamic effects). (3) LEFM requirements are assumed to be fulfilled.

The latter requires that linear-elasticity and superposition applies. Numerical models are used to test the accuracy/limitations of the analytical models and to design the experiment with the bi-material SENB specimen in Figure 2 (C).
3. Analytical model of the homogenous four-point SENB specimen

The standard solution for the stress intensity factor of the crack in the pure bending specimen (Figure 2 (C)) is given for a homogeneous material by Tada et al. [22] as:

\[ K_I = (\sigma_{xx})_{max} \sqrt{\pi a F(a/b)}, \quad (\sigma_{xx})_{max} = \frac{6M}{b^2} \]  

(6)

where \((\sigma_{xx})_{max}\) is the maximum normal stress (bending) in the beam in the \(x\)-direction at location \(y = -\frac{b}{2}\). The moment, \(M\), is per width, \(w\). \(F(a/b)\) is a non-dimensional function determined by an empirical fit to semi-analytical data obtained by the boundary collocation procedure [24]. A fit to semi-analytical data for \(F(a/b)\) is given as [25, 22]:

\[ F(a/b) = \frac{2b}{\pi a} \tan \left( \frac{\pi a}{2b} \right) \right) 0.923 + 0.199 \left( 1 - \sin \left( \frac{\pi a}{2b} \right) \right)^4 \]

(7)

which has an accuracy of better than 0.5% for any \(a/b\). The energy release rate, \(G_I\), of a mode-I crack can be determined using equation 6 and the Irwin relation in equation 1 [42]:

\[ G_I = \frac{K_I^2}{E} = \frac{1}{E} (\sigma_{xx})_{max} \pi a [F(a/b)]^2 = \frac{1}{E} \frac{36M^2}{b^4} \pi a [F(a/b)]^2 \]

(8)

For load control, \(G_I\) in equation 8 can be written on a non-dimensional form as:

\[ \frac{G_I E b^3}{M^2} = 36\pi \frac{a}{b} [F(a/b)]^2 \]

(9)

In Appendix C an equation for the strain energy of the homogenous four-point SENB specimen is derived as a function of applied displacements (not moments). It is emphasised in equation 10 that the strain energy contributions originate from normal stresses, shear stresses and the presence of the crack according to the principle described by Rice et al. [44]:

\[ U_t = \frac{P^2}{E} \left( \frac{B}{b} \right)^2 \left[ \frac{B}{b} + \frac{3h}{b} + \frac{3}{E} \frac{b}{B} \right] + \frac{3S(a/b)}{\mu B} \]

(10)

where \(S(a/b)\) is presented in equation B.16 in Appendix B based on Tada et al. [22]. A relation between moment, \(M\), and displacement, \(\delta\), is derived by combining \(U_t = P\delta/2\), from equation C.1 and equation C.2 in Appendix C, with the moment, \(M = PB/2\), as:

\[ M = \frac{U_t B}{\delta} \]

(11)

Inserting \(U_t\) from equation 10 in equation 11 and then inserting \(M\) from equation 11 in equation 8 gives an expression for \(G_I\) as a function of applied displacement, \(\delta\), as:

\[ G_I = 36\pi \frac{a}{b} F(a/b) \left[ \frac{3\delta}{2B} \left( \frac{E}{b} + \frac{3h}{b} + \frac{3}{5} \frac{b}{B} \nu + 3S(a/b) \right) \right]^2 \]

(12)
or on a non-dimensional form as:

\[
\frac{G_I b}{E \delta^2} = \frac{ab}{B^2} \left[ \frac{3}{2} \left( \frac{h}{b} + 3 \frac{h}{b} + \frac{3 b}{B} \bar{\nu} + 3 F(a/b) \right) \right]^2
\]

where \( \bar{\nu} = 1/(1-\nu) \) is for plane strain and \( \bar{\nu} = (1+\nu) \) is for plane stress. Note, in displacement control the energy release rate is coupled to the applied displacement through the elastic constants and the geometrical parameters.

4. Introducing the finite element model

A numerical model is developed in order to test the analytical derivation for the homogeneous SENB specimen and to analyse the bi-material SENB specimen, see the specimens in Figure 2 (A-B). The FE model, simulated in Abaqus CAE 6.14 (Dassault Systemes) with eight-noded plane strain elements, is parametrized with the non-dimensional groups, \( a/b, h/b \) and \( B/b \). A symmetry condition is imposed at \( x = 0 \) to reduce the computational time. A focused mesh is applied in the region of \( 0.5b \) in the \( x \)-direction of the beam and 100 elements are used over the distance \( b \).

5. Finite element modelling of homogenous SENB specimen

5.1. Benchmark of analytical derivation with finite element simulations

In the beginning of this section, for different geometries of the FE model and analytical model, two cases are compared; displacement control/load control. The non-dimensional energy release rate results from the parametric linear-elastic FE model of the homogenous four-point SENB specimen is presented in Figure 3 (A-B) for both load- and displacement control. The energy release rate results in Figure 3 (B) for load control are compared with the results of Tada et al. [22].

For displacement control, the curves in Figure 3 (A) start from zero at \( a/b = 0 \), increase to a peak and finally decrease to zero again at \( a/b = 1 \). Thus, when \( a/b \to 1 \) then \( G_I \to 0 \) since the crack approaches a free surface (at \( y = b/2 \)) and the load is applied with fixed displacements. A significant difference is observed in Figure 3 (A) between the results of the analytical- (red curve) and the numerical model (red symbols) for the short and thick specimen loaded in displacement control. The analytical model becomes inaccurate for small values of \( h/b \) and \( B/b \) since the stress field deviates more and more from the stress field assumed in beam theory, i.e. assumption (1) used in the derivation, when the beam becomes more compact.

For load control, the curves in Figure 3 (B) start from zero at \( a/b = 0 \) and increases as the crack length becomes longer. Thus, for load control (dead load), \( G_I \to \infty \) when the crack approaches the
As mentioned in the problem definition, we aim to design the test specimen such that the criterion for stable crack growth in equation 4 is fulfilled. Thus, the energy release rate should decrease with crack length. From Figure 3 (A) it is seen that equation 4 is fulfilled when \( a \) exceeds a critical value, denoted \((a/b)_{\text{peak}}\). The crack grows stable if the start-crack length is \( a_0/b \geq (a/b)_{\text{peak}} \) hence the energy release rate decreases with crack length. It is desired that \((a/b)_{\text{peak}}\) is as small as possible hence the crack can grow far before reaching the free surface and to enlarge the design space with stable crack growth. Figure 3 (B) shows that the crack grows unstable for load control for any \( a/b \) since the stability condition in equation 5 cannot be fulfilled under a fixed loading condition.

A parameter space with \( a/b, h/b \) and \( B/b \) is explored in Figure 4-7 in order to determine the geometrical parameters effect on \( G_I \) and \((a/b)_{\text{peak}}\). Based on the dotted lines in Figure 6-7, \((a/b)_{\text{peak}}\) is presented in Figure 8. It is evident that the beam should be short (small \( h \) and \( B \)) and thick (large \( b \)) to maximise the design space with stable crack growth i.e. to reduce the value of \((a/b)_{\text{peak}}\). Figure 8 shows that \((a/b)_{\text{peak}}\) is significantly reduced for \( B/b = 2.0 \) in comparison with \( B/b = 6.0 \). Thus, it is demonstrated that crack growth stability for the four-point SENB specimen depends on the start-crack length, load span, support span, and whether the specimen is tested in load- or displacement control.
Figure 4: Energy release rate results by plane strain FE model and analytical model for different $B/b$ and $a/b$ for displacement control with $h/b = 1.0$, $E_1/E_2 = 1.0$, $\nu = 1/3$ (lines are analytical results; symbols are FE results).

Figure 5: Energy release rate results by plane strain FE model and analytical model for different $B/b$ and $a/b$ for displacement control with $h/b = 3.0$, $E_1/E_2 = 1.0$ and $\nu = 1/3$ (lines are analytical results; symbols are FE results).
Figure 6: Energy release rate results by plane strain FE model and analytical model for different $h/b$ and $a/b$ for displacement control with $B/b = 2.0$, $E_1/E_2 = 1.0$ and $\nu = 1/3$ (lines are analytical results; symbols are FE results).

Figure 7: Energy release rate results by plane strain FE model and analytical model for different $h/b$ and $a/b$ for displacement control with $B/b = 6.0$, $E_1/E_2 = 1.0$ and $\nu = 1/3$ (lines are analytical results; symbols are FE results).
6. Finite Element Modelling of Bi-material SENB Specimen

A function similar to $F(a/b)$ from Tada et al. [22] can be established for the bi-material four-point SENB specimen to account for the presence of a substrate of thickness, $c$. Thus, for the bi-material specimen, with assumed isotropic adhesive and -substrate, in load control:

$$G_I = \frac{1}{E_2} (\sigma_{xx})^2_{\text{max}} \pi a F(a/(b+c), c/b, E_1/E_2)^2, \quad (\sigma_{xx})_{\text{max}} = \frac{M E_2 \Omega}{E_1 I_1 + E_2 I_2} \quad (14)$$

where subscript 1 and 2 represent the substrate and adhesive, respectively. As shown in Figure 9, $\Omega$ is the distance from the bottom of the beam and to the global neutral axis of the beam (in the specimen without crack) [45]. $I_1$ and $I_2$ are the contributions to the total moment area of inertia from material 1 and 2, respectively:

$$I_1 = \frac{c^3}{12} + c \left( \frac{c}{2} + b - \Omega \right)^2, \quad I_2 = \frac{b^3}{12} + b \left( \Omega - \frac{b}{2} \right)^2, \quad \Omega = c \frac{1 + \frac{E_1}{E_2} \frac{c}{b} + \frac{E_1}{E_2} \left( \frac{\xi}{b} \right)^2}{2 \frac{\xi}{b} \left( 1 + \frac{E_1}{E_2} \frac{c}{b} \right)} \quad (15)$$

The function, $F$, in equation 14 is determined as shown in Figure 9 by FE simulations, which is compared for $E_1/E_2 = 1$, with the solution by Tada et al. [22]. It can be seen that independently of elastic mismatch when: $a/(b+c) \to 0$ then $F(a/(b+c)) \to 1.12$. This limit is similar to the solution for a side-crack in an infinitely large homogenous plate under uni-directional tension [22, 46]. The trend in Figure 9 is comparable to the partial cracked film problem from Beuth [47]. For compliant substrates $(E_1/E_2 \lesssim 3)$, $F$ increases monotonic, whereas for stiffer substrates $(E_1/E_2 \gtrsim 9)$, $F$ reaches a peak and subsequently starts decreasing (close to $a/(b+c) = 0.8$).

Dimension analysis reveals that the energy release rate of the crack for the bi-material specimen
presented in Figure 2 (B) can, when loaded in displacement control, be written as:

\[
\frac{G_I(b + c)}{E_2\delta^2} = \frac{9\pi a(b + c)}{4} \frac{F_3(a/(b + c), h/b, B/b, c/b, E_1/E_2, \nu_1, \nu_2)^2}{B^2}
\] (16)

where the non-dimensional function, \(F_3\), is determined numerically. \(F_3\) is introduced since it is out of the scope in the paper to derive an expression analytically for the bi-material specimen like in equation 13 for the homogenous specimen.

6.1. Parameter study using the bi-material FE model

To determine a start-crack length, \(a_0/b\), that gives stable crack growth, the requirement in equation 4 needs to be applied on the results of the bi-material FE model in Figure 10-11. From Figure 10-11 it is clear that an increase of the substrate stiffness \((E_1/E_2)\), increases the energy release rate. The effect of elastic mismatch on the magnitude of \((a/b)_{peak}\) is more complex as shown in Figure 12.

The curves in Figure 12 for \(c/b = 0.2\) are determined from \((a/b)_{peak}\) of the dotted symbols in Figure 10-11 and similarly for \(c/b = 0.1\) and \(c/b = 0.3\). Additional points for \(E_1/E_2 = 1.5, 3, 6\) are included in Figure 12 to emphasise the trend of the curves, where the data points are connected by lines. Figure 12 shows that \((a/b)_{peak}\) is sensitive to \(E_1/E_2\) and \(h/b\). The graphs in Figure 12 for \(h/b = 0.9\) are marked with red colors and the graphs for \(h/b = 1.8\) are marked with blue colors. It is seen that \((a/b)_{peak}\) is reduced for the group of \(h/b = 0.9\) (red curves) compared with the group of \(h/b = 1.8\) (blue curves). This study shows the complexity of the problem since several parameters \((h/b, B/b, c/b, E_1/E_2)\) affects the resulting value of \((a/b)_{peak}\).
Figure 10: Energy release rate results from FE model for displacement control for $E_1/E_2$. Other parameters are:
$\nu_1 = \nu_2 = 1/3$, $h/b = 0.9$, $B/b = 1.35$ and $c/b = 0.2$ (lines are analytical results; symbols are FE results, #1 is substrate, #2 is adhesive).

Figure 11: Energy release rate results from FE model for displacement control for $E_1/E_2$. Other parameters are:
$\nu_1 = \nu_2 = 1/3$, $h/b = 1.8$, $B/b = 1.35$ and $c/b = 0.2$ (lines are analytical results; symbols are FE results, #1 is substrate, #2 is adhesive).
Figure 12: Peak values of $a/b$ from Figure 10-11 are used to determine, $(a/b)_{peak}$, for the bi-material specimen with varying $E_1/E_2$, $h/b$, and $c/b$. The limit value for $(a/b)_{peak}$ is determined for $E_1/E_2 = 100$ as indicated by the colored arrows. (#1 is substrate, #2 is adhesive)

7. Determination of mode-I cohesive strength of interfaces

For the cracking mechanism in Figure 1 (C), at the time where the debond crack at the interface initiates (onset of interface separation), the cohesive strength of the interface, $\sigma_i$, is equal to the stress across the interface, $\sigma_{yy,i}$, i.e. $\dot{\sigma}_i = \sigma_{yy,i}$. Using the bi-material FE model of the four-point SENB specimen, $\sigma_{yy,i}$ can be determined by:

$$\frac{\sigma_{yy,i}b^2}{M} = f(a/b, c/b, E_1/E_2, \nu_1, \nu_2)$$  

(17)

where the non-dimensional function, $f$, is determined by the FE modelling result in Figure 13-14.

The calculation procedure to determine the cohesive strength of the interface, $\dot{\sigma}_i$, should be read in details in a related paper [21], but it is listed in short here for convenience:

- During the fracture experiment, capture the time of interface crack initiation (onset of interface separation) e.g. by digital image correlation, visually or by other methods.

- From images recorded during the experiment, determine $M$ and $a/b$ at the time of interface crack initiation (onset of interface separation).

- Use measured $M$ and $a/b$ with the FE results in Figure 13-14 and equation 17 to determine the stress across the interface, $\sigma_{yy,i}$ at the onset of interface separation.

Moment, $M$ and crack length, $a/b$ are the only parameters varying during the test. $\sigma_{yy,i}$ scales linearly with $M$, but non-linearly with $a/b$ as shown in Figure 13-14. Thus, using the measured values of $M$
and $a/b$ together with the FE simulation results in Figure 13-14, the resulting cohesive strength of a material interface can be determined by equation 17.

Figure 13: Results for normalised interface stress determined numerically for different $E_1/E_2$. Other parameters are: $\nu_1 = \nu_2 = 1/3$, $h/b = 0.9$, $B/b = 1.35$ and $c/b = 0.2$.

Figure 14: Results for normalised interface stress determined numerically for different $E_1/E_2$ and $c/b$. Other parameters are: $\nu_1 = \nu_2 = 1/3$, $h/b = 0.9$ and $B/b = 1.35$. 
8. Experimental test of model predictions

In order to test the model predictions for stable growth of the crack, homogenous four-point SENB specimens cast of adhesive were manufactured and tested experimentally in the laboratory.

8.1. Experimental procedure

The four-point SENB specimens were cast by injection of an adhesive in-between two glass plates to create a homogenous plate of adhesive. Prior to the injection, the surfaces of the glass plates were waxed to ease the removal of the cast adhesive plate. The adhesive plate was post-cured and cut into beams with similar outer dimensions ($w/b \approx 0.7$). The adhesive type and processing conditions are proprietary. Start-cracks were cut using an ultra thin razor blade of thickness 76 microns (Ultra-thin single edge, stainless steel blade from Ted Pella, Inc.). The test setup were adjusted with $h/b = 0.9$ and $B/b = 1.35$. The specimens were placed in a four-point bend fixture and loaded by a MTS 858 Mini Bionix II servo-hydraulic test machine at a constant displacement-rate of 0.015 mm/min, where the load is measured by a 1.5 kN load cell. For selected tests, the crack length is measured on images taken during the test by help from a Python script [48, 49].

8.2. Experimental results

The measured bending moment as a function of load point displacement for a selection of test specimens is presented in Figure 15 (A). The curves (bending moment vs load point displacement) show similar trend as those curves reported in the literature [15, 33]. All curves, except of the first $\sim 0.05$ mm, starts with a linear part until the onset of crack propagation where the moment drops. Onset of crack propagation is identified near the measured maximum moment (the peak) according to images taken during the test. The smooth decrease in $M$ is associated with stable crack propagation. All curves decrease smoothly until zero moment is measured at displacement, $\delta \approx 0.9-1.0$ mm, except of the test with a start-crack length of $a_0/b = 0.34$ (Test 1 in Figure 15 (A)). For this test, the moment initially increases faster with applied displacement since the start-crack length is short. The moment drops instantaneously to zero at displacement, $\delta \approx 0.29$ mm, due to unstable crack growth. In accordance with the model in Figure 15 (B), the sudden drop in moment for this test exemplifies the response of the crack growth when the start-crack length is made too short i.e. unstable crack growth.

The observed unstable crack growth for Test 1 with $a_0/b = 0.34$, shown in Figure 15 (A), agrees fairly well with the prediction by the analytical model presented in Figure 15 (B). The analytical model in Figure 15 (B) at $a_0/b = 0.34$ illustrates that the crack stability criterion in equation 4 is not fulfilled. Thus, at the onset of crack propagation the crack grows unstable. As opposed to the model, in the experiment crack propagation does not become stable again at higher $a/b$ since the
unstable crack growth activates dynamic effects. For the other test specimens with \( a_0/b \geq 0.47 \), the crack propagates stable as shown by the smoothly decreasing curves in Figure 15 (A) since the crack stability criterion in equation 4 is satisfied. Thus, it is demonstrated experimentally that a proper selection of the start-crack length, e.g. by modelling, is useful for determining a start-crack length that gives stable crack propagation.

The measured bending moment and crack length of another four-point SENB specimen are presented in Figure 16 to investigate the crack propagation in further details. Overall, the bending moment decreases smoothly hence the crack propagates stable and the crack length can be measured on images during the test. However, a small instability is seen in Figure 16 at the onset of crack propagation (at displacement \( \delta \approx 0.35 \text{ mm} \)). This unexpected behaviour is attributed the manual cutting of the notch that is not as sharp as a real crack. Thus, the initiation of a crack from the notch gives abrupt crack growth in the beginning i.e. "pop-in" effect [50]. Except of this initiation phenomenon, the remaining growth of the crack is stable as shown by the measured crack length in Figure 16.

9. Discussions

9.1. Discussion of modelling results

When the length of the beam is short, the boundary effects are significant meaning that the moment and shear force distributions are not as idealised as shown in Figure C.19 in Appendix Appendix C. In the analytical model, it is assumed that the normal stress is perfectly linear across the beam thickness and the shear stress distribution is perfectly parabolic, but near the boundaries the shear stress field is not as assumed in the analytical model. While these localised boundary effects are not included in the
Figure 16: Experimental test of homogenous four-point SENB specimen with stable crack growth and parameters: \(a_0/b = 0.47\), \(h/b = 0.9\), \(B/b = 1.35\).

analytic model, they are included in the numerical model. This is one of the reasons for the deviations between the analytical- and numerical model in Figure 3. This effect is especially significant when the beam is short and thick.

9.2. Small scale yielding assumption

The assumptions from LEFM must be satisfied for the derived analytical model to be accurate for the experimental test specimens. To satisfy brittle fracture conditions (small-scale yielding), the plastic zone size near the crack tip must be small in comparison with the characteristic length scale in the problem i.e. the start-crack length. The first order estimation of the radius of the plastic zone, \(r_p\), is determined by [43, 22, 51]:

\[
 r_p = \frac{1}{2\pi} \left( \frac{K_{IC}}{\sigma_{YS}} \right)^2 \tag{18}
\]

The critical stress intensity factor, \(K_{IC}\), was measured by the experimental four-point SENB test in Figure 16, and the yield stress, \(\sigma_{YS}\), was measured by a dog bone specimen using the standard "ISO 527-2: 2012". Based on equation 18, it is determined that \(r_p/a_0 \approx 0.03\). According to the results presented by Charalambides et al. [52], a 10% deviation of the stress field from the singular form (K-dominant zone) for a SENB specimen are found to be \(r_p/a_0 \approx 0.04\). Thus, a plastic zone size of \(r_p/a_0 \approx 0.03\) is judged to be acceptable since it will be embedded in the K-dominant zone.
9.3. Homogenous four-point SENB experiments - uncertainties

The test-to-test variations observed for the initial linear slope in the results of the four-point SENB experiments are primarily attributed to the differences in start-crack length. Material variations may affect the measured maximum load. The experiments were conducted under a relatively low displacement-rate of 0.015 mm/min. Therefore, creep of the adhesive could be a phenomena influencing the stress field of the beam specimen, especially near the crack tip and the load introduction points. Furthermore, air voids in the adhesive can give variations in the measured curves in Figure 15 (A). The maximum diameter of the voids in the adhesive is measured to about 0.04b. Voids of this size may accelerate the crack growth in certain regions of the specimens e.g. indicated by the fluctuations and bumps on the smooth curves in Figure 15 (A). Adjustment and alignment of the load rig may affect the shape of the whole curve in Figure 15 (A).

9.4. Stability of crack growth

Typically in the literature, the four-point SENB specimen is used for characterising materials that have a constant fracture toughness (i.e. no R-curve), which not necessarily requires stable crack propagation. However, for many applications, as mentioned in the introduction, stable crack propagation is essential to avoid dynamic effects and enable monitoring crack growth. Furthermore, a test with stable crack growth means that a sharp crack is formed from the start-notch, which in practice cannot be made as sharp as a real crack. According to the test standards ISO-15024:2001(E) and ASTM D5528-01, the start notch should not exceed width of 13 \( \mu \text{m} \) when measuring fracture toughness of uni-directional composites. If the start notch is machined it should not exceed a width of 10 \( \mu \text{m} \) according to the study of fine grained alumina by Nishida et al. [38].

When a sharp crack propagates stable, a more accurate fracture toughness value can be measured since several measurements can be made using just a single test specimen. In any case, stable crack growth is attractive, as it can help uncovering unexpected behavior such as rate dependent- or R-curve behavior. Also, possible errors introduced by having a machined notch as a starter crack, in contrast to a truly sharp crack, is eliminated.

9.5. Discussion of results in relation to existing literature

From the results in Figure 8, the test specimen design by Brinckmann et al. [12] can be improved such that crack propagation are stable and can be monitored. For this particular specimen design \((B/b = 6.9)\), it is recommended to use a smaller \(B/b\) in order to reduce the elastic strain energy of the beam and thereby enhance the probability of stable crack propagation. However, if the substrate is a laminate, the four-point SENB specimen might fail in shear (mode-II shear crack or delamination) in between the support- and load points if \(B\) is too small and a weak interface exists in the substrate of the bi-material specimen [27].
9.6. Bi-material model with orthotropic substrate:

As mentioned in the introduction, the methods are also aimed to be applicable for designing experiments with stable crack growth for adhesive-composite joints. Therefore, it is tested, for the bi-material four-point SENB specimen, how accurate it is to model a uni-directional glass-fiber laminate as an isotropic substrate. The in-plane orthotropy can be described by the dimensionless parameters [53]:

\[
\lambda = \frac{E_T}{E_L}, \quad \rho = \frac{(E_L E_T)^{1/2}}{2G_{LT}} - (\nu_{LT} \nu_{TL})^{1/2}
\]

where subscripts \( L \) and \( T \) stands for longitudinal \((x)\) and transverse \((y)\) directions, respectively. \( \lambda = 0.3 \) and \( \rho = 2.5 \) are representative for a uni-directional glass-fiber laminate where material directions are following the coordinate system shown in Figure 9. For the FE model with isotropic substrate, \( E_L \) and \( \nu_{LT} \) are the only input of stiffness parameters i.e. \( \lambda = \rho = 1 \) [53]. Results from a bi-material FE model with a substrate of an isotropic material \( (\lambda = \rho = 1) \) were compared with a bi-material FE model with a substrate of an orthotropic material \( (\lambda = 0.3 \) and \( \rho = 2.5) \) for geometrical parameters of \( h/b = 0.9, B/b = 1.35 \) and \( c/b = 0.2 \). The energy release rates of the cracks and \( (a/b)_{peak} \) are comparable (within 5\%) for the isotropic and orthotropic cases. This suggests that the primary stiffness in the \( x \)-direction, \( E_L \), is the main stiffness parameter that governs the energy release rate of the crack in the model with orthotropic substrate.

10. Conclusions

An analytical model of the displacement loaded four-point SENB test specimen was derived and found to agree well with FE simulations under certain geometrical conditions. The models (analytical and numerical) were efficient to design the experiment with stable crack growth since stability depends on load configuration, crack length and geometry. These models suggest that the beam should be short and thick, and the start-crack length should be relatively deep for the main crack to propagate stable.

The model predictions for stable crack growth were tested experimentally by homogenous four-point SENB specimens cast of pure adhesive. The experiments showed that crack growth is stable if the start-crack length was made sufficiently long and unstable if not. The stable crack propagation was documented by a series of images captured in-situ during loading. Thus, for the material systems used in the present work, the displacement loaded four-point SENB specimen was found appropriate for studying crack deflection at interfaces since stable crack growth could be achieved.

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Appendix A. Strain energy of cantilever beam

The strain energy density for a three dimensional linear-elastic solid in terms of engineering stresses and strains is [54]:

\[
\Phi = \frac{1}{2} \left( \sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \sigma_{zz} \varepsilon_{zz} + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz} \right) \tag{A.1}
\]

where \(\sigma\) is the normal stress, \(\tau\) is the shear stress, \(\varepsilon\) is the normal strain and \(\gamma\) is the shear strain. For the cantilever beam loaded by force, \(P_0\), and moment, \(M_0\), shown in Figure A.17, the strain energy density expression in (A.1) reduces to:

\[
\Phi = \frac{1}{2} \left( \sigma_{xx} \varepsilon_{xx} + \tau_{xy} \gamma_{xy} \right) \tag{A.2}
\]

In the following the two terms in (A.2) will be assessed individually.

Figure A.17: Cantilever beam with applied bending moment, \(M_0\), and applied force, \(P_0\).
Appendix A.1. Strain energy of cantilever beam due to normal stresses

The strain energy density for the first term in (A.2) due to normal stress is:

$$\Phi(x, y) = \frac{1}{2} \sigma_{xx} \epsilon_{xx}$$  \hspace{1cm} (A.3)

Based on the stress across the beam section, $\sigma_{xx}(y) = -M_{zz} y / I$, the strain distribution, $\epsilon_{xx}(y) = -\kappa y$, and Hooke's law, $\sigma_{xx} = E \epsilon_{xx}$, it can be shown that the total strain energy in a beam under pure bending moments is [55]:

$$U_M = \int_0^{L_0} \left( \frac{M_{zz}^2}{2EI} \right) dx$$  \hspace{1cm} (A.4)

where the area moment of inertia per width, $I$, for a rectangular cross section is; $I = b^3/12$. If the beam is loaded by force, $P_0$, and moment, $M_0$, then the cross-sectional moment, $M_{zz}(x)$, varies linearly over the beam length, $L_0$, as shown in Figure A.17. The strain energy of the cantilever beam in Figure A.17 with an applied moment per width, $M_0$, and an applied force per width, $P_0$, is (see p. 726-731 in [55]):

$$U_M = \int_0^{L_0} \left( \frac{M_{zz}(x)^2}{2EI} \right) dx = \int_0^{L_0} \left( \frac{(P_0 x + M_0)^2}{2EI} \right) dx = \frac{6}{b^3 E} \left( \frac{P_0^2}{3} + P_0 M_0 L_0^2 + M_0^2 L_0 \right)$$  \hspace{1cm} (A.5)

The strain energy in the four-point bend specimen under applied bending forces is ($M_0 = 0$):

$$U_{P_0} = \frac{2}{3} P_0^2 L_0^3$$  \hspace{1cm} (A.6)

whereas the strain energy in the beam under pure bending moments is ($P_0 = 0$):

$$U_{M_0} = \frac{6}{b^3 E} M_0^2 L_0$$  \hspace{1cm} (A.7)

Appendix A.2. Strain energy of cantilever beam due to shear stresses

The strain energy density for the second term in (A.2) due to shear stress is:

$$\Phi(x, y) = \frac{1}{2} \tau_{xy} \gamma_{xy} = \frac{1}{2} \frac{\tau_{xy}^2}{\mu}$$  \hspace{1cm} (A.8)

where the shear stress, $\tau_{xy}$, is given by Hooke's law for shear, $\tau_{xy} = \mu \gamma_{xy}$ [55]. Here, $\mu$ is the shear modulus and $\gamma_{xy}$ is the shear strain. For a rectangular cross section, the through thickness shear stress distribution is assumed parabolic on the form (see p. 192 in [46] or p. 392 in [55]):

$$\tau_{xy}(x, y) = \frac{3}{2} \frac{T(x)}{b} \left( 1 - \left( \frac{y}{b} \right)^2 \right)$$  \hspace{1cm} (A.9)

where the applied shear force is denoted $T$. The strain energy density becomes:

$$\Phi(x, y) = \frac{1}{2} \frac{\tau_{xy}(x, y)^2}{\mu} = \frac{1}{2\mu} \left( \frac{3}{2} \frac{T(x)}{b} \right)^2 \left( 1 + \left( \frac{2y}{b} \right)^4 - 2 \left( \frac{2y}{b} \right)^2 \right)$$  \hspace{1cm} (A.10)
The strain energy of the entire beam due to shear force, \( T \), is determined by integration of the strain energy density over the beam volume. First, the strain energy per unit length of the beam (in the \( x \)-direction) is determined by integration of the strain energy density across the beam thickness:

\[
\frac{dU_T}{dx} = \int_{-b/2}^{+b/2} \Phi dy = \frac{1}{2\mu} \left( \frac{3T(x)}{b} \right)^2 \int_{-b/2}^{+b/2} \left( 1 + \left( \frac{2y}{b} \right)^4 - 2 \left( \frac{2y}{b} \right)^2 \right) dy = \frac{3}{5} \frac{1}{\mu} \frac{T(x)^2}{b^2} \tag{A.11}
\]

Next, the strain energy of the beam is obtained by integrating the strain energy per unit length over the entire beam length:

\[
U_T = \frac{3}{5} \frac{1}{\mu} \int_0^{L_0} T(x)^2 dx \tag{A.12}
\]

Thus, the strain energy for a constant shear force, \( T \), over the beam length, \( L_0 \), see Figure A.17, becomes:

\[
U_T = \frac{3}{5} \frac{T^2 L_0}{b\mu} \tag{A.13}
\]

**Appendix B. Analytical model of a pure bending specimen**

The simplest case possible, the pure bending specimen sketched in Figure B.18 (A), can be described by two non-dimensional geometrical parameters: \( a/b, h/b \). For beam rotations (Figure 2 (C)), the total rotation, \( \theta_t \), is the sum of the rotation without a crack, \( \theta_{no,crack} \), plus the rotation introduced by the presence of a crack, \( \theta_{crack} \), as described by the work of Rice et al. [44]:

\[
\theta_t = \theta_{no,crack} + \theta_{crack} \tag{B.1}
\]

Equivalently, the total strain energy of the beam can be written [44, 22]:

\[
U_t = U_{no,crack} + U_{crack} \tag{B.2}
\]

Equation B.1 can be related to equation B.2 using the relation of \( U = M\theta/2 \) as:

\[
\frac{1}{2} M\theta_t = \frac{1}{2} M(\theta_{no,crack} + \theta_{crack}) = \frac{1}{2} M\theta_{no,crack} + \frac{1}{2} M\theta_{crack} \tag{B.3}
\]

The rotation angle for the pure bending specimen with no crack, \( \theta_{no,crack} \), is derived by elementary beam theory based on p. 726 in Gere and Goodno [55]:

\[
\theta_{no,crack} = \frac{M2h}{ET} = \frac{24M h}{b^2E b} \tag{B.4}
\]

The term including the crack in (B.1-B.3) can be evaluated using the Griffith theory of fracture [56] for perfectly brittle elastic solids. The Griffith theory of fracture [56] is based on the principle of energy conservation [57]. The mode-I energy release rate is defined as the decrease in potential energy of the body, \( \Pi_{PE} \), with crack advance, \( da \), as [22, 43, 56, 58]:

\[
G_I = -\left( \frac{d\Pi_{PE}}{da} \right)_M \tag{B.5}
\]
where $M$ indicates that the loading is applied as a prescribed moment. As described by Hutchinson [58] for prescribed moment, the potential energy of the body is equal to the elastic strain energy of the body minus the work done by the external moment:

$$\Pi_{PE} = \frac{1}{2} M \theta_t - M \theta_t$$  \hspace{1cm} (B.6)

$$\Pi_{PE} = -\frac{1}{2} M \theta_t$$  \hspace{1cm} (B.7)

Inserting equation B.7 into equation B.5 and since $M$ is constant:

$$G_I = - \left( \frac{d(-\frac{1}{2} M \theta_t)}{da} \right)_M = \frac{1}{2} M \frac{d\theta_t}{da}$$  \hspace{1cm} (B.8)

As explained by Rice et al. [44], equation B.1 can be differentiated wrt. crack extension to give:

$$d\theta_t/da = d\theta_{crack}/da.$$  Inserting this in equation B.8 and rearranging gives:

$$G_I da = \frac{1}{2} M d\theta_{crack}$$  \hspace{1cm} (B.9)

Integration leads to ($M$ being constant):

$$\int_0^a G_I da = \frac{1}{2} M \int_0^{\theta_{crack}} d\theta_{crack} = \frac{1}{2} M \theta_{crack}$$  \hspace{1cm} (B.10)

where $\theta_{crack}$ can be interpreted as the additional rotation due to the crack. Furthermore, the right hand side of equation B.10 can be referred to as $U_{crack}$. Thus, the strain energy due to the introduction of the crack can be written as:

$$U_{crack} = \frac{1}{2} M \theta_{crack}$$  \hspace{1cm} (B.11)

From equation B.10 it is evident that the elastic strain energy per unit extension of the crack is [47]:

$$U(a') = \int_0^{a'} G_I(a) da$$  \hspace{1cm} (B.12)

where $U(a') = U_{crack}$ is the change of strain energy when the crack extends from $a = 0$ to $a = a'$. Inserting $G_I$ from equation 8 into equation B.12, the strain energy of the crack becomes:

$$U_{crack} = \frac{1}{2} M \theta_{crack} = \frac{36 M^2 \pi}{b^4 E} \int_0^{a'} \left[ \frac{a F(a/b)^2}{b^2} \right] da$$  \hspace{1cm} (B.13)

From equation B.13 the rotation angle from the presence of the crack, $\theta_{crack}$, can be determined:

$$\theta_{crack} = \frac{72 M}{b^2 E} \pi \int_0^{a'} \left[ \frac{a F(a/b)^2}{b^2} \right] da$$  \hspace{1cm} (B.14)

Inserting $M = (\sigma_{xx})_{max} b^2/6$ from equation 6 into equation B.14 gives:

$$\theta_{crack} = \frac{12 (\sigma_{xx})_{max}}{E} \pi \int_0^{a'} \left[ \frac{a F(a/b)^2}{b^2} \right] da = \frac{4 (\sigma_{xx})_{max}}{E} \left( 3 \pi \int_0^{a'} \left[ \frac{a F(a/b)^2}{b^2} \right] da \right) = \frac{4 (\sigma_{xx})_{max} S(a/b)}{E}$$  \hspace{1cm} (B.15)
Thus, from equation B.15 it follows that \( S(a/b) = 3\pi \int_0^{a'} \left[ \frac{a'F(a/b)^2}{b^4} \right] da \). \( S(a/b) \) is also given as a fit by Tada et al. [22]:

\[
S(a/b) = \left( \frac{a/b}{1 - a/b} \right)^2 [5.93 - 19.69(a/b) + 37.14(a/b)^2 - 35.84(a/b)^3 + 13.12(a/b)^4] \quad (B.16)
\]

The rotation angle, \( \theta_t \), can be determined from equation B.1 by adding \( \theta_{\text{crack}} \) (equation B.15) and \( \theta_{\text{no,crack}} \) (equation B.4). Subsequently, \( \theta_t \) in equation B.1 can be used in combination with the general compliance relation, \( C_t = \theta_t/M \), to derive the compliance, \( C_t \), for the pure bending specimen in Figure B.18 (A) as:

\[
C_t = \frac{24}{b^2E} \left( \frac{b}{b} + S(a/b) \right) \quad (B.17)
\]

The energy release rate, \( G_I \), for the pure bending specimen can be expressed in terms of an applied rotation by inserting \( C_t \) in equation B.17 in the general relation for compliance, \( M = \theta_t/C_t \), and finally inserting \( M \) into equation 8 for \( G_I \):

\[
G_I = \frac{1}{E} (\sigma_{xx})_{\text{max}}^2 \pi a[F(a/b)]^2 = \frac{1}{E} \frac{36M^2}{b^4} \pi aF(a/b)^2 = \frac{1}{16} \frac{\theta_t^2 E \pi aF(a/b)^2}{(\frac{b}{b} + S(a/b))^2} \quad (B.18)
\]

The expression for \( G_I \) (equation B.18) can also be written on a non-dimensional form as:

\[
\frac{G_I E b^3}{M^2} = 36\pi \frac{a}{b} [F(a/b)]^2 \quad \text{for load control} \quad (B.19)
\]

\[
\frac{G_I}{Eb\theta_t^2} = \frac{1}{16} \frac{a}{b} \pi [F(a/b)]^2 \left( \frac{b}{b} + S(a/b) \right)^2 \quad \text{for displacement control} \quad (B.20)
\]

Equation B.19 and equation B.20, plotted in the graphs in Figure B.18, represent the non-dimensional energy release rate for a pure bending specimen with applied moment and -rotation, respectively. Figure B.18 (B) shows that \( G_I \) increases monotonic with \( a/b \). Figure B.18 (C) shows that \( G_I \) increases until a peak values is attained, hereafter \( G_I \) decreases smoothly towards zero. Note, with increasing \( h/b \), the shape of the curves approaches the shape of the curve for moment control although the specimen is loaded in rotation control. On the other hand, stability is enhanced when \( h/b \) is small. Note, the crack approaches a free surface at \( a/b = 1.0 \). For rotation control; \( G_I \to 0 \) when \( a/b \to 1.0 \), and for moment control; \( G_I \to \infty \) when \( a/b \to 1.0 \).

**Appendix C. Analytical model of a homogenous four-point SENB specimen**

The homogeneous four-point SENB specimen in Figure C.19 with applied point loads can be described by three non-dimensional geometrical parameters: \( a/b \), \( h/b \) and \( B/b \). Equation 8 presents \( G_I \) as a function of applied load i.e. load control (dead load), but when designing experiments to be conducted under displacement control (fixed grip) it is more convenient to express \( G_I \) as a function of displacement, \( \delta \), of the force(s).
Figure B.18: (A) Analytical model of pure bending specimen. (B) Energy release rate result (equation B.19) in moment control (dashed line). (C) Energy release rate result (equation B.20) in rotation control (solid lines) for different $h/b$.

Figure C.19: Four-point bending setup including moment and shear distributions.
For a linear relationship between load and deflection, the work done by the external forces is:

\[ V_t = \frac{1}{2} P \delta \]  

(C.1)

where \( P \) is the load and \( \delta \) is the load point deflection. In an elastic beam structure, without energy dissipation i.e. a conservative system, the work done by the external force is equivalent to the total elastic strain energy [55]:

\[ U_t = V_t \]  

(C.2)

which is also known as Clapeyron’s theorem in linear elasticity theory [59, 60]. The total strain energy, \( U_t \), stored in the body of the four-point SENB specimen in Figure C.19 is the sum of the strain energy for the specimen without crack plus the strain energy for the introduction of the crack while holding the forces constant, see Tada et al. [22], Appendix B, Beuth [47] or Rice et al. [44]:

\[ U_t = U_{no,crack} + \int_0^a \frac{\partial U_t}{\partial a} da \]  

(C.3)

where \( G_I(a) = \frac{\partial U_t}{\partial a} [22, 57] \). The latter term in equation C.3 is the change in elastic energy per unit cracked area due to the introduction of the crack and related to the ideas by Beuth [47]. The strain energies from part 1 and part 2 of the beam specimen, marked with encircled numbers in Figure C.19 are added to the strain energy of the crack to form the total strain energy as [44]:

\[ U_t = U_{no,crack} + U_{crack} = (U_1 + U_2) + U_{crack} \]  

(C.A)

Part 1 of the beam spans over the length of, \(- (h + B) \leq x \leq - h \) and \( h \leq x \leq h + B \), according to Figure C.19. The strain energy of part 1 of the beam, designated \( U_1 \), consists of a strain energy contribution from the force, \( P \), causing both bending deformation and shear deformation of the beam i.e. normal stresses and shear stresses. These strain energies are denoted \( U_P \) and \( U_T \), respectively. The strain energy contribution based on normal stress in part 1 of the beam (from the force, \( P \)) is given by equation A.6 as:

\[ U_P = \frac{2(P/2)^2 B^3}{Eb^3} \]  

(C.5)

The strain energy based on the shear stresses in part 1 of the beam (from shear force, \( T \)) is determined in equation A.13 as:

\[ U_T = \frac{3 (P/2)^2 B}{5 \mu b} \]  

(C.6)

where \( \mu \) is the shear modulus that for an isotropic material is \( \mu = E/(2(1+\nu)) \). The strain energy for part 1 of the beam becomes:

\[ U_1 = 2(U_P + U_T) = 2 \left( \frac{2(P/2)^2 B^3}{Eb^3} + \frac{3 (P/2)^2 B}{5 \mu b} \right) = \frac{P^2 B}{Eb} \left( \frac{B}{b} \right)^2 + \frac{3 \bar{E}}{5 \mu} \]  

(C.7)
where the factor "2" outside the parenthesis in equation C.7 is included since two regions of the specimen are numbered 1, see Figure C.19.

Part 2 of the beam in Figure C.19 spans over $-h \leq x \leq h$. The strain energy for part 2 of the beam, with length $2h$ and a pure bending moment applied, is derived using equation A.7 for $U_{M_0}$. Here, the moment, $M = BP/2$, and the length, $L_0 = 2h$, are inserted in equation A.7 for $U_{M_0}$ to determine the strain energy for part 2 of the beam:

$$U_2 = 3\frac{P^2 B^2 h}{E b^3}$$

(C.8)

The strain energy contribution from the crack is determined by inserting $(\sigma_{xx})_{\text{max}} = 6M/b^2$ and $M = BP/2$ into $\theta_{\text{crack}} = (4(\sigma_{xx})_{\text{max}}S(a/b))/E$, which is the rotation due to the presence of the crack from Tada et al. [22], to give:

$$U_{\text{crack}} = \frac{1}{2}M\theta_{\text{crack}} = \frac{1}{2}M\frac{A(\sigma_{xx})_{\text{max}}}{E}S(a/b) = 3\frac{P^2 B^2}{E b^2}S(a/b)$$

(C.9)

where $U_{\text{crack}} = M\theta_{\text{crack}}/2$ comes from equation B.13 and $S(a/b)$ is given by an empirical fit in Tada et al. [22] and presented in equation B.16 for convenience. Thus, for the homogenous four-point SENB specimen, the strain energy of part 1, part 2, and the crack (equation C.7, C.8, C.9) can now be added to give a single expression for the total strain energy of the beam specimen volume as:

$$U_t = U_1 + U_2 + U_{\text{crack}} = \frac{P^2 B}{Eb} \left[ \left( \frac{B}{b} \right)^2 + \frac{3}{5} \frac{E}{2\mu} \right] + 3\frac{P^2 B^2 h}{E b^3} + 3\frac{P^2 B^2}{E b^2}S(a/b)$$

(C.10)

This expression (equation C.10) is used in section 3 to determine $G_I$ in displacement control (fixed grip).

References


