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A Multiscale Direct Solver for the Approximation of Flows in High Contrast Porous Media

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Abstract

We consider a non-overlapping domain decomposition approach to approximate the solution of elliptic boundary value problems with high contrast in their coefficients. We propose a method such that initially local solutions subject to Robin boundary conditions in each primal subdomain are constructed with (locally conservative) finite element or finite volume methods. Then, a novel approach is introduced to obtain a (discontinuous) global solution in terms of linear combination of the local subdomain solutions. In the proposed algorithm the computation of local solutions for unions of subdomains are localized at nearest-neighbor subdomain boundaries, thus avoiding the solution of global interface problems. We remove discontinuities in a smoothing step that is defined on a staggered grid or dual subdomains. The resulting algorithm is naturally parallelizable and can be employed as a parallel direct solver, offering great potential for the numerical solution of large problems. In fact, subdomains can be considered small enough to fit well in GPUs and the proposed procedure can handle adaptive (in space) simulations effectively. Numerical simulations are presented and discussed. We demonstrate the effectiveness of the proposed approach with two and three dimensional high contrast and channelized coefficients, that lead to challenging approximation problems. The new procedure, although designed for parallel processing, is also of value for serial calculation.

Key words. Elliptic equations; Combination algorithm; Multiscale method; Robin boundary condition; Parallel computing; High contrast media.

1 Introduction

Despite great advances in hardware and software, some boundary value problems arising in important applications are still computationally challenging, especially in the high-resolution approximation of large problems. For example, reservoir simulation involve large-scale time-dependent coupled systems of partial differential equations with heterogeneous, anisotropic permeability fields, horizontal wells, fractures, and so on. There are some very large oil reservoirs of interest to the industry, such as those in Brazil's pre-salt layer, that require a few billion cells in their discretization so that acceptable accuracy in numerical predictions is obtained. Classical solution methods based on global discretizations and iterative schemes that are built in reservoir simulators can not handle problems of this magnitude. The numerical solution of problems with these features has motivated the development of several multiscale (MS) and multilevel techniques [1, 2, 23, 24, 25, 28]. For a review of some earlier developments of multiscale methods we refer the reader to [11].

In MS methods, the global fine-scale problem is decomposed into several local problems, called local solutions or multiscale basis functions, that are numerical solutions of local boundary value problems. They are employed to construct accurate coarse-scale quantities. Therefore, the goal of MS methods is to have data exchange between fine and coarse quantities in order to improve the accuracy of upscaled coarse grid calculations. This strategy is useful in applications such as uncertainty quantification, where thousands of simulations have to be performed [5]. For a wide range of applications these methodologies have been proven to be effective in terms of accuracy and improved efficiency. However, it has been shown that they suffer, in the context of reservoir simulation, in the presence of high contrast media, e.g. channelized fields, or highly anisotropic problems [11].

The Multiscale Finite Volume (MSFV) method along with its algebraic formulation are widely used schemes in the MS framework to overcome the above-mentioned challenges [20, 22, 26, 27]. In these procedures local solutions are defined in terms of Neumann boundary condition. The MSFV is based on a localization assumption that ignores relations between edge cells of subdomains. No matter how many basis functions are considered, the MSFV does not converge to the fine grid solution. However, it has shown to be of value as an accelerator for iterative linear solvers or, in more detail, as a preconditioning matrix for iterative solvers such as GMRES or Defect Correction schemes [3].

For non-overlapping multiscale methods we mention initially, among others, two well-known schemes. The first one is the Multiscale Mortar Mixed Finite Element Method [4] that constructs local solutions with Dirichlet boundary conditions. The resulting velocity fields produced with this method show lack of accuracy (discontinuous normal component of velocity at subdomain boundaries), although a post processing step improves such fields [18, 19]. The second one is the Multiscale Mixed Hybrid method that constructs local solutions with Neumann boundary conditions. In this case, the accuracy issue that arises is related to the discontinuity of pressure [17, 16]. A domain decomposition based on the Robin boundary condition was considered in [7] and its multiscale counterpart was developed initially in [10] for mixed finite element methods. More recently, the Multiscale Robin Coupled Method was introduced in [9] where it was established that the Mortar and Mixed Hybrid methods are extreme cases of a family of multiscale methods, when parametrized by the coefficient in Robin condition of local solutions. We remark that the Robin boundary condition determines the relative importance of Dirichlet

or Neumann boundary conditions in the coupling of subdomains. This point has been carefully investigated in [9].

In order to introduce the proposed algorithm, we adopt the framework of the Multiscale Mixed Method (see [10]), that can be viewed as one of the methods considered in [9] with constant-by-parts interface spaces. Thus, we employ Robin boundary conditions to construct local solutions on primal subdomains. The new algorithm, called recursive (or, combination) algorithm, provides local solutions on domains defined as the union of some of the subdomains. In our method such solutions are obtained without recalculating solutions of boundary value problems on the larger subdomains by solving small linear systems (the continuity systems in (21)). This is a promising strategy for adaptive simulation [13] where relevant variability of coefficients of the governing equations occur only in small parts of the domain. In this class of problems, local solutions of those parts of the domain must be updated, while the remaining parts of the domain will be left intact, allowing for reduction in computational time. Primal subdomains can be selected small enough so that we can take advantage of GPUs (Graphics Processing Units) that are much faster than CPUs. However, we do not address here the implementation of the algorithm in multi-core or GPU devices, that we intend to consider in the near future. Our focus is in the presentation of the new recursive algorithm, and in the study of its approximation properties through the solution of some challenging problems. **We remark that the technique introduced here is also directly applicable to other MS methods, such as [4, 9].** Also, the proposed algorithm can be viewed as a static condensation procedure [14], which is well known in structural engineering.

Besides the multiscale mixed methods discussed above there are other lines of work that also aim at approximating equations with high-contrast coefficients. We mention, among others, the Generalized Multiscale Finite Element Method (GMsFEM). This framework was initially developed by [8]. In this method the construction of enriched coarse spaces requires the solution of local eigenvalue problems that may be computationally expensive. We also mention multiscale methods that have been developed for challenging time dependent problems, possibly involving fractures [6, 21, 12].

The paper is organized as follows. In section 2 we introduce a model boundary value problem, which is followed by definition of local solutions and a description of the strategy to combine them for two nearest-neighboring subdomains. Next, in section 3, we generalize the algorithm to an arbitrary number of subdomains. In section 4 we show how to improve efficiency of the algorithm by balancing the accuracy to a satisfactory level. As proof of concept we test the proposed schemes on several high contrast and channelized fields in section 5. Our conclusions are presented in section 6.

2 Two subdomains

Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, with Lipschitz boundary $\Gamma = \partial\Omega$. The vector function $\mathbf{u}$ and scalar function P are related in the form

$$\mathbf{u} = -\kappa(\mathbf{x})\nabla P, \quad \text{in } \Omega, \quad (1)$$

where κ is a symmetric positive definite tensor and determines the high contrast setting of the problem. Equation (1) is called Darcy's equation that relates the pressure P and the velocity

u. Considering a source term f , define the partial differential equation

$$L(\mathbf{u}, P) = f, \quad (2)$$

where a typical choice for L would be

$$L(\mathbf{u}, P) = \nabla \cdot \mathbf{u} + \mu(\mathbf{x})P, \quad (3)$$

for a given a non-negative function μ .

Let $\boldsymbol{\nu}$ be the outward unit normal vector on Γ . By denoting $\mathbf{u} \cdot \boldsymbol{\nu}$ different boundary conditions on Γ can be considered. For example, Dirichlet ($P = q_D$ on Γ_D), Neumann ($u = q_N$ on Γ_N) or Robin ($-\beta u + P = q_R$ on Γ_R) boundary conditions can be set. We summarize these possibilities in the form

$$B(\mathbf{u}, P) = q, \quad \text{on } \Gamma. \quad (4)$$

We approximate numerically the solution of the boundary value problem consisting of (2) and (4) with a combination of (locally conservative) finite element/volume methods and multiscale techniques.

Consider a discretization of Ω , say Ω_h , where h refers to the fine grid size; approximate operators in (2) and (4) by finite element/volume methods and look for a solution $\{\mathbf{u}_h, P_h\}$ governed by

$$L_h(\mathbf{u}_h, P_h) = f_h, \quad \text{in } \Omega_h, \quad (5.i)$$

$$B_h(\mathbf{u}_h, P_h) = q_h, \quad \text{on } \Gamma_h. \quad (5.ii)$$

Under reasonable restrictions, (5) has a unique solution. For a pure Neumann problem (with $\mu \equiv 0$ in (3)) the solution is unique up to a constant. If we rewrite Equations (5) for an element of Ω_h , say c , and f_{h_c} and q_{h_c} denote the restriction of f_h and q_h on c and $\partial c \cap \Gamma_h$, respectively then

$$\begin{aligned} L_h(\mathbf{u}_h, P_h) &= f_{h_c}, \quad \text{in } c, \\ B_h(\mathbf{u}_h, P_h) &= q_{h_c}, \quad \text{on } \partial c \cap \Gamma_h. \end{aligned}$$

Moreover, if c' is a nearest-neighbor element of c that shares the interface $\gamma_{cc'}$, then $\{\mathbf{u}_h, P_h\}$ must also satisfy

$$P = P_{c'} \quad \text{on } \gamma_{cc'}, \quad \text{and} \quad u_c + u_{c'} = 0 \quad \text{on } \gamma_{cc'}, \quad (6)$$

to be considered a continuous solution (the pair $\{u_c, P_c\}$ stands for edge values of the normal component of the flux and pressure, respectively). Consider a non-zero value on $\gamma_{cc'}$, say β . Then, Equation (6) is equivalent to

$$\mp \beta u_c + P_c = \pm \beta u_{c'} + P_{c'}, \quad \text{on } \gamma_{cc'}, \quad (7)$$

that we call the continuity condition.

Definition 1 *A continuous solution $\{\mathbf{u}_h, P_h\}$ of (5) we mean a solution that satisfies continuity condition (7) on all edges of elements.*

To solve (5) on smaller subdomains than the original Ω (with the hope of fitting each one of them in GPUs), consider a simple partition $\Omega_h = \hat{\Omega} \cup \tilde{\Omega}$ such as the one illustrated in Figure 1.a. Henceforth we remove the subscript h for subdomains, and we refer to them as primal subdomains. Later, in the discussion of the smoothing step, we will consider dual subdomains as well. The common boundary of $\hat{\Omega}$ and $\tilde{\Omega}$, call it γ , includes some interfaces (element edges), $\gamma = \bigcup \gamma_j$. Thus, each γ_j is a common interface edge between an element of $\hat{\Omega}$ and an element of $\tilde{\Omega}$ (a typical $\gamma_{cc'}$ in (7)). In $\tilde{\Omega}$ and corresponding to each $\gamma_j \in \gamma$, we solve a boundary value problem (see [10]):

$$L_h(\tilde{\mathbf{u}}_j, \tilde{P}_j) = 0, \quad \text{in } \tilde{\Omega}, \quad (8.i)$$

$$B_h(\tilde{\mathbf{u}}_j, \tilde{P}_j) = 0, \quad \text{on } \tilde{\Gamma}_o = \partial\tilde{\Omega} \cap \Gamma, \quad (8.ii)$$

$$-\beta \tilde{u}_j + \tilde{P}_j = g, \quad \text{on } \gamma, \quad (8.iii)$$

where β takes a fixed, non zero value on each γ_j . For simplicity we use the index j instead of γ_j and keep it if no confusion arises. g is a function defined on γ and takes non-zero values on γ_j . In line with [10] our choice for g is

$$g = \mathbf{1}_{\gamma_j} = \begin{cases} 1, & \text{on } \gamma_j \\ 0, & \text{on } \gamma \setminus \gamma_j, \end{cases} \quad (9)$$

that leads to

$$(-\beta \tilde{u}_l + \tilde{P}_l)|_{\gamma_j} = \delta_{jl}, \quad \begin{cases} \gamma_l = \gamma_j, (l \neq 0) \\ \gamma_l = \gamma, l = 0 \text{ or } l \neq j. \end{cases} \quad (10)$$

In addition to $\{\tilde{\mathbf{u}}_j, \tilde{P}_j\}_{j \in \gamma}$ we compute $\{\tilde{\mathbf{u}}_0, \tilde{P}_0\}$ which is the solution of

$$L_h(\tilde{\mathbf{u}}_0, \tilde{P}_0) = f_h|_{\tilde{\Omega}}, \quad \text{in } \tilde{\Omega}, \quad (11.i)$$

$$B_h(\tilde{\mathbf{u}}_0, \tilde{P}_0) = q_h|_{\Gamma_o}, \quad \text{on } \tilde{\Gamma}_o, \quad (11.ii)$$

$$-\beta \tilde{u}_0 + \tilde{P}_0 = 0, \quad \text{on } \gamma. \quad (11.iii)$$

Local solutions (or local basis functions) are defined by (8) and (11). These boundary value problems can be solved by either (locally conservative) finite element or finite volume methods. Then, these solutions are continuous solutions within $\tilde{\Omega}$, as well as any linear combination of them. Define

$$\tilde{P} = \tilde{P}_0 + \sum_{j \in \gamma} \tilde{\alpha}_j \tilde{P}_j, \quad (12)$$

$$\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_0 + \sum_{j \in \gamma} \tilde{\alpha}_j \tilde{\mathbf{u}}_j, \quad (13)$$

where $\tilde{\alpha}_j$ are unknown scalars, and will be determined later. Extend trivially both $\tilde{P}$ and $\tilde{\mathbf{u}}$ to Ω_h . The above construction for $\hat{\Omega}$ along with the zero extension gives

$$\hat{P} = \hat{P}_0 + \sum_{j \in \gamma} \hat{\alpha}_j \hat{P}_j, \quad (14)$$

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}_0 + \sum_{j \in \gamma} \hat{\alpha}_j \hat{\mathbf{u}}_j, \quad (15)$$

is defined in Ω_h , but is a continuous solution only in $\hat{\Omega}$. Note that Equations (8) and (11) for $\hat{\Omega}$ are independent of $\hat{\Omega}$ and can be solved in parallel (possibly in GPUs, if $\hat{\Omega}$ is sufficiently small). Now we can see that $\bar{P} := \tilde{P} + \hat{P}$, $\bar{\mathbf{u}} := \tilde{\mathbf{u}} + \hat{\mathbf{u}}$ satisfies

$$L_h(\bar{\mathbf{u}}, \bar{P}) = f_h, \quad \text{in } \Omega_h, \quad (16.i)$$

$$B_h(\bar{\mathbf{u}}, \bar{P}) = q_h, \quad \text{on } \Gamma_h, \quad (16.ii)$$

where $\{\bar{\mathbf{u}}, \bar{P}\}$ is a continuous solution on Ω_h except on γ . Now, we will explain how to make a $\{\bar{\mathbf{u}}, \bar{P}\}$ a continuous solution on Ω_h (or, on each $\gamma_j \in \gamma$, that we refer to as a *fine grid interface edge*) with an appropriate choice of the coefficients $\{\hat{\alpha}_j\}$ and $\{\tilde{\alpha}_j\}$. In line with (7), $\{\bar{\mathbf{u}}, \bar{P}\}$ and $\{\hat{\mathbf{u}}, \hat{P}\}$ must satisfy

$$\mp \beta \tilde{u} + \tilde{P} = \pm \beta \hat{u} + \hat{P}, \quad \text{on each } \gamma_j \in \gamma. \quad (17)$$

Substituting (12), (13) and (14), (15) into (17), taking (10) into account and by defining

$$\theta_{lj} = \beta u_l + P_l|_{\gamma_j} = \pm \beta u_l \mp P_l|_{\gamma_j}, \quad (18)$$

we obtain

$$\tilde{\alpha}_l = \hat{\theta}_{0l} + \sum_{j \in \gamma} \alpha_j \hat{v}_{lj}, \quad l \in \gamma, \quad (19)$$

$$\hat{\alpha}_l = \tilde{\theta}_{0l} + \sum_{j \in \gamma} \alpha_j \tilde{q}_{lj}, \quad l \in \gamma, \quad (20)$$

or, in matrix form

$$\begin{pmatrix} I & -\hat{\alpha} \\ -\tilde{\alpha} & I \end{pmatrix} \begin{pmatrix} \tilde{\alpha} \\ \hat{\alpha} \end{pmatrix} = \begin{pmatrix} \hat{\theta}_0 \\ \tilde{\theta}_0 \end{pmatrix}, \quad (21)$$

that we call the *continuity system*. The matrix entries $\hat{\theta}$ and $\tilde{\theta}$ and vectors $\hat{\theta}_0$ and $\tilde{\theta}_0$ are given in (18), hence we only need $\mathbf{u}$ on γ to construct the linear system (21), and the values of P on γ play the role of auxiliary variables. By solving the linear system (21) we obtain $\{\tilde{\alpha}_j\}$ and $\{\hat{\alpha}_j\}$ such that continuity condition (7) is satisfied on γ . Thus, $\{\bar{P}, \bar{\mathbf{u}}\}$ is the desired continuous solution of (5).

Remark 2 In general the linear system (21) has a unique solution since (5) has a unique solution. In the case of an approximation of (16)-(17) by the lowest order Raviart-Thomas spaces for pressure and velocity its well-posedness has been established [9]. For a pure Neumann problem (with $\mu \equiv \infty$ in (3)), the solution is unique up to a constant, and therefore (21) has one degree of freedom.

3 More than two subdomains

Next, we consider multiple subdomains (with many elements) that makes an effective use of multi-core parallel machines. Thus, in line with Figure 1.b, consider a partition of $\hat{\Omega}$, say $\hat{\Omega}_a \cup \hat{\Omega}_b$,

Figure 1: Partition of Ω_h . (a) $\Omega_h = \tilde{\Omega} \cup \hat{\Omega}$ (b) $\tilde{\Omega} = \tilde{\Omega}_a \cup \tilde{\Omega}_b$ (c) $\tilde{\Omega} = \tilde{\Omega}_a \cup \tilde{\Omega}_b \cup \tilde{\Omega}_c$.

their common boundary, $\tilde{\gamma}$, and γ^a and γ^b that correspond, respectively, to their intersections with γ :

$$\tilde{\gamma} = \partial\tilde{\Omega}_a \cap \partial\tilde{\Omega}_b, \quad \gamma^a = \partial\tilde{\Omega}_a \cap \gamma, \quad \gamma^b = \partial\tilde{\Omega}_b \cap \gamma, \quad \gamma = \gamma^a \cup \gamma^b.$$

For each $\gamma_j^a \in \tilde{\gamma} \cup \gamma^a$ and $\gamma_j^b \in \tilde{\gamma} \cup \gamma^b$ we solve (in parallel) local problems given by Equation (8) and (11) defined on smaller subdomains $\tilde{\Omega}_a$ and $\tilde{\Omega}_b$. We need to address the question of how to represent a local solution of $\tilde{\Omega}$, say $\tilde{P}_j$ for $j \in \gamma^a \subset \gamma$ in terms of $\{P_{a_j}\}$ where $a_j \in \gamma^a \cup \tilde{\gamma}$ and $\{P_{b_j}\}$ where $b_j \in \gamma^b \cup \tilde{\gamma}$. For simplicity in the discussion below we refer only to the pressure variables, and the corresponding equations for the flux variables (such as Equation (15) that corresponds to Equation (14)) are omitted. We show that

$$\tilde{P}_j = \tilde{P}_a + \tilde{P}_b, \text{ where } \tilde{P}_a = \sum_{k \in \tilde{\gamma}} \tilde{\alpha}_{a_k} \tilde{P}_{a_k}, \text{ and } \tilde{P}_b = \sum_{k \in \tilde{\gamma}} \tilde{\alpha}_{b_k} \tilde{P}_{b_k}. \quad (22)$$

It is clear that $\tilde{P}_j$ satisfies Equation (8) and we need to address its continuity over $\tilde{\gamma}$. In line with Equation (21), the solution of the continuity system

$$\begin{pmatrix} I & -\tilde{b}_b \\ -\tilde{b}_a & I \end{pmatrix} \begin{pmatrix} \tilde{\alpha}_a \\ \tilde{\alpha}_b \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{\theta}_{a_j} \end{pmatrix}, \quad (23)$$

gives the coefficients $\tilde{\alpha}_a = \{\tilde{\alpha}_{a_j}\}$ and $\tilde{\alpha}_b = \{\tilde{\alpha}_{b_j}\}$ such that $\tilde{P}_j$ becomes continuous over $\tilde{\gamma}$ and hence in the subdomain Ω_a . As in Equation (18) the matrix entries and vectors in Equation (23) are given by

$$[\tilde{\theta}_a]_{lk} = 2\beta u_l|_{\tilde{\gamma}_l^a} + \delta_{lk}, \quad [\tilde{\theta}_{a_j}]_l = 2\beta u_{a_j}|_{\tilde{\gamma}_l^a}.$$

Thus, we have determined $\{\tilde{P}_j\}_{j \in \gamma}$ in terms of solutions computed in smaller subdomains. We remark that if we were solving a time dependent problem and if the changes (in time) of the coefficient κ in (1) were localized in $\tilde{\Omega}_a$ from a time step to the next, then it would suffice to update only the local functions in $\tilde{\Omega}_a$, and then compute $\tilde{P}_j$ with the new matrix entries in (23). It is easy to see that $\{\tilde{P}_j\}$ for $j \in \gamma^b$ and $\tilde{P}_0$ are given by

$$\tilde{P}_j = \tilde{P}_{b_j} + \sum_{k \in \tilde{\gamma}} \tilde{\alpha}_{b_k} \tilde{P}_{b_k} + \sum_{k \in \tilde{\gamma}} \tilde{\alpha}_{a_k} \tilde{P}_{a_k}, \quad (24)$$

and

$$\tilde{P}_0 = \tilde{P}_{b_0} + \sum_{k \in \tilde{\gamma}} \tilde{\alpha}_{b_k} \tilde{P}_{b_k} + \tilde{P}_{a_0} + \sum_{k \in \tilde{\gamma}} \tilde{\alpha}_{a_k} \tilde{P}_{a_k}. \quad (25)$$

Note that coefficients in (22), (24) and (25) are not the same and, again, for simplicity, we have kept the same notation. Moreover, $\{P_j\}$ is the solution of a Robin boundary value problem (see Equation (8)) and, hence, the coefficients are the unique solution of their corresponding linear system, no matter if the underlying problem is a pure Neumann boundary value problem or not.

To sum up, we have combined two subdomains $\tilde{\Omega}_a$ and $\tilde{\Omega}_b$ or, equivalently, we have obtained a local solution in terms of local solutions of smaller subdomains. In parallel numerical simulation this can happen for any pair of subdomains, or even for more than two, as we discuss next.

In line with Figure 1.c consider a partition of $\tilde{\Omega} = \tilde{\Omega}_a \cup \tilde{\Omega}_b \cup \tilde{\Omega}_c$ and set

$$\tilde{\gamma}^{ab} = \partial\tilde{\Omega}_a \cap \partial\tilde{\Omega}_b, \quad \tilde{\gamma}^{bc} = \partial\tilde{\Omega}_b \cap \partial\tilde{\Omega}_c, \quad \tilde{\gamma}^z = \partial\tilde{\Omega}_z \cap \tilde{\gamma}, \quad z = a, b, c.$$

We show that $\tilde{P}_j$ for $j \in \gamma^a \subset \gamma$ is given by

$$\tilde{P}_j = \underbrace{\tilde{P}_{a_j} + \sum_{k \in \tilde{\gamma}^{ab}} \tilde{\alpha}_{a_k} \tilde{P}_{a_k}}_{\tilde{P}_a} + \underbrace{\sum_{k \in \tilde{\gamma}^{ab}} \tilde{\alpha}_{abk} \tilde{P}_{b_k}}_{\tilde{P}_b} + \underbrace{\sum_{k \in \tilde{\gamma}^{bc}} \tilde{\alpha}_{b_k} \tilde{P}_{b_k} + \sum_{k \in \tilde{\gamma}^{bc}} \tilde{\alpha}_{c_k} \tilde{P}_{c_k}}_{\tilde{P}_c}. \quad (26)$$

Clearly $\tilde{P}_j$ satisfies (8). Continuity on $\tilde{\gamma}^{ab}$ and $\tilde{\gamma}^{bc}$ requires that

$$\mp \beta \tilde{u}_a + \tilde{P}_a = \pm \beta \tilde{u}_b + \tilde{P}_b, \text{ on } \tilde{\gamma}^{ab} \quad \text{and} \quad \mp \beta \tilde{u}_b + \tilde{P}_b = \pm \beta \tilde{u}_c + \tilde{P}_c, \text{ on } \tilde{\gamma}^{bc}. \quad (27)$$

Substituting terms of (26) in (27) gives

$$\begin{pmatrix} I & -\tilde{\theta}_{ab} & -\tilde{\theta}_{bc} & 0 \\ -\tilde{\theta}_a & I & 0 & 0 \\ 0 & 0 & I & -\tilde{\theta}_c \\ 0 & -\tilde{\theta}'_{ab} & -\tilde{\theta}'_{bc} & I \end{pmatrix} \begin{pmatrix} \tilde{\alpha}_a \\ \tilde{\alpha}_{ab} \\ \tilde{\alpha}_{bc} \\ \tilde{\alpha}_c \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{\theta}_{a_j} \\ 0 \\ 0 \end{pmatrix}, \quad (28)$$

where, for example,

$$[\tilde{\theta}_{ab}]_{lk} = 2\beta u_{b_k}|_{\tilde{\gamma}^{ab}} + \delta_{lk} \text{ for } l, k \in \tilde{\gamma}^{ab}, \quad [\tilde{\theta}'_{ab}]_{lk} = 2\beta u_{b_k}|_{\tilde{\gamma}^{bc}} \text{ for } l \in \tilde{\gamma}^{bc} \text{ and } k \in \tilde{\gamma}^{ab}.$$

This procedure can be extended to any number of subdomains. However, it is important to note that it is *not* our intention to construct solutions on larger subdomains at all. We have computed local solutions on the smallest subdomains and want to know what would be their coefficients in the global solution, i.e., P_h in (5), that is a linear combination of all local solutions. Thus, we now focus on how to determine $P_h|_{\tilde{\Omega}_a}$. For simplicity, instead of a linear combination $\sum_{k \in \gamma} \alpha_k P_k$, we introduce a matrix $\mathbf{P}$ such that the P_k s are set to be its columns and we can write $\sum_{k \in \gamma} \alpha_k P_k = \mathbf{P}\alpha$. Hence,

$$P_h|_{\tilde{\Omega}_a} = (\tilde{P}_h|_{\tilde{\Omega}})|_{\tilde{\Omega}_a} = \tilde{P}_{a_0} + (\tilde{\mathbf{P}}\tilde{\alpha})|_{\tilde{\Omega}_a} = \tilde{P}_{a_0} + \tilde{\mathbf{P}}|_{\tilde{\Omega}_a} \tilde{\alpha} = \tilde{P}_{a_0} + \tilde{\mathbf{P}}_a \mathbf{A}_a \tilde{\alpha}, \quad (29)$$

where $\tilde{\alpha}$ is the solution of (21). $\tilde{P}_{a_j}$, that is one column of $\tilde{\mathbf{P}}|_{\tilde{\Omega}_a}$, is computed in (28), and if we put the coefficients of (28) in a matrix $\mathbf{A}_a$, then one can derive the last equality. Equation (29) displays the global solution in terms of local solutions P_{a_0} and $\tilde{\mathbf{P}}_a$, this is the key point of our method that needs to be carefully implemented. We summarize our findings as follows:

Remark 3 After computing local solutions of (8) (corresponding to each fine grid interface edge) and (11) and obtaining their coefficients by solving the continuity system (7) and (29), we get the desired continuous solution of (5). Local solutions can be computed in parallel and combined as needed efficiently. The continuity system require values (such as f_{ux}) only at subdomain boundaries, and with a careful implementation, its solution can be handled in parallel. Thus, we have developed a parallel direct solver. We refer to it as the recursive (or, combination) approach and denote the solution by $\{P_{cl}, \mathbf{u}_{cl}\}$, where cl refers to (linear) combination of local solutions.

We remark that, although we can compute P_h (the exact, fine grid solution of (5)), in applications such as simulation of porous media flows with billions of degrees of freedom, one may be able to use approximate solutions provided that they can be obtained more efficiently, at the cost of reduced accuracy. Next we discuss a strategy to find an approximation to P_h , that leads to improved efficiency in the computation of the recursive solution.

4 Efficiency

Up to this point in the development of our method a family of local solutions is computed for each subdomain corresponding to each fine grid interface element edge of each subdomain. Then, the fine grid continuous solution P_h is obtained. Note, that even in parallel implementation on many-core devices, this might be a very expensive calculation (possibly computed in an offline stage of the computation).

Efficiency of our method can be improved by relaxing the continuity requirement. To relax this restriction, we construct less local solutions within each subdomain, each one of them corresponding to a set of interface edges. Consequently, the continuity systems becomes smaller, more subdomains can be combined at once, less solutions and coefficients must be saved, and less data needs to be interchanged between processes in parallel devices.

We explain the recursive approach for the partition shown in Figure 1.a and it can be easily derived for other cases. The procedure described here is motivated by the introduction of an intermediate $\bar{H}$ length scale in [10]. Let $n \geq 1$ to be the number of interface edges employed in the construction of a local solution. Then, if we refer to Figure 1.a, we write $\gamma = \cup_j \psi_j$ where each ψ_j consists of m interface edges or γ_j s and we write $\psi_j = \cup_{i=1}^m \gamma_{ji}$. Obviously, each ψ_j may consist of a distinct number of interface edges, say m_j . However, for simplicity, we have considered only one value for m_j to be m . If we consider $m = 1$ then $\psi_j = \gamma_j$, then Remark 3 holds.

As before, we must fix a non-zero value for β on each ψ_j to compute local solutions on both $\tilde{\Omega}$ and $\hat{\Omega}$. We set β to be a constant on γ_j (see, however, [9] for other choices of this parameter). Extreme values of β might be of interest, as discussed also in [9]. If we set $\beta = 0$ then the continuity condition (7) implies continuity of P in (6) and the local solutions are defined in terms of Dirichlet boundary value problems. On the other hand, very large values for β imply continuity of u , and if it significantly dominates P then (8) becomes close to a Neumann boundary value problem. We also set a non zero value g on each ψ_j and for simplicity we use (9). In summary, the local solutions for $m \geq 1$ are the same as (8) and (11), except for the boundary values in (9) and (10) where γ_j is replaced by ψ_j , thus changing $\tilde{P}$ and $\hat{P}$ in (12) and (14), respectively:

$$\tilde{P}_m = \tilde{P}_0 + \sum_{\psi_j \in \gamma} \tilde{\alpha}_j \tilde{P}_j, \quad \hat{P}_m = \hat{P}_0 + \sum_{\psi_j \in \gamma} \hat{\alpha}_j \hat{P}_j. \quad (30)$$

We have introduced the subscript m to indicate that we are referring to ψ_j . A first result concerning this construction is that for $m > 1$ there are no coefficients $\{\alpha_j\}_{\psi_j}$ and $\{\hat{\alpha}_j\}_{\psi_j}$ that give the desired continuous solution P_h . In other words we can find $\tilde{P}$ continuous on each ψ_j (but not on each γ_j). Thus, we get a discontinuous approximation, in contrast with case $m = 1$. However, we still refer to it as $\{P_{cl}, \mathbf{u}_{cl}\}$.

Remark 4 *Our goal is to find a continuous approximation of $\{P_h, \mathbf{u}_h\}$. Thus, a discontinuous approximation $\{P_{cl}, \mathbf{u}_{cl}\}$ must be refined. This is the subject of post processing or smoothing step to be discussed in next subsection.*

Note that different values for the coefficients $\{\tilde{\alpha}_j\}_{\psi_j}$ and $\{\hat{\alpha}_j\}_{\psi_j}$ can be obtained. For example, one might consider minimizing the overall discontinuity on γ . Here we address continuity on each ψ_j , so in (17) we replace γ_j with ψ_j . A question that arises is how to compute θ_{lj} in (18). Actually, when we compute a local solution, a solution on a fine grid is obtained, hence we can only compute $2\beta u_l + \delta_{lj}$ for each γ_j , and not for a ψ_j . Thus, we assign the (arithmetic) average of $2\beta u_l + \delta_{lj}$ of m interface edges to θ_{lj} . The remaining portion of the argument for combining subdomains remains unaltered for $m > 1$ and $m = 1$. We obtain $\{P_{cl}, \mathbf{u}_{cl}\}$, an approximation for (5), that is discontinuous at the boundaries of the primal subdomains, and has to be post-processed to become a continuous approximation.

4.1 Smoothing step

Starting from $\{P_{cl}, \mathbf{u}_{cl}\}$ (discontinuous at the boundaries of primal subdomains) we describe a method (motivated by a related procedure introduced in [10]) that gives a continuous approximation to either P_h or $\mathbf{u}_h$. It is based on a primal/dual iteration, such that the boundaries of primal/dual subdomains take boundary values from the interior of each other. If only one iteration is considered we call it *dual smoothing* solution that is denoted by $\{P_{ds}, \mathbf{u}_{ds}\}$. One can take the dual solution as the final result. However, at the cost of one additional local solution in primal subdomains, the approximation can be refined one more time. We call it *primal smoothing*, denoted by $\{P_{ps}, \mathbf{u}_{ps}\}$. Thus, the cost of smoothing step just described consists of two local solutions. Next we discuss the effect of both dual and primal smoothing steps in numerical experiments.

5 Numerical experiments

The recursive algorithm described above to construct a global solution is independent of the contrast of the underlying coefficients of the governing equations. We discuss some examples to assess the impact of $\bar{\kappa}$ (or, equivalently, the number $m \geq 1$ of interface edges employed in the construction of a local solution) and the choice of boundary conditions (Dirichlet or Neumann) for the smoothing steps in the accuracy of the solution produced by our method.

We consider two and three dimensional problems with several degrees of variability in κ in (3). In Example 4, μ is non zero, and it is set to zero in the remaining examples. The

permeability data for all examples considered here (except for Example 4) are extracted from the SPE 10 project [15], that is one of the most challenging databases regarding the high contrast in the data. In all examples first we compute local solutions with constant $\beta \equiv 1$ (for other possibilities, see [9]), g as in (9), and then they are combined with our proposed approach to obtain the approximation $\mathbf{u}_{cl}$. Then, we refine it with a dual followed by a primal smoothing steps, to obtain $\mathbf{u}_{ds}$ and $\mathbf{u}_{ps}$. For plotting purposes we compute for each cell, say c , either cell-centred values for P or the magnitude of the flux on the cell edges, and then we plot the cell-wise relative error

$$e_c = \frac{\|\mathbf{u} - \mathbf{u}_h\|_c}{\|\mathbf{u}_h\|_c}. \quad (31)$$

The reported relative errors refer to the l^2 norm of this function.

5.1 Example 1

We consider a 48×48 high contrast field (κ) that is plotted in Figure 2.1. The reference solution for this example, P_h , and the magnitude of the reference $\mathbf{u}_h$ are displayed in Figures 2.2 and 2.3, respectively. The boundary conditions are set as Dirichlet on the left and right boundaries, with values zero and one, respectively. Zero Neumann condition is imposed on the top and bottom boundaries. We consider a 16×16 local grid for each primal subdomains such that we have a partition of the domain in 3×3 primal subdomains. By setting $m = 8$ in the construction of local solutions and applying the recursive approach we obtain a recursive solution $\mathbf{u}_{cl}$. The corresponding relative error (31) is shown in Figure 2.4.

We assigned the maximum error to cells with values above a threshold to highlight the largest values of errors and, as expected, larger error values are concentrated close to the green lines (the boundaries of primal subdomains). We set the values of $\mathbf{u}_{cl}$ along the white line as Neumann boundary conditions for dual subdomains to obtain $\mathbf{u}_{ds}$. The corresponding error (computed using (31)) is shown in Figure 2.5. As expected, errors around green lines have been reduced and the largest errors are concentrated around white lines. However, they are much smaller than the error associated with $\mathbf{u}_{cl}$. Finally, we apply primal smoothing. It gives a satisfactory solution, $\mathbf{u}_{ps}$, that is shown in Figure 2.6. In Figure 3 the relative error for different values of m s are plotted in log scale. For $m = 1$ the relative error is of the order of machine precision.

5.2 Example 2

The 56×200 high contrast permeability field considered in this study is plotted in Figure 4.1. Boundary conditions are pure Neumann for all boundary faces except for the bottom left (non-zero inward Neumann) and top right (Dirichlet) cells. We take 8 subdomains as primal subdomains with partitioning only in the x direction. We construct local solutions for $m = 8$ and $m = 4$ and report the result in Figures 4.2 and 4.3, respectively. We set the values of $\mathbf{u}_{cl}$ and $\mathbf{u}_{ds}$ as Neumann boundary conditions for the dual and primal smoothing steps, respectively. By looking at κ as a matrix with 56 rows and 200 columns, we compute the relative error in log scale for each column j ,

$$\frac{\|\mathbf{u} - \mathbf{u}_h\|_{Col_j}}{\|\mathbf{u}_h\|_{Col_j}} \quad (32)$$

Figure 2: Example 1. A 43×48 model with Dirichlet boundary values on left and right. Primal and dual subdomains are determined with green and white lines respectively. (1) Logarithmic scale of field which shows high contrast media. (2) Reference P_h . (3) Magnitude of reference $\mathbf{u}_h$. (4) Error of combination of local solutions. The largest errors are highlighted and around of boundary of primal subdomains. (5) Effect of dual smoothing that removes error around primal subdomains and introduces small errors on boundary of dual subdomains. (6) Effect of primal smoothing.

Figure 3: Example 1. Relative error in log scale for combination, dual and primal smoothing, for $m = 16, 8, 4, 2$. For $m = 1$ the relative error is of the order of machine precision.

and plot it. Hence at each one of the 25 columns, that are the boundaries of primal subdomains (position of the discontinues), we expect a peak for relative errors as can be seen with green lines in figures. However the error is smaller in the middle region of primal subdomains. We set these middle values as boundary conditions for dual subdomains, and we expect to see a peak for relative errors, shown in red curves. Primal smoothing gives the final result.

5.3 Example 3

We consider a 56×200 high contrast channelized field that is plotted in Figure 5.1. This type of field is very challenging for multi-scale techniques and we show the efficiency of the proposed method in this case. Boundary conditions are set as pure Neumann for all boundary faces except for the bottom left (non-zero inward Neumann) and top right (Dirichlet) cells. We consider 8 primal subdomains with partitioning only in the x direction. Local solutions are constructed for $m = 8$ and $m = 4$ and we set the values of P_{cl} and P_{ds} as Dirichlet boundary conditions for dual and primal smoothing, respectively. By looking at κ as a matrix with 56 rows and 200 columns, we compute the relative error in log scale for each column j for P as in (32), and plot it. Results are reported in Figures 5.2 and 5.3 for $m = 8$ and $m = 4$, respectively. At integer multiples of 25 columns, that are the positions of discontinues, we expect a peak for relative errors, that are smoothed in smoothing step. As an assessment of the accuracy of the proposed method we show, for $m = 4$, the cell-wise error (31) for P_{ps} .

5.4 Example 4

We consider in this example a non zero μ in (3). The domain Ω is $(0, 1) \times (0, 1)$ discretized by a 48×48 mesh. We set the source term and boundary conditions such that $P = x \cos(\pi y)$ is the exact solution of (3) for

$$\kappa = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad \mu = 0.01. \quad (33)$$

The reference solutions for P_h and the magnitude of $\mathbf{u}_h$ are plotted in Figures 6.1 and 6.2, respectively. The recursive solution is plotted in 6.3 and, as in Example 1, the largest values of

Figure 4: Example 2. A 56×200 high contrast field with 8 primal subdomains. (1) Permeability field shown in log scale. The black lines are the boundaries of primal subdomains. (2) Column-wise relative error (32) for $n = 8$. (3) Column-wise relative error (32) for $m = 4$.

Figure 5: Example 2. A 53×200 high contrast channelized field with 8 primal subdomains. (1) Permeability field shown in log scale. Black lines indicate the boundaries of primal subdomains. (2) Column-wise relative error (32) for P for $m = 8$. (3) Column-wise relative error (32) for P for $m = 4$. (4) Cell-wise relative error (31) for P_{ps} for $m = 4$.

error are highlighted and concentrated close to the primal subdomain boundaries (green lines). By setting $\mathbf{u}$ as Neumann boundary conditions, the effect of dual and primal smoothing steps can be clearly observed in 6.4 and 6.5, respectively. Finally, the relative error is shown in log scale in Figure 6.6.

5.5 Example 5

We consider in this study a three-dimensional example with 40 elements in each direction. The permeability field includes channelized and high contrast layers along with stretched cells that poses a difficult problem for numerical approximation. The values of κ are the same in x and y direction, but (in average) 1000 times less permeable in z direction, that are plotted in Figure 7. The element size in x and y is the same, but it is 5 times larger than in the z direction, thus introducing highly stretched cells. We consider 4 subdomains with partitioning in each direction separately. For example for partitioning in x direction, the boundaries of primal subdomains are layers in the $y - z$ plane, and we compute relative errors of layers in the $y - z$ plane to see error peaks and the smoothing effect. Setting values of P as Dirichlet boundary conditions, the results are reported in Figure 8. In this Figure we considered $m = 16$ (top) and $m = 4$ (down).

5.6 Example 6

Finally we consider a 60×200 permeability field with $\kappa_y = 10^4 \kappa_x$, although this magnitude of anisotropy (10^4) rarely occurs in reality. Considering 6 subdomains by partitioning the domain in the y direction and by taking 10 interface edges for constructing one local solution ($m = 10$) yields the result plotted in Figure 9. We plot 30 streamlines to visualize flow field. For such high contrast, channelized and anisotropic problem the relative errors for P_{ps} and $\mathbf{u}_{ps}$ are $4.64E - 8$ and $4.19E - 6$, respectively, in line with the results reported in Figure 9.

6 Conclusions

We present a proof of concept for a new recursive algorithm based on the combination of local finite element/volume solutions, to obtain global accurate solutions for challenging problems, in particular in the simulation of flows in porous media. In all steps of this approach, i.e. solving local solutions, combining and smoothing steps, can be handled in parallel, making the new procedure very promising for the numerical approximation large scale problems. We tested the algorithm on different two and three dimensional permeability fields with high contrast media. Thanks to the quality of local solutions and the proposed algorithm, our method provides a reliable new approach that can be seen as a multiscale direct solver.

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Figure 6: Example 4. The field κ along with μ are reported in (33). (1) The reference solution for P_h . (2) The reference solution for the magnitude of $\mathbf{u}_h$. (3) The error magnitude of the solution $\mathbf{u}_{cl}$. (4) The effect of dual smoothing. (5) The effect of primal smoothing. (6) The relative error in log scale for $m = 16, 8, 4, 2$.

Figure 7: Example 5. Permeability field in in log scale in x and y directions (Left) and z direction (Right).

Figure 8: Example 5. Relative error in log scale with partitioning in x (left), y (middle) and z (right) directions for $m = 16$ (top) and $m = 4$ (down). In all figures the vertical axis is $\log_{10}$ of the relative error of the layer in the horizontal axis. The legends are set accordingly.

Figure 9: Example 6. The field k_x in log scale displayed along with 30 streamlines to show the flow field for the reference solution (top), $\mathbf{u}_{ls}$ (middle) and $\mathbf{u}_{ps}$ (down). $\kappa_y = 10^4 \kappa_x$.

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