Gabor Frames in \(2(\mathbb{Z})\) and Linear Dependence

Christensen, Ole; Hasannasab, Marzieh

Published in:
Journal of Fourier Analysis and Applications

Link to article, DOI:
10.1007/s00041-017-9572-4

Publication date:
2018

Document Version
Early version, also known as pre-print

Citation (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
Gabor frames in $\ell^2(\mathbb{Z})$ and linear dependence

Ole Christensen, Marzieh Hasannasab

October 24, 2017

Abstract

We prove that an overcomplete Gabor frame in $\ell^2(\mathbb{Z})$ generated by a finitely supported sequence is always linearly dependent. This is a particular case of a general result about linear dependence versus independence for Gabor systems in $\ell^2(\mathbb{Z})$ with modulation parameter $1/M$ and translation parameter $N$ for some $M, N \in \mathbb{N}$, and generated by a finite sequence $g$ in $\ell^2(\mathbb{Z})$ with $K$ nonzero entries.

Keywords: Frames, Gabor system in $\ell^2(\mathbb{Z})$, linear dependency of Gabor systems

2010 Mathematics Subject Classifications: 42C15

1 Introduction

Linear dependence versus linear independence is a well-studied topic in Gabor analysis. In particular Linnell [11] proved that any Gabor system in $L^2(\mathbb{R})$ generated by a nonzero function and a time-frequency lattice $a\mathbb{Z} \times b\mathbb{Z}$ is linearly independent, hereby confirming a conjecture by Heil, Ramanathan and Topiwala [4]. The analogous problem based on time-frequency shifts on a general locally compact abelian group was studied by Kutyniok in [9] and Gabor systems on finite groups were analyzed in the paper [10] by Lawrence, Pfander, and Walnut. Results by Jitomirskaya [8] imply that the conjecture would fail on $\ell^2(\mathbb{Z})$, as explained by Demeter and Gautam in [3].

The purpose of this short note is to give a more detailed discussion of frame properties and linear independence versus linear dependence for Gabor systems in $\ell^2(\mathbb{Z})$. In particular we prove that an overcomplete Gabor frame in $\ell^2(\mathbb{Z})$ generated by a finite sequence is always linearly dependent. Furthermore we collect and apply various methods for analysis of such frames, e.g., the duality
principle, sampling of Gabor frames for $L^2(\mathbb{R})$, and perturbation methods. For $g \in \ell^2(\mathbb{Z})$ we denote the $j$th coordinate by $g(j)$. For $M \in \mathbb{N}$, define the modulation operators $E_{m/M}, m = 0, \ldots, M - 1$, acting on $\ell^2(\mathbb{Z})$ by $E_{m/M}g(j) := e^{2\pi ij m/M}g(j)$; also, define the translation operators $T_n, n \in \mathbb{Z}$, by $T_ng(j) = g(j - n)$. The Gabor system generated by a fixed $g \in \ell^2(\mathbb{Z})$ and some $M, N \in \mathbb{N}$ is $\{E_{m/M}T_{nN}g\}_{n \in \mathbb{Z}, m = 0, \ldots, M - 1}$; specifically, $E_{m/M}T_{nN}g$ is the sequence in $\ell^2(\mathbb{Z})$ whose $j$th coordinate is $E_{m/M}T_{nN}g(j) = e^{2\pi ij m/M}g(j - nN).

In the rest of this note we will write $\{E_{m/M}T_{nN}g\}$ instead of $\{E_{m/M}T_{nN}g\}_{n \in \mathbb{Z}, m = 0, \ldots, M - 1}$. It is well-known [2] that $\{E_{m/M}T_{nN}g\}$ can only be a frame for $\ell^2(\mathbb{Z})$ if $N/M \leq 1$. We prove that if $N/M < 1$, such frames can be constructed with windows $g$ having any number $K \geq N$ of nonzero entries; in contrast to the case of Gabor frames in $L^2(\mathbb{R})$ these frames are always linearly dependent. Similarly, for $M = N$ we can construct Riesz bases for $\ell^2(\mathbb{Z})$ with windows $g$ having any number $K \geq N$ of nonzero entries; however, for exactly the same parameter choices there also exist linearly dependent Gabor systems. More generally, we characterize the parameters $M, N, K$ for which the Gabor system is automatically linearly independent, linear dependent, resp. that both cases can occur depending on the choice of $g \in \ell^2(\mathbb{Z})$.

2 Gabor systems in $\ell^2(\mathbb{Z})$

For a finitely supported sequence $g \in \ell^2(\mathbb{Z})$, let $|\text{supp}g|$ denote the number of nonzero entries of $g$. For illustrations and concrete examples we will often use the sequences $\delta_k \in \ell^2(\mathbb{Z}), k \in \mathbb{Z}$, given by

$$\delta_k(j) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

It was observed already by Lopez & Han [12] that for any $M, N \in \mathbb{N}$ with $N \leq M$ there exist frames $\{E_{m/M}T_{nN}g\}$ for $\ell^2(\mathbb{Z})$ generated by windows with $N$ nonzero elements. We will need the following extension, characterizing the existence of Gabor frames $\{E_{m/M}T_{nN}g\}$ for $\ell^2(\mathbb{Z})$ with a given support size $K$.

**Theorem 2.1** Let $M, N, K \in \mathbb{N}$. Then the following hold:

(i) There exists a Gabor frame $\{E_{m/M}T_{nN}g\}$ for $\ell^2(\mathbb{Z})$ generated by a window $g$ with $|\text{supp}g| = K$ if and only if $N \leq M$ and $K \geq N$. 

2
(ii) There exists a Riesz sequence \( \{E_m/M T_{nN}g\} \) in \( \ell^2(\mathbb{Z}) \) generated by a window \( g \) with \( |\text{supp} \, g| = K \) if and only if \( N \geq M \) and \( K \geq M \).

**Proof.** For the proof of (i), the necessity of the condition \( N \leq M \) is obvious. We will now show that if \( K < N \) then \( \{E_m/M T_{nN}g\} \) cannot be complete in \( \ell^2(\mathbb{Z}) \). We do this by identifying some \( k \in \mathbb{Z} \) such that \( E_m/M T_{nN}g(k) = 0 \) for all \( n \in \mathbb{Z} \) and \( m \in \{0, \ldots, M - 1\} \). Consider \( I := \{1, \ldots, N\} \); then, for any \( j \in \mathbb{Z} \), there exists exactly one value of \( n \in \mathbb{Z} \) such that \( j + nN \in I \). Since \( g(j) \neq 0 \) only occur for \( K < N \) values of \( j \), there exists some \( k \in I \) such that \( j + nN \neq k \) for all \( n \in \mathbb{Z} \) and all \( j \in \mathbb{Z} \) such that \( g(j) \neq 0 \). That is, \( k - nN \neq j \) for all \( n \in \mathbb{Z} \) and all \( j \in \mathbb{Z} \) such that \( g(j) \neq 0 \). Thus for all \( n \in \mathbb{Z} \), we have that \( g(k - nN) = 0 \). This proves that \( E_m/M T_{nN}g(k) = 0 \) for all \( n \in \mathbb{Z} \) and \( m \in \{0, \ldots, M - 1\} \) and thus \( \{E_m/M T_{nN}g\} \) can not be complete if \( K < N \); in other words, \( K \geq N \) is necessary for \( \{E_m/M T_{nN}g\} \) to be a frame for \( \ell^2(\mathbb{Z}) \).

Now assume that \( N \leq M \) and consider any \( g \in \ell^2(\mathbb{Z}) \) for which

\[
g(j) \neq 0 \text{ for } j \in \{1, \ldots, N\} \text{ and } g(j) = 0 \text{ for } j \notin \{1, \ldots, N\}.
\]

All the vectors in \( \{E_m/M g\}_{m=0, \ldots, M-1} \) have support in \( \{1, \ldots, N\} \). Writing the coordinates for these vectors for \( j \in \{1, \ldots, N\} \) as rows in an \( M \times N \) matrix, we get

\[
\mathcal{A} = \begin{pmatrix}
ge(1) & g(2) & \cdots & g(N) \\
e^{2\pi i/\delta} g(1) & e^{2\pi i/\delta} g(2) & \cdots & e^{2\pi i/\delta} N g(N) \\
e^{2\pi i/\delta^2} g(1) & e^{2\pi i/\delta^2} g(2) & \cdots & e^{2\pi i/\delta^2} N g(N) \\
\vdots & \vdots & \ddots & \vdots \\
e^{2\pi i/\delta^{N-1}} g(1) & e^{2\pi i/\delta^{N-1}} g(2) & \cdots & e^{2\pi i/\delta^{N-1}} g(N) 
\end{pmatrix}.
\]

Thus, letting \( \omega := e^{2\pi i/\delta} \),

\[
\mathcal{A} = [u^{(k-1)j}]_{k=1,\ldots,M,j=1,\ldots,N} \text{ Diag}(g(1), \ldots, g(N)).
\]

Proposition 1.4.3 in [1] shows that the rows in the matrix \( \mathcal{A} \) form a frame for \( \text{span}\{\delta_k\}_{k=1}^N \) if and only if the columns in \( \mathcal{A} \) are linearly independent; since \( g(j) \neq 0 \) for \( j = 1, \ldots, N \) the linear independence of the columns follows from (2.2). Applying the translation operators \( T_{nN} \) it now follows that \( \{E_m/M T_{nN}g\}_{n \in \mathbb{Z}, m=0, \ldots, M-1} \) is a frame for \( \ell^2(\mathbb{Z}) \), with \( K = N \).

Now, consider any \( K > N \) and any \( \epsilon > 0 \) and let \( \tilde{g} := g + \epsilon \sum_{k=N+1}^K \delta_k \). It is easy to see that \( \{E_m/M T_{nN} \delta_k\} \) is a Bessel sequence with bound \( M \); it follows
that for any finite sequence \( \{c_{m,n}\} \in \ell^2(\{1, \ldots, M - 1\} \times \mathbb{Z}) \),
\[
\left\| \sum c_{m,n} E_{m/M} T_{nN}(\tilde{g} - g) \right\| = \left\| \epsilon \sum_{k=N+1}^{K} \sum c_{m,n} E_{m/M} T_{nN} \delta_k \right\|
\leq \epsilon \sum_{k=N+1}^{K} \left\| \sum c_{m,n} E_{m/M} T_{nN} \delta_k \right\|
\leq \epsilon (K - N) \sqrt{M} \left( \sum |c_{m,n}|^2 \right)^{1/2}.
\]

Let \( A \) denote a lower frame bound for \( \{E_{m/M} T_{nN} g\}_{n \in \mathbb{Z}, m=0, \ldots, M-1} \). If we choose \( \epsilon > 0 \) such that \( \epsilon (K - N) \sqrt{M} < A \), it follows from Theorem 22.1.1 in [1] that \( \{E_{m/M} T_{nN} \tilde{g}\}_{m=0, \ldots, M-1, n \in \mathbb{Z}} \) is a frame for \( \ell^2(\mathbb{Z}) \). By construction, \( K = |\text{supp } g| \).

The result in (ii) is a consequence of the duality principle [7], stating that a Bessel sequence \( \{E_{m/M} T_{nN} g\} \) is a frame for \( \ell^2(\mathbb{Z}) \) if and only if the Gabor system \( \{E_{m/N} T_{nM} g\} \) is a Riesz sequence; in particular the finitely supported windows \( g \) generating frames in (i) are precisely the ones that generate Riesz sequences in (ii). A direct proof of the existence can be given along the lines of the proof of (i), as follows. Assume that \( M \leq N \) and consider any \( g \in \ell^2(\mathbb{Z}) \) for which \( g(j) \neq 0 \) for \( j \in \{1, \ldots, M\} \) and \( g(j) = 0 \) for \( j \notin \{1, \ldots, M\} \). Then \( \{E_{m/M} g\}_{m=0, \ldots, M-1} \) is a basis for \( \text{span}\{\delta_k\}_{k=1}^{M} \); since \( N \geq M \) this implies that \( \{E_{m/M} T_{nN} g\} \) is a Riesz sequence in \( \ell^2(\mathbb{Z}) \). A similar perturbation argument as in (i) now yields the conclusion. \( \square \)

Let us mention yet another way of proving the existence of Gabor frames \( \{E_{m/M} T_{nN} g\} \) for \( N/M < 1 \), using sampling of B-spline generated Gabor frames for \( L^2(\mathbb{R}) \). Recall that the B-splines \( B_K, K \in \mathbb{N} \), are defined recursively by convolutions, \( B_1 := \chi_{[0,1]}, B_{K+1}(x) := (B_K * B_1)(x) = \int_0^1 B_K(x - t) \, dt, x \in \mathbb{R} \).

**Example 2.2** Assume that \( N < M \) and consider the B-spline \( B_{N+1} \). Since \( 1/M \leq 1/(N + 1) \), the system \( \{e^{2\pi imx/M} B_{N+1}(x - nN)\}_{n,m \in \mathbb{Z}} \) is a Gabor frame for \( L^2(\mathbb{R}) \) by Corollary 11.7.1 in [1]. Define the discrete sequence \( B_{N+1}^D = \{B_{N+1}(j)\}_{j \in \mathbb{Z}} \). Since \( B_{N+1} \) is a continuous function with compact support, the sampling results in [10] imply that the discrete Gabor system \( \{E_{m/M} T_{nN} B_{N+1}^D\}_{n \in \mathbb{Z}, m=0, \ldots, M-1} \) is a frame for \( \ell^2(\mathbb{Z}) \). Note that \( \text{supp } B_{N+1}^D = \{1, 2, \ldots, N\} \), i.e., \( |\text{supp } B_{N+1}^D| = N \). \( \square \)

The main body of Gabor analysis in \( L^2(\mathbb{R}) \) has a completely parallel version in \( \ell^2(\mathbb{Z}) \), but with regard to linear dependence the two cases are very different. In fact, certain choices of the parameters \( M, N, K \in \mathbb{N} \) imply that the Gabor
Theorem 2.3 Let $M, N \in \mathbb{N}$. Then the following hold:

(i) If $M = 1$, the system $\{E_{m/M}T_{nN}g\}$ is linearly independent for all $g \in \ell^2(\mathbb{Z}) \setminus \{0\}$.

(ii) If $M > |\text{supp } g|$ the Gabor system $\{E_{m/M}T_{nN}g\}$ is linearly dependent.

(iii) If $N < M$, the Gabor system $\{E_{m/M}T_{nN}g\}$ is linearly dependent for any finitely supported $g \in \ell^2(\mathbb{Z})$.

(iv) For all $M, N, K \in \mathbb{N}$ there exists a linearly dependent Gabor system $\{E_{m/M}T_{nN}g\}$ with $K = |\text{supp } g|$.

(v) If $N \geq M$, then there exists for any $K \geq M$ a linearly independent Gabor system $\{E_{m/M}T_{nN}g\}$ with $K = |\text{supp } g|$.

Proof. For $M = 1$ the system $\{E_{m/M}T_{nN}g\}$ equals the shift-invariant system $\{T_{nN}g\}_{n \in \mathbb{Z}}$ and is thus linearly independent whenever $g \in \ell^2(\mathbb{Z}) \setminus \{0\}$; this proves (i). For the proof of (ii), the vectors $\{E_{m/M}g\}_{m=1,\ldots,M-1}$ can be considered as $M$ vectors in a space of dimension $|\text{supp } g|$; thus they are linearly dependent if $M > |\text{supp } g|$, and hence $\{E_{m/M}T_{nN}g\}$ is linearly dependent.

For the proof of (iii), consider any finitely supported $g \in \ell^2(\mathbb{Z})$. Without loss of generality, assume that $g(j) = 0$ for $j \notin \{1, 2, \ldots, L\}$. Now, if $L < M$, then the finite collection of vectors $\{E_{m/M}g\}_{m=0,\ldots,M-1}$ is clearly linear dependent. Thus, we now consider the case $M \leq L$. Considering a finite number of translates of $g$, i.e., $\{T_{nN}g\}_{n=0,\ldots,\ell}$ for some $\ell \in \mathbb{N}$, there are at most $L + \ell N$ coordinates where one or more of the vectors are nonzero; thus the system $\{T_{nN}g\}_{n=0,\ldots,\ell}$ belongs to an $(L + \ell N)$-dimensional space. Therefore the collection $\{E_{m/M}T_{nN}g\}_{m=0,\ldots,M-1,n=0,\ldots,\ell}$ consists of $(\ell + 1)M$ vectors in an $(L + \ell N)$-dimensional space. Clearly they are linearly dependent if we choose $\ell \in \mathbb{N}$ such that $(\ell + 1)M > L + \ell N$, i.e., $\ell > \frac{L-M}{M-N}$. Thus the Gabor system $\{E_{m/M}T_{nN}g\}$ is linearly dependent, as claimed.

For the proof of (iv), given $M \in \mathbb{N}$, let $g := \sum_{k=1}^{K} \delta_{kM}$; then for any $m' \in \mathbb{N}$,

$$E_{m'/M}g(j) = e^{2\pi im'j/M} \sum_{k=1}^{K} \delta_{kM}(j) = \sum_{k=1}^{K} \delta_{kM}(j) = g(j), \ \forall j \in \mathbb{Z},$$
\[ i.e., \quad E_{m'/M} g = g; \quad \text{thus the Gabor system} \quad \{E_{m/M} T_{nN} g\} \quad \text{is linearly dependent.} \]

The result in (v) is a consequence of Theorem 2.3 (ii). \qed

Let us single out the particular result that indeed motivated us to write this short note. Recall that a frame that is not a basis is said to be overcomplete; for a frame \( \{E_{m/M} T_{nN} g\} \) in \( \ell^2(\mathbb{Z}) \) this is the case if and only if \( N < M \) [2].

**Corollary 2.4** Any overcomplete Gabor frame \( \{E_{m/M} T_{nN} g\} \) with a finitely supported window \( g \in \ell^2(\mathbb{Z}) \) is linearly dependent.

**Proof.** The result follows immediately from Theorem 2.3 (iii). \qed

The picture changes if we allow windows with infinite support: linearly independent and overcomplete Gabor frames with infinitely supported windows exist, as we show now. Our construction is inspired by a calculation for Hermite functions in \( L^2(\mathbb{R}) \) given in [4].

**Proposition 2.5** Define \( g \in \ell^2(\mathbb{Z}) \) by \( g(j) = e^{-j^2} \). Then \( \{E_{m/M} T_{nN} g\} \) is linearly independent for all \( M, N \in \mathbb{N} \) and a frame for \( \ell^2(\mathbb{Z}) \) if \( N < M \). The result follows immediately from Theorem 2.3 (iii). \qed

It is well-known that a Gabor system \( \{e^{2\pi ibx} \varphi(x - na)\}_{m,n} \) in \( L^2(\mathbb{R}) \) is a Gabor frame for \( L^2(\mathbb{R}) \) whenever \( \varphi(x) = e^{-x^2} \) and \( 0 < ab < 1 \). Applying the sampling results by Janssen (see Proposition 2 in [6]) it follows that the sequence \( g \) generates a Gabor frame \( \{E_{m/M} T_{nN} g\} \) for \( \ell^2(\mathbb{Z}) \) whenever \( N/M < 1 \). Note that this argument uses that the Gaussian satisfies the so-called condition R; we refer to [6] for details.

Now consider any \( M, N \in \mathbb{N} \). In order to show that \( \{E_{m/M} T_{nN} g\} \) is linearly independent, assume that there is a finite scalar sequence \( \{c_{n,m}\}_{n=-L,\ldots,L, m=0,\ldots,M-1} \) such that \( \sum_{n=-L}^{L} \sum_{m=0}^{M-1} c_{n,m} E_{m/M} T_{nN} g = 0 \). Thus, for all \( j \in \mathbb{Z} \),

\[
0 = \sum_{n=-L}^{L} \sum_{m=0}^{M-1} c_{n,m} e^{2\pi j m/M} e^{-(j-nN)^2} = e^{-j^2} \sum_{n=-L}^{L} \left( \sum_{m=0}^{M-1} c_{n,m} e^{2\pi j m/M} \right) e^{2nNj-(nN)^2}
\]

For \( n = -L, \ldots, L \), defining the functions \( \mathcal{E}_n \) on \( \mathbb{Z} \) by \( \mathcal{E}_n(j) = \sum_{m=0}^{M-1} c_{n,m} e^{2\pi j m/M} \), \( j \in \mathbb{Z} \), we thus have

\[
\sum_{n=-L}^{L} \mathcal{E}_n(j) e^{2nNj-(nN)^2} = 0, \quad \forall \, j \in \mathbb{Z}. \tag{2.3}
\]

Note that \( \mathcal{E}_n \) is a bounded and \( M \)-periodic function on \( \ell^2(\mathbb{Z}) \). We will first prove that \( \mathcal{E}_n = 0 \) for all \( n = -L, \ldots, L \). Assume that there is some \( n > 0 \)
such that $\mathcal{E}_n(j) \neq 0$ for some $j \in \mathbb{Z}$. Then take the largest such $n$ and a corresponding $j_0 \in \{1, \ldots, M - 1\}$ such that $\mathcal{E}_n(j_0) \neq 0$. Then

$$\sum_{n=-L}^{L} \mathcal{E}_n(j_0 + \ell M)e^{-(nN)^2}e^{2nN(j_0 + \ell M)} \to \infty \quad \text{as } \ell \to \infty$$

which is contradicting (2.3). Therefore for all $0 < n \leq L$, $\mathcal{E}_n = 0$. A similar argument shows that for all $-L \leq n < 0$, we have $\mathcal{E}_n = 0$. Now (2.3) implies that also $\mathcal{E}_0 = 0$, as claimed.

Considering now any $n = -L \ldots, L$, we thus have $\sum_{m=0}^{M-1} c_{n,m}e^{2\pi jm/M} = 0$ for all $j = 0, \ldots, M - 1$. Writing this set of equations in matrix form, the matrix describing the system is a Vandermonde matrix and thus invertible; it follows that $c_{n,m} = 0$ for $m = 0, \ldots, M - 1$. Since $n \in \{-L, \ldots, L\}$ was arbitrary, this proves that the Gabor system is linearly independent. 

Let us also give a construction of a linearly dependent Gabor frame for $\ell^2(\mathbb{Z})$ with an infinitely supported window.

**Example 2.6** Assume that $N < M$ and consider the sequence $g \in \ell^2(\mathbb{Z})$ given by $g(j) = 1$ for $j \in \{1, \ldots, N\}$ and $g(j) = 0$ for $j \notin \{1, \ldots, N\}$. As we have seen in the proof of Theorem 2.1 (i), the system $\{E_{m/M}T_nNg\}$ is a frame for $\ell^2(\mathbb{Z})$. For $\epsilon > 0$, let $\tilde{g} = g + \sum_{\ell=1}^{\infty} \frac{\epsilon}{\ell} \delta_{\ell M+1}$. Then $\tilde{g}$ has infinite support and a similar calculation as in the proof of Theorem 2.1 (i) shows that for any finite sequence $\{c_{m,n}\}$, $\| \sum c_{m,n}E_{m/M}T_nN(g - \tilde{g})\| \leq \sqrt{M} \left( \sum |c_{m,n}|^2 \right)^{1/2}$.

Applying again the perturbation results for frames (Theorem 22.1.1 in [1]), it follows that for sufficiently small $\epsilon$, the system $\{E_{m/M}T_nNg\}$ is a frame for $\ell^2(\mathbb{Z})$. Now, since $N < M$ and the support of $g$ has length $N$, the system $\{E_{m/M}g\}_{m=0, \ldots, M-1}$ is linearly dependent; thus, we can choose a nonzero scalar sequence $\{c_{m}\}_{m=0}^{M-1}$ such that $\sum_{m=0}^{M-1} c_{m}E_{m/M}g = 0$, i.e., $\sum_{m=0}^{M-1} c_{m}e^{2\pi im/M} = 0$ for $j = 1, \ldots, N$. It follows that for any $\ell \in \mathbb{N},$

$$\sum_{m=0}^{M-1} c_{m}E_{m/M}\delta_{\ell M+1}(\ell M + 1) = \sum_{m=0}^{M-1} c_{m}e^{2\pi i(\ell M+1)m/M} = \sum_{m=0}^{M-1} c_{m}e^{2\pi im/M} = 0,$$

and thus $\sum_{m=0}^{M-1} c_{m}E_{m/M}\delta_{\ell M+1} = 0$. The construction of the sequence $\tilde{g}$ now shows that $\sum_{m=0}^{M-1} c_{m}E_{m/M}g = 0$; it follows that the Gabor system $\{E_{m/M}T_nNg\}$ is linearly dependent, as claimed. 

**Acknowledgment:** The authors would like to thank Guido Janssen, Chris Heil and Shahaf Nitzan for useful comments and references.
References


