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Gabor frames in $\ell^2(\mathbb{Z})$ and linear dependence

Ole Christensen, Marzieh Hasannasab

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Abstract

We prove that an overcomplete Gabor frame in $\ell^2(\mathbb{Z})$ generated by a finitely supported sequence is always linearly dependent. This is a particular case of a general result about linear dependence versus independence for Gabor systems in $\ell^2(\mathbb{Z})$ with modulation parameter $1/M$ and translation parameter N for some $M, N \in \mathbb{N}$, and generated by a finite sequence g in $\ell^2(\mathbb{Z})$ with K nonzero entries.

Keywords: Frames, Gabor system in $\ell^2(\mathbb{Z})$, linear dependency of Gabor systems

2010 Mathematics Subject Classifications: 42C15

1 Introduction

Linear dependence versus linear independence is a well-studied topic in Gabor analysis. In particular Linnell [11] proved that any Gabor system in $L^2(\mathbb{R})$ generated by a nonzero function and a time-frequency lattice $a\mathbb{Z} \times b\mathbb{Z}$ is linearly independent, hereby confirming a conjecture by Heil, Ramanathan and Topiwala [4]. The analogous problem based on time-frequency shifts on a general locally compact abelian group was studied by Kutyniok in [9] and Gabor systems on finite groups were analyzed in the paper [10] by Lawrence, Pfander, and Walnut. Results by Jitomirskaya [8] imply that the conjecture would fail on $\ell^2(\mathbb{Z})$, as explained by Demeter and Gautam in [3].

The purpose of this short note is to give a more detailed discussion of frame properties and linear independence versus linear dependence for Gabor systems in $\ell^2(\mathbb{Z})$. In particular we prove that an overcomplete Gabor frame in $\ell^2(\mathbb{Z})$ generated by a finite sequence is always linearly dependent. Furthermore we collect and apply various methods for analysis of such frames, e.g., the duality

principle, sampling of Gabor frames for $L^2(\mathbb{R})$, and perturbation methods. For $g \in \ell^2(\mathbb{Z})$ we denote the j th coordinate by $g(j)$. For $M \in \mathbb{N}$, define the *modulation operators* $E_{m/M}$, $m = 0, \dots, M-1$, acting on $\ell^2(\mathbb{Z})$ by $E_{m/M}g(j) := e^{2\pi i j m/M}g(j)$; also, define the *translation operators* T_n , $n \in \mathbb{Z}$, by $T_n g(j) = g(j-n)$. The *Gabor system* generated by a fixed $g \in \ell^2(\mathbb{Z})$ and some $M, N \in \mathbb{N}$ is $\{E_{m/M}T_{nN}g\}_{n \in \mathbb{Z}, m=0, \dots, M-1}$; specifically, $E_{m/M}T_{nN}g$ is the sequence in $\ell^2(\mathbb{Z})$ whose j th coordinate is

$$E_{m/M}T_{nN}g(j) = e^{2\pi i j m/M}g(j - nN).$$

In the rest of this note we will write $\{E_{m/M}T_{nN}g\}$ instead of $\{E_{m/M}T_{nN}g\}_{n \in \mathbb{Z}, m=0, \dots, M-1}$. It is well-known [2] that $\{E_{m/M}T_{nN}g\}$ can only be a frame for $\ell^2(\mathbb{Z})$ if $N/M \leq 1$. We prove that if $N/M < 1$, such frames can be constructed with windows g having any number $K \geq N$ of nonzero entries; in contrast to the case of Gabor frames in $L^2(\mathbb{R})$ these frames are always linearly dependent. Similarly, for $M = N$ we can construct Riesz bases for $\ell^2(\mathbb{Z})$ with windows g having any number $K \geq N$ of nonzero entries; however, for exactly the same parameter choices there also exist linearly dependent Gabor systems. More generally, we characterize the parameters M, N, K for which the Gabor system is automatically linearly independent, linear dependent, resp. that both cases can occur depending on the choice of $g \in \ell^2(\mathbb{Z})$.

2 Gabor systems in $\ell^2(\mathbb{Z})$

For a finitely supported sequence $g \in \ell^2(\mathbb{Z})$, let $|\text{supp } g|$ denote the number of nonzero entries of g . For illustrations and concrete examples we will often use the sequences $\delta_k \in \ell^2(\mathbb{Z})$, $k \in \mathbb{Z}$, given by

$$\delta_k(j) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

It was observed already by Lopez & Han [12] that for any $M, N \in \mathbb{N}$ with $N \leq M$ there exist frames $\{E_{m/M}T_{nN}g\}$ for $\ell^2(\mathbb{Z})$ generated by windows with N nonzero elements. We will need the following extension, characterizing the existence of Gabor frames $\{E_{m/M}T_{nN}g\}$ for $\ell^2(\mathbb{Z})$ with a given support size K .

Theorem 2.1 *Let $M, N, K \in \mathbb{N}$. Then the following hold:*

- (i) *There exists a Gabor frame $\{E_{m/M}T_{nN}g\}$ for $\ell^2(\mathbb{Z})$ generated by a window g with $|\text{supp } g| = K$ if and only if $N \leq M$ and $K \geq N$.*

(ii) *There exists a Riesz sequence $\{E_{m/M}T_{nN}g\}$ in $\ell^2(\mathbb{Z})$ generated by a window g with $|\text{supp } g| = K$ if and only if $N \geq M$ and $K \geq M$.*

Proof. For the proof of (i), the necessity of the condition $N \leq M$ is obvious. We will now show that if $K < N$ then $\{E_{m/M}T_{nN}g\}$ can not be complete in $\ell^2(\mathbb{Z})$. We do this by identifying some $k \in \mathbb{Z}$ such that $E_{m/M}T_{nN}g(k) = 0$ for all $n \in \mathbb{Z}$ and $m \in \{0, \dots, M-1\}$. Consider $I := \{1, \dots, N\}$; then, for any $j \in \mathbb{Z}$, there exists exactly one value of $n \in \mathbb{Z}$ such that $j + nN \in I$. Since $g(j) \neq 0$ only occur for $K < N$ values of j , there exists some $k \in I$ such that $j + nN \neq k$ for all $n \in \mathbb{Z}$ and all $j \in \mathbb{Z}$ such that $g(j) \neq 0$. That is, $k - nN \neq j$ for all $n \in \mathbb{Z}$ and all $j \in \mathbb{Z}$ such that $g(j) \neq 0$. Thus for all $n \in \mathbb{Z}$, we have that $g(k - nN) = 0$. This proves that $E_{m/M}T_{nN}g(k) = 0$ for all $n \in \mathbb{Z}$ and $m \in \{0, \dots, M-1\}$ and thus $\{E_{m/M}T_{nN}g\}$ can not be complete if $K < N$; in other words, $K \geq N$ is necessary for $\{E_{m/M}T_{nN}g\}$ to be a frame for $\ell^2(\mathbb{Z})$.

Now assume that $N \leq M$ and consider any $g \in \ell^2(\mathbb{Z})$ for which

$$g(j) \neq 0 \text{ for } j \in \{1, \dots, N\} \text{ and } g(j) = 0 \text{ for } j \notin \{1, \dots, N\}. \quad (2.1)$$

All the vectors in $\{E_{m/M}g\}_{m=0, \dots, M-1}$ have support in $\{1, \dots, N\}$. Writing the coordinates for these vectors for $j \in \{1, \dots, N\}$ as rows in an $M \times N$ matrix, we get

$$\mathcal{A} = \begin{pmatrix} g(1) & g(2) & \cdot & \cdot & g(N) \\ e^{\frac{2\pi i}{M}}g(1) & e^{\frac{2\pi i}{M}2}g(2) & \cdot & \cdot & e^{\frac{2\pi i}{M}N}g(N) \\ e^{\frac{2\pi i}{M}2}g(1) & e^{\frac{2\pi i}{M}2 \cdot 2}g(2) & \cdot & \cdot & e^{\frac{2\pi i}{M}2 \cdot N}g(N) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ e^{\frac{2\pi i}{M}(M-1)}g(1) & e^{\frac{2\pi i}{M}(M-1)2}g(2) & \cdot & \cdot & e^{\frac{2\pi i}{M}(M-1)N}g(N) \end{pmatrix}.$$

Thus, letting $\omega := e^{\frac{2\pi i}{M}}$,

$$\mathcal{A} = [w^{(k-1)j}]_{k=1, \dots, M, j=1, \dots, N} \text{Diag}(g(1), \dots, g(N)). \quad (2.2)$$

Proposition 1.4.3 in [1] shows that the rows in the matrix $\mathcal{A}$ form a frame for $\text{span}\{\delta_k\}_{k=1}^N$ if and only if the columns in $\mathcal{A}$ are linearly independent; since $g(j) \neq 0$ for $j = 1, \dots, N$ the linear independence of the columns follows from (2.2). Applying the translation operators T_{nN} it now follows that $\{E_{m/M}T_{nN}g\}_{n \in \mathbb{Z}, m=0, \dots, M-1}$ is a frame for $\ell^2(\mathbb{Z})$, with $K = N$.

Now, consider any $K > N$ and any $\epsilon > 0$ and let $\tilde{g} := g + \epsilon \sum_{k=N+1}^K \delta_k$. It is easy to see that $\{E_{m/M}T_{nN}\delta_k\}$ is a Bessel sequence with bound M ; it follows

that for any finite sequence $\{c_{m,n}\} \in \ell^2(\{1, \dots, M-1\} \times \mathbb{Z})$,

$$\begin{aligned} \left\| \sum c_{m,n} E_{m/M} T_{nN} (\tilde{g} - g) \right\| &= \left\| \epsilon \sum_{k=N+1}^K \sum c_{m,n} E_{m/M} T_{nN} \delta_k \right\| \\ &\leq \epsilon \sum_{k=N+1}^K \left\| \sum c_{m,n} E_{m/M} T_{nN} \delta_k \right\| \\ &\leq \epsilon (K - N) \sqrt{M} \left(\sum |c_{m,n}|^2 \right)^{1/2}. \end{aligned}$$

Let A denote a lower frame bound for $\{E_{m/M} T_{nN} g\}_{n \in \mathbb{Z}, m=0, \dots, M-1}$. If we choose $\epsilon > 0$ such that $\epsilon (K - N) \sqrt{M} < A$, it follows from Theorem 22.1.1 in [1] that $\{E_{m/M} T_{nN} \tilde{g}\}_{m=0, \dots, M-1, n \in \mathbb{Z}}$ is a frame for $\ell^2(\mathbb{Z})$. By construction, $K = |\text{supp } g|$.

The result in (ii) is a consequence of the duality principle [7], stating that a Bessel sequence $\{E_{m/M} T_{nN} g\}$ is a frame for $\ell^2(\mathbb{Z})$ if and only if the Gabor system $\{E_{m/N} T_{nM} g\}$ is a Riesz sequence; in particular the finitely supported windows g generating frames in (i) are precisely the ones that generate Riesz sequences in (ii). A direct proof of the existence can be given along the lines of the proof of (i), as follows. Assume that $M \leq N$ and consider any $g \in \ell^2(\mathbb{Z})$ for which $g(j) \neq 0$ for $j \in \{1, \dots, M\}$ and $g(j) = 0$ for $j \notin \{1, \dots, M\}$. Then $\{E_{m/M} g\}_{m=0, \dots, M-1}$ is a basis for $\text{span}\{\delta_k\}_{k=1}^M$; since $N \geq M$ this implies that $\{E_{m/M} T_{nN} g\}$ is a Riesz sequence in $\ell^2(\mathbb{Z})$. A similar perturbation argument as in (i) now yields the conclusion. $\square$

Let us mention yet another way of proving the existence of Gabor frames $\{E_{m/M} T_{nN} g\}$ for $N/M < 1$, using sampling of B-spline generated Gabor frames for $L^2(\mathbb{R})$. Recall that the B-splines $B_K, K \in \mathbb{N}$, are defined recursively by convolutions, $B_1 := \chi_{[0,1]}$, $B_{K+1}(x) := (B_K * B_1)(x) = \int_0^1 B_K(x-t) dt$, $x \in \mathbb{R}$.

Example 2.2 Assume that $N < M$ and consider the B-spline B_{N+1} . Since $1/M \leq 1/(N+1)$, the system $\{e^{2\pi i m x/M} B_{N+1}(x - nN)\}_{n,m \in \mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ by Corollary 11.7.1 in [1]. Define the discrete sequence $B_{N+1}^D = \{B_{N+1}(j)\}_{j \in \mathbb{Z}}$. Since B_{N+1} is a continuous function with compact support, the sampling results in [6] imply that the discrete Gabor system $\{E_{m/M} T_{nN} B_{N+1}^D\}_{n \in \mathbb{Z}, m=0, \dots, M-1}$ is a frame for $\ell^2(\mathbb{Z})$. Note that $\text{supp } B_{N+1}^D = \{1, 2, \dots, N\}$, i.e., $|\text{supp } B_{N+1}^D| = N$. $\square$

The main body of Gabor analysis in $L^2(\mathbb{R})$ has a completely parallel version in $\ell^2(\mathbb{Z})$, but with regard to linear dependence the two cases are very different. In fact, certain choices of the parameters $M, N, K \in \mathbb{N}$ imply that the Gabor

system $\{E_{m/M}T_{nN}g\}$ is linearly dependent for all windows $g \in \ell^2(\mathbb{Z})$ with $|\text{supp } g| = K$; for other choices of the parameters there exist linearly dependent as well as linearly independent Gabor systems. The precise statement is as follows.

Theorem 2.3 *Let $M, N \in \mathbb{N}$. Then the following hold:*

- (i) *If $M = 1$, the system $\{E_{m/M}T_{nN}g\}$ is linearly independent for all $g \in \ell^2(\mathbb{Z}) \setminus \{0\}$.*
- (ii) *If $M > |\text{supp } g|$ the Gabor system $\{E_{m/M}T_{nN}g\}$ is linearly dependent.*
- (iii) *If $N < M$, the Gabor system $\{E_{m/M}T_{nN}g\}$ is linearly dependent for any finitely supported $g \in \ell^2(\mathbb{Z})$.*
- (iv) *For all $M, N, K \in \mathbb{N}$ there exists a linearly dependent Gabor system $\{E_{m/M}T_{nN}g\}$ with $K = |\text{supp } g|$.*
- (v) *If $N \geq M$, then there exists for any $K \geq M$ a linearly independent Gabor system $\{E_{m/M}T_{nN}g\}$ with $K = |\text{supp } g|$.*

Proof. For $M = 1$ the system $\{E_{m/M}T_{nN}g\}$ equals the shift-invariant system $\{T_{nN}g\}_{n \in \mathbb{Z}}$ and is thus linearly independent whenever $g \in \ell^2(\mathbb{Z}) \setminus \{0\}$; this proves (i). For the proof of (ii), the vectors $\{E_{m/M}g\}_{m=1, \dots, M-1}$ can be considered as M vectors in a space of dimension $|\text{supp } g|$; thus they are linearly dependent if $M > |\text{supp } g|$, and hence $\{E_{m/M}T_{nN}g\}$ is linearly dependent.

For the proof of (iii), consider any finitely supported $g \in \ell^2(\mathbb{Z})$. Without loss of generality, assume that $g(j) = 0$ for $j \notin \{1, 2, \dots, L\}$. Now, if $L < M$, then the finite collection of vectors $\{E_{m/M}g\}_{m=0, \dots, M-1}$ is clearly linearly dependent. Thus, we now consider the case $M \leq L$. Considering a finite number of translates of g , i.e., $\{T_{nN}g\}_{n=0, \dots, \ell}$ for some $\ell \in \mathbb{N}$, there are at most $L + \ell N$ coordinates where one or more of the vectors are nonzero; thus the system $\{T_{nN}g\}_{n=0, \dots, \ell}$ belongs to an $(L + \ell N)$ -dimensional space. Therefore the collection $\{E_{m/M}T_{nN}g\}_{m=0, \dots, M-1, n=0, \dots, \ell}$ consists of $(\ell + 1)M$ vectors in an $(L + \ell N)$ -dimensional space. Clearly they are linearly dependent if we choose $\ell \in \mathbb{N}$ such that $(\ell + 1)M > L + \ell N$, i.e., $\ell > \frac{L-M}{M-N}$. Thus the Gabor system $\{E_{m/M}T_{nN}g\}$ is linearly dependent, as claimed.

For the proof of (iv), given $M \in \mathbb{N}$, let $g := \sum_{k=1}^K \delta_{kM}$; then for any $m' \in \mathbb{N}$,

$$E_{m'/M}g(j) = e^{2\pi i m' j/M} \sum_{k=1}^K \delta_{kM}(j) = \sum_{k=1}^K \delta_{kM}(j) = g(j), \quad \forall j \in \mathbb{Z},$$

i.e., $E_{m'/M}g = g$; thus the Gabor system $\{E_{m/M}T_{nN}g\}$ is linearly dependent. The result in (v) is a consequence of Theorem 2.1 (ii). $\square$

Let us single out the particular result that indeed motivated us to write this short note. Recall that a frame that is not a basis is said to be *overcomplete*; for a frame $\{E_{m/M}T_{nN}g\}$ in $\ell^2(\mathbb{Z})$ this is the case if and only if $N < M$ [2].

Corollary 2.4 *Any overcomplete Gabor frame $\{E_{m/M}T_{nN}g\}$ with a finitely supported window $g \in \ell^2(\mathbb{Z})$ is linearly dependent.*

Proof. The result follows immediately from Theorem 2.3 (iii). $\square$

The picture changes if we allow windows with infinite support: linearly independent and overcomplete Gabor frames with infinitely supported windows exist, as we show now. Our construction is inspired by a calculation for Hermite functions in $L^2(\mathbb{R})$ given in [4].

Proposition 2.5 *Define $g \in \ell^2(\mathbb{Z})$ by $g(j) = e^{-j^2}$. Then $\{E_{m/M}T_{nN}g\}$ is linearly independent for all $M, N \in \mathbb{N}$ and a frame for $\ell^2(\mathbb{Z})$ if $N < M$.*

Proof. It is well-known that a Gabor system $\{e^{2\pi i b x} \varphi(x - na)\}_{m,n \in \mathbb{Z}}$ in $L^2(\mathbb{R})$ is a Gabor frame for $L^2(\mathbb{R})$ whenever $\varphi(x) = e^{-x^2}$ and $0 < ab < 1$. Applying the sampling results by Janssen (see Proposition 2 in [6]) it follows that the sequence g generates a Gabor frame $\{E_{m/M}T_{nN}g\}$ for $\ell^2(\mathbb{Z})$ whenever $N/M < 1$. Note that this argument uses that the Gaussian satisfies the so-called condition R; we refer to [6] for details.

Now consider any $M, N \in \mathbb{N}$. In order to show that $\{E_{m/M}T_{nN}g\}$ is linearly independent, assume that there is a finite scalar sequence $\{c_{n,m}\}_{n=-L, \dots, L, m=0, \dots, M-1}$ such that $\sum_{n=-L}^L \sum_{m=0}^{M-1} c_{n,m} E_{m/M} T_{nN} g = 0$. Thus, for all $j \in \mathbb{Z}$,

$$0 = \sum_{n=-L}^L \sum_{m=0}^{M-1} c_{n,m} e^{2\pi i j m / M} e^{-(j-nN)^2} = e^{-j^2} \sum_{n=-L}^L \left(\sum_{m=0}^{M-1} c_{n,m} e^{2\pi i j m / M} \right) e^{2nNj - (nN)^2}$$

For $n = -L, \dots, L$, defining the functions $\mathcal{E}_n$ on $\mathbb{Z}$ by $\mathcal{E}_n(j) = \sum_{m=0}^{M-1} c_{n,m} e^{2\pi i j m / M}$, $j \in \mathbb{Z}$, we thus have

$$\sum_{n=-L}^L \mathcal{E}_n(j) e^{2nNj - (nN)^2} = 0, \quad \forall j \in \mathbb{Z}. \quad (2.3)$$

Note that $\mathcal{E}_n$ is a bounded and M -periodic function on $\ell^2(\mathbb{Z})$. We will first prove that $\mathcal{E}_n = 0$ for all $n = -L, \dots, L$. Assume that there is some $n > 0$

such that $\mathcal{E}_n(j) \neq 0$ for some $j \in \mathbb{Z}$. Then take the largest such n and a corresponding $j_0 \in \{1, \dots, M-1\}$ such that $\mathcal{E}_n(j_0) \neq 0$. Then

$$\sum_{n=-L}^L \mathcal{E}_n(j_0 + \ell M) e^{-(nN)^2} e^{2nN(j_0 + \ell M)} \rightarrow \infty \quad \text{as } \ell \rightarrow \infty$$

which is contradicting (2.3). Therefore for all $0 < n \leq L$, $\mathcal{E}_n = 0$. A similar argument shows that for all $-L \leq n < 0$, we have $\mathcal{E}_n = 0$. Now (2.3) implies that also $\mathcal{E}_0 = 0$, as claimed.

Considering now any $n = -L \dots, L$, we thus have $\sum_{m=0}^{M-1} c_{n,m} e^{2\pi i j m / M} = 0$ for all $j = 0, \dots, M-1$. Writing this set of equations in matrix form, the matrix describing the system is a Vandermonde matrix and thus invertible; it follows that $c_{n,m} = 0$ for $m = 0, \dots, M-1$. Since $n \in \{-L, \dots, L\}$ was arbitrary, this proves that the Gabor system is linearly independent. $\square$

Let us also give a construction of a linearly dependent Gabor frame for $\ell^2(\mathbb{Z})$ with an infinitely supported window.

Example 2.6 Assume that $N < M$ and consider the sequence $g \in \ell^2(\mathbb{Z})$ given by $g(j) = 1$ for $j \in \{1, \dots, N\}$ and $g(j) = 0$ for $j \notin \{1, \dots, N\}$. As we have seen in the proof of Theorem 2.1 (i), the system $\{E_{m/M} T_{nN} g\}$ is a frame for $\ell^2(\mathbb{Z})$. For $\epsilon > 0$, let $\tilde{g} = g + \sum_{\ell=1}^{\infty} \frac{\epsilon}{2^\ell} \delta_{\ell M + 1}$. Then $\tilde{g}$ has infinite support and a similar calculation as in the proof of Theorem 2.1 (i) shows that for any finite sequence $\{c_{m,n}\}$, $\|\sum c_{m,n} E_{m/M} T_{nN} (g - \tilde{g})\| \leq \epsilon \sqrt{M} (\sum |c_{m,n}|^2)^{1/2}$. Applying again the perturbation results for frames (Theorem 22.1.1 in [1]), it follows that for sufficiently small ϵ , the system $\{E_{m/M} T_{nN} \tilde{g}\}$ is a frame for $\ell^2(\mathbb{Z})$. Now, since $N < M$ and the support of g has length N , the system $\{E_{m/M} g\}_{m=0, \dots, M-1}$ is linearly dependent; thus, we can choose a nonzero scalar sequence $\{c_m\}_{m=0}^{M-1}$ such that $\sum_{m=0}^{M-1} c_m E_{m/M} g = 0$, i.e., $\sum_{m=0}^{M-1} c_m e^{2\pi i j m / M} = 0$ for $j = 1, \dots, N$. It follows that for any $\ell \in \mathbb{N}$,

$$\sum_{m=0}^{M-1} c_m E_{m/M} \delta_{\ell M + 1}(\ell M + 1) = \sum_{m=0}^{M-1} c_m e^{2\pi i (\ell M + 1) m / M} = \sum_{m=0}^{M-1} c_m e^{2\pi i m / M} = 0,$$

and thus $\sum_{m=0}^{M-1} c_m E_{m/M} \delta_{\ell M + 1} = 0$. The construction of the sequence $\tilde{g}$ now shows that $\sum_{m=0}^{M-1} c_m E_{m/M} \tilde{g} = 0$; it follows that the Gabor system $\{E_{m/M} T_{nN} \tilde{g}\}$ is linearly dependent, as claimed. $\square$

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