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Damping of Torsional Vibrations in Thin-Walled Beams by Viscous Bimoments

David Hoffmeyer^{*a} and Jan Høgsberg^b

Department of Mechanical Engineering
Technical University of Denmark
Kongens Lyngby, Denmark

Abstract

Torsional vibrations in beams are damped by discrete, axial viscous dampers in configurations constituting bimoments. The bimoments act on warping displacements and partial restraining of warping suggests the inclusion of a flexibility parameter, calibrated from numerical results. Correcting for distortion is incorporated in a similar manner. The use of the governing differential equation with calibrated parameters as an alternative to a full complex analysis by a FE model is justified by an example. It is demonstrated that the differential equation determines the damped frequencies and damping ratios with great accuracy. Significant damping ratios of the lowest torsional mode are obtained.

Keywords: Torsional vibrations; damping; warping; thin-walled beams; bimoment; finite element method.

1 Introduction

In slender thin-walled beam structures, torsional vibrations may be induced by e.g. wind, earthquakes - and in the case of bridges - also by excitations from pedestrians or traffic. For beams with symmetric cross-sections torsional vibrations may be initiated when the load acts with an eccentricity to the cross-sectional shear centre. For cross-sections without double symmetry, the elastic and shear centres do not coincide, whereby the torsional and flexural vibrations inherently couple. In either case, the mitigation of vibrations may lead to a reduction in fatigue stresses [1], improvements in comfort levels and even removal of aerodynamic instabilities such as flutter for structures exposed to wind. During flutter an initial small or moderate motion develops over time to vibrations with large amplitudes, possibly damaging a structure to failure. Flutter is usually a combination of torsion and flexure, but also pure torsional flutter may occur [2]. The phenomenon is associated with apparent negative damping that typically cannot be sufficiently compensated by the inherent material damping.

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^{*a}Corresponding author: PhD student, davhoff@mek.dtu.dk, Nils Koppels Alle, Building 404, DK-2800, Kongens Lyngby, Denmark. ORCID: 0000-0003-2499-1964.

^bAssociate Professor, jhg@mek.dtu.dk, Nils Koppels Alle, Building 404, DK-2800, Kongens Lyngby, Denmark. ORCID: 0000-0002-1098-3245

In the design procedure for slender cable-stayed bridges substantial effort is put into avoiding flutter, as it may be a limiting factor for the overall span of suspension bridges [3]. A remedy for avoiding the onset of flutter is by simply increasing the governing flutter limit, e.g. by separating the main girder in two with connecting beams [4]. For an aerodynamic analysis of bridges with slotted box girders see [5], while an active control strategy for bridge cable vibrations has been proposed in [6], resulting in an increase in the critical flutter wind speed. If geometric changes with the purpose of reducing the flutter limit are deemed inappropriate, flutter may instead be compensated by adding a supplemental damping mechanism or device. For example, damping the critical torsional mode of a bridge deck may be of particular interest when it is economically cheaper than producing a closed box girder [3]. The phenomenon of flutter in bridge decks is treated in greater detail in [7, 8, 9, 10, 11]. Recently it was shown in [12] that the flutter critical rotational speed of a wind turbine rotor may be increased by the use of a viscous type eddy current damper working on the torsional motion of the turbine blades.

Torsional vibrations in thin-walled beams are associated with inhomogeneous torsion that generates out-of-plane, axial warping displacements. For beams with open cross-sections these displacements are often non-vanishing at the boundaries of the beam. For very long beams, inhomogeneous torsion has little influence and solutions based on pure homogeneous torsion are therefore adequate. However, for beams of short to moderate length, relative to the size of the cross-section, the contribution from inhomogeneous torsion may be significant [13]. The vibration characteristics of uncoupled torsional vibrations, including the effect of warping, were originally investigated by Gere [14], who solved for the natural frequencies and the associated mode shapes of beams with various natural boundary conditions. Subsequently, Gere and Lin [15] studied the coupling between flexural and torsional vibrations in beams with non-coinciding elastic and shear centres. A thorough review of torsion in beams has been given in [16].

The problem of torsional vibrations not only occurs in civil and wind engineering structures, but may also yield structure-borne sound and noise in power trains inside ships and in vehicles [17], while even in drillstrings used for drilling of oil or gas wells the torsional dynamic behaviour may cause severe problems [18]. Furthermore, in the aerospace and aviation industry, composite materials are vastly used and torsional vibrations in for example plate-like structures may be damped and/or mitigated by viscoelastic constraining layers [19] or piezoelectric patches [20, 21].

The warping effect was taken into account by Christiano and Salmela [22], expanding the original work of Gere [14] by introducing the concept of partially restrained warping at the boundary, specifically by adding supplemental stiffness locally to the out-of-plane axial warping displacements. A similar approach, with a localized viscous effect acting on the warping motion was used to introduce supplemental damping in [23], where the viscous boundary condition was applied directly to the pure bimoments at the beam supports. In [23] the eigenvalue problem associated with the free vibrations of the damped beam was solved with respect to the complex-valued natural frequency, whereby the corresponding damping ratio could be determined subsequently. It was demonstrated that significant

damping ratios could be realized by applying viscous bimoments at the beam supports. Recently, Hoffmeyer and Høgsberg [24] showed by a finite element comparison that the location of discrete axial viscous dampers on the cross-section in fact has a great influence on the attainable damping ratios when the dampers are balanced properly [25]. For general details on viscous dampers see e.g. [25, 26, 27, 28]. Even though the finite element method may be the most popular numerical method, Sapountzakis [29] applied a boundary element method for solving the torsion problem for beams with variable cross-section.

In the present paper the efficiency of torsional damping by viscous warping control is investigated by detailed numerical analysis, where a supplemental flexibility parameter is introduced in the damper model to take into account the inability of local forces on a beam cross-section to fully restrain axial warping. Thus, the flexibility parameter describes the remaining flexibility of the beam due to partly restrained warping at a boundary when the applied viscous dampers locks at infinite damping. The locking changes the infinitely damped natural frequency of the beam, reflecting the presence of the axial dampers and their particular spatial configuration. The inclusion of such a flexibility parameter corresponds to the effect of flexible boundary conditions as described in e.g. [30, 31, 32, 33, 34]. The supplemental flexibility parameter is calibrated by dedicated finite element results, and thus an additional correction parameter may even be included in the spatial solution of the differential equation to account for in-plane distortion of the cross-section. The resulting analytical model for torsional vibrations of thin-walled beams is therefore accurate and includes important supplemental effects. The linear finite element model used for both calibration and comparison is based on isoparametric elements with cubic-linear interpolation [35], especially suitable for modelling of thin-walled beams. Configurations with pure bimoments are easily represented by symmetrically located axial dampers, representing a pure viscous damping term in the governing equation due to direct proportionality with conjugated velocity. The damped eigenvalue problem is obtained by setting up the numerical equations of motion in state-space form, whereby the complex natural frequencies and the corresponding damping ratios can be obtained directly from the eigenvalues. It is demonstrated by an example how different bimoment configurations influence the damping properties of the first torsional mode and how an optimal location of the dampers on the cross-section may be estimated. The example furthermore shows that when designed properly, viscous bimoments are a very effective means for damping torsional vibrations in thin-walled beams.

2 Uncoupled torsional vibrations

The design and calibration of beam structures is conveniently based on an analytical approach and thus the theoretical solutions are initially derived. The characteristics of torsional vibrations in thin-walled beams may be analysed by considering the free response of a beam element. The beam has the length ℓ , axial coordinate z and in-plane coordinates $\{x_1, x_2\}$. The angle of twist $\theta(z, t)$ and coordinate system are seen in Fig. 1a. In the remainder of the paper, the independent spatial

Figure 1: (a) Beam with length ℓ , transverse coordinate axes $\{x_1, x_2\}$ and angle of twist θ . (b) Modified I-profile cross-section with sector-coordinate ψ and indicated shear center.

coordinates are not depicted to maintain a compact notation.

2.1 Governing equation

The governing differential equation may be established by the elastic and kinetic energy. In order to take the warping effect into account the elastic strain energy receives contributions from both homogeneous torsion (St. Venant) and inhomogeneous torsion (Vlasov). It then takes the form,

$$U(t) = \frac{1}{2} \int_{\ell} \theta'(t) G K \theta'(t) dz + \frac{1}{2} \int_{\ell} \theta''(t) E I_{\psi} \theta''(t) dz \quad (1)$$

where G is the shear modulus, K is the torsion stiffness parameter, E is Young's modulus, I_{ψ} is the warping moment of inertia with ψ indicating the warping function, or sector-coordinate, of the cross-section and $(\cdot)' = \partial(\cdot)/\partial z$ indicates differentiation with respect to z . The part of the elastic energy describing inhomogeneous torsion relates to the out-of-plane axial displacements described by the sector-coordinate. For a modified I-profile with added vertical flanges, the sector-coordinate is shown in Fig. 1b. The kinetic energy contains the torsional inertia,

$$T(t) = \frac{1}{2} \int_{\ell} \dot{\theta}(t) \rho J \dot{\theta}(t) dz \quad (2)$$

where ρ is the mass density, J is the polar moment of inertia and $(\dot{\cdot}) = \partial(\cdot)/\partial t$ indicates differentiation with respect to t . Energy is dissipated through idealized discrete viscous dampers located at an end cross-section. Each damper j is associated with a particular viscous damping coefficient c_j , while several dampers may act on the same cross-section. Hereby the dissipated energy D may be expressed as

$$D(t) = \sum_j \dot{\theta}_j(t) c_j \dot{\theta}_j(t) \quad (3)$$

with the summation taking into account the individual contributions from all dampers acting on the cross-section. In case of damped vibrations the change in mechanical energy is balanced by the dissipated energy: $\dot{U} + \dot{T} = -D$. However, as the dissipated energy enters the equilibrium equation

through the boundary conditions, the governing differential equation for the beam remains undamped as

$$EI_\psi \theta''''(t) - GK\theta''(t) + \rho J \ddot{\theta}(t) = 0 \quad (4)$$

A time harmonic solution of the form $\theta(t) = \varphi e^{i\omega t}$ may in the following be assumed with ω as the natural frequency and φ representing the spatial solution. Following the classic approach in [14] the spatial solution may be written as

$$\varphi = D_1 \cosh(\alpha z) + D_2 \sinh(\alpha z) + D_3 \cos(\beta z) + D_4 \sin(\beta z) \quad (5)$$

where D_1 to D_4 are integration constants. The two apparent wave numbers are written as in [23],

$$\alpha^2 = \sqrt{\frac{1}{4}k^4 + \lambda^4} + \frac{1}{2}k^2, \quad \beta^2 = \sqrt{\frac{1}{4}k^4 + \lambda^4} - \frac{1}{2}k^2 \quad (6)$$

defining the warping length scale k , wave speed v and scaled wave number λ , respectively, as

$$k^2 = \frac{GK}{EI_\psi}, \quad v^2 = \frac{GK}{\rho J}, \quad \lambda^4 = \left(\frac{k\omega}{v}\right)^2 \quad (7)$$

Depending on the boundary conditions the integration constants in (5) are determined, yielding the transcendental equation for the specific beam problem. This equation must be solved iteratively for a combined set of α and β together with the relations in (6) and (7). See [23] for transcendental equations for various beam configurations.

2.2 Bimoment from axial forces

With the differential equation defined in (4), damping is now introduced as a boundary condition to be used together with the spatial solution in (5). The damping effect is applied by placing concentrated viscous forces on the end cross-section acting in the axial direction of the beam, as illustrated in Fig. 2a. A number of viscous forces may be placed symmetrically or even asymmetrically, whereby a given set of axial damping forces may result in both a normal force and bending moments alongside the desired bimoment, see e.g. [36]. However, if the force configuration yields vanishing normal force and bending moments, the produced bimoment will in the following be referred to as a pure bimoment. Besides the resulting normal force and bending moments, a general geometrically asymmetric configuration of a set of axial dampers may furthermore contribute with additional effects like distortion or higher order warping modes. These effects are not captured by the differential equation but will inherently be a part of a three-dimensional solid (finite element) model. It is in the present work not of interest to set-up configurations that introduce a normal force or a bending moment as this complicates the comparison with the solutions derived from the uncoupled differential equation (4). Therefore, only symmetric configurations without a resulting normal force and bending moment will be considered in the present analysis.

The principle of virtual work may be used to quantify whether a certain damper configuration also introduces a normal force or bending moment. Let the j 'th damper force be $F_{d,j}$ and multiply

Figure 2: (a) Modified I-profile with axial, viscous forces in a pure bimoment configuration and warping functions (b) for $c_j = 0$ and (c) for $c_j = \infty$.

with a virtual displacement δu_j to obtain the virtual work $F_{d,j}\delta u_j$. The total virtual work is therefore obtained as the sum of the individual contributions from all dampers acting on the cross-section,

$$\delta W_d(t) = \sum_j F_{d,j}(t)\delta u_j(t) \quad (8)$$

where subscript d indicates the cross-section with the damper(s) applied at either end of the beam. The contribution to a normal force N_d is derived from (8) by assuming that $\delta u_j = \delta u_c$, where δu_c is the axial virtual displacement at the elastic center $C = \{c_1, c_2\}$ of the cross-section. Thereby, the relation $\delta W_d = N_d\delta u_c$ defines the normal force as

$$N_d(t) = \sum_j F_{d,j}(t) \quad (9)$$

Similarly, a contribution to the bending moment with respect to a principle cross-sectional axis can be obtained by introducing the virtual damper displacement $\delta u_j = (x_1 - c_1)\delta\theta_1 + (x_2 - c_2)\delta\theta_2$ expressed in terms of the damper coordinates $\{x_1, x_2\}$ and the cross-section inclinations θ_1 and θ_2 in the two corresponding in-plane directions, see Fig. 1. The virtual work relation $\delta W_d = M_{d,1}\delta\theta_1 + M_{d,2}\delta\theta_2$ then identifies the bending moments as

$$M_{d,\alpha}(t) = \sum_j F_{d,j}(t)(x_{\alpha,j} - c_\alpha) \quad , \quad \alpha = 1, 2 \quad (10)$$

For the configuration in Fig. 2a the resulting normal force and bending moments vanish as the forces and moments add up to zero according to the summations in (9) and (10). Thus, for a symmetric cross-section a corresponding symmetric configuration of passive or collocated active dampers produce a pure bimoment without coupling to section forces associated with extension and bending.

In the present work only pure bimoments will be generated for pure torsional motion. The virtual work produced by a bimoment B_d is given as

$$\delta W_d(t) = -B_d(t)\delta\theta'(t) \quad (11)$$

The axial out-of-plane displacement is expressed in terms of the sector coordinate ψ as

$$\delta u_j(t) = -\psi_j\delta\theta'(t) \quad (12)$$

where ψ_j is the intensity of the sector coordinate at the location of the j 'th damper. Substitution of (8) into (11) then defines the bimoment as

$$B_d(t) = \sum_j \psi_j F_{d,j}(t) \quad (13)$$

From (13) it is evident that even a single applied axial force will generate a bimoment, while only force configurations satisfying the conditions in (9) and (10) will result in a pure torsional moment with vanishing normal force and bending moments.

2.3 Boundary condition

The set of axial viscous forces that produce a pure bimoment works on the axial warping displacements generated by the twisting motion of the beam. This viscous bimoment constitutes the boundary condition and the axial viscous force $F_{d,j}$ at the boundary is therefore given as

$$F_{d,j}(t) = \pm \alpha_j c_0 \dot{u}_{d,j}(t) \quad (14)$$

where $\dot{u}_{d,j}$ is the axial velocity at the location of the damper, defined positive in the axial z -direction as shown in Fig. 1a. Furthermore c_0 is the viscous coefficient and α_j is a balancing factor in the case of multiple dampers not placed symmetrically. In that case all viscous damping coefficients are balanced relative to the reference coefficient c_0 . The axial force is defined positive in tension, and the '+' in (14) therefore refers to the left end of the beam at $z = 0$, while the '-' indicates the right end of the beam at $z = \ell$. According to (12) the axial velocity is represented by the sector-coordinate ψ as

$$\dot{u}_{d,j}(t) = -\psi_j \dot{\theta}'_d(t) \quad (15)$$

The rate of external work produced by the viscous forces are found by multiplying the viscous force $F_{d,j}$ with the energy conjugate velocity $\dot{u}_{d,j}$ followed by summation over the applied viscous forces. This work is equated with the work produced by a bimoment: $D = -\dot{\theta}'_d B_d$. This gives the resulting bimoment produced by the axial viscous forces as

$$B_d(t) = \mp \dot{\theta}'_d(t) \sum_j \alpha_j c_0 \psi_j^2 \quad (16)$$

This bimoment must be counteracted by the section bimoment when EI_ψ is the warping stiffness as

$$B_d(t) = -EI_\psi \theta''_d(t) \quad (17)$$

By assuming the frequency solution $\theta(t) = \varphi e^{i\omega t}$ and inserting (16) into (17) the boundary condition appears as

$$\varphi'' \mp i\omega\eta\varphi' = 0 \quad (18)$$

where

$$\eta = \sum_j \alpha_j c_0 \frac{\psi_j^2}{EI_\psi} \quad (19)$$

is an effective damping parameter that collects the damping forces. The boundary condition (18) may be applied directly to determine the integration constants in (5), thereby establishing a transcendental equation to be solved for the wave number or natural frequency. In the case of infinite damping $\eta \rightarrow \infty$ and for the damper configuration in Fig. 2a the warping function in Fig. 2b, associated with no damping, will approach the one in Fig. 2c.

3 Finite element model

The differential equation (4) governs the torsional vibrations of the continuous beam model with damping applied as a boundary condition. It is effectively solved and may conveniently be used for calibration and optimization of the damper design. However, as the dampers act locally on the cross-section at discrete positions, some effects may occur which are not captured by the differential equation derived on the basis of simplified kinematic assumptions with a resulting bimoment (16) acting proportionally to the gradient of the angle of twist. These local effects could include in-plane distortion of the cross-section, local bending of flanges etc. When modelling a complex structure it is of interest to be able to do simple, yet fairly accurate preliminary analyses, rather than performing the relevant investigations on a full three-dimensional finite element model - which may be computationally very expensive or even intractable. Therefore a finite element model is set up in order to investigate the effect of applying local dampers, and to compare with the results from solving the differential equation.

The finite element model is based on a discretization of the three-dimensional beam structure into finite elements, as indicated in Fig. 2b, where the geometry is defined by a number of nodes. At these nodes concentrated damping forces F_d may be placed, introducing damping at discrete locations on the cross-section. The present model considers viscous dampers, as defined in (14), but in a finite element formulation general passive or active dampers may as well be applied. In the structural model for the beam any inherent structural damping is neglected, whereby the model only consists of the stiffness matrix $\mathbf{K}$, the mass matrix $\mathbf{M}$ and n degrees-of-freedom collected in the column vector $\mathbf{q}$. For the structure without dampers, the equations of motion can therefore be written in the general form as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0} \quad (20)$$

Free vibration solutions are obtained by assuming the harmonic representation $\mathbf{q}(t) = \tilde{\mathbf{q}}e^{i\omega t}$, where the tilde indicates harmonic amplitudes and ω is the free vibration angular frequency. Substitution of the harmonic representation into (20) yields the associated eigenvalue problem

$$(\mathbf{K} - \omega_{0,n}^2 \mathbf{M})\tilde{\mathbf{q}}_n = \mathbf{0} \quad (21)$$

which determines the undamped natural frequency $\omega_{0,n}$ and the corresponding mode shape $\tilde{\mathbf{q}}_n$ for the n 'th vibration form of the structure.

Concentrated and discrete dampers are now applied in the numerical model, as indicated by the

arrows with F_d in Fig. 2a. These forces are added to the system by a connectivity vector $\mathbf{w}_j$ for the j 'th damper, with all zero entries except for a unit value at the single degree-of-freedom where the specific damper is located. Thus, the connectivity vector governs the location, direction and scaling of the damper force $F_{d,j}$, as discussed in greater detail for viscous dampers in [25]. For the structure with dampers the equation of motion (20) is extended to

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = - \sum_j \mathbf{w}_j F_{d,j}(t) \quad (22)$$

In (14) the local damper velocity $\dot{u}_{d,j} = \mathbf{w}_j^T \dot{\mathbf{q}}$ for the j 'th damper is as well determined by the connectivity vector. Hereby the dissipative term on the right hand side of (22) can be expressed as

$$\sum_j \mathbf{w}_j F_{d,j}(t) = \mathbf{C}_d \dot{\mathbf{q}}(t) \quad (23)$$

introducing the damping matrix

$$\mathbf{C}_d = \sum_j \alpha_j c_0 \mathbf{w}_j \mathbf{w}_j^T \quad (24)$$

with c_0 being the viscous reference value and α_j the balancing factor for the j 'th damper - thus $c_j = \alpha_j c_0$ and c_0 may be chosen arbitrarily among all dampers. Substitution of (23) into the right hand side of (22) then yields the equations of motion with added local dampers as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}_d \dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0} \quad (25)$$

This equation constitutes a quadratic eigenvalue problem for the solution of the natural frequency, conveniently expressed in state-space form as

$$\frac{d}{dt} \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}_d \end{bmatrix} \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} \quad (26)$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. These coupled equations may be written in compact form by introducing the state vector as $\mathbf{z} = [\mathbf{q}, \dot{\mathbf{q}}]^T$ and the state matrix in (26) as $\mathbf{A}$. When assuming frequency dependent harmonic solutions, the eigenvalue problem for the damped structure can be written in the standard form

$$(\mathbf{A} - i\omega_n \mathbf{I}) \tilde{\mathbf{z}}_n = \mathbf{0} \quad (27)$$

with $\tilde{\mathbf{z}}$ being the eigenvector and $i\omega_n$ the eigenvalue. For finite values of the viscous coefficients c_j , the solution to (27) will be complex-valued, with the imaginary part of the natural frequency ω_n representing the damping associated with the n 'th vibration form. For vanishing viscous damping, ω_n retains $\omega_{0,n}$ determined by (21), while for all $c_j \rightarrow \infty$ the damped natural frequency ω_n approaches $\omega_{n,\infty}$, representing the real-valued natural frequency of mode n associated with full restraintment of the degrees-of-freedom connected to a damper.

A disadvantage of the state-space form is that the state-matrix is $2n \times 2n$ whereby the size of the system is doubled. Thus solving the system for large structures imposes a substantial numerical effort.

Figure 3: (a) Bi-cubic-linear and (b) cubic-bi-linear isoparametric Lagrange-elements for thin-walled flanges.

For instance, a three-dimensional finite element model may easily have several thousand degrees-of-freedom, whereby this method may be computationally heavy or even infeasible. In a design scenario, it may therefore be necessary to establish a faster - yet approximate - tool, such as solving the transcendental equation associated with the differential equation in Section 2.

With the damping matrix $\mathbf{C}_d$ added to the equations of motion in (25) the damped frequency may be expressed through the damping ratio ζ as

$$\omega_n = |\omega_n| \left(\sqrt{1 - \zeta_n^2} + i\zeta_n \right) \quad (28)$$

The damping ratio can therefore be obtained directly from (28) as the relative imaginary part of the damped frequency,

$$\zeta_n = \frac{\text{Im}[\omega_n]}{|\omega_n|} \quad (29)$$

The mode shapes obtained from (27) are in general also complex, with the real part representing the physical vibration form and the imaginary part representing the spatial phase shift due to the presence of the energy dissipating dampers. The remaining part of the paper only considers the first torsional mode, whereby the subscript $n = 1$ is omitted.

An effective and systematic way of modelling thin-walled beams is by use of the isoparametric Lagrange-elements shown in Fig. 3. They consist of a bi-cubic-linear element with eight internal nodes for flange parts, and a cubic-bi-linear element for corners and junctions. They have four nodes in the flange-wise direction and two across the thickness. As long as the flanges are thin or of moderate thickness, the linear interpolation across the thickness is assumed sufficient. In a static analysis, a single element across a web or flange part yields a cubic interpolation, capturing the sufficient order of the shear stress distribution, see e.g. [35]. In the present context, the advantage of using such solid elements rather than shell elements is that they represent the warping displacements more accurately as the warping across the flange thickness is not necessarily constant. In the top row of Fig. 6 it may be seen how a section of the modified I-profile may be discretized with these elements.

Figs. 4a and b show the convergence of the first undamped and infinitely damped torsional frequencies ω_0 and ω_∞ relative to the corresponding analytical frequencies ω^{ana} from Section 2.1 as a function of the number of elements in the lengthwise direction. The beam has 'simple' supports that restrain rotation but allow warping, with a geometry as described in Section 5. The black

Figure 4: Convergence of ω_0 (a) and ω_∞ (b) as a function of the number of lengthwise elements and local discretization. Black line is one in-plane element per flange part and blue line is two.

line indicates an in-plane discretization of one element per flange part as in Fig. 6, while the blue line represents two elements. As seen the frequency converges with relatively few elements, and the difference between using one or two elements per flange part is minimal. In the top row of Fig. 6 it may be seen how local damping forces are applied as concentrated forces acting at specific nodes. When the viscous coefficient for these dampers is infinitely high the dampers lock and act as axial supports. These discrete supports have a very local nature and a significant influence on the convergence of the infinitely damped frequency ω_∞ . The element topology and discretization of the structural domain in the lengthwise direction then becomes important. If the beam was to be discretized with equally sized elements in the lengthwise direction, a very high amount of elements are needed for sufficient convergence. Therefore, it is proposed to perform a finer discretization close to the damped end of the beam. Fig. 5 shows such a local discretization, for dampers acting at the right end cross-section with one fourth of the beam with length ℓ being discretized with the same amount of elements as the remaining part of the beam. The three discretizations (a)-(c) in Fig. 5 are depicted with (o) in Fig. 4b. Within the densely discretized zone, the element lengths decrease exponentially towards the beam end. Fig. 4b shows the convergence of the infinitely damped frequency ω_∞ as a function of the number of lengthwise elements. The two groups of curves represent in-plane discretization with one (black) or two (blue) elements per flange part. Each group contains four curves, obtained for different relative lengths of the zone with finer discretization: $\ell/10$ (—), $\ell/15$ (— ·), $\ell/20$ (— · ·) and $\ell/25$ (· · ·). It is

Figure 5: Discretization with lengthwise elements. One fourth of the beam is discretized with an equal amount of elements as the remaining part, but with lengths of decaying exponential type.

seen that the convergence is much slower. For the modified I-profile sufficient convergence is obtained with one in-plane element per flange part, 30 lengthwise elements and $\ell/25$ with finer discretization.

4 Damper design

The placing and tuning of external dampers is often a crucial process during the design. If the damper is placed in a position with no motion at all or if the damping parameter is chosen so that it provides too little resistance, it will have no effect. On the contrary if it provides too large resistance it will act as a rigid link. In a tuning procedure it is obviously of interest to reach maximum modal damping, which is reached by maximizing ω_∞ when $c_j \rightarrow \infty$ for the particular damper placement.

4.1 Balancing of dampers

In the case with more than one damper mounted on the structure, the dampers should be balanced such that they work optimally together. For symmetric structures and symmetric locations they should all have equal gain, whereas in the case of asymmetry they should not. An approximate yet accurate procedure is described in [25], balancing a set of viscous dampers against an arbitrarily chosen reference damper. The method is based on a reduction technique using the undamped mode n associated with the eigenvalue problem in (21). In the other limiting case where $c_j \rightarrow \infty$ for all dampers they lock and act as rigid links providing constraint forces at the damper locations which enters the eigenvalue problem. When using the papers two-component approximation technique the constraint force $\rho_{n,j}$ for the j 'th rigid link in vibration mode n may be determined as

$$\rho_{n,j} = \mathbf{w}_j^T (\omega_{\infty,n}^2 \mathbf{M} - \mathbf{K}) \tilde{\mathbf{q}}_{\infty,n} \quad (30)$$

The idea is then to minimize the residual forces leading to the following rule for distributing the relative gains of the viscous dampers acting together,

$$\frac{c_j}{c_1} = \frac{\rho_{n,j}}{\rho_{n,1}} \frac{\mathbf{w}_1^T \tilde{\mathbf{q}}_{0,n}}{\mathbf{w}_j^T \tilde{\mathbf{q}}_{0,n}} \quad (31)$$

When all dampers are balanced relative to damper $j = 1$, then $c_0 = c_1$ in (14) and (24), whereby balancing factor $\alpha_j = c_j/c_1$ is given directly as the expression in (31).

4.2 Restraining warping

An important measure for damping efficiency is the ability of the dampers to fully restrain warping. Restraining warping of the entire cross-section at one or possibly both ends of a beam by the chosen set of dampers yields the highest possibly value of ω_∞ . This damper configuration will therefore provide the largest attainable damping ratio for a particular beam vibrating in pure torsion. For the analytical beam solution, the fully locked limit is represented by a support condition that fully restrains the rate of twist: $\theta'(t) = 0$. In reality and in the numerical finite element model, the limit $c_j \rightarrow \infty$ implies that only the degrees-of-freedom connected to a damper are fully restrained by the

Figure 6: Section of the modified I-profile with three different configurations of dampers, and the associated warping displacements (scaled for visual reason).

locking of the dampers, thus allowing the cross-section to warp between the damper locations with a non-vanishing warping function different from the kinematic representation used in the analytical beam solution. In the numerical finite element model an optimal distribution of the local dampers is therefore associated with minimizing the residual warping when $c_j \rightarrow \infty$ and thereby maximizing the locked natural frequency ω_∞ .

Figure 6 illustrates the warping functions associated with locked dampers for different damper configurations acting on the modified I-profile. The top row of the figure shows a quarter of the double symmetric cross-section with three different configurations of axial damping forces, that by construction are placed symmetrically to avoid a resulting normal force or bending moments. The forces therefore produce a pure bimoment in all three configurations. The second row of Fig. 6 shows plots of the corresponding out-of-plane warping displacements (artificially amplified) with restrained axial displacement at the damper locations. The axial displacement is averaged across the thickness of the flanges to illustrate representative center line distributions of the axial warping function. It is initially verified that the axial displacements are in fact zero at the location of the dampers. Two or four local forces are applied in the finite element model to avoid unnecessary distortion across the wall thickness. It is seen that finite out-of-plane warping still is present, although the dampers are fully locked. Thus, a certain degree of flexibility in the locked limit $c_j \rightarrow \infty$ remains in the finite element model, while for the analytical boundary condition in (18) the rate of twist $\varphi'_d = 0$ by a supporting bimoment governed by (17).

Assume a continuous beam model with 'simple-simple' support conditions for the torsional problem and dampers acting on the warping displacement at the right free end. The cross-section of the beam is shown in Fig. 6 and the length of the slender beam is equal to thirty times the cross-section height, while the remaining cross-section properties are given in Section 5. Table 1 provides the natural

Table 1: Relative frequency increments due to damper locking.

	(a)	(b)	(c)
ω_∞/ω_0 [-]	1.318	1.368	1.225

frequency ω_∞ for the first torsional mode obtained by the numerical finite element model. The results are presented relative to the corresponding natural frequency ω_0 without dampers. The warping function without dampers is shown in Fig. 2b, which shows that the largest warping occurs at the free end of the small vertical flange. Thus, the damper placement depicted in Fig. 6a may be expected to provide the best authority and thereby the largest relative frequency increase. It is seen in the table that for this configuration (a) the frequency ratio is $\omega_\infty/\omega_0 = 1.318$, providing almost a 32% increase due to locking of the dampers. However, it is seen from the warping distribution in Fig. 6a that the out-of-plane displacement is fairly large at the corner and non-vanishing along the flanges. When instead placing the dampers as in Fig. 6b the displacement is almost fully restrained along the horizontal flanges, while only non-vanishing along the small vertical flanges. Apparently, this configuration is better at restraining the warping displacement and Table 1 also confirms a small increase in the frequency ratio $\omega_\infty/\omega_0 = 1.368$ for case (b). The optimal placement is therefore not necessarily dictated by the largest value of the sector coordinate ψ , but instead by the ability to restrain warping when the dampers are fully locked. In Fig. 6c the dampers are placed closer to the junction with the web, whereby the warping function appears to be qualitatively similar to that in Fig. 1b, although with visible displacements at the horizontal flange. Therefore, the frequency increase for this configuration (c) is less than for the previous case (b). Another location of dampers may however be even better than case (b), indicating that finding the optimal placement is a non-trivial task. The efficiency of the individual damper configurations are analyzed in much greater detail Section 5.

Figure 7: (a) In-plane distortion of the modified I-profile when vibrating in pure torsion and (b) frequency locus in the complex plane and consequence of parameters κ and k_c .

4.3 Calibration of parameters

When modelling thin-walled beams by the use of beam theory, it is commonly assumed that the beam is sufficiently slender for the beam theory to apply and be represented with sufficient accuracy by the associated cross-sectional parameters. However, for short or very flexible beams, distortional effects, as illustrated by the in-plane deformations in Fig. 7a, may have great influence on the vibration characteristics [37, 38]. The influence of distortional effects on the torsional vibration characteristics can be corrected in the differential equation by adding relevant terms. Unfortunately, this severely complicates the analytical work and increase the complexity of the transcendental equations that will then be much more difficult to solve. The warping length scale k in (7) solely determines the undamped natural frequency ω_0 . Thus, a way to account for the possible effects from distortion is to calibrate k by comparison with either experimental results or accurate finite element solutions.

The attainable damping depends very much on the half-circular trajectory of the natural frequency in the complex plane, as indicated in Fig. 7b. Thus, in order for the differential equation to represent the finite element solutions as accurately as possible, the complex frequency locus should initiate and terminate at the correct frequencies ω_0 and ω_∞ , respectively. As demonstrated in the previous section, the placement of the dampers has a great impact on the locked natural frequency ω_∞ . Thus, the warping length scale is conveniently calibrated to capture the exact natural frequency ω_0 , containing all distortional effects from the finite element model. The natural frequency ω_0 for the analytical beam model is obtained by solving the continuous eigenvalue problem based directly on the differential equation in Section 2.1. A modified warping length scale k_c is then determined by for example a simple numerical search until the exact natural frequency ω_0 from the finite element model is retained.

In the sketch in Fig. 7b the solid curve represents the half-circular root locus associated with the analytical solution to the continuous eigenvalue problem when the original wave length scale $k = \sqrt{GK/EI_\psi}$ is used. The dashed locus instead indicates the exact complex root trajectory obtained by solving the full eigenvalue problem (27) in state-space form for the finite element structure. The analytical beam problem slightly overestimates ω_0 , because distortional effects constitute non-represented flexibility, whereby the corrected wave length parameter k_c must be larger than the original k determined by the basic cross-section parameters. This effect is analyzed in more detail in the subsequent example, which shows that this correction due to distortional effects is mainly required for relatively short beams.

In the case of infinite damping the out-of-plane warping is completely restrained at the particular points on the cross-section where the dampers are placed. Thus, between these points the cross-section will still be able to warp, as illustrated in Fig. 6. This excess deformation is associated with additional flexibility relative to the full restraintment of the cross-section when solving the differential equation. This additional flexibility is represented by the artificial stiffness κ , introduced to the boundary condition in (18) as

$$\varphi'' \mp \left(\frac{1}{i\omega\eta} + \frac{1}{\kappa} \right)^{-1} \varphi' = 0 \quad (32)$$

The effect of the additional flexibility $1/\kappa$ is illustrated in Fig. 7b, resulting in a reduction of the natural frequency from the differential equation. The correction constant κ may be calibrated as for k_c , based on a simple numerical search routine that retains the correct natural frequency ω_∞ obtained from the finite element model. The correct κ is calibrated so that the continuous eigenvalue problem with the flexible boundary condition

$$\varphi'' \mp \kappa \varphi' = 0 \quad (33)$$

exactly recovers the numerically determined ω_∞ associated with $\eta \rightarrow \infty$. In practice, the state-space problem in (27) is not conveniently solved to get ω_∞ . Instead axial support conditions are added to the finite element model at damper locations, whereby the natural frequency ω_∞ may be readily obtained from a generalized eigenvalue problem similar to (21).

In the limiting case of infinite damping with $\eta \rightarrow \infty$, the boundary condition (32) dictates that $\varphi' = 0$, thus without excess warping displacement, when neglecting the correction term $1/\kappa$ in the parenthesis. However, with the concentrated dampers applied at discrete points on the cross-section, the partially restrained warping will inherently reduce the infinitely damped frequency due to the remaining flexibility, as the cross-section will obviously warp between damper locations. This further implies that the warping displacements are linear combinations of the sector-coordinates associated with the undamped beam in Fig. 2b and the remaining warping function in Fig. 2c for infinite damping. The axial displacements in (15) are therefore only an approximation, leaving a slightly altered warping stiffness to be used for the section bimoment in (17). However, as the actual warping displacements at infinite damping vanish, the current model actually predicts the damping behaviour with great accuracy, as demonstrated by the numerical example in the next section. Thus, instead of conducting a full root locus analysis with the large finite element model, the eigenvalue solution from the partial differential equation may effectively be used to design the damper system, as long as the two correction parameters k_c and κ are calibrated accurately.

5 Damping of a 'simple-simple' beam

To illustrate the attainable damping properties by restraining the warping displacements - and to demonstrate the accuracy of using the differential equation with calibrated parameters rather than doing a full root locus analysis - a beam is analysed with the two different bimoment configurations in Figs. 6a and 6b, from now denoted as B_a and B_b , respectively. The beam in Fig. 8a is with 'simple-simple' supports which represent restrained rotation and free warping at both ends. Viscous damping is applied at $z = \ell$. The cross-section in Fig. 8b has a height and width a , thickness $t = a/40$ and added vertical flanges of length $a/4$. Poisson's ratio is $\nu = 0.3$. According to [23] the transcendental equation governing the simple-simple beam may be written as

$$\begin{aligned} & ((\alpha\ell)^2 + (\beta\ell)^2) \sinh(\alpha\ell) \sin(\beta\ell) - \\ & \left(\frac{1}{i\omega\eta\ell} + \frac{1}{\kappa\ell} \right)^{-1} \cosh(\alpha\ell) \cos(\beta\ell) [\beta\ell \tanh(\alpha\ell) - \alpha\ell \tan(\beta\ell)] = 0 \end{aligned} \quad (34)$$

Figure 8: (a) 'Simple-simple' beam with damping treatment at $z = \ell$, and (b) modified I-profile.

The first step is to calibrate k_c and κ based on the established finite element model of the beam. Fig. 9a shows the variation of k_c relative to the original $k = \sqrt{GK/EI_\psi}$ for different lengths of the beam. The graph basically shows the influence of cross-section distortion on the undamped frequencies, as a function of the beam length to the height of the cross-section. It shows that distortion has a significant effect when the beam is sufficiently short. With increasing length the ratio k_c/k approaches unity, without the distortional effect. Based on the calibrated k_c the stiffness parameter κ may subsequently be calibrated. Fig. 9b shows the variation of $\kappa\ell$ for the two bimoment configurations as they give different infinitely damped frequencies. The specific value of $\kappa\ell$ ensures that the infinitely damped frequency ω_∞ obtained by the transcendental equation matches the corresponding frequency from the finite element analysis. Fig. 9b shows that κ increases with the slenderness ratio ℓ/a as the warping effect is less important in long beams, whereby the term $1/\kappa\ell \rightarrow 0$ in (32) as $\kappa\ell \rightarrow \infty$.

5.1 Optimal balancing of dampers

The damper configurations B_a in Fig. 6a and B_b in Fig. 6b involve 8 and 16 dampers, respectively. The dampers are placed asymmetrically on the quarter cross-section as shown in Fig. 10 where c_{a1} - c_{a2} are associated with B_a and c_{b1} - c_{b4} are associated with B_b . As the warping amplitudes are different at all six locations in Fig. 10 the dampers must be balanced relative to each other according to (31). The configuration B_a is balanced relative to c_{a1} and B_b is balanced relative to c_{b1} . The balancing factors

Figure 9: (a) Calibrated k_c and (b) $\kappa\ell$ for B_a (blue line) and B_b (red line) based on finite element results.

Figure 10: Order of the damping coefficients.

α_j are seen in Table 2. For B_a the balancing factors are within the same order of magnitude, and only a vanishing deviation would be observed if the dampers were not balanced. In the case of B_b the necessity of balancing is obvious as c_{b3} and c_{b4} are much larger than c_{b1} and c_{b2} and if all dampers had the same damping coefficient the root locus would consist of multiple smaller semicircles as illustrated in [25] with less effect.

5.2 Root locus analysis

With the parameters k_c and κ calibrated, a full root locus analysis may be performed by solving the transcendental equation (34) by varying the viscous damping parameter in the interval $c_0 \in [0; \infty[$. The real part of the complex frequency will then shift from ω_0 to ω_∞ . The frequency loci and damping ratios obtained from the transcendental equation are compared to similar analyses in the finite element format, see Fig. 11. The analyses are based on $\ell/a = 30$. The results are not dependent of the size of the cross-section if the ratio ℓ/a is kept constant, neither does the material have an influence on the relative frequency increment $(\omega_\infty - \omega_0)/\omega_0$, and thereby the maximum attainable damping ratio. However, the optimal value of the viscous damping parameter and absolute frequencies are sensitive to changes in geometry and material and will therefore change accordingly.

The black dashed line indicates a fully restrained cross-section, therefore representing the maximum frequency and damping ratio. The parameters needed for solving the transcendental equation are given in Table 2. Considering the frequency loci it may be observed that the frequencies coincide at ω_0 and ω_∞ as expected due to the calibration of k_c and $\kappa\ell$. It may furthermore be observed by the blue lines, indicating that the dampers are placed in the outermost points of the cross-section (Fig. 6a) yield

Table 2: Parameters used for analysis.

Bimoment	B_a		B_b			
	c_{a1}	c_{a2}	c_{b1}	c_{b2}	c_{b3}	c_{b4}
α_j	1.0	0.8	1.0	7.5	70.2	141.9
$\kappa\ell$	0.385		0.645			
k_c/k	1.004		1.004			

Figure 11: Complex frequency loci (a) and damping ratio (b). Results from finite element analysis (blue circle) and (red circle) and differential equation (blue line, blue plus symbol) and (red line, red plus symbol). Blue and red corresponds to B_a and B_b respectively. (—) indicates fully restrained cross-sections.

a lower ω_∞ than having the dampers placed at the corners (Fig. 6b), represented by the red lines. The damping ratio is determined based on the imaginary part of the frequency, see (29), and thus the larger the imaginary part the larger the damping ratio. The (o)-markers and solid lines with the (+)-markers representing FE results and results from solving the transcendental equation respectively match perfectly.

Table 3 presents the relative frequency increments and maximum damping ratios, ζ_{max} . The results from the full finite element analysis and from solving the transcendental equation are presented for comparison. From the relative frequency increments $(\omega_\infty - \omega_0)/\omega_0$ it may be seen that warping has a significant influence on the natural frequency. Relative increments of 30-50 % are observed in this particular case, merely by restraining warping partially at one end of the beam. The maximum damping ratio ζ_{max} corresponds approximately to the top point of the frequency loci in Fig. 11a and thus to half of the relative frequency increment [26, 39],

$$\zeta^* = \frac{1}{2} \frac{\omega_\infty - \omega_0}{\omega_0} \quad (35)$$

From Table 3 it may be seen that this estimate fits well with the results from the root locus analyses. Thus, in a preliminary design situation the maximum damping ratio is easily estimated from ω_0 and ω_∞ , obtained without solving the full complex problem in (27).

Table 3: Frequency increments and damping ratios from the root locus analysis.

Bimoment	$(\omega_\infty - \omega_0)/\omega_0$	ζ_{max}^{FEM}	$\zeta_{max}^{Diff.eq.}$	ζ^*
B_a	0.318	0.160	0.161	0.159
B_b	0.368	0.186	0.187	0.184
∞	0.480	—	0.248	0.240

Figure 12: Undamped (a, c) and infinitely damped (b, d) mode shapes. (a, b) Shows the rotation relative to the shear center from the FE results ($\circ$) and from the differential equation ($—, +$) and (c, d) shows patch plots from the FE model.

If transient analyses are to be performed in a preliminary design stage, the motion of the beam must also be sufficiently represented by the differential equation. The undamped and infinitely damped mode shapes from the differential equation are shown with (+)-markers in Figs. 12a and b, in which the rotations from the finite element model are plotted with ($\circ$)-markers. These rotations are determined based on an average displacement of the eight junction nodes at the connection between the cross-section web and the two flanges, see Fig 2b. In Fig. 12 it may be seen that the rotations from the two methods match almost perfectly, and the differential equation therefore accurately represents the vibration form as well. From the curves it is also evident that restraining warping partially at the right end of the beam causes the gradient of the rotation to decrease, which also can be seen in the mode shapes from the finite element model in Figs. 12c and d, with colors indicating the absolute displacements. If the right end cross-section was fully prevented from warping the rotation curve would have been flat at the right end of Fig. 12b as the gradient would then be zero, $\varphi'(\ell) = 0$.

5.3 Optimal damper location

From the root locus analysis it is evident that the maximum damping ratio depends very much on the damper location, and it has previously been demonstrated that the optimal position is not directly proportional with the magnitude of the warping displacements. The largest damping ratio is however obtained when placing the dampers such that the flexibility of the beam is lowered as much as possible, thereby increasing the infinitely damped frequency ω_∞ . Therefore the bimoment configuration B_b yields the largest damping ratios during the root locus analysis. However, determining the position of the dampers such that ω_∞ is increased as much as possible is as mentioned earlier not a trivial task.

An optimal position of the dampers is robustly estimated by simply sweeping over locations along the flanges of the profile and determining ω_∞ when replacing the dampers with axial supports. Com-

Figure 13: Relative frequency increments for different damper locations.

paring ω_∞ with ω_0 as in (35) gives a good estimate of an optimal position that maximizes the damping ratio. In the current example the dampers may be placed at six different locations, either as a horizontal pair as in Fig. 6a, as four dampers as in Fig. 6b or as a vertical pair as in Fig. 6c. For each of the six positions, ω_∞ may be found by either solving the transcendental equation or by the finite element model, which only requires solving a real-valued problem. Figure 13 shows the ratios ω_∞/ω_0 of which two are recovered in Table 3. This analysis again shows that placing the dampers at the corner is more effective than at the bottom of the vertical flange. However, the actual optimal damper location is seemingly on the vertical flange just below the corner element.

6 Conclusions

A method for damping torsional vibrations in thin-walled beams by restraining the out-of-plane axial warping displacements has been expanded by including a flexibility term in the viscous boundary condition, to be used in relation with the governing differential equation. The flexibility term alters the infinitely damped frequency for a given bimoment configuration and may be calibrated by supplemental finite element results. Furthermore, the warping length scale k has been calibrated according to finite element results to take the in-plane distortional effects associated with short beams into account in the modified value k_c . Hereby the frequency loci produced by the differential equation are obtained, which initiate and terminate at the correct frequencies, as obtained by the finite element model. Almost identical results in terms of maximum damping ratios and the associated viscous damping coefficients are obtained by the differential equation and the finite element model. As the axial dampers have a very local nature caution must be taken when meshing the structure. It is proposed to mesh $25/\ell$ of the beam end with the dampers applied with as many elements as the rest of the beam. Furthermore, the eight internal nodes of the bi-cubic-linear element ensure a consistent convergence behaviour.

The finite element model of the beam from the example has more than 18.500 degrees-of-freedom. A full root locus analysis of a three-dimensional finite element model may therefore be computationally very heavy and time consuming, and even in some cases practically impossible. This suggests that preliminary analyses may be performed by using the differential equation with calibrated parameters obtained by solving only two undamped eigenvalue problems. This allows for an investigation of

damper configurations, different geometries etc., retrieving damping ratios, frequencies and optimal values of viscous damping coefficients with great accuracy. With a single or few final designs a more detailed analysis may then be conducted in the full finite element format.

It has furthermore been demonstrated how different damper configurations on the cross-section may influence the attainable damping properties. For a chosen beam configuration the dampers should be placed to minimize the flexibility of the beam and thereby increase the relative frequency increment between the undamped and infinitely damped frequencies. A substantial amount of damping has been demonstrated, and the differential equation and the finite element model shows very good conformity. Though, it still remains a challenge to determine the flexibility parameter κ explicitly. As it requires information about the beam configuration and the cross-section, it may be achieved with the hierarchical Carrera Unified Formulation [40] exploiting some higher-order theory to capture the altered warping of the cross-section. Finally a simply way of estimating the optimal location of the dampers was presented.

As the warping displacements are generally quite small, damping could be realized with e.g. piezoelectric transducers as presented in [41], where large forces may be generated at small stroke levels. If the beam is small, piezoelectric patches [21] could be placed in pairs on the beam flanges where one acts as the sensor and the other as actuator. Although these would act on the relative displacements, they should still be able to control warping and thus introduce damping.

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