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Hamilton cycles in sparse locally connected graphs

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ABSTRACT

A graph G is *locally connected* if for every $v \in V(G)$ the open neighbourhood $N(v)$ of v is nonempty and induces a connected graph in G . We characterize locally connected graphs of order n with less than $2n$ edges and show that for any natural number k the Hamilton Cycle Problem for locally connected graphs of order n with m edges is polynomially solvable if $m \leq 2n + k \log_2 n$, but NP-complete if $m = 2n + \lfloor n^{1/k} \rfloor$.

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1. Introduction and background

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The cardinalities of $V(G)$ and $E(G)$ are called the *order* and *size* of G and denoted by $n(G)$ and $m(G)$, respectively. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. When the context is clear we omit the G and simply write n, m, Δ etc. For undefined concepts and notation we refer the reader to [1].

A graph G is *locally connected* (LC) if for every $v \in V(G)$ the open neighbourhood $N(v)$ of v is nonempty and induces a connected graph. The concept of a locally connected graph was introduced by Chartrand and Pippert [2]. One of the first results proved on the topic is the following.

Theorem 1.1 ([2]). *If G is a connected LC graph of order at least 3 with $\Delta \leq 4$, then G is either hamiltonian or isomorphic to the complete tripartite graph $K_{1,1,3}$.*

Global cycle properties of locally connected graphs with $\Delta = 5$ were investigated by Kikust [8], Clark [3], Hendry [5] and Gordon, Orlovich, Pots and Strusevich [4]. Hendry [6] defined a graph G to be *fully cycle extendable* if every vertex of G lies on a 3-cycle and for every nonhamiltonian cycle C in G there is a cycle C^* in G that contains all the vertices of C plus a single new vertex. The combined results of the mentioned authors yield the following.

Theorem 1.2 ([3–5,8]). *If G is a connected LC graph with $\Delta = 5$ and $\delta \geq 3$, then G is fully cycle extendable. However, there exist infinitely many nonhamiltonian connected LC graphs with $\Delta = 5$ and $\delta = 2$.*

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Fig. 1. The translation of vertices and edges in G to nodes and borders in G^* .

It is well-known that the Hamilton Cycle Problem (the problem of deciding whether a given graph is hamiltonian) is NP-complete.

Henceforth we abbreviate Hamilton Cycle Problem to HCP. [Theorem 1.1](#) shows that the HCP for LC graphs with $\Delta \leq 4$ is fully solved. Gordon et al. [4] proved that it is NP-complete for LC graphs with $\Delta = 7$. Irzhavski [7] improved on this result by proving the following:

Theorem 1.3 ([7]). *The HCP for LC graphs with minimum degree 2 and maximum degree 5 is NP-complete. Moreover, for any two integers d and D such that $D \geq 6$ and $2 \leq d \leq D$, the HCP for locally connected graphs with minimum degree d and maximum degree D is NP-complete.*

We would like to thank the referee who made us aware of Irzhavski's theorem.

If $\mathcal{R}$ is a finite set of natural numbers, we say a graph G is $\mathcal{R}$ -regular if the degrees of the vertices in G are all elements of $\mathcal{R}$. In Section 2 we state a corollary of Irzhavski's theorem in terms of $\mathcal{R}$ -regular graphs. We also use his technique to prove one of our main results in Section 4. Since Irzhavski's paper [7] has only been published in Russian, we include the main steps of the proof for easy access.

[Theorem 1.1](#) prompted us to investigate the HCP for LC graphs with average degree at most 4, i.e., graphs with $m \leq 2n$. In Section 3 we characterize LC graphs with $m < 2n$. In Section 4 we show that for any natural number k the HCP is NP-complete for LC graphs with $m = 2n + \lfloor n^{1/k} \rfloor$, but polynomially solvable for LC graphs with $m \leq 2n + k \log n$.

2. Hamilton cycles in $\mathcal{R}$ -regular LC graphs

Picouleau [9] proved the following result regarding the complexity of the HCP for r -regular graphs.

Theorem 2.1 ([9]). *For any fixed integer $r \geq 3$ the HCP for r -regular graphs is NP-complete.*

The problem of the complexity of the HCP for $\mathcal{R}$ -regular LC graphs is settled by [Theorem 1.2](#) and the following result, which is a corollary of [Theorem 1.3](#), proved by Irzhavski [7].

Theorem 2.2 ([7]). *The HCP for $\mathcal{R}$ -regular LC graphs is NP-complete if $\mathcal{R} = \{2, 5\}$ or $\mathcal{R}$ is any set of natural numbers with $\max \mathcal{R} \geq 6$.*

Proof. It is sufficient to prove the cases $\mathcal{R} = \{2, 5\}$ and $\mathcal{R} = \{r\}$, $r \geq 6$.

Consider any cubic graph G . By [Theorem 2.1](#) it is sufficient to show that G can be transformed in polynomial time to an $\mathcal{R}$ -regular LC graph G^* such that G^* is hamiltonian if and only if G is hamiltonian. We briefly explain Irzhavski's transformation for the cases $\mathcal{R} = \{2, 5\}$ and $\mathcal{R} = \{6\}$.

For the case $\mathcal{R} = \{2, 5\}$, the graph G^* is constructed by replacing each vertex in G with a triangle (henceforth called a *node* of G^*) and each edge in G with a *border* isomorphic to the graph B shown in [Fig. 1](#), so that neighbouring nodes of G^* are joined by borders as shown in [Fig. 2](#). It is easily seen that G^* is LC and $\{2, 5\}$ -regular.

For a vertex v_1 in G with $N(v_1) = \{u_1, u_2, u_3\}$, let V_1 and U_i , $i = 1, 2, 3$ be the nodes in G^* corresponding to v_1 and u_i , $i = 1, 2, 3$, respectively and let B_i be the borders in G^* corresponding to the edges $v_1 u_i$, $i = 1, 2, 3$. Let the vertices a, b, c, d, x be as shown in [Fig. 3](#).

Without loss of generality, suppose G has a Hamilton cycle C that contains the path $u_1 v_1 u_2$. Then C also contains the path $v_2 u_3 v_3$, where $N(u_3) = \{v_1, v_2, v_3\}$. Those two paths correspond, for example, to the two paths in G^* indicated by thick lines in [Fig. 3](#). Hence, for any Hamilton cycle in G , there is a corresponding Hamilton cycle in G^* .

Now suppose G^* has a Hamilton cycle C^* . If C^* contains the pair ad, bc of "parallel edges", then C^* obviously contains the path $adxc b$. It follows that C^* contains at most one pair of parallel edges incident with any given node of G^* .

Now suppose C^* contains no pair of parallel edges incident with V_1 . Since $cx d$ is a subpath of C^* , at most one of the edges $\{ad, bd, bc\}$ of B_3 lies on C^* . The same is true for the corresponding edges of the borders B_1 and B_2 . The vertices of V_1 therefore

Fig. 2. A node in a $\{2, 5\}$ -regular LC graph G^* connected with borders to three other nodes in G^* .

Fig. 3. Illustrating how a Hamilton cycle in G translates to a Hamilton cycle in G^* .

lie on a P_3 in C^* , and in one of the borders, say B_1 , the cycle C^* has no edge of the form ad , bc or db incident with V_1 . But then C^* contains the pair of parallel edges of B_1 incident with U_1 . Thus C^* translates to a Hamilton cycle in G . This proves the result for the case $\mathcal{R} = \{2, 5\}$.

Fig. 4 illustrates how two nodes are joined by a border to effect the transformation in the case $\mathcal{R} = \{6\}$. (The sketch that appears in [7] has been redrawn so as to show clearly how the construction for the $\{2, 5\}$ -case has been modified to obtain a 6-regular LC graph.)

For $r \geq 7$ it is not difficult to obtain an r -regular LC graph with the desired properties by expanding the nodes and borders in the $\{2, 5\}$ -construction. $\square$

3. Characterization of the connected LC graphs with $m < 2n$

We first prove the following.

Proposition 3.1. *If G is a connected LC graph of order n and size m , then $m \geq 2n - 3$.*

Proof. The proof is by induction on n . The result is obviously true if $n \leq 3$. Now let G be a connected LC graph of order $n \geq 4$ and size m . Let x be a vertex of minimum degree in G . We may assume $d(x) = 2$ or 3 . Let $H = G - x$.

If $d(x) = 2$, then H is obviously a connected LC graph of order $n - 1$ and size $m - 2$. Hence, by our induction hypothesis, $m - 2 \geq 2(n - 1) - 3 = 2n - 5$, so then $m \geq 2n - 3$.

Fig. 4. A 6-regular LC construction.

If $d(x) = 3$, there are two cases to consider.

Case 1. $G[N(x)] \cong K_3$: In this case H is a connected LC graph of size $m - 3$ and hence, by our induction hypothesis, $m - 3 \geq 2n - 5$, so $m \geq 2n - 2$.

Case 2. $G[N(x)]$ is a path $v_1v_2v_3$: In this case the graph $H + v_1v_3$ is connected and LC and has size $m - 2$. Hence, by our induction hypothesis, $m - 2 \geq 2n - 5$, so $m \geq 2n - 3$. $\square$

Suppose that for $i = 1, 2$, the graph G_i has a proper induced subgraph H_i that is isomorphic to a given graph H . If we identify the subgraphs H_1 and H_2 while retaining all other edges present in G_1 and G_2 , the resulting graph G of order $n(G) = n(G_1) + n(G_2) - n(H)$ is said to be obtained from G_1 and G_2 by *pasting* G_1 and G_2 together on a copy of H . We observe the following.

Observation 3.2. Let H be any graph with $\delta(H) \geq 1$ and suppose G is a graph consisting of two LC graphs pasted together on a copy of H . Then G is also LC.

If an LC graph G consists of two LC graphs of order at least 3 each, pasted together on an edge (i.e., on a copy of K_2), we say that G is a *reducible* LC graph. If G is an LC graph that consists of more than one component, we also consider G to be reducible, since each of its components is a connected LC graph. We define an *irreducible* LC graph G as a graph that is not reducible.

A set of vertices S in a connected graph G is a *vertex cut* if $G - S$ is disconnected. If $\{x, y\}$ is a vertex cut in a connected LC graph G , then it is easily seen that $xy \in E(G)$. This implies the following.

Observation 3.3. If G is an irreducible LC graph of order n , then $G \in \{K_2, K_3\}$ if $n \leq 3$, and G is 3-connected if $n \geq 4$.

In order to characterize the connected LC graphs with $m < 2n$ it suffices to characterize the irreducible ones, as the next result shows.

Proposition 3.4. If G is a connected reducible LC graph with $m < 2n$, then G is obtained by pasting irreducible LC graphs of size less than twice the order together along edges.

Proof. The proof is by induction on n . By assumption, G is the union of two LC graphs G_1, G_2 having two vertices x, y and the edge xy in common. For $i = 1, 2$, let n_i, m_i be the number of vertices and edges, respectively, in G_i . Proposition 3.1 implies that $m_i \geq 2n_i - 3$ for $i = 1, 2$. As $n = n_1 + n_2 - 2$ and $m = m_1 + m_2 - 1$, it follows that $m_i < 2n_i$ for $i = 1, 2$, and now the result follows by induction. $\square$

A vertex u in a graph G is called a *universal vertex* if u is adjacent to every vertex in $G - u$. A graph is called a *wheel* if it consists of a cycle (called the *rim* of the wheel) plus a universal vertex (called the *centre* of the wheel). Fig. 5 depicts a wheel of order 9.

Let W be a wheel with centre w and rim $w_1w_2 \dots w_t$. The edges $ww_i, i = 1, \dots, t$ are called the *spokes* of W . A wheel of order n may be *extended* to a wheel of order $n + 1$ by *inserting a spoke*, i.e., by subdividing an edge on the rim of W and joining the inserted vertex to the centre of W . The reverse of this action is called *eliminating a spoke*.

If an edge on the rim of a wheel is deleted, the resulting graph is a *fan*. Fig. 6 depicts a fan of order 9. Thus a fan is a path plus a universal vertex. Let F be a fan with central vertex v and rim $v_1 \dots v_p$. The edges vv_1 and vv_p are the *outer spokes* of F , and $vv_2, \dots, vv_{p-1}$ the *inner spokes*. A fan may be *extended* by inserting a spoke between two of its spokes, or by adding a

Fig. 5. A wheel of order 9.

Fig. 6. A fan of order 9.

Fig. 7. Two depictions of an extended θ -graph.

new vertex and joining it to the central vertex and an end vertex of the rim. A fan may be *reduced* by eliminating an internal spoke or deleting an end vertex of the rim (together with its two incident edges).

A θ -graph is the union of three internally disjoint paths with shared end-vertices. An *extended θ -graph* consists of a θ -graph plus a universal vertex.

An extended θ -graph may be viewed as two wheels W and Y pasted together on a fan F such that the centre of Y is pasted to the centre of W . If F is a triangle (a fan with just two spokes) the result is a wheel plus an edge. An example of an extended θ -graph is depicted in Fig. 7.

Note that a wheel of order 4 is a K_4 , and any vertex in a K_4 may be regarded as its centre. Thus pasting a K_4 and any other wheel together on a triangle necessarily results in an extended θ -graph.

If two wheels W and Y of order at least 5 each are pasted together on a triangle such that the centre of Y is pasted to a vertex on the rim of W , the result is called a *triangle bicycle*.

If two wheels W and Y of order at least 5 each are pasted along a diamond (that is, a K_4 minus an edge) such that the centre of Y is pasted to a vertex on the rim of W , the result is called a *diamond bicycle*.

For example, let W be a wheel with centre w and rim $w_1w_2 \dots w_8w_1$ and let Y be a wheel with centre y and rim $y_1y_2 \dots y_5y_1$. Fig. 8 depicts a triangle bicycle, obtained from W and Y by pasting the triangle $Y[\{y, y_1, y_2\}]$ to the triangle $W[\{w_1, w_2, w\}]$. Fig. 9 depicts a diamond bicycle, obtained from W and Y by pasting the diamond $Y[\{y, y_2, y_1, y_3\}]$ to the diamond $W[\{w_2, w, w_3, w_1\}]$.

Theorem 3.5. G is an irreducible LC graph with $m(G) \leq 2n(G) - 1$ if and only if $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, where

- $\mathcal{T}_3 = \{K_2, K_3\}$.
- $\mathcal{T}_2$ consists of all the wheels.
- $\mathcal{T}_1$ consists of the extended θ -graphs, the diamond bicycles and the triangle bicycles.

Fig. 8. A triangle bicycle.

Fig. 9. A diamond bicycle.

Proof. Adding a universal vertex to any connected graph obviously results in an LC graph. This, together with [Observation 3.2](#), implies that every graph in $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ is LC. It is easily seen that every graph in $\mathcal{T}_1 \cup \mathcal{T}_2$ is 3-connected and that every graph in $\mathcal{T}_i$ satisfies $m = 2n - i$, $i = 1, 2, 3$. Thus, if $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, then G is an irreducible LC graph with $m(G) \leq 2n(G) - 1$.

To prove the converse, suppose G is an irreducible LC graph with $m(G) \leq 2n(G) - 1$. We prove by induction on the order that $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$.

If $n(G) \leq 3$, then $G \in \mathcal{T}_3$, so the basis step is proved.

Now suppose $n(G) \geq 4$. Then G is 3-connected by [Observation 3.3](#). Since $m(G) < 2n(G)$, it therefore follows that G has a vertex, say x , of degree 3. Let $H = G - x$. Since G is locally connected, $G[N(x)]$ is isomorphic to either K_3 or P_3 .

We now consider three cases, depending on the relative size of G . In each case we use the fact that if $G[N(x)]$ is a K_3 , then H is an irreducible LC graph, and if $G[N(x)]$ is a path $v_1v_2v_3$, then $H + v_1v_3$ is an irreducible LC graph. Note that in the latter case, G can be retrieved from H by subdividing the edge v_1v_3 with a vertex x and then adding the edge xv_2 .

Case 1. $m(G) = 2n(G) - 3$:

If $G[N(x)]$ is a K_3 , then $m(H) = m(G) - 3 = 2n(H) - 4$, contradicting [Proposition 3.1](#). Hence we may assume that $G[N(x)]$ is a path $v_1v_2v_3$. Then $m(H + v_1v_3) = 2n(H) - 3$. It therefore follows from our induction hypothesis that $H + v_1v_3 \in \mathcal{T}_3$, so $H + v_1v_3$ is a K_3 . But then G is a diamond, contradicting that G is irreducible. Hence K_2 and K_3 are the only irreducible LC graphs with $m = 2n - 3$.

Case 2. $m(G) = 2n(G) - 2$:

If $G[N(x)]$ is a K_3 , then $m(H) = 2n(H) - 3$, so then it follows from our induction hypothesis that $H \in \mathcal{T}_3$ and hence $H \cong K_3$. Then $G \cong K_4$, so $G \in \mathcal{T}_2$.

Now suppose $G[N(x)]$ is a path $v_1v_2v_3$. Then $m(H + v_1v_3) = 2n(H + v_1v_3) - 2$. Hence, by our induction hypothesis, $H + v_1v_3$ is a wheel W . Since the set $\{v_1, v_2, v_3\}$ induces a triangle in W , the centre w of W is either v_1, v_2 or v_3 .

If $w = v_2$, then v_1 and v_3 are consecutive vertices on the rim of W . Then G is the wheel obtained from W by inserting the spoke wx . Hence $G \in \mathcal{T}_2$.

Now suppose $w = v_1$. Then we may suppose that v_3 is the successor of v_2 on the rim of W . Let v_3^+ be the successor of v_3 on the rim of W . If $n(W) \geq 5$, then since $v_1v_3 \notin E(G)$, it follows that $G[N(v_3^+)]$ is not connected, contradicting that G is LC. Hence $n(W) = 4$, so $W \cong K_4$. This implies that G is a wheel of order 5, centred at v_2 . Hence $G \in \mathcal{T}_2$.

Case 3. $m(G) = 2n(G) - 1$:

Suppose $G[N(x)]$ is a K_3 with vertex set v_1, v_2, v_3 . Then $m(H) = 2n(H) - 2$. Hence, by our induction hypothesis, H is a wheel. Then $G[\{x, v_1, v_2, v_3\}]$ is a K_4 , and G is obtained by pasting this K_4 and the wheel H together on the triangle $G[\{v_1, v_2, v_3\}]$. Thus G is an extended θ -graph, so $G \in \mathcal{T}_1$.

Now suppose $G[N(x)]$ is a path $v_1v_2v_3$. Then $m(H + v_1v_3) = 2n(H + v_1v_3) - 1$, so by our induction hypothesis $H + v_1v_3 \in \mathcal{T}_1$. There are three cases to consider.

(a) $H + v_1v_3$ is an extended θ -graph: Let u be the universal vertex of $H + v_1v_3$. Then $H + v_1v_3 - u$ consists of two vertices a and z and three internally disjoint $a - z$ paths P_1, P_2, P_3 .

Fig. 10. Extending a border in G^* to construct G_N .

If $u = v_2$, then v_1 and v_3 are two consecutive vertices on one of the paths P_1, P_2, P_3 , and G is obtained from $H + v_1v_3$ by inserting x on the edge v_1v_3 and joining x to u . So in this case G is also an extended θ -graph.

If $u = v_1$, then since $G[N(v_3)]$ is connected, we may assume that $P_1 = az$ and P_2 and P_3 have length 2 each. Let y and w be the internal vertices of P_2 and P_3 , respectively. First, suppose $v_2 = a$. Then v_2 is a universal vertex of G . If $v_3 = z$, then $G - v_2$ consists of the three internally disjoint $v_1 - v_3$ paths v_1xv_3 , v_1yv_3 and v_1wv_3 . If $v_3 = y$, then $G - v_2$ consists of the three internally disjoint $v_1 - z$ paths v_1z , v_1xv_3z and v_1wz . In either case, G is an extended θ -graph. By symmetry, this proves all the cases when $v_2 \in \{a, z\}$. If $v_2 \notin \{a, z\}$, it is sufficient to consider the case $v_2 = w$, $v_3 = z$. Then G consists of the wheels $G[\{v_2, a, v_1, x, v_3\}]$ and $G[\{a, v_1, v_2, v_3, y\}]$, pasted together on the diamond $G[\{a, v_2, v_1, v_3\}]$, so then G is a diamond bicycle.

Now suppose $u \notin \{v_1, v_2, v_3\}$. Then we may assume that $P_1 = az$ and $P_2 = ayz$, where $\{a, y, z\} = \{v_1, v_2, v_3\}$. If $y = v_2$, then G consists of the wheel $G[\{y, u, a, x, z\}]$ and the wheel $G[\{u, y\} \cup V(P_3)]$, pasted together on the diamond $G[\{y, u, a, z\}]$. So in this case G is a diamond bicycle. If $y = v_1$, then G consists of the wheel $G[\{z, u, y, x, a\}]$ and the wheel $G[\{u\} \cup V(P_3)]$, pasted together on the triangle $G[\{u, z, a\}]$, so then G is a triangle bicycle. In either case $G \in \mathcal{T}_1$.

(b) $H + v_1v_3$ is a triangle bicycle: Then $H + v_1v_3$ consists of two wheels W and Y , each of order at least 5, pasted together on a triangle T such that w and y , the respective centres of W and Y , are distinct vertices in T . We may assume that $G[v_1, v_2, v_3]$ is a triangle in W . First, suppose v_1 or v_3 is the centre of W , say $v_1 = w$. Then we may assume that v_3 is the successor of v_2 on the rim of W . Let v_3^+ be the successor of v_3 on the rim of W . Now $v_1 \notin N_G(v_3)$ and since $n(W) \geq 5$, there is no path from v_2 to v_3^+ in $G[N(v_3)]$. Hence $G[N(v_3)]$ is not connected, contradicting that G is LC. This proves that $\{v_1, v_3\} \cap \{w, y\} = \emptyset$ and hence $G[\{v_1, v_2, v_3\}] \neq T$.

We may therefore assume that $v_2 = w$ and v_1v_3 is an edge on the rim of W . Since $y \notin \{v_1, v_2, v_3\}$, it follows that $v_1v_3 \notin E(T)$. Now let W^* be the wheel obtained from W by subdividing the edge v_1v_3 with the vertex x and adding the edge xw . Then G is the triangle bicycle consisting of the wheels W^* and Y , pasted together on the triangle T . Hence $G \in \mathcal{T}_1$.

(c) $H + v_1v_3$ is a diamond bicycle: Then $H + v_1v_3$ consists of two wheels W and Y , each of order at least 5, that share a diamond D such that w and y , the respective centres of W and Y , are distinct vertices in D .

Suppose $v_1 = w$. Then we may assume that v_3 is the successor of v_2 on W . Let v_3^+ be the successor of v_3 on W . If $v_3 = y$, then $G[N(v_3^+)]$ is not connected, since there is no path from v_1 to v_3 in $N_G(v_3^+)$. If $v_3 \neq y$, then there is no path from v_3^+ to v_2 , so then $G[N(v_3)]$ is not connected. This proves that $\{v_1, v_3\} \cap \{w, y\} = \emptyset$ and that $G[\{v_1, v_2, v_3\}]$ is not a triangle in D .

We may therefore assume that $v_2 = w$ and v_1v_3 is an edge on the rim of W such that v_1v_3 is not an edge of the diamond D . Hence G is a diamond bicycle, consisting of two wheels pasted together on D . $\square$

Corollary 3.6. Every irreducible LC graph with $n \geq 3$ and $m \leq 2n - 1$ is hamiltonian.

4. The hamilton cycle problem for sparse LC graphs

A modification of the construction that produced the $\{2, 5\}$ -regular graph G^* in the proof of Theorem 1.3 (see Fig. 3), yields the following result.

Theorem 4.1. For every natural number $k > 1$ the Hamilton Cycle Problem for locally connected graphs with $m = 2n + \lfloor n^{1/k} \rfloor$ is NP-complete.

Proof. Consider any cubic graph G of order n . Let G^* be the $\{2, 5\}$ -regular graph obtained from G as in [Theorem 1.3](#). If $k > 1$, we extend the border B by successively attaching diamonds as shown in [Fig. 10](#), until the resulting graph G_N has $N = (3n/2)^k$ vertices. (Note that the order of G^* is $3n + (9n/2)$, which has the same parity as $3n/2$.)

This can be done in polynomial time. Each added vertex has degree 4 and the neighbours of vertices of degree 2 are of degree 5 in G_N . Hence the average degree of the vertices that are not in the node-set of G_N is 4. Note that G_N has n nodes and each node has three vertices, each of which has degree 5. Hence

$$\sum_{v \in V(G_N)} d(v) = 5(3n) + 4(N - 3n) = 4N + 3n,$$

so

$$|E(G_N)| = 2N + 3n/2 = 2N + N^{1/k}.$$

G_N is clearly LC, and it follows easily from the proof of [Theorem 1.3](#) that G_N is hamiltonian if and only if G is. $\square$

Our next theorem shows that if the value of m in [Theorem 4.1](#) is slightly decreased, the complexity of the problem is significantly reduced.

Theorem 4.2. *For every natural number k the Hamilton Cycle Problem for LC graphs is in $\mathbb{P}$ for graphs with $2n + k \log_2 n$ edges or fewer.*

The proof of [Theorem 4.2](#) relies on the following reduction procedure, which decreases the order of an LC graph, while retaining local connectivity.

Reduction procedure. Let G be an LC graph of order n and size $2n + p$, where p is a positive integer. We choose a set of edges in G , which we refer to as *special edges* of G . We describe three types of reductions that may be performed on G .

Reduction 1: Suppose $v \in V(G)$ has degree 2, and let $N(v) = \{x, y\}$. Now delete the vertex v . The resulting graph G' is LC and $|E(G')| = 2|V(G')| + p$. We observe that G does not have a Hamilton cycle containing the edge xy . We also note that G is hamiltonian if and only if G' has a Hamilton cycle containing the edge xy . We therefore add xy to the set of special edges of the reduced graph.

Reduction 2: Suppose G has a vertex v of degree 3 such that $\langle N(v) \rangle$ is a path $x_1x_2x_3$. Then we delete v and add the edge x_1x_3 . The resulting graph G' is LC and $|E(G')| = 2|V(G')| + p$. We call v a *nice vertex* of degree 3. We note that G' is hamiltonian if G is, but the converse need not be true.

Reduction 3: Suppose G has a vertex v of degree 3 such that $\langle N(v) \rangle$ is a triangle. Then we delete v . The resulting graph G' is LC and $|E(G')| = 2|V(G')| + p - 1$. We call v a *problematic vertex* of degree 3. It follows from [Proposition 3.1](#) that we meet at most $p + 3$ problematic vertices of degree 3 when we successively delete vertices of degree 3. We note that G' is hamiltonian if G is, but the converse need not be true.

Next, we present some observations, lemmas and definitions that will be used to prove [Theorem 4.2](#).

Observation 4.3. *If we reduce a graph G by removing a vertex v by means of a Type 1, 2 or 3 reduction, then the neighbours of v are mutually adjacent in the resulting graph (even if two of the neighbours were not adjacent in G).*

If v is a vertex in a connected graph G such that the neighbours of v are mutually adjacent, then clearly $G - v$ is connected, so by [Observation 4.3](#), Reductions 1, 2 and 3 do not create new vertex cuts. We shall apply this observation in the special version described in [Lemma 4.4](#).

Lemma 4.4. *Let G be a connected LC graph and let F be a graph obtained from G by means of a number of consecutive reductions of Type 1, 2 or 3. Let z be the last vertex of G that was removed to obtain F and suppose z had exactly two neighbours x, y in F . Then $\{x, y\}$ is a vertex cut of G .*

The square of an n -cycle, denoted by C_n^2 , is the 4-regular graph obtained from the n -cycle C_n by adding an edge between each pair of vertices at distance 2 from one another. The graph C_{12}^2 is depicted in [Fig. 11](#).

Gordon et al. [4] proved the following.

Lemma 4.5. *[[4]] Every connected 4-regular LC graph is the square of a cycle.*

Let $v_1v_2 \dots v_{2n}v_1$ denote the underlying $2n$ -cycle of the graph C_{2n}^2 . If the three edges between the sets $\{v_1, v_2\}$ and $\{v_{2n-1}, v_{2n}\}$ are deleted from C_{2n}^2 , the result is a *zig-zag strip* of order $2n$. If the vertex v_{2n} is also deleted, the result is a zig-zag strip of order $2n - 1$. Thus a zig-zag strip consists of two “parallel” paths whose lengths differ by at most one, with edges between the paths in a zig-zag pattern. [Fig. 12](#) depicts zig-zag strips of order 11 and 12.

Each triangle in a zig-zag strip may be viewed as a fan with two spokes, having its apex on one of the two parallel paths and its rim on the other. We call a graph obtained from a zig-zag strip by inserting spokes into such fans (as described in [Section 3](#)) a *nice generalized strip*. Note that any nice generalized strip is locally connected. [Fig. 13](#) depicts an example of a nice generalized strip.

Fig. 11. The square of a 12-cycle.

Fig. 12. Zig-zag strips of order 11 and 12.

Fig. 13. An example of a nice generalized strip.

A fan in a graph G is a *nice fan* if every internal vertex on the rim of the fan has degree 3 in G , i.e., only the end vertices of the rim and the centre may have neighbours outside the fan. We note that every fan in a nice generalized strip is a nice fan, provided its rim is on one of the two parallel paths.

A *path cover* of a graph is a set of vertex disjoint paths that contain all the vertices of the graph. An r -*path cover* is a path cover consisting of r vertex disjoint paths.

Lemma 4.6. *Let S be a nice generalized strip and let a_1, a_2 and b_1, b_2 be the end vertices of the two parallel paths of S . Suppose Q is a path cover of S such that the end vertices of all the paths in Q are in $\{a_1, a_2, b_1, b_2\}$. Then there are at most 26 possibilities for the end vertices of the paths in Q .*

Proof. If Q is a 1-path cover (i.e., a Hamilton path), then the two end vertices can be chosen from the set $\{a_1, a_2, b_1, b_2\}$ in $\binom{4}{2} = 6$ ways.

Now suppose Q is a 2-path cover $\{Q_1, Q_2\}$. If neither Q_1 nor Q_2 is a singleton, then there are only 2 possibilities for their end vertices, since S is planar. If Q_1 is a single vertex v , then there are 4 choices for v and for each such choice there are 3 possibilities for the end-vertices of Q_2 . Thus the total number of possibilities in the case of a 2-path cover is 14.

If Q is a 3-path cover, then two of the three paths in Q are singletons, so in this case there are only 6 possibilities for the end vertices.

Thus the total number of possibilities is $6 + 14 + 6 = 26$. $\square$

If G is a graph with a chosen set of special edges, we call a Hamilton cycle of G that contains all the special edges of G a *special Hamilton cycle* of G .

For simplicity, we only prove Theorem 4.2 for $k = 1$, since it easily generalizes to an arbitrary natural number k . We actually prove the following slightly more general result for $k = 1$.

Theorem 4.7. *The problem of deciding whether an LC graph of order n and size at most $2n + \log_2 n$ has a Hamilton cycle containing a set of prescribed edges is in $\mathbb{P}$.*

We precede the proof with a brief outline, so that the reader can keep track of the main steps while going through the details of the proof.

Outline of proof. Let G be an LC graph of order n and size at most $2n + \log_2 n$, with a prescribed set of special edges. We use our reduction procedure to iteratively remove vertices of degree 2 and 3, until we obtain a graph H that has exactly $\lfloor \log_2 n \rfloor$ vertices or has minimum degree 4, whichever occurs first. We choose the special edges of H to consist of the special edges inherited from G , together with every edge that became special due to a Type 1 reduction. Then any special Hamilton cycle of G reduces to a special Hamilton cycle of H , but the converse is not necessarily true.

We now reverse the reductions by first returning vertices of degree 3 as far as possible without returning vertices of degree 2, to obtain the graph G^* . When reversing a reduction, any edges that were added during the reduction are deleted, so G^* is an induced subgraph of G .

Fig. 14. Reversing a reduction of Type 3 divides a maximal nice fan into 2 maximal nice fans.

An analysis of the structure, order and size of G^* allows us to shrink G^* to a graph H' of sufficiently small size so that it can be checked in $O(n^c)$ time (for some constant c) whether H' has a special Hamilton cycle, and which is such that H' has a special Hamilton cycle if and only if G^* has one.

If $G = G^*$, we are done. If not, there is at least one vertex z in $G - V(G^*)$ that has two neighbours x, y in G^* and xy is an edge that became special due to a Type 1 reduction. By Lemma 4.4, $\{x, y\}$ is a vertex cut of G , and since G is locally connected, $G - \{x, y\}$ has exactly two components. Let Z be the component of $G - \{x, y\}$ that contains z . Let $G_{xy} = G[V(Z) \cup \{x, y\}]$ and let $G_1 = G[V(G^*) \cup V(G_{xy})]$. Then G_1 has a special Hamilton cycle if and only if G^* has a special Hamilton cycle and G_{xy} has a special Hamilton cycle containing the edge xy . To check whether G_{xy} has such a Hamilton cycle, we repeat our algorithm, starting with the 3-cycle $zxyz$ instead of the graph H . Again this is an $O(n^c)$ algorithm.

Since the number of Type 1 reductions that we need to reverse is less than n , repeated applications of the procedure result in an $O(n^{c+1})$ algorithm for determining whether G has a special Hamilton cycle.

Now we present the detailed proof.

Proof. Let G be an LC graph of order n and size at most $2n + \log_2 n$, with a given set of special edges. We may assume G is connected, since otherwise the problem is trivial. We perform reductions 1, 2 and 3 in any order until the resulting graph H has precisely $\lfloor \log_2 n \rfloor$ vertices or has minimum degree at least 4. In either case it follows from the discussions of the reductions above that $|E(H)| \leq 2|V(H)| + \log_2 n$.

Case 1. H has precisely $\lfloor \log_2 n \rfloor$ vertices:

In this case $|E(H)| \leq 2 \log_2 n + \log_2 n = 3 \log_2 n$. We consider each subset of $E(H)$ and check if that subset induces a special Hamilton cycle of H . This algorithm has time complexity $O(2^{|E(H)|} \log_2 n) = O(2^{3 \log_2 n} \log_2 n) = O(n^3 \log_2 n)$.

We now focus on the maximal nice fans. We create the graph G^* by reversing all the reductions of Types 2 and 3 that can be reversed without reversing any reductions of Type 1. We note that $|E(G^*)| \leq 2|V(G^*)| + \log_2 n$.

If a nice vertex whose neighbours form a triangle in a nice fan of order greater than 3 is returned by reversing a Type 2 reduction, then the edge of the triangle that is on the rim of the fan is deleted. (For, if a spoke were deleted we would not get an LC graph.) Thus, reversing a Type 2 reduction can extend a maximal nice fan but does not create additional maximal nice fans. Hence, if no Type 3 reductions were reversed to create G^* from H , then the number of maximal nice fans in G^* would be at most the number of edges of H , i.e., at most $3 \log_2 n$.

However, we may need to return problematic vertices of degree 3 to create G^* . Returning a problematic vertex of degree 3 whose neighbours are a triangle in a maximal nice fan divides that fan into two edge disjoint maximal nice fans — see Fig. 14.

Since the number of problematic vertices that were deleted from G to create H is at most $\log_2 n + 3$ (as explained in the description of Reduction 3), the addition of problematic vertices to H creates at most $\log_2 n + 3$ additional maximal nice fans. Thus the number of maximal nice fans in G^* is at most $|E(H)| + \log_2 n + 3 = 4 \log_2 n + 3$.

We now reduce any maximal nice fan in G^* that has more than 4 spokes to a nice fan with exactly four spokes, by eliminating internal spokes (as described in Section 3). Call the resulting graph H' . Since eliminating a spoke amounts to reducing the number of vertices by one and the number of edges by two, it follows that $|E(H')| \leq 2|V(H')| + \log_2 n$.

We may assume that the set of special edges in G^* does not contain two non-consecutive internal spokes of any nice fan (since otherwise G^* , and hence G , will be nonhamiltonian). It is therefore possible to choose edges to be special in H' so that H' has a special Hamilton cycle if and only if G^* has a special Hamilton cycle.

We note that if F is a nice fan in G^* that was obtained from a nice fan in H by adding only nice vertices of degree 3, the two outer spokes of F are both in H . Thus, if F is reduced to a fan with four spokes in H' , it contains at most 2 vertices that were not in H .

If a problematic vertex of degree 3 is inserted into a triangle in a nice fan while we are building G^* , then two of the three the neighbours of that vertex lie on the outer spokes of two new maximal nice fans (see Fig. 14). Thus returning a problematic vertex results in at most 3 more vertices in H' that were not in H . Thus

$$|V(H')| \leq |V(H)| + 2(4 \log_2 n + 3) + 3(\log_2 n + 3) = 12 \log_2 n + 15$$

and

$$|E(H')| \leq 2|V(H')| + \log_2 n \leq 25 \log_2 n + 30,$$

so all the Hamilton cycles in H' can be found by means of an $O(2^{25 \log_2 n + 30} \log_2 n) = O(n^{25} \log_2 n)$ algorithm.

Fig. 15. Structure of H .

Thus, in this case there is an $O(n^{26})$ algorithm for determining whether G^* has a special Hamilton cycle.

If $G = G^*$, we are done. If not, we reverse a Type 1 reduction and carry on as explained in the last paragraph of the outline of the proof. The result is an $O(n^{27})$ algorithm for determining whether G has a special Hamilton cycle.

Case 2. H has more than $\log_2 n$ vertices (and $\delta(H) \geq 4$):

Recall that also in this case, $|E(H)| \leq 2|V(H)| + \log_2 n$. Let

$$X = \{v \in V(H) : d_H(v) > 4\}, \quad N = N_{H-X}(X), \quad M = G[X \cup N].$$

We assume that X is nonempty. (Otherwise, the argument below is simpler.) Note that

$$\sum_{v \in V(H)} d_H(v) \leq 4|V(H)| + 2 \log_2 n.$$

Hence, since $\delta(H) \geq 4$, it follows that $\sum_{v \in X} (d_H(v) - 4) \leq 2 \log_2 n$, and hence $|X| \leq 2 \log_2 n$. Now

$$|N| \leq \sum_{v \in X} d_H(v) = \sum_{v \in V(H)} d_H(v) - \sum_{v \in V(H)-X} d_H(v).$$

But every vertex in $H - X$ has degree exactly 4 in H , so

$$\begin{aligned} |N| &\leq 4|V(H)| + 2 \log_2 n - 4(|V(H)| - |X|) \\ &= 4|X| + 2 \log_2 n \\ &\leq 10 \log_2 n \end{aligned}$$

and

$$|V(M)| \leq 12 \log_2 n.$$

Now

$$\begin{aligned} \sum_{v \in V(M)} d_H(v) &= 4|V(M)| + \sum_{v \in X} (d_H(v) - 4) \\ &\leq 4|V(M)| + 2 \log_2 n. \end{aligned}$$

Hence

$$|E(M)| \leq 2|V(M)| + \log_2 n \leq 25 \log_2 n.$$

Since every vertex in $H - X$ has exactly 4 neighbours in H , it can easily be proved using the same reasoning as in [Lemma 4.5](#) that every component in $H - V(M)$ together with its neighbours in N forms a zig-zag strip that has its two end edges in N and all the other vertices in $H - V(M)$ – see [Fig. 15](#).

Since every vertex in N has degree 4 in H and has at least one neighbour in X , no vertex in N lies in more than one zig-zag strip of H , and each zig-zag strip in H contains 4 vertices of N . Since $|N| \leq 10 \log_2 n$, it follows that H has at most $5/2 \log_2 n$ zig-zag strips.

We now create the graph G^* by reversing all the reductions of Types 2 and 3 that can be reversed without reversing a reduction of Type 1. We first return vertices that have three neighbours in a zig-zag strip. If only nice vertices are added to a zig-zag strip, that strip becomes a nice generalized strip. If a problematic vertex is added in a nice generalized strip, then the nice generalized strip becomes two nice generalized strips and we then add the problematic vertex and its neighbours to M , as illustrated in [Fig. 16](#). Thus each addition of a problematic vertex to a nice generalized strip creates one more nice generalized strip and increases the number of vertices in M by 4 and the number of edges by 6.

Fig. 16. Reversing reductions of Types 2 and 3 to create G^* .

Since there are at most $\log_2 n + 3$ problematic vertices, $|V(M)|$ increases to at most

$$12 \log_2 n + 4(\log_2 n + 3) = 16 \log_2 n + 12$$

and $|E(M)|$ increases to at most

$$25 \log_2 n + 6(\log_2 n + 3) = 31 \log_2 n + 18,$$

while the number of nice generalized strips increases to at most

$$5/2 \log_2 n + \log_2 n + 3 = 7/2 \log_2 n + 3.$$

The intersection of a Hamilton path of G^* with each nice generalized strip is a path cover of which the end vertices of all the paths are in N . By Lemma 4.6 there are 26 possibilities for the end vertices of such path covers. For each of those, there is an $O(n)$ algorithm for deciding whether there exists a corresponding path cover containing all the special edges in the strip. Therefore, since there are at most $7/2 \log_2 n + 3$ nice generalized strips, there is an $O(n) \cdot 26^{7/2 \log_2 n + 3} < O(n) \cdot (2^5)^{7/2 \log_2 n + 3} < O(n^{19})$ algorithm for deciding whether there exist corresponding path covers containing all the special edges in all the strips.

So far we have returned vertices that have three neighbours in a zig-zag strip. We now return vertices whose neighbours induce triangles or paths with 3 vertices in M . Denote the resulting extension of M by M^* . Using the method of Case 1, we find that M^* contains at most $|E(M)| + \log_2 n + 3 = 32 \log_2 n + 21$ maximal nice fans.

Now we reduce M^* to the graph M' by reducing all the maximal nice fans in M^* to fans with at most 4 spokes and choose appropriate special edges for M' as in Case 1. Then

$$|V(M')| \leq |V(M)| + 2(32 \log_2 n + 21) + 3(\log_2 n + 3)$$

$$= 16 \log_2 n + 12 + 67 \log_2 n + 51 = 83 \log_2 n + 63,$$

and

$$|E(M')| \leq 2|V(M')| + \log_2 n \leq 166 \log_2 n + 126.$$

Let H' denote the graph consisting of M' together with all the nice generalized strips. Then H' has a special Hamilton cycle if and only if G^* has a special Hamilton cycle. For each subset of $E(M')$ and each choice of path covers of the strips we check, in $O(n)$ time, if the subsets of $E(M')$ and the path covers of the strips form a special Hamilton cycle of H' . The complexity of this algorithm is at most $O(n^{19}) \cdot 2^{166 \log_2 n + 126} \cdot O(n) = O(n^{176})$.

Thus, in this case there is an $O(n^{176})$ algorithm for determining whether G^* has a special Hamilton cycle.

If $G = G^*$, we are done. If not, we reverse a Type 1 reduction and carry on as explained in the last paragraph of the outline of the proof. The result is an $O(n^{177})$ algorithm for determining whether G has a special Hamilton cycle. $\square$

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