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Sensitivity filtering from the non-local perspective

Anton Evgrafov · José C. Bellido

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Abstract We present a mathematical model of topology optimization for non-local linear diffusion equation. The model provides a rigorous and constructive answer to the challenge of interpreting sensitivity filtering as an internal feature of a specific continuum mechanics model, which has been raised in Sigmund O., Maute K.: Sensitivity filtering from a continuum mechanics perspective. *Struct. Multidiscip. O.* 46(4),471–475 (2012). Our model enjoys two very interesting properties. On the one hand, the governing non-local integral equations of this model approximate the well-known local generalized Laplace equation with diffusion coefficient obeying SIMP (Solid Isotropic Material with Penalization) law. On the other hand, the topology optimization problem admits optimal solutions for a range of penalization parameters of practical interest without the need for any external regularization techniques, something which is well known to be false in the case of topology optimization with SIMP for classical local description of continuum mechanics.

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A. Evgrafov
Department of Mechanical Engineering
Technical University of Denmark
DK-2800 Kgs Lyngby, Denmark
Tel.: +45 45 25 42 05
E-mail: aaev@mek.dtu.dk

J.C. Bellido
Department of Mathematics
University of Castilla-La Mancha
13.071 Ciudad Real, Spain
Tel.: +34 926295261
E-mail: JoseCarlos.Bellido@uclm.es

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1 Introduction

Sigmund and Maute (2012) have raised a curious research question: can the sensitivity filtering, which has been utilized as a heuristic regularization scheme in the earliest work on compliance minimization with SIMP model (see Bendøse and Sigmund, 2003; Allaire, 2012, and references therein), be viewed as a natural byproduct of applying homogenization based topology optimization to a non-local mechanical model? Whereas the answer provided in (Sigmund and Maute, 2012) is unsatisfactory, the question has generated enough interest that other researchers (see for example Li et al., 2017; Bellido, 2015) followed the suit. Yet to this point a plausible interpretation of sensitivity filtering has remained elusive. Correcting this state of affairs is the main objective of the present brief note.

We build upon the recent work on non-local optimal design by Andrés and Muñoz (2015), which utilizes an integral, derivative-free description of continuum mechanics (Kunin, 1975; Silling, 2000). Our optimization problem will employ a non-local linear diffusion model, which can represent a broad range of physical phenomena such as the steady state heat conduction, Darcy flow through the porous medium, or small deflection of an elastic membrane; generalizing our results to linear elasticity involving vector fields is high on our research agenda. We will stick to the vocabulary of heat conduction henceforth referring to the design param-

ters as conductivities and to the corresponding states as temperatures.

2 The local problem

We set the stage for a forthcoming discussion by recollecting the classical compliance minimization problem with SIMP model for the generalized Laplace equation (by our convention modelling the steady state heat conduction). Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be an open bounded domain with Lipschitz continuous boundary. In this domain we would like to distribute isotropic heterogeneous heat conducting material with conductivity κ^{loc} satisfying the SIMP model $\kappa^{\text{loc}}(x) = \rho^p(x)$, $x \in \Omega$, where $p \geq 1$ is a penalty parameter, and ρ is a design variable taken from the admissible set

$$\mathcal{A}^{\text{loc}} = \left\{ \rho \in L^\infty(\Omega) \mid \rho(\cdot) \in [\underline{\rho}, \bar{\rho}], \int_{\Omega} \rho(x) dx = \gamma |\Omega| \right\}.$$

We will assume that the bounds satisfy the inequalities $0 < \underline{\rho} < \bar{\rho} < +\infty$, and that the volume fraction $\gamma \in]\underline{\rho}, \bar{\rho}[$ to avoid trivial situations. Let $f \in L^2(\Omega)$ be the volumetric heat source applied in Ω . It is well known that the steady state temperature in Ω , assuming homogeneous Dirichlet boundary conditions, is the unique $u \in \mathcal{U}_0^{\text{loc}} = H_0^1(\Omega)$, which minimizes the quadratic energy functional

$$v \mapsto I_{\kappa^{\text{loc}}}^{\text{loc}}(v) := \frac{1}{2} \int_{\Omega} \kappa^{\text{loc}} |\nabla v|^2 dx - \int_{\Omega} f v dx. \quad (1)$$

The compliance minimization problem in this notation can be stated as a saddle-point problem

$$\max_{\rho \in \mathcal{A}^{\text{loc}}} \min_{u \in \mathcal{U}_0^{\text{loc}}} I_{\rho^p}^{\text{loc}}(u). \quad (2)$$

This problem famously attains its maximum for $p = 1$ (convex case), and does not attain its maximum for $p > 1$, see (Allaire, 2012; Bendøse and Sigmund, 2003; Céa and Malanowski, 1970).

Let us denote by $\rho \mapsto c_{\text{loc}}(\rho) := -2I_{\rho^p}^{\text{loc}}(u_\rho)$ the (scaled) reduced objective function of (2), where u_ρ is the steady state temperature corresponding to the material distribution $\rho \in \mathcal{A}^{\text{loc}}$. The first variation of this function in the direction $\xi \in L^\infty(\Omega)$ is given by the formula

$$c'_{\text{loc}}(\rho; \xi) = -p \int_{\Omega} \rho^{p-1}(x) \xi(x) |\nabla u_\rho(x)|^2 dx,$$

and therefore the function $\nabla_\rho c_{\text{loc}} = -p\rho^{p-1}|\nabla u_\rho|^2$ may be thought of as the gradient of the (reduced) compliance. ‘‘Sensitivity filtering’’ can be viewed as regularizing the derivatives c'_{loc} by taking a convolution of the underlying gradient $\nabla_\rho c_{\text{loc}}$ with a suitable

Lipschitz continuous radial kernel $A_\delta^{\text{loc}} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with a small compact support (often $A_\delta^{\text{loc}}(|x - x'|) := c_{\text{norm}} \max\{\delta^{\text{loc}} - |x - x'|, 0\}$, for some $\delta^{\text{loc}} > 0$ and a normalization constant $c_{\text{norm}} > 0$), that is,

$$\begin{aligned} c'_{\text{loc}}(\rho; \xi) &= -p \int_{\Omega} \xi(x) \times \\ &\times \int_{\Omega} A_\delta^{\text{loc}}(|x - x'|) \underbrace{\rho^{p-1}(x') |\nabla u_\rho(x')|^2}_{= -\nabla_\rho c_{\text{loc}}(x')/p} dx' dx. \end{aligned} \quad (3)$$

Interestingly enough, there is convincing numerical evidence that the variational inequality $c'_{\text{loc}}(\rho^*; \rho - \rho^*) \geq 0$, $\forall \rho \in \mathcal{A}^{\text{loc}}$, which is a perturbed version of first order necessary optimality conditions for (2), admits solutions $\rho^* \in \mathcal{A}^{\text{loc}}$, and these solutions are of practical value for the users of topology optimization method.

3 The non-local problem

The non-local model of steady state heat conduction that we consider can be conveniently obtained by replacing the quadratic energy functional (1) containing first order derivatives of the temperature with a non-local (double integral) quadratic functional, which only involves function values. More precisely, let $\delta > 0$ be the non-local interaction horizon; only the points which are closer than δ can directly interact with each other. Let $\Omega_\delta = \cup_{x \in \Omega} B(x, \delta)$, where $B(x, \delta) = \{x' \in \mathbb{R}^n : |x - x'| < \delta\}$. Ω_δ is therefore the set of points, which points in Ω can interact with, and we denote by $\Gamma^{\text{nl}} = \Omega_\delta \setminus \Omega$ the ‘‘non-local boundary’’ of Ω . In the model we consider interaction of two points $x, x' \in \Omega_\delta$ with temperatures $u(x)$ and $u(x')$ results in the bond energy density of

$$i_{\delta, \kappa}(u) := \kappa(x, x') A_\delta(|x - x'|) \frac{(u(x) - u(x'))^2}{|x - x'|^2}, \quad (4)$$

where $\kappa : \Omega_\delta^2 \rightarrow [\underline{\kappa}, \bar{\kappa}] \subset \mathbb{R}_{++}$ is the non-local conductivity, and $A_\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non-negative radial function, which vanishes for $|x - x'| \geq \delta$ (points outside the horizon do not interact) and behaves like $|x - x'|^{-(n+2s-2)}$, $0 < s < 1$, for small $|x - x'|$. The non-local Dirichlet principle is that the steady state of this thermal system is the state of the minimal energy as defined by the following quadratic energy functional

$$v \mapsto I_{\delta, \kappa}(v) := \frac{1}{2} \int_{\Omega_\delta} \int_{\Omega_\delta} i_{\delta, \kappa}(v) dx dx' - \int_{\Omega} f v dx. \quad (5)$$

The minimum of $I_{\delta, \kappa}(u)$ is attained when taken over the natural energy space $\mathcal{U}_0 \subset L^2(\Omega_\delta)$ of functions satisfying the non-local homogeneous Dirichlet boundary conditions $u|_{\Gamma^{\text{nl}}} = 0$ and having the finite energy

Fig. 1 Left: optimal design for the local problem (2) corresponding to $p = 1$. Middle: optimal design for the non-local problem (6) corresponding to $p = 1$. Right: locally optimal (the problem is not convex) design for the non-local problem (6) corresponding to $p = 2$. In all figures, $\kappa^{\text{loc}} = \rho^p$ is plotted.

$\int_{\Omega_\delta} \int_{\Omega_\delta} i_{\delta, \kappa}(u) dx dx' < +\infty$; see for example (Bellido and Mora-Corral, 2014) for more details and generalizations.

We now assume that the non-local conductivity is a geometric mean of local conductivities obeying SIMP, that is $\kappa(x, x') = \sqrt{\kappa^{\text{loc}}(x)\kappa^{\text{loc}}(x')} = \rho^{p/2}(x)\rho^{p/2}(x')$. Similarly to $\mathcal{A}^{\text{loc}}$, we define the admissible set for the non-local problem by

$$\mathcal{A} = \{ \rho \in L^\infty(\Omega_\delta) \mid \rho(\cdot) \in [\underline{\rho}, \bar{\rho}], \int_{\Omega_\delta} \rho(x) dx = \gamma |\Omega| \},$$

where the volume fraction $\gamma \in [\underline{\rho}|\Omega_\delta|/|\Omega|, \bar{\rho}|\Omega_\delta|/|\Omega|]$ to avoid trivial cases. Also, similarly to (2) we define the non-local compliance minimization problem as the following saddle point problem:

$$\max_{\rho \in \mathcal{A}} \min_{u \in \mathcal{U}_0} I_{\delta, \rho^{p/2}(x)\rho^{p/2}(x')}(u). \quad (6)$$

4 Discussion of the non-local problem

Existence of optimal solutions: Recall that the local compliance minimization problem (2) admits optimal solutions in the convex case corresponding to $p = 1$, and lacks optimal solutions (without further regularization) in the non-convex case $p > 1$, see (Allaire, 2012; Bendøse and Sigmund, 2003; Ca and Malanowski, 1970). The non-local compliance minimization problem (6) behaves in a remarkably different fashion.

Theorem 1 *For each $p \in [1, 2]$ the problem (6) admits an optimal solution.*

Proof We note that the mapping $L^\infty(\Omega_\delta \times \Omega_\delta) \ni \kappa \mapsto I_{\delta, \kappa^{p/2}}(u) \in \mathbb{R}$ is concave and continuous (for $0 < \underline{\kappa} \leq \kappa \leq \bar{\kappa} < \infty$) for each $p \in [1, 2]$ and $u \in \mathcal{U}_0$, whence also weak* sequentially upper semicontinuous. The latter also holds true for the function $\kappa \mapsto \min_{u \in \mathcal{U}_0} I_{\delta, \kappa^{p/2}}(u)$

because the upper-semicontinuity is preserved under taking the infimum. Additionally, the map $\mathcal{A} \ni \rho \mapsto \rho(x)\rho(x') \in L^\infty(\Omega_\delta \times \Omega_\delta)$ maps weak* converging sequences into weak* converging sequences. It remains to note that the set $\mathcal{A}$ is non-empty, bounded, closed, and convex in $L^\infty(\Omega_\delta)$ (hence also weak* sequentially compact), therefore the generalized Weierstrass' theorem is applicable. $\square$

Relation to “sensitivity filtering”: We now return to the issue of the sensitivity filtering raised in (Sigmund and Maute, 2012). Indeed, taking the variation of the compliance functional $c(\rho) = -2I_{\delta, \rho^{p/2}(x)\rho^{p/2}(x')}(u_\rho)$ at a point $\rho \in \mathcal{A}$ in some direction $\xi \in L^\infty(\Omega_\delta)$, we obtain the expression

$$\begin{aligned} c'(\rho; \xi) = & -\frac{p}{2} \int_{\Omega_\delta} \int_{\Omega_\delta} [\rho^{p/2-1}(x)\rho^{p/2}(x')\xi(x) \\ & + \rho^{p/2}(x)\rho^{p/2-1}(x')\xi(x')] \times \\ & \times A_\delta(|x-x'|) \frac{(u_\rho(x) - u_\rho(x'))^2}{|x-x'|^2} dx' dx, \end{aligned}$$

where $u_\rho \in \mathcal{U}_0$ is the steady state temperature corresponding to the material distribution $\rho \in \mathcal{A}$. In particular, for the interesting case $p = 2$ we obtain (note the striking similarity with (3) for $p = 2$)

$$\begin{aligned} c'(\rho; \xi) = & -2 \int_{\Omega_\delta} \xi(x) \times \\ & \times \underbrace{\int_{\Omega_\delta} A_\delta(|x-x'|) \rho(x') \frac{(u_\rho(x) - u_\rho(x'))^2}{|x-x'|^2} dx'}_{= -\nabla_\rho c(x)/2} dx, \end{aligned}$$

and therefore we can interpret the underlined expression (up to the scaling factor of $-1/2$) as the gradient of the compliance. Note that this expression can be viewed as a weighed non-local average of a local bond energy density, precisely as desired in Sigmund and Maute (2012). It should also be noted that

$\nabla_\rho c(x) \approx -2\rho(x)|\nabla u_\rho(x)|^2 = \nabla_\rho c_{\text{loc}}(x)$ under appropriate assumptions on u_ρ and ρ for small δ , see Ponce (2004), thus the “filtering” disappears in the limit of small horizons and only the standard gradient of the local compliance remains.

We would like to remark that the multiplicative structure of the non-local conductivity model is crucial both for obtaining existence of optimal solutions for $p \in (1, 2]$ and for recovering the the sensitivity filtering from the non-local model. None of these features hold if one considers the additive parametrization proposed in Andrés and Muñoz (2015).

Relation to the local problem: The local compliance optimization problem can be recovered from the the non-local compliance problems in the limit of vanishing interaction horizons, in the sense that minimizers of the non-local problem converge to minimizers of the local one as $\delta \rightarrow 0$. This was shown for a “SIMP-less” parametrization model in Andrés and Muñoz (2015), and we plan to address this question for our model in the near future.

5 Numerical example

For illustrative purposes we include a numerical example, where we use a uniform heat source $f \equiv 1$, $n = 2$, $\Omega =]0, 1]^2$, $\delta = 0.05$, $s = 1/3$, $\gamma = 0.4$, $\underline{\rho} = 10^{-3}$, $\bar{\rho} = 1.0$. The problem is solved on a regular grid, where Ω is decomposed into 320^2 squares, and each square is split into two triangles. Standard piecewise-linear continuous Lagrange elements are used for approximating functions in $\mathcal{U}_0$, and piecewise-constants for approximating functions in $\mathcal{A}$. Galerkin–Riesz method is used for discretizing the non-local state equations corresponding to minimizing the quadratic energy functional (5). The resulting optimal solutions to the non-local and local problems (6), (2) are shown in Fig. 1.

6 Concluding remarks

We have presented a brief introduction of a non-local topology optimization model which, unlike its local counterpart, admits globally optimal solutions for the compliance minimization problem with SIMP for penalty values $p \in [1, 2]$. The gradient of the compliance in this framework has a structure, which bears a striking resemblance to “sensitivity filtering” thus addressing the issue raised in Sigmund and Maute (2012). In this fashion we have addressed not one but two important open research questions in the topology optimization literature.

7 Replication of results

The format of a brief note does not allow us to go into details of the numerical results, included as an illustration of our work. We are currently preparing a manuscript dedicated to a significantly more detailed exploration of the presented non-local topology optimization model from both the theoretical and the numerical angles.

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