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On chromatic number and minimum cut

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ABSTRACT

For a graph G , the tree Kneser graph $KG(G, \mathcal{T}_t)$ has all tree subgraphs of G with t vertices as vertex set and two vertices are adjacent if their corresponding trees are edge-disjoint. Also, the r -th cut number of G is the minimum number of edges between the parts of a partition of the vertex set of G into two parts such that each part has size at least r . In this paper, we investigate the chromatic number of tree Kneser graphs. Roughly speaking, we prove that for any nonnegative integer function $t = t(n)$, where $n - t = o(n)$, if n is sufficiently large, then for any dense graph G with n vertices, the chromatic number of the tree Kneser graph $KG(G, \mathcal{T}_t)$ is equal to the $(n - t + 1)$ -th cut number of G . In particular, as a consequence of this result, if n is large enough, then for any dense graph G with n vertices, the chromatic number of the tree Kneser graph $KG(G, \mathcal{T}_n)$ is equal to the minimum cut (minimum degree) of G . This result can be somehow considered a reminiscent of the Nash-Williams disjoint trees theorem. The proof is based on a lower bound for the chromatic number of graphs found by the present authors Alishahi and Hajiabolhassan (2015) [1] which is inspired by Tucker's lemma, an equivalent combinatorial version of the Borsuk-Ulam theorem. Finally, motivated by the aforementioned results, we close the

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paper by a discussion on the complexity of determining the r -th cut number of graphs.

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1. Introduction

Throughout the paper, the symbol $[n]$ stands for the set $\{1, \dots, n\}$. The *Kneser graph* $\text{KG}(n, k)$ is a graph whose vertex set consists of all k -subsets of $[n]$ and two vertices are adjacent if their corresponding k -subsets are disjoint. In 1955, Kneser [10] posed a conjecture about the chromatic number of $\text{KG}(n, k)$. He proved that $\text{KG}(n, k)$ can be properly colored by $n - 2k + 2$ colors and conjectured that this number of colors is the least possible. In a breakthrough paper, Lovász [13] proved this conjecture with the help of algebraic topology. Actually, Lovász presented a general lower bound for the chromatic number of graphs which is tight in the case of Kneser graphs $\text{KG}(n, k)$.

Throughout the paper, by a *nonempty graph*, we mean a graph with nonempty edge set. For a graph G and a nonempty family $\mathcal{F}$ of nonempty graphs, the *general Kneser graph* $\text{KG}(G, \mathcal{F})$ is a graph whose vertex set consists of all subgraphs of G isomorphic to some members of $\mathcal{F}$ and two vertices are adjacent if their corresponding subgraphs are edge-disjoint. When $\mathcal{F} = \{H\}$, we would prefer to use $\text{KG}(G, H)$ instead of $\text{KG}(G, \mathcal{F})$. Several well-known families of graphs can be represented in the form of general Kneser graphs. For instance, the general Kneser graphs $\text{KG}(nK_2, kK_2) \cong \text{KG}(K_{1,n}, K_{1,k})$, $\text{KG}(C_n, kK_2)$, and $\text{KG}(C_n, P_{d+1})$, are respectively isomorphic to the Kneser graph $\text{KG}(n, k)$, the Schrijver graph $\text{SG}(n, k)$ and the circular complete graph $K_{\frac{n}{d}}$, where P_{d+1} is a path with $d + 1$ vertices.

1.1. Alternating partial 2-coloring

In this subsection, unless otherwise stated, we shall consider graphs which are not necessarily simple. For a graph G and its subgraph H , throughout the paper, we write $G - E(H)$ and $G - H$ for the subgraphs of G obtained by removing the edges and the vertices of H , respectively. Let G be a graph and $\sigma : e_1 < \dots < e_{|E(G)|}$ be a linear ordering of $E(G)$. A *partial 2-coloring of $E(G)$* is an assignment of two colors red and blue to some edges of G . Note that some edges might receive no color. For an edge $e \in E(G)$, if the red (resp. blue) color is assigned to e , then we call this edge a *red edge* (resp. *blue edge*) and if no color is assigned to e , then we call it a *neutral edge*. Moreover, the *red subgraph* G^R (resp. *blue subgraph* G^B) is a spanning subgraph of G whose edge set is the set of all red (resp. blue) edges of G . A partial 2-coloring is called σ -*alternating* if either it has no colored edge or any two consecutive (respect to ordering σ) colored edges of it receive different colors, i.e., if $e_{i_1}, \dots, e_{i_k}$ are the colored edges, where $i_1 < \dots < i_k$, then

for each $j \in [k-1]$, the edges e_{i_j} and $e_{i_{j+1}}$ have different colors. If there is no ambiguity in the ordering σ , then such a coloring is simply called an alternating partial 2-coloring. The *length* of a σ -alternating partial 2-coloring of $E(G)$ is the number of colored edges, i.e., $|E(G^R)| + |E(G^B)|$. Note that, in view of the definition of σ -alternating, for any σ -alternating partial 2-coloring, we have $||E(G^R)| - |E(G^B)|| \leq 1$. For a subgraph H of G , the set of neutral edges of $E(H)$ is denoted by $\text{NEU}(H)$. Moreover, the set of neutral edges of $E(H)$ incident with a vertex v of H is denoted by $\text{NEU}(v, H)$. Using the topological tools, the present authors proved the following theorem.

Theorem A. [1,2] *Let G be a nonempty graph and $\mathcal{F}$ be a nonempty family containing some subgraphs of G . Also, let $\sigma : e_1 < \dots < e_{|E(G)|}$ be a linear ordering. If for any σ -alternating partial 2-coloring of $E(G)$ of length k , at least one of G^R and G^B contains a subgraph isomorphic to some member of $\mathcal{F}$, then*

$$\chi(\text{KG}(G, \mathcal{F})) \geq |E(G)| - k + 1.$$

It should be mentioned that the lower bound presented in Theorem A is originally proved in [1] stated in a different language.

1.2. The i -th cut number

Let G be a graph with n vertices. For two disjoint nonempty subsets A, B of $V(G)$, by $E(A, B)$, we mean the set of edges in $E(G)$ with one end in A and the other end in B . Let i be an integer, where $1 \leq i \leq \frac{n}{2}$. A partition (X, Y) of $V(G)$ is called an i -cut of G if $\min\{|X|, |Y|\} \geq i$. For any integer $1 \leq i \leq \frac{n}{2}$, define the i -th cut number of a graph G , denoted $\text{cut}_i(G)$, to be the minimum of $|E(X, Y)|$, where the minimum is taken over all i -cuts (X, Y) of G , i.e.,

$$\text{cut}_i(G) = \min \{|E(S, V(G) \setminus S)| : S \subseteq V(G), |S| \geq i, |V(G) \setminus S| \geq i\}.$$

To clarify the definition, we compute the i -th cut numbers of K_n and $K_{n,n}$. Let (S, T) be an i -cut of K_n , where $|S| \leq |T|$ and $1 \leq i \leq \frac{n}{2}$. Clearly, $|E(S, T)| = s(n-s)$, where $s = |S|$. Since $f(s) = s(n-s)$ is increasing on $[i, \frac{n}{2}]$, it takes its minimum in $s = i$. Accordingly, $\text{cut}_i(K_n) = i(n-i)$ for each $1 \leq i \leq \frac{n}{2}$. Now, let $K_{n,n} = (X, Y)$ and (S, T) be an i -cut of $K_{n,n}$, where $|S| \leq |T|$ and $1 \leq i \leq n$. For $x = |S \cap X|$ and $y = |S \cap Y|$, we have $|E(S, T)| = x(n-y) + y(n-x)$. To compute $\text{cut}_i(K_{n,n})$, we need to find the minimum of the function $g(x, y) = (x+y)n - 2xy$ over $\{(x, y) : x, y \in \{0, \dots, n\}, i \leq x+y \leq n\}$. It is not hard to see that the minimum is $ni - 2\lfloor \frac{i}{2} \rfloor \lceil \frac{i}{2} \rceil$ taking place when $x = \lceil \frac{i}{2} \rceil$ and $y = \lfloor \frac{i}{2} \rfloor$. Therefore, $\text{cut}_i(K_{n,n}) = ni - 2\lfloor \frac{i}{2} \rfloor \lceil \frac{i}{2} \rceil$ for each $1 \leq i \leq n$. Note that $\text{cut}_1(G)$ is a well-known parameter known as the minimum cut of G , denoted by $\text{cut}(G)$.

Also, for a partition $P = (V_1, \dots, V_m)$ of $V(G)$, any edge of G whose ends are in different parts of P is called a *cross-edge* of P . Tutte [15] and Nash-Williams [14] independently proved that a graph G contains k pairwise edge-disjoint spanning trees if

and only if for any partition $P = (V_1, \dots, V_m)$ of $V(G)$, there are at least $k(m-1)$ cross-edges of P . For $n \geq 2$, let $\mathcal{T}_n$ be the family of all pairwise non-isomorphic trees with n vertices. Clearly, the aforementioned result by Tutte and Nash-Williams can be restated as follows:

For any graph G with n vertices, $\omega(\text{KG}(G, \mathcal{T}_n)) \geq k$ if and only if there are at least $k(m-1)$ cross-edges of any partition $P = (V_1, \dots, V_m)$ of $V(G)$.

In other words, it shows that the clique number of $\text{KG}(G, \mathcal{T}_n)$ is the floor of the minimum possible ratio of the number of cross-edges of a partition $P = (V_1, \dots, V_m)$ to $m-1$, where the minimum is taken over all partitions $P = (V_1, \dots, V_m)$ of $V(G)$ with $m > 1$. Accordingly, this value is at least $\left\lceil \frac{\text{cut}_1(G)-1}{2} \right\rceil$, see [11]. Moreover, for an integer r , where $0 \leq r \leq \frac{n}{2} - 1$, let (X, Y) be an $(r+1)$ -cut of G with $E(X, Y) = \{e_1, \dots, e_C\}$. For any vertex T of $\text{KG}(G, \mathcal{T}_{n-r})$, define $c(T)$ to be the minimum i such that $e_i \in E(T)$. One can check that c is a proper C -coloring of $\text{KG}(G, \mathcal{T}_{n-r})$. Hence, we have

$$\chi(\text{KG}(G, \mathcal{T}_{n-r})) \leq \text{cut}_{r+1}(G). \quad (1)$$

Since the clique number of a graph can be considered as an estimation of its chromatic number, the prior discussion implies that

$$\left\lceil \frac{\text{cut}_1(G)-1}{2} \right\rceil \leq \chi(\text{KG}(G, \mathcal{T}_n)) \leq \text{cut}_1(G).$$

Note that we have found an upper bound for $\chi(\text{KG}(G, \mathcal{T}_{n-r}))$ in terms of $\text{cut}_{r+1}(G)$ while the lower bound has been proved just for $r = 0$. Although it is valuable to have a similar lower bound in terms of $\text{cut}_{r+1}(G)$ for the chromatic number of $\text{KG}(G, \mathcal{T}_{n-r})$, to prove it, in our approach we need an appropriate generalization of Tutte and Nash-Williams theorem providing a sufficient condition for the existence of k pairwise edge-disjoint subtrees of G with $n-r$ vertices. As far as we know, no one has proved such a generalization though it is a nice question to investigate.

Motivated by the preceding discussion, it is interesting to investigate the chromatic number of $\text{KG}(G, \mathcal{T}_{n-r})$ for $0 \leq r \leq \frac{n}{2} - 1$. In this regard, the present authors (see Theorem 2 in [2]) proved the following theorem which is originally stated in terms of Turán number.

Theorem B. [2] *Let $t \in \{0, \dots, n-2\}$ be an integer and G be a nonempty multi-graph with n vertices such that each of its edges has the multiplicity at least 2. The chromatic number of $\text{KG}(G, \mathcal{T}_t)$ is equal to $m(G, t)$, that is, the minimum number of edges should be removed from G so that any connected component of the resulting graph has at most $t-1$ vertices.*

One can see that this result implies the following statement;

If G is a nonempty connected multi-graph, each of its edges has the multiplicity at least 2, then $\chi(\text{KG}(G, \mathcal{T}_{n-r})) = \text{cut}_{r+1}(G)$ for each $r \in \left\{0, \dots, \left\lceil \frac{|V(G)|}{3} \right\rceil - 1\right\}$.

Clearly, to prove the assertion, it suffices to show that $m(G, n-r) = \text{cut}_{r+1}(G)$ for each $r \in \left\{0, \dots, \left\lceil \frac{|V(G)|}{3} \right\rceil - 1\right\}$. In view of the definition of $\text{cut}_{r+1}(G)$, it is clear that $m(G, n-r) \leq \text{cut}_{r+1}(G)$. Now, consider $F \subseteq E(G)$ with $|F| = m(G, n-r)$ and such that any component of $G - F$ has at most $n-r-1$ vertices. Also, let $Q_1, \dots, Q_m$ be the components of $G - F$, where $|Q_1| \leq \dots \leq |Q_m|$. Consider the smallest $l \in [m]$ such that $|Q_1| + \dots + |Q_l| \geq r+1$. First note that $l < m$. We claim that $|Q_1| + \dots + |Q_l| \leq n-r-1$. On the contrary, suppose that $|Q_1| + \dots + |Q_l| \geq n-r$. Due to the minimality of l , we clearly have $|Q_1| + \dots + |Q_{l-1}| \leq r$, implying that $|Q_l| \geq n-2r$. On the other hand, we have

$$n-2r \leq |Q_l| \leq |Q_{l+1}| \leq n - \left| \bigcup_{i=1}^l Q_i \right| \leq n - (n-r) = r,$$

concluding that $3r \geq n$ which is a contradiction. Now, set $S = \bigcup_{i=1}^l Q_i$. In view of the definition of S and since $r+1 \leq |S| \leq n-r-1$, we have

$$\text{cut}_{r+1}(G) \leq |E(S, V(G) \setminus S)| \leq m(G, n-r),$$

completing the proof.

1.3. Main results

Unfortunately, the idea used in the proof of Theorem B does not work when we deal with the simple graphs. Hereafter, unless otherwise stated, we consider only simple graphs and by a graph we mean a simple graph. The next theorem is the main result of this paper which presents some sufficient conditions for an n -vertex simple graph G to satisfy the equation $\chi(\text{KG}(G, \mathcal{T}_{n-r})) = \text{cut}_{r+1}(G)$.

Theorem 1. Let δ be a real number and $r = r(n)$ be a nonnegative integer function, where $\frac{5}{6} < \delta < 1$ and $r = o(n)$. There is a constant $N(\delta, r)$ such that if $n \geq N(\delta, r)$, then for any graph G with n vertices and $\delta(G) \geq \delta n$, we have

$$\chi(\text{KG}(G, \mathcal{T}_{n-r})) = \text{cut}_{r+1}(G).$$

One can simply see that for $n \geq 6$, the cut number of a graph G , satisfying the conditions of Theorem 1, is $\delta(G)$. In view of this observation, we shall prove the next theorem.

Theorem 2. Let n be a positive integer and δ be a real number, where $\frac{5}{6} < \delta < 1$ and $n \geq \max\{4716, \frac{828}{6\delta-5}\}$. If G is a graph with n vertices and $\delta(G) \geq \delta n$, then $\chi(\text{KG}(G, \mathcal{T}_n)) = \delta(G)$.

By the aforementioned results, one can introduce some graphs whose chromatic numbers are about twice of their clique numbers. To see this, note that if $n \geq 4716$, then $\chi(\text{KG}(K_n, \mathcal{T}_n)) = n - 1$. Also, if t is an even integer, then the complete graph K_t is decomposable into edge-disjoint Hamiltonian paths. Hence, for any integer $n \geq 2$, the clique number of $\text{KG}(K_n, \mathcal{T}_n)$ is $\lfloor \frac{n}{2} \rfloor$, while for $n \geq 4716$, we have $\chi(\text{KG}(K_n, \mathcal{T}_n)) = n - 1$.

It would be of interest to know if we can drop some conditions from the hypothesis of Theorem 1. Simply, one can see that Theorem 1 does not hold for sparse graphs. To see this, consider the tree Kneser graph $\text{KG}(G, \mathcal{T}_n)$, where G is a connected graph with $n \geq 6$ vertices and $|E(G)| \leq 2n - 3$. Clearly, the graph $\text{KG}(G, \mathcal{T}_n)$ has no edge, and consequently, its chromatic number is equal to 1, while the cut number of G can be equal to 3.

In addition, it should be noticed that for each $r \geq \left\lceil \frac{|V(G)|}{3} \right\rceil$, not only Theorem 1, but also the statement right after Theorem B does not hold in general. To see this, let k be an integer, where $k \geq 5$. Set $n = 3k$ and consider the graph H with n vertices, consisting of three disjoint copies of K_k and such that for any two of these three copies, there is exactly one edge between them. Also, let G be the multi-graph obtained from H , by multiplying each edge of H . One can simply check that $\chi(\text{KG}(H, \mathcal{T}_{n-r})) \leq 3$ and $\chi(\text{KG}(G, \mathcal{T}_{n-r})) \leq 6$, while $\text{cut}_{r+1}(H) \geq (k - 1) \geq 4$ and $\text{cut}_{r+1}(G) \geq 2(k - 1) \geq 8$ for each $r \in \{k, \dots, \lfloor \frac{n}{2} \rfloor - 1\}$.

However, it is interesting to extend Theorem 1 by finding some other conditions or improving the current conditions such that we still have the conclusion of this theorem.

1.4. Plan

This paper is organized as follows. Section 2 is devoted to the proof of Theorems 1 and 2. In Section 3, we end the paper by a discussion on the complexity of finding the t -th cut number of graphs. Roughly speaking, we prove that the problem of finding the t -th cut number of graphs is solvable in polynomial time if t is a constant and it is an NP-hard problem even if t and the input graphs satisfy the hypotheses of Theorem 1.

2. Proofs of Theorem 1 and Theorem 2

For two graphs G and H , an H -packing of G is a set $\{H_1, \dots, H_t\}$ of pairwise edge-disjoint subgraphs of G such that for each $1 \leq i \leq t$, the graph H_i is isomorphic to H . The H -packing number of G is the maximum cardinality of an H -packing of G . An H -packing of G is called a *monogamous H -packing* if every pair of vertices of G appears in at most one copy of H in the packing. In [12], it was shown that for positive even integers m and n , the complete bipartite graph $K_{m,n}$ has a monogamous C_4 -decomposition

if and only if $(m, n) = (2, 2)$ or $6 \leq n \leq m \leq 2n - 2$. Clearly, any K_r -packing of any graph G is monogamous. The problem of finding as many as possible vertex-disjoint complete subgraphs isomorphic to K_r has been studied in several articles. Hajnal and Szemerédi [7] proved the next theorem.

Theorem C. [7] *Let G be a graph with n vertices. If $\delta(G) \geq (1 - \frac{1}{r})n$, then G contains $\lfloor \frac{n}{r} \rfloor$ vertex-disjoint copies of K_r .*

Let G be a graph and $\sigma : e_1 < \dots < e_{|E(G)|}$ be a linear ordering of $E(G)$. For a vertex $v \in V(G)$, a path uvw is called a (σ, v) -consecutive path if the edges uv and vw are consecutive in the ordering σ . Since in any σ -alternating partial 2-coloring of $E(G)$, we do not assign the same color to the edges of any (σ, v) -consecutive path, we have the following observation.

Lemma 1. *Let G be a graph, $v \in V(G)$, and σ be a linear ordering of $E(G)$. Consider a σ -alternating partial 2-coloring of $E(G)$ and let E_v^R be a subset of $E(G)$ which contains all red edges incident to v . Moreover, suppose that there are t_1 edge-disjoint (σ, v) -consecutive paths such that t_2 number of them intersect E_v^R . Then there are at least $t_1 - t_2$ neutral edges incident with v in $E(G) \setminus E_v^R$.*

Now, we are ready for the proof of Theorem 1.

2.1. Proof of Theorem 1

In view of Inequality 1, to fulfill Theorem 1, it suffices to show that

$$\chi(\text{KG}(G, \mathcal{T}_{n-r})) \geq \text{cut}_{r+1}(G).$$

Therefore, using Theorem A, we shall have proved Theorem 1, if we prove the next lemma.

Lemma 2. *Let δ be a real number and $r = r(n)$ be a nonnegative integer function, where $\frac{5}{6} < \delta < 1$ and $r = o(n)$. There is a constant $N(\delta, r)$ such that if $n \geq N(\delta, r)$, then for any graph G with n vertices and $\delta(G) \geq \delta n$, there is an ordering $\sigma : e_1 < \dots < e_{|E(G)|}$ such that for any σ -alternating partial 2-coloring of $E(G)$ with length $|E(G)| - \text{cut}_{r+1}(G) + 1$, at least one of G^R and G^B has a connected component with at least $n - r$ vertices.*

It is worth mentioning that although there are some other topological lower bounds for the chromatic number of general Kneser graphs, Theorem A has a privilege since it can be seen as a pure combinatorial tool (even if it also relies on a topological proof) for finding a lower bound for the chromatic number of general Kneser graphs. For using the power of Theorem A, one needs to propose an appropriate ordering and somehow prove a Turán type result with respect to the ordering. This approach can be employed by any

one even with no background in topological combinatorics. The proof of Lemma 2 can be considered as an example of such application which is divided into two main parts. In the first part, we will define an ordering σ while in the second part, we will prove that the ordering σ satisfies the assertion of Lemma 2.

Proof of Lemma 2. Assume that $n \geq N$, where $N = N(\delta, r)$ is a constant which will be determined during the proof. Let G be a graph with $n \geq N$ vertices and $\delta(G) \geq \delta n$.

Part (I): Definition of the ordering σ .

Let $V(G) = \{v_1, v_2, \dots, v_n\}$. Set $l = \lceil \frac{26(r+1)}{\delta} \rceil = o(n)$ and let $n = 4a + b$, where a and b are positive integers and $1 \leq b \leq 4$. Join $4 - b$ new vertices to G in order to obtain the graph $G' = G \vee K_{4-b}$. Also, let F be a graph with exactly $a + 1$ connected components each isomorphic to K_4 . By Theorem C, there are at least $\frac{1}{36}n$ edge-disjoint copies of F in G' . Let N be large enough such that $\frac{1}{36}n \geq 4l + 3$. Let $\{F_1, F_2, \dots, F_{4l+3}\}$ be $4l + 3$ edge-disjoint copies of F in G' . For each F_i and for each connected component of F_i which is a subgraph of G , choose a four cycle C_4 of this component. Clearly, these C_4 's form a monogamous C_4 -packing of G . Call each C_4 of this packing a 4-block. Also, one can see that every vertex of G appears in at least $4l$ and at most $4l + 3$ number of 4-blocks.

Construct a bipartite graph with two parts W and W' , where W is a set consisting of l copies of each vertex of G and W' is the set of all 4-blocks. For each vertex $v \in V(G)$, join each of l copies of v to any 4-block in W' containing v . One can check that the degree of every vertex in the part W is at least $4l$ and also the degree of every vertex in the part W' is exactly $4l$. In view of Hall's Theorem, this bipartite graph has a matching saturating all vertices in W . For any vertex $v_i \in V(G)$, set A_i to be the set of 4-blocks assigned to l copies of v_i through the aforementioned matching. Note that $|A_i| = l$ and since 4-blocks form a monogamous C_4 -packing, the intersection of any two 4-blocks in A_i is v_i . For each $i \in [n]$, define H_i to be the subgraph of G consisting of all 4-blocks in A_i . In other words, H_i is the union of 4-blocks in A_i . Clearly, $H_1, H_2, \dots, H_n$ are pairwise edge-disjoint subgraphs of G and for each $i \in [n]$, we have $v_i \in V(H_i)$. Also, every vertex of $V(H_i) \setminus \{v_i\}$ has degree 2 in H_i . Since any vertex of G appears in at most $4l + 3$ numbers of 4-blocks, the degree of every vertex of the graph $G - \bigcup_{i=1}^n E(H_i)$ is at least $\delta n - 8l - 6$. Note that since $8l + 6 = o(n)$, we can consider N large enough to have $\delta n - 8l - 6 \geq \frac{n}{2} + 2$. Now, by the well-known result of Dirac, the graph $G - \bigcup_{i=1}^n E(H_i)$ contains two edge-disjoint Hamiltonian cycles C, C' . Without loss of generality, we can assume that $E(C) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_1 v_n\}$. Consider the family M defined as follows:

- If G has no odd degree vertex, then set $M = \emptyset$.

- Otherwise, let M be a family of vertex-disjoint paths in C' such that every odd vertex of G is a leaf (vertex of degree one) of these paths and also any leaf of these paths has an odd degree in G . One can simply check that such a family exists.

Let L be the spanning subgraph of G whose edge set is $E(M) \cup E(C) \cup (\bigcup_{i=1}^n E(H_i))$.

Clearly, we have $y = |E(L)| \leq (4l + 4)n$. Consider a maximum K_3 -packing of $G - E(L)$ and call each of its triangles a 3-block. Let q be the number of these 3-blocks. By a result of Huu Hoi (see page 8 in [9]), there are at least

$$\frac{\epsilon}{3\nu^2}(4\epsilon - \nu^2)\frac{(\nu - 3)(\nu - 4)}{(\nu - 1)(\nu - 2)} = \frac{\epsilon}{3\nu(\nu - 2)}(4\epsilon - \nu^2) - O(\nu) \quad (2)$$

edge-disjoint triangles in a graph with ν vertices and ϵ edges, where $\nu \geq 5$. Therefore, the number of 3-blocks is at least

$$\frac{|E(G)| - y}{3n(n - 2)}(4|E(G)| - 4y - n^2) - O(n) = \frac{|E(G)|}{3n(n - 2)}(4|E(G)| - n^2) - o(n^2). \quad (3)$$

Let G'' be the graph obtained from $G - E(L)$ by removing the edges of the aforementioned 3-blocks. Clearly, any vertex of G'' is an even vertex, and therefore, every connected component of G'' is an Eulerian graph. Note that the way that we choose the family M grants this property. Let $Q_1, Q_2, \dots, Q_s$ be the connected components of G'' .

Also, note that the graph $G - E(M)$ is Eulerian. By the following procedure, we will construct an Eulerian tour for $G - E(M)$ and using it, we will introduce the desired ordering σ .

For $i \in [n]$, at the i -th stage, do the following steps:

- Traverse an Eulerian tour of H_i started at v_i .
- Traverse every untraversed 3-block containing v_i started at v_i . Any two consecutively traversed 3-blocks in this stage are called *consecutive*.
- If there is a $j \in [s]$ such that $v_i \in Q_j$ and the edge set of Q_j is still untraversed, then consider an Eulerian tour for Q_j starting at v_i and traverse it.
- Traverse the edge $v_i v_{i+1}$ (the indices are taken modulo n).
- If $i < n$, then start the $(i + 1)$ -th stage.

Construct an ordering π for $E(G) - E(M)$ such that the edges are ordered corresponding to their ordering in the aforementioned Eulerian tour, i.e., if we traverse the edge e before the edge e' in the Eulerian tour, then in the ordering π , we have $e < e'$. Now, let the ordering σ be an ordering obtained from π by putting the edges of M at the end of the ordering π such that any two incident edges in M are consecutive.

Hereafter, during the rest of the proof, by ordering σ , we mean the resulting ordering mentioned in the preceding discussion.

Part (II): For any σ -alternating partial 2-coloring of $E(G)$ with length $|E(G)| - \text{cut}_{r+1}(G) + 1$, at least one of G^R and G^B has a connected component with at least $n - r$ vertices.

Consider an arbitrary fixed σ -alternating partial 2-coloring of $E(G)$ with length $|E(G)| - \text{cut}_{r+1}(G) + 1$. Since

$$\text{cut}_{r+1}(G) = \min\{|E_G(S, V(G) \setminus S)| : S \subseteq V(G), |S| \geq r+1, |V(G) \setminus S| \geq r+1\},$$

we have

$$\text{cut}_{r+1}(G) \leq (r+1)(n-r-1) = o(n^2). \quad (4)$$

Also, since $||E(G^R)| - |E(G^B)|| \leq 1$ and the number of neutral edges is $\text{cut}_{r+1}(G) - 1$, we have

$$\min\{|E(G^R)|, |E(G^B)|\} \geq \frac{|E(G)| - \text{cut}_{r+1}(G)}{2} > \frac{5}{24}n^2 - \frac{1}{2}(r+1)(n-r-1). \quad (5)$$

In view of the definition of σ , for any vertex $u \in V(G)$, the number of red (resp. blue) edges incident with u in $G - E(M)$ is at most $\frac{\deg_G(u) - \deg_M(u) + 2}{2}$. Also, according to the ordering of the edges of M in σ , at most one edge in $E(M)$ incident with u (if there is such an incident edge) is red (resp. blue). Consequently, for each vertex $u \in V(G)$, we have

$$\max\{\deg_{G^R}(u), \deg_{G^B}(u)\} \leq \frac{\deg_G(u) + 3}{2}. \quad (6)$$

Note that we shall have established Part (II), if we prove the following claim.

Claim 1. G^R or G^B has a connected component with at least $n - r$ vertices.

Proof of Claim 1. On the contrary, suppose that every connected component of G^R and also of G^B has at most $n - r - 1$ vertices. Let Ω_R and Ω_B be the largest connected component of G^R and G^B , respectively. Note that we have $\max\{|V(\Omega_R)|, |V(\Omega_B)|\} \leq n - r - 1$. Also, in view of Inequality 5, the number of red edges and the number of blue edges in G are both $O(n^2)$, i.e., $|E(G^R)| = O(n^2)$ and $|E(G^B)| = O(n^2)$. Therefore, since $r = o(n)$, if N is large enough, then we should have $\min\{|V(\Omega_R)|, |V(\Omega_B)|\} \geq r + 1$. Let H_R be the bipartite subgraph of G with two parts U_R and V_R , where $\{U_R, V_R\} = \{V(\Omega_R), V(G) \setminus V(\Omega_R)\}$ and $|U_R| \leq |V_R|$. A vertex of U_R is adjacent to a vertex of V_R if they are adjacent in G . Similarly, the bipartite subgraph H_B of G has U_B and V_B as its part, where $\{U_B, V_B\} = \{V(\Omega_B), V(G) \setminus V(\Omega_B)\}$ and $|U_B| \leq |V_B|$. Set $\alpha = |U_R|$ and $\beta = |U_B|$. In view of the former discussion, note that $r + 1 \leq \alpha, \beta \leq \frac{n}{2}$. Consider the bipartite graph H_R . Since there is no red edge in $E(H_R) = E_G[U_R, V_R]$, each of red

edges incident with u is in $E(G[U_R])$ for each $u \in U_R$. Hence, for any vertex $u \in U_R$, at most $\alpha - 1$ red edges are adjacent to v . Therefore, in view of Inequality 6, we have

$$\begin{aligned}
 |\text{NEU}(G)| &\geq \sum_{u \in U_R} (\deg_G(u) - \deg_{G[U_R]}(u) - \deg_{G^B}(u)) \\
 &\geq \sum_{u \in U_R} (\deg_G(u) - (\alpha - 1) - \deg_{G^B}(u)) \\
 &\geq \sum_{u \in U_R} (\deg_G(u) - (\alpha - 1) - \frac{\deg_G(u) + 3}{2}) \\
 &\geq \sum_{u \in U_R} (\frac{\deg_G(u)}{2} - \alpha - \frac{1}{2}) \\
 &\geq \alpha (\frac{1}{2}\delta n - \alpha - \frac{1}{2}).
 \end{aligned}$$

Since there are $\text{cut}_{r+1}(G) - 1$ neutral edges in G , we have

$$\alpha \left(\frac{1}{2}\delta n - \alpha - \frac{1}{2} \right) \leq \text{cut}_{r+1}(G) - 1 \leq (n - r - 1)(r + 1) - 1.$$

One can check that if N is large enough, then the inequality $\frac{1}{2}\delta n\alpha - \alpha^2 - \frac{1}{2}\alpha \leq (n - r - 1)(r + 1) - 1$ implies that we have either $r + 1 \leq \alpha \leq \frac{3(r+1)}{\delta}$ or $\frac{\delta}{2}n - o(n) \leq \alpha \leq \frac{n}{2}$. Similarly, we have either $r + 1 \leq \beta \leq \frac{3(r+1)}{\delta}$ or $\frac{\delta}{2}n - o(n) \leq \beta \leq \frac{n}{2}$. These observations lead us to investigate the following two cases. In what follows, we will show that these cases do not hold.

Case I. $\frac{\delta}{2}n - o(n) \leq \alpha \leq \frac{n}{2}$ or $\frac{\delta}{2}n - o(n) \leq \beta \leq \frac{n}{2}$.

Because of similarity, we just discuss the case $\frac{\delta}{2}n - o(n) \leq \alpha \leq \frac{n}{2}$. For a contradiction, suppose that $\frac{\delta}{2}n - o(n) \leq \alpha \leq \frac{n}{2}$. Consider the bipartite graph H_R and note that

$$|E(H_R)| \geq \alpha(\delta n - \alpha + 1) \geq \alpha(\delta n - \alpha) \geq \left(\frac{2\delta - 1}{4}\right)n^2 - o(n^2),$$

provided that N is large enough. To see this, note that the function $f(\alpha) = \alpha(\delta n - \alpha)$ for $\alpha \in [\frac{\delta}{2}n - o(n), \frac{n}{2}]$, takes its minimum in $\frac{\delta}{2}n - o(n)$ or $\frac{n}{2}$. Comparing two values $f(\frac{\delta}{2}n - o(n))$ and $f(\frac{n}{2})$, we have the aforementioned inequality. Since, there is no red edge between two parts U_R and V_R , and moreover, there are at most $(n - r - 1)(r + 1) - 1 = o(n^2)$ neutral edges between them, one can conclude that there are at least $(\frac{2\delta - 1}{4})n^2 - o(n^2)$ blue edges in $E(H_R)$.

We call an edge of G *unusual*, if it is either a neutral edge of G or a blue edge in $E(G) \setminus E(H_R)$. We remind that two 3-blocks are called consecutive if they are traversed

consecutively in the Eulerian tour of $G - E(M)$ used for introducing the ordering σ . In other words, if two 3-blocks are consecutive, then there is no other edge between the edges of them in the ordering σ . Set q to be the number of 3-blocks. For any $i \in [n]$, let q_i be the number of 3-blocks which are traversed in the i -th stage. One can check that amongst the edges of any two consecutive 3-blocks, there exists at least one unusual edge. Therefore, there are at least $\frac{q_i-1}{2}$ unusual edges amongst the edges of these q_i 3-blocks. Hence, we have at least $\frac{q-n}{2}$ unusual edges in G . In view of Equality 3 and the aforementioned discussion, the number of unusual edges is at least

$$\frac{1}{2} \left(\frac{|E(G)|}{3n(n-2)} (4|E(G)| - n^2) - o(n^2) \right) - \frac{n}{2} = \frac{|E(G)|}{6n(n-2)} (4|E(G)| - n^2) - o(n^2).$$

Also, note that there are at most $(n-r-1)(r+1) - 1 = o(n^2)$ neutral edges in $E(G)$. Accordingly, the number of blue edges in G is

$$|E(G^B) \setminus E(H_R)| + |E(G^B) \cap E(H_R)| \geq \frac{|E(G)|}{6n^2} (4|E(G)| - n^2) + \frac{2\delta-1}{4} n^2 - o(n^2).$$

Since $\delta > \frac{5}{6}$, one can see that there is a $\mu = \mu(\delta) > 0$ for which

$$\frac{|E(G)|}{6n^2} (4|E(G)| - n^2) + \frac{2\delta-1}{4} n^2 > (1+\mu) \frac{|E(G)|}{2}.$$

To see this, set $e = |E(G)| \in \left(\frac{5n^2}{12}, \frac{n^2}{2} \right]$ and note that the function

$$g(e) = \frac{e}{6n^2} (4e - n^2) + \frac{2\delta-1}{4} n^2 - (1+\mu) \frac{e}{2}$$

is decreasing for $e \in \left(\frac{5n^2}{12}, \frac{n^2}{2} \right]$. Consequently, it takes its minimum in $e = \frac{n^2}{2}$ which is

$$g\left(\frac{n^2}{2}\right) = \left(2\delta - \frac{5}{3} - \mu\right) \frac{n^2}{4}.$$

Choosing μ sufficiently small, for instance $\mu = \delta - \frac{5}{6}$, we clearly have $g\left(\frac{n^2}{2}\right) > 0$. Therefore, since $\frac{\mu}{2}|E(G)| = O(n^2)$, for large enough N , we have

$$|E(G^B) \setminus E(H_R)| + |E(G^B) \cap E(H_R)| > \frac{|E(G)|}{2} + 1,$$

contradicting the fact that the number of blue edges in G is at most $\frac{|E(G)|}{2} + 1$. Similarly, if $\frac{\delta}{2}n - o(n) \leq \beta \leq \frac{n}{2}$, then we get a contradiction.

Case II). $r+1 \leq \alpha \leq \frac{3(r+1)}{\delta}$ or $r+1 \leq \beta \leq \frac{3(r+1)}{\delta}$.

Because of similarity, we just discuss the case $r+1 \leq \alpha \leq \frac{3(r+1)}{\delta}$. For a contradiction, suppose that $r+1 \leq \alpha \leq \frac{3(r+1)}{\delta}$. Note that for any vertex $x \in U_R \cap U_B$, all edges between x and the vertices in $V(G) \setminus (U_R \cup U_B)$ are neutral, i.e., each edge in $E_G(U_R \cap U_B, V(G) \setminus (U_R \cup U_B))$ is neutral. Clearly, since $r = o(n)$, if $|U_R \cap U_B| \geq \frac{2(r+1)}{\delta}$ and N is sufficiently large, then the number of neutral edges in G is at least

$$|U_R \cap U_B|(\delta n - |U_R \cup U_B|) \geq \frac{2(r+1)}{\delta}(\delta n - \frac{6(r+1)}{\delta}) > (r+1)(n-r-1)$$

which is impossible. Now, suppose $|U_R \cap U_B| \leq \frac{2(r+1)}{\delta}$.

Consider a vertex $v_i \in U_R \setminus U_B$ (if there is such a vertex). Since all 4-blocks are chosen from a monogamous packing, the number of 4-blocks in H_i having some vertex in $U_R \cup U_B \setminus \{v_i\}$ is at most $|U_R \cup U_B| - 1 \leq \frac{6(r+1)}{\delta} - 1$.

Also, it is clear that any 4-block in H_i which has no red edge incident with v_i contains at most one red edge. Note that if a 4-block in H_i has a red edge incident with v_i , then it contains some vertex in $U_R \cup U_B \setminus \{v_i\}$. Suppose that there are γ consecutive 4-blocks of H_i (with respect to the ordering σ) such that each of them has no vertex in $U_R \cup U_B \setminus \{v_i\}$. Clearly, these 4-blocks have no red edge incident with v_i . Consequently, they have at most γ red edges. Also, since the edges of these 4-blocks are consecutive in σ , they contain at most $\gamma + 1$ blue edges. This implies that there are at least $4\gamma - (2\gamma + 1) = 2\gamma - 1$ neutral edges amongst the edges of these 4-blocks. Since 4-blocks form a monogamous C_4 -packing of G , the number of 4-blocks in H_i containing some vertex of $(U_R \cup U_B) \setminus \{v_i\}$ is at most $|U_R \cup U_B| - 1$. This observation implies that

$$|\text{NEU}(H_i - (U_R \cup U_B \setminus \{v_i\}))| \geq 2l - 3|U_R \cup U_B| + 2. \quad (7)$$

Moreover, in view of the definition of σ , there are at least $\frac{\deg_G(v_i) - 3}{2}$ edge-disjoint (σ, v_i) -consecutive paths in G . Amongst these (σ, v_i) -consecutive paths, at most $l + 1$ (resp. $|U_R \cup U_B| - 1$) number of them have nonempty intersection with $E(H_i)$ (resp. $E(G[U_R \cup U_B])$). If we set $E_{v_i}^R = E(H_i) \cup E(G[U_R \cup U_B])$, then in view of Lemma 1, the number of neutral edges incident with v_i which are not in $E(H_i) \cup E(G[U_R \cup U_B])$, i.e., $|\text{NEU}(v_i, G - (E(H_i) \cup E(G[U_R \cup U_B])))|$, is at least

$$\frac{\deg_G(v_i) - 3}{2} - (l + 1) - (|U_R \cup U_B| - 1) = \frac{1}{2}\deg_G(v_i) - l - |U_R \cup U_B| - \frac{3}{2}. \quad (8)$$

Similarly, with the same argument, we can obtain the same assertions for any vertex $v_j \in U_B \setminus U_R$. Hence, in view of Inequality 7 and since $l = \lceil \frac{26r+26}{\delta} \rceil$, for any vertex $v_i \in (U_R \setminus U_B) \cup (U_B \setminus U_R) = U_R \cup U_B \setminus U_R \cap U_B$, we have

$$\begin{aligned}
& |\text{NEU}(v_i, G - (E(H_i) \cup E(G[U_R \cup U_B])))| + |\text{NEU}(H_i - (U_R \cup U_B \setminus \{v_i\}))| \\
& \geq \frac{1}{2} \deg_G(v_i) - l - |U_R \cup U_B| - \frac{3}{2} + 2l - 3|U_R \cup U_B| + 2 \\
& = \frac{\deg(v_i)}{2} + l - 4|U_R \cup U_B| + \frac{1}{2} \\
& > \frac{\deg(v_i)}{2} + \frac{2(r+1)}{\delta}.
\end{aligned}$$

If $u, u' \in U_R \cap U_B$ and $v_i, v_j \in (U_R \cup U_B) \setminus (U_R \cap U_B)$, such that $u \neq u'$ and $v_i \neq v_j$, then

$$\begin{aligned}
& \text{NEU}(u, G - (U_R \cup U_B \setminus \{u\})) \cap \text{NEU}(u', G - (U_R \cup U_B \setminus \{u'\})) = \emptyset, \\
& \text{NEU}(u, G - (U_R \cup U_B \setminus \{u\})) \cap \text{NEU}(v_i, G - (E(H_i) \cup E(G[U_R \cup U_B]))) = \emptyset, \\
& \text{NEU}(u, G - (U_R \cup U_B \setminus \{u\})) \cap \text{NEU}(H_i - (U_R \cup U_B \setminus \{v_i\})) = \emptyset, \\
& \text{NEU}(v_i, G - (E(H_i) \cup E(G[U_R \cup U_B]))) \cap \text{NEU}(v_j, G - (E(H_j) \cup E(G[U_R \cup U_B]))) = \emptyset, \\
& \text{NEU}(H_i - (U_R \cup U_B \setminus \{v_i\})) \cap \text{NEU}(v_j, G - (E(H_j) \cup E(G[U_R \cup U_B]))) = \emptyset,
\end{aligned}$$

and

$$\text{NEU}(H_i - (U_R \cup U_B \setminus \{v_i\})) \cap \text{NEU}(H_j - (U_R \cup U_B \setminus \{v_j\})) = \emptyset.$$

In what follows, by using the preceding equations, we shall present a lower bound for the number of neutral edges in G , leading us to a contradiction. Before this, we remind that all edges between vertices in $U_R \cap U_B$ and the vertices in $V(G) \setminus (U_R \cup U_B)$ are neutral.

$$\begin{aligned}
|\text{NEU}(G)| & \geq \sum_{u \in U_R \cap U_B} |\text{NEU}(u, G - (U_R \cup U_B \setminus \{u\}))| \\
& + \sum_{v_i \in (U_R \cup U_B) \setminus U_R \cap U_B} |\text{NEU}(v_i, G - (E(H_i) \cup E(G[U_R \cup U_B])))| \\
& + \sum_{v_i \in (U_R \cup U_B) \setminus U_R \cap U_B} |\text{NEU}(H_i - (U_R \cup U_B \setminus \{v_i\}))| \\
& \geq |E_G(U_R \cap U_B, V(G) \setminus (U_R \cup U_B))| + \\
& + \sum_{v \in (U_R \cup U_B) \setminus (U_R \cap U_B)} \left(\frac{\deg_G(v)}{2} + \frac{2(r+1)}{\delta} \right).
\end{aligned}$$

Define

$$\begin{aligned}
 J &= \sum_{v \in (U_R \cup U_B) \setminus (U_R \cap U_B)} \left(\frac{\deg_G(v)}{2} + \frac{2(r+1)}{\delta} \right) \\
 &= \frac{2(r+1)}{\delta} |(U_R \cup U_B) \setminus (U_R \cap U_B)| + \sum_{v \in (U_R \cup U_B) \setminus (U_R \cap U_B)} \frac{\deg_G(v)}{2}.
 \end{aligned}$$

Let $(U_R \cup U_B) \setminus (U_R \cap U_B) = \{u_1, u_2, \dots, u_d\}$, where $\deg_G(u_1) \leq \deg_G(u_2) \leq \dots \leq \deg_G(u_d)$. Note that $d \geq 2r + 2 - 2|U_R \cap U_B|$. Set

$$d' = \max\{0, r + 1 - |U_R \cap U_B|\}, \quad Y = \{u_1, u_2, \dots, u_{d'}\}, \quad \text{and} \quad Z = Y \cup (U_R \cap U_B),$$

where $Y = \emptyset$ if $d' = 0$. Note that $r + 1 \leq |Z| \leq \frac{6(r+1)}{\delta} \leq n - r - 1$. Consequently, we have

$$\begin{aligned}
 |\text{NEU}(G)| &\geq |E_G(U_R \cap U_B, V(G) \setminus (U_R \cup U_B))| + J \\
 &\geq |E_G(U_R \cap U_B, V(G) \setminus (U_R \cup U_B))| + \frac{2(r+1)}{\delta} |(U_R \cup U_B) \setminus (U_R \cap U_B)| \\
 &\quad + \sum_{u \in (U_R \cup U_B) \setminus (U_R \cap U_B)} \frac{\deg_G(u)}{2}.
 \end{aligned}$$

Clearly, since $|U_R \cap U_B| \leq \frac{2(r+1)}{\delta}$, we have

$$\begin{aligned}
 &|E_G(U_R \cap U_B, V(G) \setminus (U_R \cup U_B))| + \frac{2(r+1)}{\delta} |(U_R \cup U_B) \setminus (U_R \cap U_B)| \\
 &\geq |E_G(U_R \cap U_B, V(G) \setminus (U_R \cap U_B))|.
 \end{aligned}$$

Accordingly, since $2d' \leq d$, we have

$$\begin{aligned}
 |\text{NEU}(G)| &\geq |E_G(U_R \cap U_B, V(G) \setminus (U_R \cap U_B))| + \sum_{u \in (U_R \cup U_B) \setminus (U_R \cap U_B)} \frac{\deg_G(u)}{2} \\
 &\geq |E_G(U_R \cap U_B, V(G) \setminus (U_R \cap U_B))| + \sum_{u \in Y} \deg_G(u) \\
 &\geq |E_G(Z, V(G) \setminus Z)| \\
 &\geq \text{cut}_{r+1}(G),
 \end{aligned}$$

a contradiction. $\square$

Clearly the aforementioned claim implies the desired assertion of Part (II) which completes the proof of lemma. $\square$

2.2. Proof of Theorem 2

To prove Theorem 2, we use the same approach as in the proof of Theorem 1 with $r = 0$. This assumption makes the computation easier.

Proof of Theorem 2. Let G be a graph with $n \geq 4716$ vertices and $\delta(G) \geq \delta n$. To prove the assertion, we follow the proof of Theorem 1 with $r = 0$. Note that $l = \lceil \frac{26}{\delta} \rceil \leq 32$. In the definition of σ , we need the inequality $n \geq 36(4l + 3)$ which is valid for $n \geq 4716$. Also, for using Dirac's theorem, the inequality $\delta n - 8l - 6 \geq \frac{n}{2} + 2$ should be satisfied which is due to the assumption $n \geq 4716$. Continuing the proof of Theorem 1, we meet the following inequalities:

$$\frac{1}{2}\delta n\alpha - \alpha^2 - \frac{1}{2}\alpha \leq n - 2 \text{ and } \frac{1}{2}\delta n\beta - \beta^2 - \frac{1}{2}\beta \leq n - 2.$$

For $n \geq 27$, these inequalities lead us to the following cases:

- (I) Either $1 \leq \alpha \leq \frac{3}{\delta}$ or $\frac{\delta}{2}n - 7 \leq \alpha \leq \frac{n}{2}$.
- (II) Either $1 \leq \beta \leq \frac{3}{\delta}$ or $\frac{\delta}{2}n - 7 \leq \beta \leq \frac{n}{2}$.

Same as in the proof of Theorem 1, we distinguish two different cases, each of them leads us to a contradiction.

Case I). $\frac{\delta}{2}n - 7 \leq \alpha \leq \frac{n}{2}$ or $\frac{\delta}{2}n - 7 \leq \beta \leq \frac{n}{2}$.

We have at least $(\frac{2\delta-1}{4}n^2 - 49)$ edges in $E(H_R)$. Also, the number of 3-blocks is at least

$$\frac{e-y}{3n^2}(4e-4y-n^2)\frac{(n-3)(n-4)}{(n-1)(n-2)} \geq \frac{e-y}{3n^2}(4e-4y-n^2)(1-\frac{5}{n}),$$

where $e = |E(G)|$ and $y = |E(L)|$. Since $y \leq (4l+4)n \leq 132n$, we have at least

$$\begin{aligned} \frac{e-y}{3n^2}(4e-4y-n^2)(1-\frac{5}{n}) &\geq \frac{e-132n}{3n^2}(4e-528n-n^2)(1-\frac{5}{n}) \\ &\geq \frac{e}{3n^2}(4e-n^2) - \frac{797}{6}n \end{aligned}$$

3-blocks. Hence, we have at least $\frac{e}{6n^2}(4e-n^2) - \frac{797}{12}n - \frac{n}{2}$ unusual edges. Note that since there is no red edge in $E(H_R)$ and there is at most $(n-2)$ neutral edges in G , we have at least

$$(\frac{2\delta-1}{4}n^2 - 49 - (n-2) + \frac{e}{6n^2}(4e-n^2) - \frac{803}{12}n - (n-2)$$

blue edges in G . We claim that for $n \geq \frac{828}{6\delta-5}$,

$$(\frac{2\delta-1}{4}n^2 + \frac{e}{6n^2}(4e-n^2) - \frac{827}{12}n - 45 > \frac{e}{2} + 1 \quad (9)$$

which is a contradiction. To see this, note that the function

$$g(e) = \left(\frac{2\delta-1}{4}\right)n^2 + \frac{2}{3n^2}e^2 - \frac{2}{3}e - \frac{827}{12}n - 46$$

takes its minimum in $e = \frac{n^2}{2}$ which is

$$g\left(\frac{n^2}{2}\right) = \left(\frac{6\delta-5}{12}\right)n^2 - \frac{827}{12}n - 46 > 0$$

provided that $n \geq \frac{828}{6\delta-5}$. The rest of the proof runs as before. $\square$

3. Complexity of finding the t -th cut number

In Theorem 1, it is proved that the chromatic number of $\text{KG}(G, \mathcal{T}_{n-r})$ is equal to $\text{cut}_{r+1}(G)$ provided that G satisfies some conditions. This motivates us to consider the problem of finding the t -th cut number of a given graph G . Note that, in addition to being motivated by Theorem 1, this problem is interesting for its own sake. In this regard, we consider the following problem. Let $t = t(n)$ be a positive function such that $t(n) \leq \frac{n}{2}$ for each $n \in \mathbb{N}$. For a given graph G with n vertices, what is the value of

$$\text{cut}_t(G) = \min \{|E(S, V(G) \setminus S)| : S \subseteq V(G), \min\{|S|, |V(G)| - |S|\} \geq t\}$$

Hereafter, by *Min t -Cut problem*, we refer to this problem. We remind that for $t \equiv 1$, $\text{cut}_1(G) = \text{cut}(G)$ is called the cut number of G . Also, we prefer to use Min-Cut problem instead of Min 1-Cut problem.

The famous max-flow min-cut theorem says that the maximum network flow from the source s to the sink t in a network N equals to the minimum weighted cut-number of N , denoted $\text{cut}(N)$ which is the smallest total weight of the edges which if removed would disconnect the source from the sink (there is no directed path from s to t). Hao and Orlin 1992 [8] showed how to use the maximum flow algorithm by Goldberg and Tarjan 1988 [6] in order to find the minimum weighted cut-number of a network N in polynomial time of $|V(N)|$. There are other polynomial time methods for finding the minimum weighted cut-number in N , notably the Edmonds-Karp algorithm [4]. Edmonds-Karp algorithm ensures that we can find the maximum flow in any network N with n vertices and m edges in $O(nm^2)$ time. In what follows, with the similar idea as Goldberg and Tarjan used, we prove that the problem of finding $\text{cut}_t(G)$ can be solved in a polynomial time of $|V(G)|$, where t is a constant function. For a given graph G with $|V(G)| = n$, define $\vec{G}$ to be a directed graph with the same vertex set as G and for any edge $e = uv \in E(G)$, put two edges (u, v) and (v, u) in $\vec{G}$. Also, for each edge in $\vec{G}$, define its capacity to be 1. Now, let (A, B) be a pair of disjoint t -subsets of $V(G)$. Define $N_{(A, B)}$ to be a network with vertex set $V(G) \cup \{s, t\}$ and edge set

$$E(N_{(A,B)}) = E(\vec{G}) \cup \{(s, a) : a \in A\} \cup \{(b, t) : b \in B\}.$$

Also, set the capacity of each edge incident with s or t to be infinite. Note that the max-flow min-cut theorem and Edmonds-Karp algorithm imply that we can find the minimum cut of $N_{(A,B)}$ in $O(n^5)$ time. On the other hand, one can simply see that

$$\text{cut}_t(G) = \min_{(A,B)} \text{cut}(N_{(A,B)}),$$

where the minimum is taken over all pairs (A, B) of disjoint t -subsets of $V(G)$. Since t is a constant and the number of such pairs (A, B) is $\binom{n}{t} \binom{n-t}{t} = O(n^{2t})$, the minimum can be found in polynomial time. Hence, we can find $\text{cut}_t(G)$ in polynomial time of n . However, it is proved that the complexity of finding the t -th cut number when t is a part of the input is NP-hard. In this regard, the Min $\lfloor \frac{n}{2} \rfloor$ -Cut problem, known as the minimum bisection problem in the literature, is proved to be an NP-hard problem by Garey, Johnson, and Stockmeyer [5]. Bui and Jones [3] showed that allowing the two sets to be unequal does not make the problem easier while we have a lower bound on the size of smaller set. To be more specific, let b be a constant, where $0 < b \leq \frac{1}{2}$. In a graph G , a cut (S, T) is called b -balanced if $\min\{|S|, |T|\} \geq b|V(G)|$. The minimum number of edges in the b -balanced cuts is called the b -balanced cut number of G . Bui and Jones [3] proved that the problem of finding the b -balanced cut number is an NP-hard problem.

However, it would be desirable for us to know the complexity of Min t -Cut problem whenever the function $t(n)$ and the given graphs satisfy the conditions of Theorem 1. In the rest of this section, we focus on investigating this problem. By the next theorem, we shall prove that the Min t -Cut problem still remains NP-hard even if the given graphs are satisfying the conditions of Theorem 1 and the function $t(n)$ is too small with respect to n . Let us first introduce a famous NP-complete problem in graphs, known as Max-Cut problem which has a key role in the proof of the next theorem. Finding the maximum size of a cut in a graph G is known as *Max-Cut problem*. Also, the decision version of the Max-Cut problem is the following problem: Given a graph G and an integer $k \in \mathbb{N}$, is there a cut of size at least k in G ? It is proved to be an NP-complete problem by Garey, Johnson, and Stockmeyer [5]. Therefore, the above computational version has to be an NP-hard problem. One should notice that the proof of the following theorem is similar to a result in [5].

Theorem 3. *Let $k \geq 2$ be an arbitrary fixed positive integer and n be a positive integer. The Min $n^{\frac{1}{k}}$ -Cut problem is an NP-hard problem even for the family of graphs with n vertices satisfying the hypotheses of Theorem 1.*

Proof. To prove the theorem, we introduce a reduction from Max-Cut problem to Min $n^{\frac{1}{k}}$ -Cut problem. To this end, let G be a graph with m vertices which is an input for Max-Cut problem. As an input for Min $n^{\frac{1}{k}}$ -Cut problem, define $H = \vec{G} \vee K_q$, where $q = m^k - m$ and $V(K_q) = \{1, \dots, q\}$. Since H has $n = m^k$ vertices and $\delta(H) \geq n - n^{\frac{1}{k}}$, it satisfies the conditions of Theorem 1 provided that n is large enough.

Let (U_0, V_0) be a partition of $V(G)$ for which we have

$$|E_G(U_0, V_0)| = \max \{|E_G(U, V)| : (U, V) \text{ is a cut of } G\}.$$

Set $S = U_0 \cup \{1, \dots, n^{\frac{1}{k}} - |U_0|\}$ and $T = V(H) \setminus S$. Note that $|S| = n^{\frac{1}{k}}$ and $|T| = n - n^{\frac{1}{k}} \geq n^{\frac{1}{k}}$ for $n \geq 4$. Therefore, (S, T) is an $n^{\frac{1}{k}}$ -cut of H . Moreover, we clearly have

$$\text{cut}_{n^{\frac{1}{k}}}(H) \leq |E_H(S, T)| = n^{\frac{1}{k}}(n - n^{\frac{1}{k}}) - |E_G(U_0, V_0)|. \quad (10)$$

Conversely, let (S, T) be an $n^{\frac{1}{k}}$ -cut of H for which we have $|E_H(S, T)| = \text{cut}_{n^{\frac{1}{k}}}(H)$. Define $U = S \cap V(G)$ and $V = V(G) \setminus U$. Clearly, since $\min\{|S|, |T|\} \geq n^{\frac{1}{k}}$, we have

$$|E_G(U_0, V_0)| \geq |E_G[U, V]| = |S||T| - |E_H[S, T]| \geq n^{\frac{1}{k}}(n - n^{\frac{1}{k}}) - \text{cut}_{n^{\frac{1}{k}}}(H). \quad (11)$$

Therefore, in view of Equations (10) and (11), we have $|E(U_0, V_0)| = n^{\frac{1}{k}}(n - n^{\frac{1}{k}}) - \text{cut}_{n^{\frac{1}{k}}}(H)$. Consequently, solving Min $n^{\frac{1}{k}}$ -Cut problem, even for graphs satisfying the conditions of Theorem 1, results polynomially in solving Max-Cut problem which is an NP-hard problem, completing the proof. $\square$

In view of the previous theorem, for any fixed number $\epsilon \in (0, 1)$, Max-Cut problem and Min t -Cut problem with $t \geq n^\epsilon$ are both NP-hard while, as an interesting comparison, Min t -Cut problem with constant t is solvable in polynomial time (see the discussion before Theorem 3). As a result of Theorem 1 and the preceding observation, one should note that, for the family of graphs G satisfying the conditions of Theorem 1, the problem of computing the chromatic number of tree Kneser graphs $\text{KG}(G, \mathcal{T}_t)$ is polynomially solvable if t is fixed, while it is an NP-hard problem provided that $t = n^{\frac{1}{k}}$ for some fixed integer $k \geq 2$.

It is worth mentioning that by a similar approach as in the proof of Theorem 3, we can prove a theorem similar to Theorem 3 for the decision version of the Min t -Cut problem.

Remark. There is an unpublished version of the present paper available on arXive ([arXive:1407.8035v2](https://arxiv.org/abs/1407.8035v2)) containing a generalization of Theorem 1. This generalization can be somehow considered as a colored version of Theorem 1 which determines the chromatic number of several induced subgraphs of $\text{KG}(G, \mathcal{T}_{n-r})$. This generalization is too technical and not easy to follow. We did not put it in the journal version of the paper since we have preferred clarity to generality and also to keep the paper of broad enough interest.

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