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Transverse vibrations induced by longitudinal excitation in beams with geometrical and loading imperfections

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Abstract

With structural health monitoring techniques based on measuring transverse vibrations of beam-like components it can be difficult to provide transverse excitation, while longitudinal excitation can be easier. We experimentally investigate how transverse vibrations in a beam can be excited by a longitudinal hammer impact. We carry out a comparative study on the measured transverse natural frequencies and frequency response coherence for different input and output locations and directions. It is shown that transverse vibrations are excited regardless of the impact locations and directions. Theoretical explanation to this counter-intuitive phenomenon is provided in terms of various imperfections associated with beams and impacts.

1 Introduction

With some vibration-based health monitoring techniques it can be beneficial to excite the structure under test in a direction different from that of the vibration aimed at. For example, when using transverse vibrations for estimating bolted joint tension in operation [1], longitudinal excitation is much easier than transverse.

In an ideal situation, purely axial loading, where the impact is realized at the center of the beam cross section, will only produce longitudinal vibration modes. However such a purely axial loading is impossible to realize in most experiments, thus resulting in both longitudinal and transverse modes being excited at the same time. Figure 1 shows the frequency

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response of a carefully machined beam excited by a *supposedly* purely axial hammer impact. The measured frequency response function (FRF) is displayed in the figure. Investigating the theoretical transverse frequencies of this beam reveals that the first three measured peaks in Figure 1 correspond to the transverse natural frequencies, indicating that there is a mechanism transferring the energy of the longitudinal impact to transverse vibrations.

The straight configuration of a rod subjected to a static longitudinal load can lose stability by deflecting laterally, when the load exceeds the Euler critical value [2]. In the case of dynamic loading, even if the load is smaller than the Euler one, the straight configuration may lose stability by exhibiting transverse vibrations. Morozov et al. [3] analytically investigated the occurrence of transverse vibrations when a rod is subjected to a sudden axial load, which is lower than Euler load. They showed, using a linear approximation, that this instability was generated by parametric resonance. However the resulting transverse vibrations amplitude increased unboundedly. To obtain finite amplitudes, Belyaev et al. [4] considered a quasilinear system, where the influence of transverse vibrations on the longitudinal ones was taken into account. A two-scale expansion was used to construct an approximate analytic solution. A short survey of the above-mentioned analytical investigations was given in [5].

The coupling between longitudinal and transverse vibrations in rods can be induced by geometric non-uniformity. Waltking [6] showed that, in the case of slightly curved beams, the longitudinal-transverse coupling influenced the natural frequencies of transverse vibrations. Longitudinal waves were experimentally shown to produce transverse waves in an L-joint model, which were propagated in the second bar [7]. Goldsmith et al. [8] experimentally investigated the generation of elastic waves by longitudinal impact strikers consisting of steel bars. When the applied longitudinal impact is eccentric, transverse waves were observed. The relation between transverse and longitudinal strains was discussed. Kuo [9] experimentally subjected a rod to the eccentric longitudinal impact of another rod, resulting in transverse as well as longitudinal waves in the struck bar. A review of experimental works until 1986 investigating transverse waves generated by longitudinal impact was given in [10].

Papadopoulos and Dimarogonas [11] showed that coupling of longitudinal and bending vibrations of a shaft can be due to an open transverse crack. Ghayesh and Amabili [12] investigated the nonlinear coupled longitudinal-transverse behavior of a geometrically imperfect microbeam. They showed that, depending on the initial imperfection, the resonant response of the system exhibits hardening or softening behavior. The alternation of hardening and softening behaviors was also observed in the coupled axial-transverse oscillations of a Timoshenko beam for different axial constraints and slenderness [13].

Coupling between longitudinal and transverse vibrations was observed in various engineering industries. In the ship research field, Huang et al. [14] studied experimentally and numerically the effect of hull deformation on the coupled transverse and longitudinal vibration of a marine propulsion shaft for different rotation speeds. Scurtu et al. [15] investigated the coupled transverse and longitudinal vibrations of an automotive serpentine belt with one end clamped and the other under periodical longitudinal excitation. The nonlinear model was solved using an analog equation method, and a comparison of the theoretical and

experimental results also given.

In this work we experimentally investigate the excitation of transverse modes by longitudinal impacts, and compare the natural frequencies to the ones excited by transverse impacts. We show that transverse vibrations are excited regardless of impact location and direction, and can be measured even longitudinally. Theoretical support of this observation is provided in terms of geometrical and loading imperfections, and simple modeling. In section 2 the experimental, procedure, and observations are described. A theoretical analysis that explains the measurements is given in Section 3, and conclusions in Section 4.

2 Experiments

2.1 Experimental setup

In the experiment we use an Ø40x434 mm massive steel cylindrical beam (Young's modulus 206 GPa, density 7850 kg/m³). The beam is supplied with three steel pins fixed at one end of the beam, and next to the same end another pin is fixed on the side of the beam as shown in Figure 2. Two of the pins are tilted, forming angle 10° and 20° with the purely axial pin fixed in the middle of the beam's cross section.

The four pins serve as energy channels for the impacts to direct the energy in a desired direction and location; they are firmly fixed to form with the beam a single elastic body. The free-free boundary conditions of the beam are approximated using two rubber bands attached around the nodal points of the first bending mode. To provide controllable impacts, we used a hung pendulum with a steel tip. The force was measured by a force transducer B&K Type 8206, mounted between the tip and the pendulum arm. To record the response two B&K accelerometers Type 4397 were mounted by a thin layer of beeswax, see Figure 2. The accelerometers and the force transducer are connected to B&K DAQ LAN-Xi module Type 3150, and the signals processed by B&K PULSE Labshop software. Figure 3 shows the experimental setup.

2.2 Experimental procedure

Each of the pins was impacted by the impact pendulum, inclining it by a certain angle and releasing after that. This guaranteed repeatability of the impacts. At the moment of impact, the cross sections of the hammer tip and the pin were parallel to each other. Therefore, at the moment of impact the position of the pendulum arm was a little different in each test, depending on the pin impacted. The data recorded from the force transducer and the two accelerometers was collected and analyzed. The values of the natural frequencies were read from the FRF plot using the B&K PULSE "resonance cursor", with readings up to the frequency resolution 1 Hz of the frequency span [0, 6.4] kHz.

Figure 1: Measured FRF for a beam ($\varnothing 6 \times 260$ mm, Young's modulus 207 GPa, density 7950 kg/m³) subjected to a longitudinal hammer impact and with an accelerometer placed transversely.

Figure 2: Experimental setup.

Figure 3: Experimental beam (a) and impact pendulum (b). Accelerometer 1 is placed to measure transversely and Accelerometer 2 longitudinally along the axis the beam.

2.3 Expected and measured natural frequencies

The natural frequencies of the beam in the frequency range of interest was computed using COMSOL FEM, with 3D solid elements, an “Extra Fine” mesh, and free-free boundary conditions.

Table 1 shows a comparison of the lowest theoretical and experimental natural frequencies for the experimental beam. The subscripts B , T and L in the natural frequencies denote bending/transverse, torsional and longitudinal vibration modes. The (unexplained) peak at 6128 Hz defines the upper frequency range of the study. Next we analyze and discuss the experimental measurements for different impact directions (input) and accelerometer locations (output).

Table 1: Theoretical and experimental natural frequencies identified for the experimental beam. Subscript B/T/L denoting Bending (transverse)/Torsional/Longitudinal. Transverse (longitudinal) natural frequencies were obtained experimentally using transverse (longitudinal) excitation and transverse (longitudinal) response measurement, while the torsional frequency was obtained using transverse impact and longitudinal response measurement.

$f[\text{Hz}]$	$f_{B,1}$	$f_{B,2}$	$f_{T,1}$	$f_{B,3}$	$f_{L,1}$	$f_{B,4}$	$f_{L,2}$
Theoretical	948	2518	3660	4705	5899	7320	7353
Experimental	947	2517	3710	4688	5928	6128	
Deviation (%)	0.1	0.04	-1.36	0.36	-0.49	N/A	

2.4 Experimental observations

2.4.1 Impact excitation and acceleration measurement

In Figure 4(a,b), an accelerometer (Accelerometer 1) is transversely placed next and parallel to the transverse pin, and the beam is impacted at the transverse pin (solid line/red) and at the purely axial middle pin (dashed/blue). In the remaining of this paper we will refer to the different input directions (impact) and output locations (accelerometer) as input/output, for example a transverse impact and longitudinally placed accelerometer will be written as Transverse/Longitudinal case. Figure 4a shows clearly that the longitudinal impact excites transverse vibration modes. The transverse and longitudinal natural frequencies are the same for the two types of impacts, transverse and longitudinal, with a deviation of 0.05% for the longitudinal natural frequency. However, the peak corresponding to the longitudinal frequency in the case Transverse/Transverse (solid line/red) is very small compared to the one in the case Longitudinal/Transverse (dashed/blue). This apparent difference can be explained by the fact that the longitudinal frequency is excited by the transverse impact and is only sensed due to transverse sensitivity of the accelerometer used. The type of accelerometer (B&K 4397) used in the experiment has transverse sensitivity of 3%, therefore it can sense motions along the axis at 90 degrees to the main sensitive axis of the accelerometer [18].

In Figure 4(c,d) the accelerometer (Accelerometer 2) is placed longitudinally. Figure 4(c,d) shows that the longitudinally placed accelerometer senses the transverse natural frequencies for both types of impacts, transverse and longitudinal. The transverse and longitudinal natural frequencies produced by the two impacts are equal. However, for the case Transverse/Longitudinal (solid line/red) there is peak at the frequency 3710 Hz that corresponds to the torsional natural frequency of the beam (cf. Table 1). This appearance of the torsional frequency could be due to imperfection in excitation or measurement direction (including non-axial accelerometer sensitivity), and/or nonlinear coupling, and this could be a study in its own right, but outside our main scope of detecting transverse natural frequencies from longitudinal impacts. The coherence for all cases in (b) and (d) is close to unity across the frequency range, except for the inherent drops at antiresonances. As mentioned above, the (unexplained) peaks at 6128 Hz in Figure 4 defines the upper frequency range of the study, and therefore not important for our present work.

2.4.2 Oblique impact

In Figure 5 we consider the case where the impact direction is not fully longitudinal but oblique. The hammer impacts each of the two tilted pins or the purely axial middle pin, and we compare the responses collected by one accelerometer placed transversely (Figure 5a) and another accelerometer placed longitudinally (Figure 5b). Figure 5a shows that the two oblique impacts, the 10° and 20° - pins, excite the transverse mode vibrations. The transverse and longitudinal natural frequencies are the same for the three impacts. However, the two oblique impacts at the 10° and 20° pins produce peaks around 3710 Hz and 3700 Hz,

Figure 4: Measured FRFs (a,c) and coherence (b,d) with transverse impact (solid line/red) and longitudinal impact (dashed/blue) from a,b) transversely and c,d) longitudinally placed accelerometer.

respectively, corresponding to the torsional natural frequency.

Figure 5b shows that the longitudinally placed accelerometer senses not only the longitudinal natural frequency, but also the transverse natural frequencies. We also note that there is a peak corresponding to the torsional natural frequency for the oblique impact at the 20^o-

pin. The reason the peak of the torsional natural frequency is not visible in all FRFs could be attributed to uncontrolled details of the impact and measurement (e.g. weak tangential / off center components of the hammer impact, combined with the transverse sensitivity of the accelerometer sensing the weak rotational component from torsion).

2.4.3 Summary of experimental observations

Transverse natural frequencies could be reliably identified with good coherence by a longitudinal impact, regardless of the location of the accelerometer, transversely or longitudinally. The transverse, longitudinal and oblique impacts produced the same transverse natural frequencies, regardless of the location of the accelerometer. In the next section, we provide a theoretical explanation of how longitudinal impact can excite transverse vibration modes.

Figure 5: Measured FRFs: Middle pin impact (solid line/red), 10°- pin impact (dashed/blue), and 20°- pin impact (dash-dot/black) with a) transversely and b) longitudinally placed accelerometer.

3 Theory

In section 2 it was demonstrated experimentally how longitudinal impact excites both

longitudinal and transverse vibration modes, and whilst the opposite was anticipated the transverse natural frequencies were found to be reliably identified with good coherence. Now we aim to explain theoretically the possible ways transverse vibration modes can be excited by longitudinal impact. In an ideal simplified theory, applying a purely axial and central impact $N(l, t) = N_l(t)$, perpendicular to the beam's cross section, results only in longitudinal vibration modes. However in the following analysis we show, even without considering nonlinearity, that there are *at least* two potential ways of exciting transverse modes by a longitudinal loading. These ways are due to imperfections, which we group into two categories, one contains loading imperfections, the other beam imperfections.

3.1 Imperfections

The two types of imperfections are illustrated in Figure 6. As mentioned, in the case of a purely axial and central loading, and with a perfect straight beam, only longitudinal vibrations $v(x, t)$ are expected to occur, Figure 6a. However when the axial loading is imperfect, the loading is forming a small angle $\epsilon \theta$ ($\epsilon \ll 1$), with the beam's center-line, or the loading is purely axial but the impact location is off-center by a small distance ϵd , thereby creating flexural moment and resulting in transverse vibrations $u(x, t)$ and $w(x, t)$ being excited. In Figure 6b, we assume that the axial loading is both oblique ($\epsilon \theta$) and off the center-line by (ϵd). The second type of imperfections, which we term *beam* imperfections, that could result in transverse deflections are related to the shape of the beam, that is when the assumed purely axial and central loading is applied to a beam already deflected with a small amount $\epsilon u_0(x)$ from the straight center-line, see Figure 6c. This could be the result of an external transverse loading having non-zero time average, say $\epsilon p(x)$, or an intrinsic geometric defect due to e.g. improper machining. In the following we obtain the equation of motion of the beam and show how these two types of imperfections can result in exciting transverse vibrations.

3.2 Equation of motion

Applying Bernoulli-Euler theory with the assumption of small slopes, we obtain the equation of motion of the beam in terms of the transverse deflection $u(x, t)$ (the one for $w(x, t)$ is similar) [16, 17]:

$$\rho A \ddot{u} + EI u'''' - N(l, t) u'' + M' = Q, \quad (1)$$

where dots and primes denote differentiation w.r.t. to t and x , respectively. We assume the line of tension $N(l, t)$ is forming an angle $\epsilon \theta$ with the beam's center-line and off the center-line by (ϵd) as in Figure 6b, such that:

Figure 6: Beam and loading imperfections - a) Perfect beam and loading ($\epsilon = 0$). b) Loading imperfection ($\epsilon \theta, \epsilon d$). c) Beam imperfection ($\epsilon u_0(x), \epsilon p(x)$).

$$N(l, t) = -N_l(t) \cos(\epsilon \theta) = -N_l(t) + O(\epsilon^2), \quad \epsilon \ll 1 \quad (2)$$

where primes and dots denote spatial and time derivatives, respectively. Damping is ignored here, since it is not relevant for the purpose of analysis and the conclusion. The two terms $Q(x, t)$ and $M(x, t)$ are the distributed transverse load and the distributed bending moment, respectively:

$$Q(x, t) = \epsilon p(x) - N_l(t) \sin(\epsilon \theta) \delta(x - l) = \epsilon (p(x) - \theta N_l(t) \delta(x - l)) + O(\epsilon^3), \quad (3)$$

$$M(x, t) = \epsilon d N_l(t) \cos(\epsilon \theta) \delta(x - l) = \epsilon d N_l(t) \delta(x - l) + O(\epsilon^3), \quad (4)$$

representing the loading imperfections in terms of $\epsilon \theta$ and ϵd , and with δ denoting Dirac's delta function. Next, considering the case of a beam imperfection where the beam has an initial curvature $\epsilon u_0(x)$, the transverse deflection in Eq. (1) becomes $u(x, t) - \epsilon u_0(x)$. If furthermore we assume $\epsilon u_0(x)$ to be the static deflection corresponding to the static (non-zero average) transverse load $\epsilon p(x)$, and substitute Eq. (3) and the derivative of Eq. (4) in Eq. (1), we obtain up to order $O(\epsilon)$:

$$\rho A \ddot{u} + E I (u'''' - \epsilon u_0''') + N_l(t) (u'' - \epsilon u_0'') + \epsilon d N_l(t) \delta'(x - l) = \epsilon (p(x) - \theta N_l(t) \delta(x - l)). \quad (5)$$

Taking into account the static deflection $u_0(x)$ is a solution of the static equation:

$$E I u_0''''(x) = -p(x), \quad (6)$$

Eq. (5) becomes

$$\rho A \ddot{u} + E I u'''' + N_l(t) u'' = \epsilon N_l(t) (u_0'' - d \delta'(x - l) - \theta \delta(x - l)). \quad (7)$$

The solution $u(x, t)$ to Eq. (7) can be expressed as a modal expansion:

$$u(x, t) = \sum_{j=1}^{\infty} q_j(t) \phi_j(x). \quad (8)$$

Since the applied tension $N_l(t)$ is only non-zero in a very short time interval, the mode shapes and natural frequencies are actually those for a free-free beam. Therefore the $\phi_j(x)$ in Eq. (8) are the free-free mode shapes constituting an infinite set of orthogonal functions, and $q_j(t)$ are the modal coefficients. The mode shapes $\phi_j(x)$ for free-free boundary conditions are [16]:

$$\phi_j(x) = \sin \alpha_j x + \sinh \alpha_j x - \frac{\sin \alpha_j l - \sinh \alpha_j l}{\cosh \alpha_j l - \cos \alpha_j l} (\cos \alpha_j x + \cosh \alpha_j x), \quad (9)$$

where $\alpha_j = \lambda_j/l$, and λ_j is a solution to

$$\cos \lambda \cosh \lambda = 1, \quad (10)$$

with roots $\lambda_1 = 4.73004334, \lambda_2 = 7.85320678, \lambda_3 = 10.99560784, \dots$ etc. Inserting Eq. (8) into Eq. (7), multiplying by $\phi_i(x)$, and integrating over the length of the beam gives:

$$\begin{aligned} & \rho A \sum_{j=1}^{\infty} \ddot{q}_j \int_0^l \phi_j \phi_i dx + \sum_{j=1}^{\infty} q_j \left(E I \int_0^l \phi_j'''' \phi_i dx + N_l(t) \int_0^l \phi_j'' \phi_i dx \right) \\ & = \epsilon N_l(t) \left(\int_0^l u_0'' \phi_i dx - d \int_0^l \delta'(x-l) \phi_i dx - \theta \int_0^l \delta(x-l) \phi_i dx \right), \end{aligned} \quad (11)$$

where $j = 1, 2, \dots$, and where the orthonormalized modes shapes give:

$$\int_0^l \phi_j \phi_i dx = l \delta_{ij} \quad (12)$$

$$\int_0^l \phi_j'''' \phi_i dx = \int_0^l \phi_j'' \phi_i'' dx = l \alpha_j^4 \delta_{ij} \quad (13)$$

$$\int_0^l \phi_j'' \phi_i dx = - \int_0^l \phi_j' \phi_i' dx = -l \alpha_j^2 \delta_{ij} \quad (14)$$

$$\int_0^l u_0'' \phi_i dx \equiv \gamma_i \quad (15)$$

$$\int_0^l \delta(x-l) \phi_i dx = \phi_i(l) \quad (16)$$

$$\int_0^l \delta'(x-l) \phi_i dx = - \int_0^l \delta(x-l) \phi_i' dx = -\phi_i'(l), \quad (17)$$

where δ_{ij} denotes Kronecker's delta. Inserting Eqs. (13)-(17) into Eq. (12) yields:

$$\ddot{q}_j + \left(\omega_j^2 - \alpha_j^2 n_l(t) \right) q_j = \epsilon n_l(t) \left(\gamma_j + d \phi_j'(l) - \theta \phi_j(l) \right), \quad j = 1, 2, \dots \quad (18)$$

where

$$\omega_j^2 = \frac{\alpha_j^4 E I}{\rho A}, \quad \text{and} \quad n_l(t) = \frac{N_l(t)}{\rho A}, \quad (19)$$

with ω_j being the j th natural frequency of the free-free beam and $n_l(t)$ the mass-normalized longitudinal force per unit length. It is assumed that the linear stiffness (coefficient of q_j) is positive, i.e. $E I > N_l(t)$ in Eq. (18).

3.3 Longitudinally excited transverse motions

The discretized/modal equations (18) can be used to estimate FRFs which could be compared to the experimental FRFs of Section 2. However, our aim here is to demonstrate and theoretically explain in principle how longitudinal impact can induce transverse vibrations. For that Eq. (18) is sufficient, giving direct insight into how various imperfections provide a pathway for the longitudinal impact ($n_l(t)$) to create transverse vibrations ($q_j(t)$).

Equations (18) show the contribution of the loading and beam imperfections in exciting transverse vibrations. These imperfections are characterized by the parameters γ_j , d and θ which represent the initial beam curvature, the off-center impact, and the oblique impact, respectively. Due to these imperfections the longitudinal impact manifest as transverse external forcing that excites transverse deflection. If at least one of these three terms is non-zero, the beam will vibrate transversely. For example, if the impact is central ($d = 0$) and purely axial ($\theta = 0$), the beam will still exhibit transverse vibrations if it is initially curved ($\gamma_j \neq 0$).

4 Conclusions

We showed experimentally how longitudinal impacts can excite transverse vibrations. The transverse natural frequencies could be reliably identified with good coherence by a longitudinal impact, regardless of the location of the accelerometer, transversely or longitudinally. The transverse and longitudinal impacts produce the same transverse natural frequencies with the accelerometer placed either transversely or also longitudinally due to the accelerometer's transverse sensitivity.

We theoretically showed different scenarios, depending on the type of imperfection, that can excite transverse vibrations by longitudinal impacts. In the presence of imperfections, the impacts manifest as transverse external forcing in the modal equations of the beam. The transverse excitation of the beam can be attributed to the presence of two types of imperfections: loading imperfection and beam imperfection. The loading imperfection is due to an oblique impact or an off-center impact, while the beam imperfection is due to non-zero beam curvature, that could be caused either by an external transverse loading having non-zero time average, or by imperfect machining.

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References

- [1] S.M. Sah, J.J. Thomsen, M. Brøns, A. Fidlin, D. Tcherniak, Estimating bolt tightness using transverse natural frequencies. *J. Sound Vib.* 431 (2018) 137-149.
- [2] S.P. Timoshenko, *History of Strength of Materials*. McGraw-Hill Book Co., Inc., New York, 1953.
- [3] N.F. Morozov, P. E. Tovstik, Dynamic Loss of Stability of a Rod under Longitudinal Load Lower Than the Eulerian Load, *Dokl. Phys.* 58 (11) (2013) 510-513.
- [4] A.K. Belyaev, N.F. Morozov, P.E. Tovstik, T.P. Tovstik, Beating in the Problem of Longitudinal Impact on a Thin Rod, *Mech. Solids* 50(4) (2015) 451-462.
- [5] A.K. Belyaev, P.E. Tovstik, T.P. Tovstik, Thin rod under longitudinal dynamic compression, *Mech. Solids* 52(4) (2017) 364-377.
- [6] F.W. Waltking, Schwingungszahlen und schwingungsformen von kreisbogenträgern [Natural frequencies and mode shapes of circular arc structures], *Ingenieur-Archiv* 5 (1934) 429-449.
- [7] K.J. Atkins, S.C. Hunter, The propagation of longitudinal elastic waves around right angled corners in rods of square cross-section, *Q. J. Mech. Appl. Math.* 28 (1975) 245-260.
- [8] W. Goldsmith, P. Y. Lee, J.L. Sackman, Pulse propagation in straight circular elastic tubes, *J. Appl. Mech.* 39 (1972) 1011-1018.
- [9] S.S. Kuo, Beam subjected to eccentric longitudinal impact, *Exp. Mech.* 1 (1961) 102-108.
- [10] M.M. Al-Mousawi, On experimental studies of longitudinal and flexural wave propagations: An annotated bibliography, *Appl. Mech. Rev.* 9(6) (1986) 853-865.
- [11] C.A. Papadopoulos, D.Dimarogonas, Coupled longitudinal and bending vibrations of a rotating shaft with an open crack, *J. Sound Vib.* 117(1) (1987) 81-93.
- [12] M.H. Ghayesh, M. Amabili, Coupled longitudinal-transverse behaviour of a geometrically imperfect microbeam, *Compos. Part B Eng.* 60 (2014) 371-377.
- [13] S. Lenci, G. Rega, Axial-transversal coupling in the free nonlinear vibrations of Timoshenko beams with arbitrary slenderness and axial boundary conditions, *Proc. R. Soc. A* 472 (2016) 20160057.
- [14] Q. Huang, X. Yan, Y. Wang, C. Zhang, Y. Jin, Numerical and experimental analysis of coupled transverse and longitudinal vibration of a marine propulsion shaft, *J. Mech. Sci. Technol.* 30 (12) (2016) 5405-5412.

- [15] P.R. Scurtu, M. Clark, J.W. Zu, Coupled longitudinal and transverse vibration of automotive belts under longitudinal excitations using analog equation method, J. Vib. Control 18(9) (2011) 1336-1352.
- [16] D.J. Inman, Engineering Vibration, Prentice Hall, Upper Saddle River, NJ 1996.
- [17] J.J. Thomsen, Vibrations and stability: Advanced Theory, Analysis, and Tools, Springer-Verlag Berlin Heidelberg, 2003.
- [18] PRODUCT DATA Miniature DeltaTron Accelerometer Type 4397A, Brüel & Kjær Sound & Vibration Measurement A/S (<https://www.bksv.com/-/media/literature/Product-Data/bp1849.ashx>)