# Personalized public transport mobility service: a journey ranking approach for route guidance

**Jiang, Yu; Ceder, Avishai**

*Published in:*
Transportation Research Procedia

*Link to article, DOI:*
10.1016/j.trpro.2019.05.048

*Publication date:*
2019

*Document Version*
Publisher's PDF, also known as Version of record



*Citation (APA):*
Jiang, Y., & Ceder, A. (2019). Personalized public transport mobility service: a journey ranking approach for route guidance. *Transportation Research Procedia*, *38*, 935-955. https://doi.org/10.1016/j.trpro.2019.05.048

Available online at www.sciencedirect.com

ScienceDirect

Transportation Research Procedia 38 (2019) 935–955

Transportation Research Procedia

www.elsevier.com/locate/procedia

23rd International Symposium on Transportation and Traffic Theory, ISTTT 23, 24-26 July 2019, Lausanne, Switzerland

# Personalized public transport mobility service: a journey ranking approach for route guidance

Avishai (Avi) Ceder[a, b,*], Yu Jiang[c]

[a]*Faculty of Civil and Environmental Engineerin Technion-Israel Institute of Technology, Haifa, 32000, Israel*
[b]*Transportation Research Centre, Department of Civil and Environmental Engineering, 20 Symonds Street, The University of Auckland, New Zealand*
[c]*Transport DTU, Danmarks Tekniske Universitet (DTU), DK-2800 Kgs. Lyngby, Denmark*

**Abstract**

The use of smartphone applications (apps) to acquire real time and readily available journey planning information is becoming an instinctive behavior of public transport (PT) users. Through the apps, a traveler not only seeks a path from origin to destination, but also a satisfactory path that caters to the traveler's preferences at the requested time of travel. In other words, to strive for a personalized PT service. The personal preferences are naturally enabled because of the existence of multiple attributes associated with alternative PT routes. For instance, preferences can be connected to attributes of time, cost and convenience. Initially this work establishes an adjusted design framework, to those existed nowadays, for a personalized PT service by integrating users' experiences using apps with operators' data sources and operations modeling. The work then focuses on its key component, namely the personalized route guidance methodology. In addition to using the classic shortest path method, two developments are suggested: a *k*-weighted shortest path method and a novel lexicographical shortest path method with a just noticeable difference (JND) consideration. The latter adopts lexicographical ordering to capture traveler preferences over different PT attributes following Ernst Weber's law of human perception threshold. However, Weber's law violates the axiom of transitivity required for an implementable algorithm, and thus a revised method is developed with correctness proven algorithms for ranking different paths. A small example is used as an explanatory device to illustrate the differences between the three route-guidance methods and to demonstrate the effects of the JND perception threshold on the order of the alternative PT routes. A simulation study was conducted using the Copenhagen PT network. The results show that the average reduction of the value of the most important PT traveler's attribute is 12.3% for the *k*-weighted shortest path method, and 13.4% for the lexicographical JND-based shortest path method in comparison with the classical shortest path method. The computation time indicates favorable potential for real life applications.


Peer-review under responsibility of the scientific committee of the 23rd International Symposium on Transportation and Traffic Theory.

* Corresponding author. Tel.: +972-50-5216084; fax: +972-153-50-5216084.
*E-mail address:* ceder@technion.ac.il

2352-1465 © 2019 The Authors. Published by Elsevier B.V.
Peer-review under responsibility of the scientific committee of the 23rd International Symposium on Transportation and Traffic Theory.
10.1016/j.trpro.2019.05.048

*Keywords:* public transport mobility; personalized service; route guidance; shortest path; just noticeable difference.

## 1. Introduction

Public transport (PT) is a key element in most major cities around the world. With the development of smartphones, real time and readily available journey planning information are becoming an integral part of the PT system (Ceder 2016). At the same time, smartphones and other devices are information sources capable of contributing to big data, and while each traveler has specific preferences when undertaking a trip, every traveler can also communicate information that might better inform another users' travel planning. Thus this PT *information coin* has two sides: (a) informing big data centers to help predict traveler mobility for providing planning services and real time adjustments, and (b) considering personalization of travelers' preferences using optimal route guidance methodology. Because of (a) and (b) being two sides of the same *coin*, the objective of this study is to develop a methodology considering an adjusted design framework, to those existed nowadays, of (a) with a new modelling of (b).

### *1.1. Travelers' preferences*

Beyond the basic mechanics of planning and paying for a journey, PT travelers are likely to have preferences over different features of a journey that might even influence their travel decisions including modal choice and route choice. A number of factors/attributes have been found to be important in affecting passengers' choices, such as the quality of a PT service, connectivity, fare cost, accessibility, and journey distance (e.g., Kingham et al. 2001, Galdames et al. 2011). These factors or attributes could play a major role in determining PT route choices (Grison et al. 2017). Moreover, passengers' preferences over these attributes could vary depending on various factors, such as time of day, mood, schedule to follow, family considerations, etc. Therefore, every trip a user makes has unique requirements. Accordingly, a well-designed smartphone application should allow travelers to interact with the operator in terms of sending preferences and receiving tailored information. In the last few years, increased attention has been and continues to be given for informative and to some extent personalized smartphone applications by the industry and researchers (e.g., Global Mass Transit 2014, Shaheen et al. 2016).

Generally speaking, as per Chorus (2012), a personalized smartphone application can be beneficial from two perspectives: (i) The services remember and learn from the traveler's choice profile and allow us to predict travelers' mobility or to issue context-sensitive personal advice (Lathia et al. 2013, Bouhana et al. 2013, Arentze 2013); and (ii) the services consider travelers' preferences over different attributes (i.e., Peng and Huang 2000, Zografos et al. 2009, Chorus et al. 2009).

As noted above, in this study we will construct an adjusted framework to capture both perspectives. It is proposed that perspective (i) be addressed by a big data center, wherein various data sources are stored and utilized to predict passengers' travel behavior and to assist operators in making real time operational decisions. Perspective (ii), the core of this study, is addressed by developing a personalized route guidance methodology that considers passengers' preferences. Related work was done by Nuzzolo et al. (2014), in which the path choice set is generated based on a set of rules that allows for defining the feasible paths and reducing the high potential number of path alternatives. Compared with their study and others, our novel development is sixfold: (1) consider passengers' order of preferences to be, for instance, specified via a future smartphone application; (2) adopting *k*-weighted shortest path algorithm to generate a set of paths to assist the traveller to make a pre-trip plan or en route adaptive decision; (3) devise a path comparsion methodoglogy that captures human perception elements combined with preferences over different PT attributes; (4) establish the theorem that the comparison method satisfies the axiom of transitivity; (5) develop a sorting algorithm and prove its correctness; and (6) develop an algorithm for the shortest path of prioritizing preferences.

Overall, this study first depicts, in this section, an architecture-type framework for future personalized smartphone applications. Secondly, a novel route guidance methodology is developed with the consideration of passengers' preferences over various PT attributes combined with human perception elements. The proposed methodology is

compared with weighted *k*-shortest path and shortest path methods to demonstrate the advantage of the novel approach in the provision of better personalized routes for future smartphone applications.

### *1.2. Design framework*

Fig. 1 depicts an adjusted design framework to what it is available nowadays. That is, the general scheme of the figure is already part of PT operation in most developed countries, but without the use of a prudent online path finding algorithms. Fig. 1 comprised of two parts on the left and right sides of the figure. The left side refers to the user experience with a smartphone application (app), and the right side refers to the operator's components:

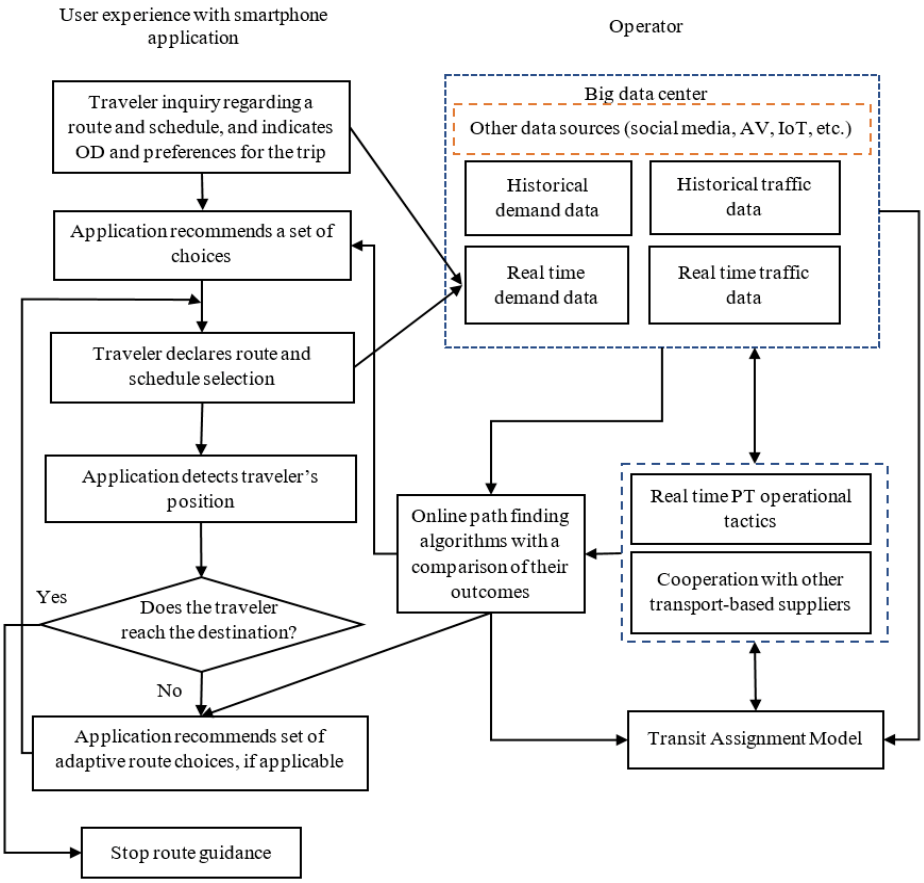



Fig 1. Design framework

1) *User experience*: For the use of personalized PT guidance application, a traveler first sets up a trip destination, along with the preferences associated with different PT attributes of this trip. The passenger, then, declares the selection of a route and a schedule. During the trip the application detects traveler's location and recommends, if applicable, a set of alternative route choices adapted to the real time circumstances. Passengers' preferences and route choice data can be stored in the big data center given passengers' consent.
2) *Operator:* The big data center stores both real time and historical data including smartcard and traffic count data. In addition it is suggested that other emerging data sources be incorporated from social networks (Xiao and Lo 2016), special events (Calabrese et al. 2010), Internet of things (IoT) (Handte et al. 2016), autonomous vehicle (AV), etc. These data will assist the operator in determining real time PT operational tactics. It is worth mentioning that for further performance improvements, it is also recommended that the operator collaborate with other transport-related suppliers, such as shared autonomous vehicles, bike sharing, on-demand transit, etc. The latter will enable the provision of a seamless multimodal travel experience and will pave the way to mobility as a service.

## 2. Route guidance methodology

No doubt that the path recommended for a traveler is important because it affects the passenger's route choice and results in different travel experiences. If the traveler is satisfied with the route suggested and experiences a pleasant journey, it improves the attractiveness and image of the PT service. This section discusses three path recommendation methods: the shortest path, *k* weighted shortest path, and lexicographical ordering using JND threshold. The overview of these three methods is shown in Fig. 2.

The basic methodology for recommending a path is the shortest path method. This method finds the best route based on one of the passenger's preferences, e.g., travel time or an OD related attribute. The merit is its computational efficiency based on well-developed shortest path algorithms in practice. It is suitable for a traveller who is only interested in one attribute, such as time or fare. However, because this approach only recommends one path, it is highly possible that different passengers would receive the same recommendation, implying that the personalization level is low, and could induce congestion in PT vehicles, if all travelers follow the suggestion.

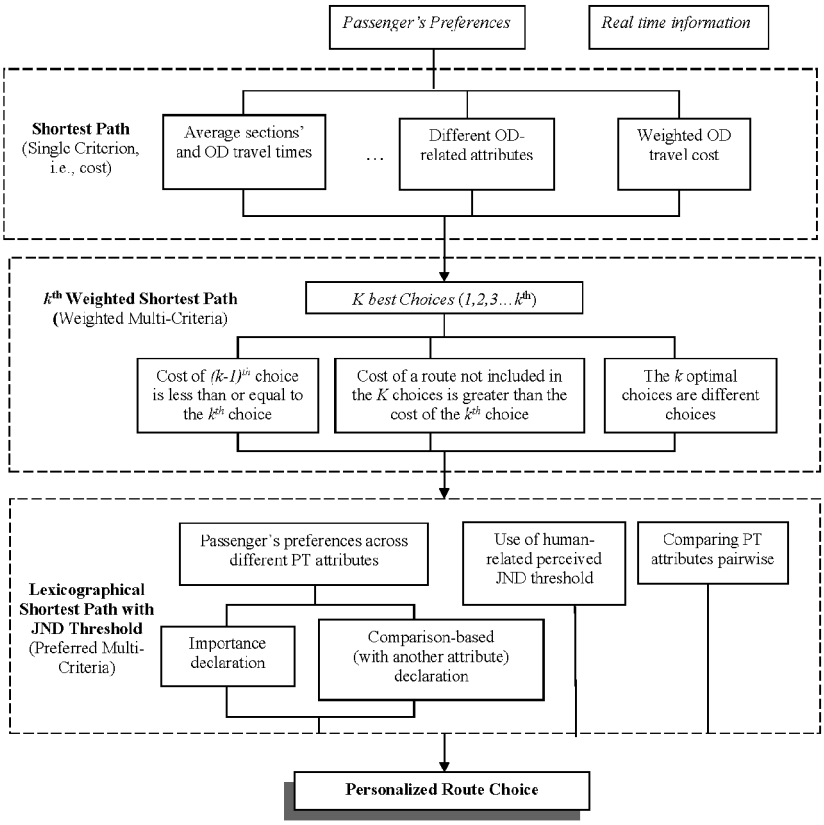



Fig 2. Three methods for path recommendation

We found a few avenues to overcome the demerits of the shortest path method. First, is the use of the *k*-weighted shortest path method that provides a set of paths sorted by weighted cost. That is, the use of weighted cost to search and order paths by incorporating the effects of various PT attributes on passengers' route choice. Generally speaking, the values of weighting parameters could be input by the passengers using default values or predicted via advance machine learning or artificial intelligence (AI) algorithms. Because of the freedom of setting weighting parameters, across time of day and day of week, different passengers could obtain different paths catering to their preferences. This feature indeed achieves a certain degree of personalization. However, the set and order of paths considered and recommended are sensitive to the values of the weighting parameters; if the parameters are not well calibrated, the path order may not reflect passenger preference order as is shown below in the example in section 4. At the same time,

the request for a traveller to input the weighting parameters may not be perceived as so user friendly. A good personalized route guidance application should, on the one hand, provide satisfactory personalized results, but, on the other hand, minimize passenger effort in inserting input parameters. In addition, using the weighted cost path method implicitly requires that all of the PT attributes be converted to monetary values. In this respect, we note that although such a requirement is generally acceptable in practice and research, there are still investigations and discussions on valuing the PT attributes, such as passenger walking, waiting, and transfer times.

Nonetheless, for personalized consideration, it is most important to understand and consider human factors in the modeling and analysis. Both the shortest path and *k*-weighted shortest path methods do not really take human factors into account. Therefore, our second avenue to overcome the demerits of the shortest path method is to propose a lexicographical ordering method with a just noticeable difference (JND) threshold. This method relies on passenger declaration of the importance of the different PT attributes and on a parameter that captures human perception as it is explicated below in section 3.3. In the following subsection, the *k*-shortest path and lexicographical ordering methods are elaborated mathematically.

### 2.1. Notations

| | |
|---|---|
| $R, N, A, M$ | Set of PT routes, stops, arcs, and PT attributes |
| $A_i^+$ | Set of PT lines leaving node *i* |
| $u^{ij}(t)$ | Minimum travel cost between nodes *i* and *j* at time *t* |
| $v_{l,m}^{ij}$ | The value of $m^{\text{th}}$ PT attributes between nodes *i* and *j* using PT line *l* at node *i* |
| $\mu_m$ | Parameter that converts the $m^{\text{th}}$ PT attribute's value $v_{l,m}^{ij}$ to monetary value |
| $\tau_l^{ij}$ | Travel time between nodes *i* and *j* via line *l* |
| $w_m$ | Weighting parameter associated with PT attribute *m* |
| $I(t)$ | Information provided at time *t* |
| $\beta_m^{JND}$ | Just noticeable difference (JND) threshold value associated with PT attribute *m* (Weber's JND) |
| P | Set of passenger's preferences for all of the PT attributes |
| $V_l^{ij}(t,\mathrm{P})$ | Ordered vector of PT attributes between nodes *i* and *j* using PT line *l* at node *i* |
| $S^{ij}$ | Candidate choice set between nodes *i* and *j* |
| $\eta_m^{\text{accept}}$ | Personal maximum/minimum acceptable value of the $m^{\text{th}}$ PT attribute |

### 2.2. K-weighted shortest path

The *k*-weighted shortest path algorithm is an extension of the Dijkstra shortest path algorithm that allows more than one path to be evaluated, that is, to find not only the shortest path, but also subsequent shortest paths (Hadas and Ceder 1996). This idea has been applied to assist PT customers in the context of schedule based PT networks (Xu et al. 2012). The proposed *k*-shortest path guidance methodology is based on computing a weighted travel cost for the consideration of multiple PT attributes.

Consider a passenger travels from node *i* to node *d* where the weighted travel cost between these nodes via line *l* is given by

$$u_l^{id}(t) = \sum_m w_m \mu_m v_{l,m}^{ij} + u^{jd}\left(t + \tau_l^{ij}\right) \Big| I(t), \forall i, d, l \in A_i^+, t, j = i(l)^+ \text{、} \tag{1}$$

$$u^{jd}(t) = \min_{l \in A_j^+} \left\{ u_l^{jd}(t) \right\}, \forall j, d \tag{2}$$

where $i(l)^+$ denotes the subsequent stop of node *i* along line *l*.

The set of *k*-weighted shortest path is defined by set $K^{id} = \{l_1, l_2, ..., l_k\}$, which contains *k* paths from node *i* to destination *d* that have the shortest weighted length (Yen 1971).

### 2.3. Lexicographical ordering using JND threshold

The proposed new route guidance methodology is comprised of three components: (1) elimination of choices that are unacceptable to the traveler; (2) selection only of choices noticeable to the traveler; and (3) sorting the noticeable choices using lexicographical ordering method. These components are explained in this section.

#### 2.3.1. Personal maximum or minimum acceptable value of an attribute

A passenger, naturally, can have preferences over given PT attributes. In addition, a passenger would be able to set up a maximum or minimum acceptable value on certain PT attributes. For example, (i) a tired traveler who prefers waiting to walking may set a 20-minute wait limit to avoid being late; (ii) a slightly disabled traveler for whom having a seat constitutes the minimum comfort level sought. Existing apps, like Google Maps, limited to the travel time attribute, can set up this feature. For other attributes, the passenger needs to check and compare manually. We envision that in the future, a highly personalized app should allow for a traveler to input an acceptable value for each PT attribute associated with a requested trip. Moreover, the comparison should be completed automatically by the app instead of bothering the traveler. To attain this function we determine a candidate set of paths, $S^{id}$, as follows:

$$S^{id} = \left\{ l \middle| v_{l,m}^{id} \le \eta_m^{\text{accept}}, \forall m \right\}, \quad \forall i, d \tag{3}$$

Eq. (3) states that the use of PT line $l$ is contained in candidate choice set $S^{id}$, if each of its attributes agrees with the personal max/min acceptable value.

#### 2.3.2 Just noticeable difference (JND)

Following the determination of the candidate set of paths we model traveller behavior in the process of deciding the order of these paths. Personalized decisions are naturally affected by human perception elements. Thus, in making a route choice, the differences between available PT routes must be perceived. This perception based component leads us to the field of experimental psychology (psychophysics). That is, if the travel time of one route is 100 minutes and the travel time of the compared route is 103 minutes, the question arises of whether these three minute differences are noticeable to the user. In other words, what is the range of non-perception minutes for the travel time of 100 minutes? Is it [95,105], or [90,110], or something else?

For this perception based component, we use the difference threshold definition of JND highlighted by Ernst Weber's law (Baird and Noma 1978, Laming 2008, Chowdhury et al. 2015). Weber's law states that when two stimuli are compared with each other, rather than simply perceiving the difference between the magnitudes of stimuli being compared, human beings perceive the ratio of difference. Mathematically, it is formulated as

$$\Delta U / U = \text{constant} \tag{4}$$

where $\Delta U$ is the change required for an individual to just notice a difference in the magnitude of an attribute, and $U$ is the current stimulus's (attribute's) magnitude. As the value of the constant decreases, the perceptual sensitivity certainly improves.

In the field of transportation, the JND literature is limited. Shi et al. (2011) examined car following distances in driver behavior using JND with the adoption of 0.3 as the constant for drivers' perception of change in headway. In Chowdhury et al. (2015), the JND is adopted to capture the minimum travel time and cost savings invoking the willingness of PT users' to take routes with transfers.

This work considers Weber's law using $\beta_m^{JND}$ as the JND threshold value associated with PT attribute $m$. Hence, the comparison between the personalized choices will consider the same if it is not perceived; i.e., the analysis will be looking at

$$\Delta U / U \le \beta_m^{JND}, \forall m \tag{5}$$

where $U$ is the best (most attractive) value of the attribute among the choices, and $\beta_m^{JND}$ may be presented in %.

### 2.3.3 Lexicographical ordering method

To incorporate the JND threshold value into the path comparison procedure, we propose using the lexicographical ordering method. The lexicographical ordering method is a method for multicriteria optimization problems when different objectives are considered in a hierarchical manner. This method is adopted because: a) the path recommendation problem is intrinsically a multicriteria optimization problem; b) it fits our assumption that passengers have preferences across different PT attributes; c) it could simplify the passenger's input on the smartphone application, i.e., passengers can either declare a PT attribute is "extremely important" (=E), "very important" (=V), "important" (=I), or "less important" (=L), or simply indicate a preference for a comparison, i.e., "travel fare is more important than travel time."

### 2.3.4 Formulation and algorithms

Instead of using an aggregated weighted cost like *k*-weighted shortest path method, the lexicographical ordering method compares each PT attribute pairwise. Accordingly, we use an ordered set, $V_l^{id}(t,\mathbf{P})$, to represent various PT attributes at time *t* given passenger preference **P**, where **P** is a set of preferences of all attributes. Mathematically, $V_l^{id}(t,\mathbf{P})$ is defined by

$$V_l^{id}(t,\mathbf{P}) = \left\{ v_{l,1}^{id}, v_{l,2}^{id}, ..., v_{l,|M|}^{id} \right\}, \forall l,i,d \tag{6}$$

The set also represents passengers' preferences over different PT attributes, i.e., $v_{l,1}^{id}$ is the most important attribute and $v_{l,|M|}^{id}$ is the least important one. Meanwhile, $v_{l,m}^{id}$ denotes the scalar value of attribute *m* rather than a monetary value or a weighted value to reduce the number of parameters to be calibrated.

Suppose that at node *i* there are two candidate options leading to destination *d*, i.e., using PT lines *l* and *l'*. We say that option *l* is better than option *l'* if the following holds:

$$l \succ l' \Leftrightarrow \begin{cases} \exists q \in (1,...,m), v_{l',q}^{id} - v_{l,q}^{id} > \beta_q^{JND} \min\left(v_{l,q}^{id}, v_{l',q}^{id}\right) \cdots\cdots\cdots\cdots\cdots\cdots (a) \\ \forall k \le q-1, \left|v_{l,k}^{id} - v_{l',k}^{id}\right| \le \beta_k^{JND} \min\left(v_{l,k}^{id}, v_{l',k}^{id}\right) \cdots\cdots\cdots\cdots\cdots\cdots (b) \end{cases} \tag{7}$$

Eq. (7a) states that the value of option *l'* is larger (less desirable) than that of *l*'s for the $q^{\text{th}}$ attribute. Eq. (7b) states that for the $k^{\text{th}}$ attribute among the first *q*-1 attributes the difference of values between options *l* and *l'* are within the JND threshold, meaning that a passenger does not notice the difference. It is worth noting that the *min* operator could be replaced by a *max* operator whenever applicable; for instance, if the attribute is getting points (similar to credit points with airlines), then, naturally one wants to maximize it.

In Eq. (7), the symbol "$\succ$" is used to denote a strictly preferred relationship; that is, $l \succ l'$ indicates that the passenger prefers option *l* over *l*'. In Eq. (7a) if no *q* exists to suit the equation, it means that travelers are indifferent to the differences between the two options *l* and *l'* ("~"). Mathematically speaking

$$l \sim l' \Leftrightarrow \forall q \in (1,...,m), \left|v_{l,q}^{id} - v_{l',q}^{id}\right| \le \beta_q^{JND} \min\left(v_{l,q}^{id}, v_{l',q}^{id}\right) \tag{8}$$

We integrate Eqs. (7) and (8), and use symbol "$\succeq$" to denote a weak preferred *l* over *l*' by

$$l \succeq l' \Leftrightarrow l \succ l' \vee l \sim l' \tag{9}$$

That is, option *l* is better or equal to option *l*', where $\vee$ means both are true or either is true, and $\sim$ means indifference. In addition, it is necessary to differentiate the symbols "$\succ$" (i.e., $l \succ l'$) and "$\le$" (i.e,. $k \le q-1$). The former denotes a preferred relationship and the latter compares between the attribute's order indexes. As is mentioned above under Eq. (6) the smaller the order index is, the more important the attribute is.

Intuitively, the comparison between lines *l* and *l'* using Eq. (7) can be clarified as follows: after determination of the set of acceptable choices, expressed by Eq. (3), we compare the choices of the acceptable set and look into each attribute according to the passenger's preference order. Suppose that two choices, the use of PT line *l* or *l'*, are compared. Then, according to the methodology developed:

a. If the values of the $k^{th}$ attribute of the two options are not noticeable to the passenger, then the passenger is indifferent to the two choices and we proceed to the comparison of the next attribute, expressed by Eq. (7b).
b. If the difference between the two options, for the $q^{th}$ attribute, is noticeable, then the passenger considers the best one of the two as a better choice, as expressed by Eq. (7a).

The following example is used as an explanatory device of Eq. (7). Consider three attributes: travel time, fare, and waiting time. A passenger ranks travel time as the most important attribute, followed by fare and waiting time. For simplicity, it is assumed that both options A and B are acceptable to the passenger. The data and computation procedures for the comparison are given in Table 1.

From Table 1(a) we conclude that option A is better than B, i.e., $A \succ B$. The last column of this table shows that the passenger does not perceive the difference of 1.5 minutes between the travel times, or in other words it is less than the JND threshold value. Then, the difference of the waiting time is noticed, thus making option A a better choice. We note that the difference of the fare is also noticed (better for B than for A), but the waiting time is more important than the fare. Therefore the conclusion drawn is that option A is better than B.

Table 1. Comparison based on Eq. (7)

(a) Compare A and B

| $m$ | Option A | Option B | $\beta_m^{JND}$ | $\lvert v_{l,m}^{id} - v_{l',m}^{id} \rvert$ | $\beta_m^{JND} \min\left(v_{l,m}^{id}, v_{l',m}^{id}\right)$ | Noticeable difference |
|---|---|---|---|---|---|---|
| Travel time (very important) | 11.5 min | 10 min | 20% | 1.5 min | 2 min | No: 1.5 min < 2.0 min |
| Waiting time (important) | 5 min | 10 min | 20% | 5 min | 1 min | Yes: 5 min > 1 min |
| Fare (least important) | $ 5 | $ 4 | 10% | $ 1 | $ 0.4 | Yes: $1 > $0.4 |

However, the personalized attribute-based option selection process above may not satisfy the axiom of transitivity, which requires that if $A \succeq B$ and $B \succeq C$, then $A \succeq C$. This can be proven by the following two continuations of Table 1 using a third option, C.

(b) Compare B and C

| $m$ | Option B | Option C | $\beta_m^{JND}$ | $\lvert v_{l,m}^{id} - v_{l',m}^{id} \rvert$ | $\beta_m^{JND} \min\left(v_{l,m}^{id}, v_{l',m}^{id}\right)$ | Noticeable difference |
|---|---|---|---|---|---|---|
| Travel time (very important) | 10 min | 9 min | 20% | 1 min | 1.8 min | No: 1 min < 1.8 min |
| Waiting time (important) | 10 min | 9 min | 20% | 1 min | 1.8 min | No: 1 min < 1.8 min |
| Fare (least important) | $ 4 | $ 5 | 10% | $ 1 | $ 0.4 | Yes: $ 1 > $ 0.4 |

(c) Compare A and C

| $m$ | Option A | Option C | $\beta_m^{JND}$ | $\lvert v_{l,m}^{id} - v_{l',m}^{id} \rvert$ | $\beta_m^{JND} \min\left(v_{l,m}^{id}, v_{l',m}^{id}\right)$ | Noticeable difference |
|---|---|---|---|---|---|---|
| Travel time (very important) | 11.5 min | 9 min | 20% | 2.5 min | 1.8 min | Yes: 2.5 min > 1.8 min |
| Waiting time (important) | 5 min | 9 min | 20% | 4 min | 1 min | Yes: 4 min > 1 min |
| Fare (least important) | $ 5 | $ 5 | 10% | $ 0 | $1 | No: $ 0 < $1 |

From Table 1(b) and Table 1(c), it is concluded that $B \succeq C$ and $C \succeq A$, respectively. Thus, we show here that the process of using Weber's law violates the axiom of transitivity.

The axiom of transitivity is crucial because of the need to ensure robust and consistent methodology. Our framework is established on the assumption that the traveler is rational, has preferences over different PT attributes and can arrange different PT attributes by order of importance. This rationalism calls for maintaining the axiom of transitivity, and for setting the personalized preferences within a feasible and rational framework. Realistically, the input of preferences are boundless and some users may input anything they want (also for their own amusement). This may violate the axiom of transitivity and thus must be checked and adjusted.

To amend this axiom violation possibility, we propose a revised comparison equation: to use the minimum value of attributes across all options as the reference point and define a noticeable threshold as the product of that minimum value and the Weber's JND parameter. We shall call this the adjusted JND. Mathematically, it is as follows:

$$l \succ l' \Leftrightarrow \begin{cases} \exists q \in (1,\ldots,m), v_{l',q}^{id} > v_{l,q}^{id} \text{ and } v_{l',q}^{id} - \min\limits_{\bar{l} \in A_i^+} \left\{ v_{\bar{l},q}^{id} \right\} > \beta_q^{JND} \min\limits_{\bar{l} \in A_i^+} \left\{ v_{\bar{l},q}^{id} \right\} \cdots\cdots\cdots\cdots\cdots\cdots (a) \\ \forall k \le q-1, \begin{cases} v_{j,k}^{id} - \min\limits_{\bar{l} \in A_i^+} \left\{ v_{\bar{l},k}^{id} \right\} \le \beta_k^{JND} \min\limits_{\bar{l} \in A_i^+} \left\{ v_{\bar{l},k}^{id} \right\} \ \forall j \in \{l, l'\}, \text{or} \\ v_{l',k}^{id} = v_{l,k}^{id}, \text{if } v_{j,k}^{id} - \min\limits_{\bar{l} \in A_i^+} \left\{ v_{\bar{l},k}^{id} \right\} > \beta_k^{JND} \min\limits_{\bar{l} \in A_i^+} \left\{ v_{\bar{l},k}^{id} \right\} \ \forall j \in \{l, l'\} \end{cases} \cdots\cdots\cdots (b) \end{cases} \tag{10}$$

Eq. (10a) states that if option $l$ is better than option $l'$, then there exists the $q^{th}$ attribute such that its value of $l'$ is larger (less desirable) than that of $l$, and the difference between this value and the minimum attribute's value across all feasible PT lines/routes is noticeable (crosses the JND threshold). Eq. (10b) states that for the $k^{th}$ attribute among the first $q$-1 attributes being more important than the $q^{th}$ attribute, the difference between its value and the minimum attribute's value across all feasible lines is not noticeable (does not cross the JND threshold) for both $l$ and $l'$. Similar to Eq. (7), the min operator could also be replaced by a max operator. In addition, we can rigorously define the indifference between optional lines, and the weak preferred line conditions based on Eqs. (8) and (9).

**Theorem 1**: The comparison method based on Eq. (10) satisfies the axiom of transitivity.
***Proof***. See Appendix A.

Based on the theorem, we can obtain the following corollaries.

*Corollary 1*: If path A is better than path B because of attribute $q$ (with a noticeable difference), the $q^{th}$ attribute values of paths B and C are not equal, and path B is better than path C because of attribute $\bar{q}$, then attribute $\bar{q}$ is more important than attribute $q$ or equally as important as attribute $q$.
***Proof***. See Appendix A.

Corollary 1 implies that for a given traveller's choice between pairwise alternatives, we can deduce the traveller's preference.

*Corollary 2*: In continuation of *corollary 1*; given $A \succ C$ because of the $q^{th}$ attribute and $B \succ C$ because of the $\bar{q}^{th}$ attribute.

(a) If $q < \bar{q}$, then $A \succ B$ (if attribute $q$ is more important, then A is better than B);
(b) If $q > \bar{q}$, then $B \succ A$ (if attribute $\bar{q}$ is more important, then B is better than A).

***Proof***. See Appendix A.

Corollary 2 can be viewed as the counterpart of corollary 1. It indicates that for a given traveler's pairwise alternatives and a partial preference, we can deduce the traveler's preferences with regards to all options. A partial preference means that we only need to know the importance of $q$ and $\bar{q}$ attributes, rather than all of the PT attributes.

The example from Table 1 has undergone changes in Table 2 to elaborate the comparison using Eq. (10) and to demonstrate that the results are in compliance with the axiom of transitivity.

Table 2. Comparison based on Eq. (10)
(a) The minimum value and JND threshold for each attribute

| $m$ | Option A | Option B | Option C | $\min\limits_{\bar{l} \in A_i^+} \left\{ v_{\bar{l},q}^{id} \right\}$ | $\beta_m^{JND}$ | $\beta_m^{JND} \min\limits_{\bar{l} \in A_i^+} \left\{ v_{\bar{l},q}^{id} \right\}$ |
|---|---|---|---|---|---|---|
| Travel time (very important) | 11.5 min | 10 min | 9 min | **9 min** | 20% | **1.8 min** |
| Waiting time (important) | 5 min | 10 min | 9 min | **5 min** | 20% | **1.0 min** |
| Fare (least important) | $ 5 | $ 4 | $ 5 | **$ 4** | 10% | **$ 0.4** |

(b) Compare A and B*

| $m$ | Option A | Option B | $v_{l',q}^{id} - \min\limits_{\bar{l} \in A_i^+} \left\{ v_{\bar{l},q}^{id} \right\}$ | Compare with $\beta_m^{JND} \min\limits_{\bar{l} \in A_i^+} \left\{ v_{\bar{l},q}^{id} \right\}$ | Noticeable difference |
|---|---|---|---|---|---|
| Travel time (very important) | 11.5 min | 10 min | A: (11.5 – 9) = 2.5 min<br>B: (10 – 9) =1.0 min | A: 2.5 >1.8 min<br>B: 1.0 < 1.8 min | A: Yes<br>B: No |
| Waiting time (important) | 5 min | 10 min | A: (5 – 5) = 0 min | A: 0 < 1.0 min | A: No |

| | | | | | |
|---|---|---|---|---|---|
| | | | B: (10 – 5) = 5 min | B: 5 >1.0 min | B: Yes |
| Fare (least important) | $ 5 | $ 4 | A: (5 – 4) = $ 1<br>B: (4 – 4) =$ 0 | A: $1 > $ 0.4<br>B: $ 0 < $ 0.4 | A: Yes<br>B: No |

*It is concluded from the table that $B \succeq A$ because of noticed difference between travel times (and the least is noticed for B).

(c) Compare B and C**

| $m$ | Option B | Option C | $v_{l',q}^{id} - \min_{\bar{l} \in A_i^+}\{v_{\bar{l},q}^{id}\}$ | Compare with $\beta_m^{JND} \min_{\bar{l} \in A_i^+}\{v_{\bar{l},q}^{id}\}$ | Noticeable difference |
|---|---|---|---|---|---|
| Travel time (very important) | 10 min | 9 min | B: (10 – 9) = 1.0 min<br>C: (9 – 9) = 0 min | B: 1.0 < 1.8 min<br>C: 0 < 1.8 min | B: No<br>C: No |
| Waiting time (important) | 10 min | 9 min | B: (10 – 5) = 5 min<br>C: (9 – 5) = 4 min | B: 5 > 1.0 min<br>C: 4 >1.0 min | B: Yes<br>C: Yes |
| Fare (least important) | $ 4 | $ 5 | B: (4 – 4) = $ 0<br>C: (5 – 4) =$ 1 | B: $1 > $ 0.4<br>C: $ 0 < $ 0.4 | B: Yes<br>C: No |

**It is concluded from the table that $C \succeq B$ because of noticed difference between waiting times (and the least is noted for C).

(d) Compare A and C***

| $m$ | Option A | Option C | $v_{l',q}^{id} - \min_{\bar{l} \in A_i^+}\{v_{\bar{l},q}^{id}\}$ | Compare with $\beta_m^{JND} \min_{\bar{l} \in A_i^+}\{v_{\bar{l},q}^{id}\}$ | Noticeable difference |
|---|---|---|---|---|---|
| Travel time (very important) | 11.5 min | 9 min | A: (11.5 – 9) = 2.5 min<br>C: (9 – 9) = 0 min | A: 2.5 >1.8 min<br>C: 0 < 1.8 min | A: Yes<br>C: No |
| Waiting time (important) | 5 min | 9 min | A: (5 – 5) = 0 min<br>C: (9 – 5) = 4 min | A: 0 < 1.0 min<br>C: 4 >1.0 min | A: No<br>C: Yes |
| Fare (least important) | $ 5 | $ 5 | A: (5 – 4) = $ 1<br>C: (5 – 4) =$ 1 | A: $1 > $ 0.4<br>C: $ 0 < $ 0.4 | A: Yes<br>C: No |

*** It is concluded from the table that $C \succeq A$ because of noticed difference between travel times (and the least is noted for C).

The three comparisons above follow the constructed Eq. (10) to assure agreement with the axiom of transitivity. This implies that if $B \succeq A$ and $C \succeq B$, then $C \succeq A$. It is also observed that $C \succeq B$ because of the waiting time and $B \succeq A$ because of the travel time. This confirms *corollary 1* to show that travel time is a more important attribute than waiting time.

*Corollary 1* and *corollary 2* imply that the perceived importance of the attributes always complies with the axiom of transitivity. These corollaries pave the way for us to develop a generalized sorting algorithm considering order of importance and the JND threshold. The algorithm is as follows (for simplicity, the superscript "*id*" is omitted).

**Algorithm 1**: Lexicographical ordering method using JND threshold (the JND sorting algorithm)
**Input**: Set of acceptable options $S$, and attributes of each option $\{v_{l.m}\}$
**Output**: Set of sorted options: $\bar{S}$
**Procedure**:

0: $\bar{S} = \varnothing$ // Initialization
1: **For** $i$ = 1 to $m$ // Compare from the most important to the least important attributes
2: $Q^i = \left\{ l \middle| v_{l,i} - \min_{l' \in S}\{v_{l',i}\} > \beta_i^{JND} \min_{l' \in S}\{v_{l',i}\} \right\}$ // Generate the noticeable set of (line) options
3: $\bar{Q}^i = \text{sort}(Q^i, i)$ // Sort $Q^i$ in ascending order based on the value of the $i^{\text{th}}$ attribute
4: $\bar{S} = \bar{Q}^i \cup \bar{S}$ // Insert $\bar{Q}^i$ into set $\bar{S}$
5: $S = S - Q^i$ // Update the remaining choice set to be sorted.
6: If $S = \varnothing$:
7: break // Stop if all the options have been added to the ordered set

**Theorem 2:** The JND sorting algorithm is correct.
***Proof***. See Appendix B.

Remarks on the above JND sorting algorihtm: (i) in step 2 of the above procedure, when two or more (line) options of set $Q^i$ have an equal value of the $i^{\text{th}}$ attribute, we need to further compare the $(i+1)^{\text{th}}$ and the $m^{\text{th}}$ attributes of these options until preference is detected; (ii) in step 4, the insert of $\bar{Q}^i$ at the beginning of $\bar{S}$ is done because of the options in $\bar{Q}^i$ being better than the options determined previously; (iii) the JND sorting algorithm applies when the set of

options is given; this set represents the scenario after enumerating all paths following the sorting of paths to satisfy passenger's preference; and (iv) the complexity of the algorithm involves also the complexity of the sorting algorithm of step 3; in our current implementation we use the genetic sort algorithm in C++ standard library with the worst-case complexity known to be $O(n \log n)$.

The preceding comparison and ordering method rely on knowing all options a priori (see Table 2(a)). This may not be efficient for large network applications because of the required path enumeration and storage. Thus, it would be more promising to develop a shortest path or *k*-shortest lexicographical ordering path algorithm based on the proposed comparison method. However, two issues are observed in the development of the shortest path algorithm. First, some PT attributes are not link additive, such as zone based fare or points from airlines, making path enumerating inevitable; which we leave for a future study. Second, the options at a node cannot be known a priori because of the label setting/correcting step of the shortest path algorithm; this step compares the existing best option and a new option and chooses the better one. Then the best path is adjusted. Nonetheless, thanks to the fact that the shortest path algorithm only determines the best option, the axiom of transitivity still holds. This is formally stated as the following corollary:

*Corollary 3*: In continuation of *corollaries 1* and *2*; if $B \succeq C$ with respect to options $\{B, C\}$ and $A \succeq B$ with respect to options $\{A, B, C\}$, then $A \succeq C$ .

***Proof***. See Appendix A.

Corollary 3 states that the axiom of transitivity holds when augmenting the option set. Determining the best option does not require knowing all the options in advance. Thus, the set of options can be gradually updated. Corollary 3 allows for developing a multi-criteria shortest path as follows.

**Algorithm 2**: Shortest path for lexicographical ordering method with JND threshold (shortest path of ordering preferences)

**Input**: Set of attributes $M$ , Set of nodes $N$ , Set of arcs $A = \{(i, j) | i, j \in N, i \neq j\}$

Attributes associated with each edge: $V = \left\{ V_{(i,j)} \middle| V_{(i,j)} = \left\{ v_{(i,j),1}, v_{(i,j),2}, .., v_{(i,j),m} \right\} \right\}$ , source node is *s*.

**Output**: Shortest path from *s* to all nodes

**Procedure**:

0: $L(s) = \{\pi_{s,q} = 0, \forall q \in M\}, L_{\min}(s) = \{\pi_{s,q}^{\min} = 0, \forall q \in M\}$, // Initialize label at source node

$L(u) = \{\pi_{u,q} = +\infty, \forall q \in M\}, L_{\min}(u) = \{\pi_{u,q}^{\min} = +\infty, \forall q \in M\}, u \neq s$ // Initialize labels at other nodes

1: $S = \{s\}$ // Initialize the queue of nodes to be scanned

3: While $S \neq \varnothing$

2: Select $i^*$ in $S$ // Popup a node from the queue

3: For each neighbor node $j$ , such that $(i^*, j) \in A$ // Select a neighbor node

4: For $q$ = 1 to $m$

5: If $\pi_{j,q}^{\min} > \pi_{i^*,q}^{\min} + v_{(i^*,j),q}$

6: $\pi_{j,q}^{\min} = \pi_{i^*,q}^{\min} + v_{(i^*,j),q}$ //Update the minimum label

7: $S = S \cup j$

8: End if

9: End for

10: If $V_{(i^*,j)} + L(i^*) \succ L(j)$

11: $L(j) = V_{(i^*,j)} + L(i^*)$ // Update the best option label

12: $S = S \cup j$ // Add node to the queue

13: End if

14: End for

15: $S = S - \{i^*\}$ // Remove scanned node

16: End while

Remarks on the algorithm above for the shortest path of ordering preferences: 1) The label associated with every node and arc is a vector of attributes. Accordingly, the "+" in Steps 10 and 11 represents vector addition. 2) We create another label $L_{\min}$ for each node in comparison with the ordinary shortest path algorithm. This label is required to store the minimum value of each attribute and add another loop, Steps 4-9, to update the minimum attribute label. 3) We compare the preference relationship "$\succ$" under Step 5.

We note that Algorithm 2 does not address the personal max/min acceptable value feature. For incorporating it, one needs to solve a constrained shortest path problem, known to be NP-complete (i.e., Handler and Zang 1980; Lorenz and Raz 2001; Dumitrescu and Boland 2001). In our study the *k*-shortest path approach finds the *k*-shortest lexicographical ordered paths with JND threshold based on Yen's *k*-shortest path algorithm (Yen 1971); that is, the routes that violate the boundary constraints are eliminated after generating the *k*-shortest lexicographical ordered paths. Because of focusing on route guidance methodologies for future personalized PT, and because of space limitation, the details of the *k*-shortest path algorithm are not presented.

## 3. Illustrative example

A schematic map of the illustrative example appears in Fig. 3. It contains a small PT network comprised of walking, use of bus and metro lines, and making transfers. There are assumed to be two passengers, A and B, departing from the origin to the destination shown in Fig. 3. The two passengers naturally have different preferences.

### *3.1. Data and assumptions*

The notations, data and assumptions of the example are shown in Table 3. Three groups of PT attributes are considered: time related, fare related, and comfort related. The group of time related attributes may include travel time, waiting time, walking time, transfer time, elevator time (if any), etc. The group of fare related attributes may include fare of trip, fare of special app (if any), special service fare (if any), etc. The group of comfort related attributes

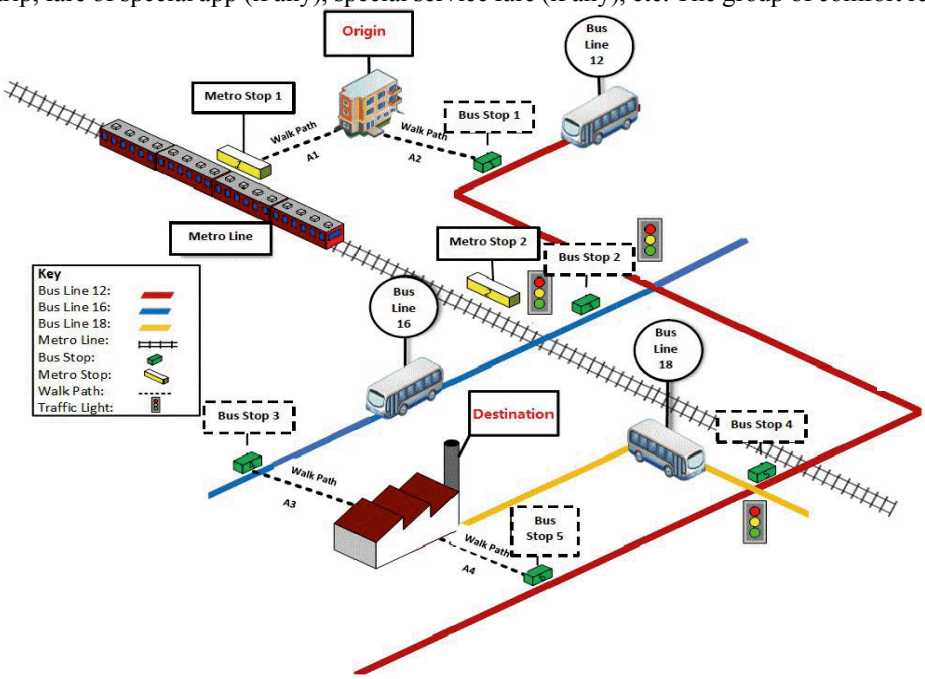



Fig. 3: Schematic map of an illustrative example

(for example, on a scale of 0-5, where 0=no comfort at all, and 5=excellent comfort) may include waiting comfort, riding comfort, transfer comfort, etc. It is worth noting that the PT attributes can be treated by individual attributes, and not by groups, as is demonstrated in the examples of Tables 1 and 2. With the use of the three groups of PT attributes we illustrate, in the following problem example, the differences among the three path recommendation methods.

Table 3. Input for the schematic map example of Fig. 3

(a) Notations used for the example

| | | |
|---|---|---|
| $WP_i$ = Walk path $i$ | $BLi$ = Bus line $i$ | $BSi$ = Bus stop $i$ |
| $MS_i$ = Metro stop $i$ | $M$ = Metro | (…) = Element in the parenthesis is a PT transfer point |

(b) Routing data

| Option | Routing | Total fare ($)* | Travel time (min)** | Comfort level *** |
|---|---|---|---|---|
| 1 | $WP_{A1}$ - $MS_1$ - $M$ - ($MS_2$/$BS_2$) - $BL_{16}$ – $BS_3$ - $WP_{A3}$ | 12 | 70 | 3 |
| 2 | $WP_{A2}$ - $BS_1$ - $BL_{12}$ – $BS_5$ -$WP_{A4}$ | 8 | 96 | 4 |
| 3 | $WP_{A2}$ - $BS_1$ - $BL_{12}$ - ($BS_4$) - $BL_{18}$ | 10 | 75 | 4 |
| 4 | $WP_{A2}$ - $BS_1$ -$BL_{12}$ - ($BS_2$) - $BL_{16}$ – $BS_3$- $WP_{A3}$ | 15 | 105 | 2 |

*of all fare related attributes; ** of all time related attributes; *** of all comfort related attributes

(c) Passenger's preferences and weights (all weights sum up to 1.0)

| Preference | Passenger A | | Passenger B | |
|---|---|---|---|---|
| | Attribute | Weight | Attribute | Weight |
| Most important | Fare related | 0.60 | Time related | 0.60 |
| Moderately important | Time related | 0.30 | Comfort related | 0.30 |
| Least important | Comfort related | 0.10 | Fare related | 0.10 |

(d) Passenger's preference and input of JND threshold (by percentage) and max/min acceptable values

| Preference | Passenger A | | | | Passenger B | | | |
|---|---|---|---|---|---|---|---|---|
| | Attribute | $\beta_m^{JND}$ | $\beta_m^{JND} \min_{\bar{l} \in A_i^+} \{v_{\bar{T},q}^{id}\}$ | Acceptable value $\eta_m^{\text{accept}}$ | Attribute | $\beta_m^{JND}$ | $\beta_m^{JND} \min_{\bar{l} \in A_i^+} \{v_{\bar{T},q}^{id}\}$ | Acceptable value $\eta_m^{\text{accept}}$ |
| Most important | Fare related | 10% | $0.8 | $15 | Time related | 10% | 7.0 min | 100 min |
| Moderately important | Time related | 30% | 21.0 min | 120 min | Comfort related | 20% | 0.4 | 3 |
| Least important | Comfort related | 40% | 0.8 | 1 | Fare related | 30% | $2.4 | $20 |

### *3.2. Clarifications and results*

To ensure clarity, following are a few comments on the example:

1) $BS_2$ is linked with $MS_2$, $BS_{12}$ and $BS_{16}$.
2) Passenger A is a student, thus placing a high weight on the fare related attributes, expressed in the example as total fare. In addition, because of being sensitive to pocket dollars, the JND threshold of the total fare for Passenger A is low (it is an input based perceived value).
3) Passenger B is a traveler in a hurry, thus placing high weight on time related attributes, expressed in the example by travel time. Also, because of being pressed for time the JND travel time threshold for Passenger B is low (it is an input based perceived value).
4) Travel time (in minutes) contains all of the time related attributes: walking, waiting and riding times.
5) Total fare (in dollars) contains the riding fare and any other fare required for using the PT service.
6) Comfort level ranges from 0 (least comfort) and 5 (maximum comfort) and refers to the overall comfort of walking, waiting, riding and making transfers with each option. The JND threshold of comfort, expressed by percentage, refers to the 0-5 scale. Accordingly, 20% means that the passenger does not perceive the difference between two

adjacent comfort levels, but a difference of two comfort levels is noticeable. For example, the comfort level for walking flat on asphalt is 4, for a low uphill gradient is 3, and for a semi-moderate uphill gradient is 2.

7) The parameter $\mu_m$ that converts the $m^{th}$ attribute's value to monetary value is set to be \$0.2 per minute for the travel time attribute, and \$(-1) per one unit of the comfort scale. We note that the higher the comfort level, the lower the travel cost, and thus $\mu_m$ is negative for the comfort scale, unless this scale is interpreted in the opposite direction. Both parameters can be based, either for each attribute on an average survey based monetary value, or on a fixed input monetary value per passenger. For the former, the travel time can be based on studies of value of time.

Subsequent to these clarifications, we present the results of the example in Table 4.

Table 4. Results of recommended paths per method used

| **Path recommendation method** | **For Passenger A** | **For Passenger B** |
|---|---|---|
| Shortest path method* | Option 3 | Option 3 |
| Shortest path based on the most important attribute | Option 2 | Option 1 |
| *k*-weighted shortest path method** | Option 3 $\succ$ 2 $\succ$ 1 $\succ$ 4 | Option 1 $\succ$ 3 $\succ$ 2 $\succ$ 4 |
| Lexicographical order with JND | Option 2 $\succ$ 3 $\succ$ 1 $\succ$ 4 | Option 3 $\succ$ 1 $\succ$ 2 |

* The shortest path of total travel cost; ** The weighted travel cost is computed by Eq. (1)

Let us elaborate on the results of Table 4. The use of the shortest path method is straightforward because it is based only on the minimum travel cost across all paths, using Eq.(1) with the weighing factor $w_m$=1.0. Thus, for both Passengers A and B it will be Option 3, because of the travel cost of path Option 1 (see Table 3(b)) is 12+0.2*70+3*(-1)=\$23, and for Options 2, 3, and 4 is \$23.2, \$21 and \$34, respectively. If the shortest path determination is based on the most important attribute, Options 2 and 1 are recommended for passengers A and B, respectively. Essentially, this case is equivalent to the lexicographical order method without the consideration of JND, or to the weighted shortest path method with a weight set up to 1 for the most important attribute.

For the *k*-weighted shortest path method we consider the weights of Table 3(c), the parameter $\mu_m$ values of \$0.2 and \$(-1) as per comment (vii) above, and the attributes' values in Table 3(b). For example, for path Option 1 for Passenger B the weighted travel cost is 0.1*12+0.6*0.2*70+0.3*(-1)*3=\$8.7, and for Options 2, 3, 4 the costs are \$11.12, \$8.8, and \$13.5, respectively. Thus, for Passenger B Option 1 $\succ$ 3 $\succ$ 2 $\succ$ 4.

For the method using lexicographical order with JND we use the information of Table 3(d) and of the attributes' values from Table 3(b). For passenger B the travel time of path Option 4 is more than the passenger's max acceptable value (105 min > 100 min) and thus this option is eliminated. Then, we first consider the most important time related attribute and look at its minimum value plus the adjusted JND threshold of Eq. (10) created to satisfy the axiom of transitivity. That is, 70 min + 7 min to find out that path Options 1 and 3 are perceived as the same (75<77min, thus the difference is not noticeable) and better than Option 2. Secondly, because Options 1 and 3 are percieved as the same, we look at the moderately important comfort related attribute to find that Option 3 is better than Option 1, where the difference of comfort units is greater than the adjusted JND threshold (1>0.4). Thus, Option 3 $\succ$ 1 $\succ$ 2. We find a similar situation to be the case for Passenger A.

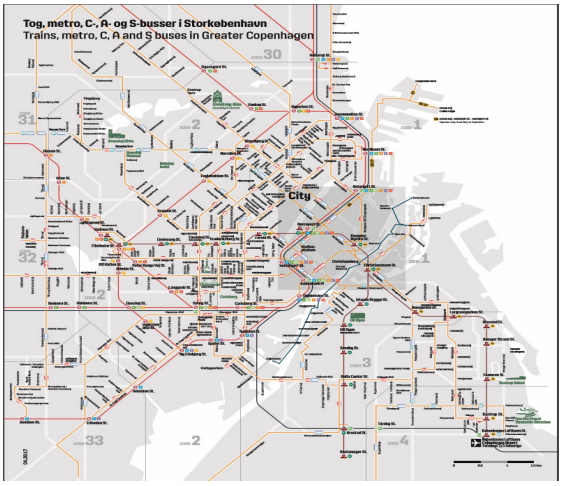
### 3.3. *Interpretation*

A few characteristics of the illustrated example are noteworthy, including explanations of the results shown in Table 4, as follows: (1) The shortest path method finds only one path based on the total travel cost, while the *k*-weighted shortest path and the JND methods recommend a set of paths; (2) The *k*-weighted shortest path and the JND methods recommend different path sets for Passengers A and B. This demonstrates that our method could provide personalized path information for different passengers; (3) For passenger B (a traveller in a hurry), the *k*-weighted shortest path ranks path Option 1 as the best, and the JND method ranks path Option 3 as the best; however, for the most important time-related attribute both options are perceived identically because of being inside the range of the JND threshold. The JND method in this case shows its advantage by allowing the next important comfort related attribute to determine the preference of Passenger B; (4) For Passenger A (a student) using the *k*-weighted shortest

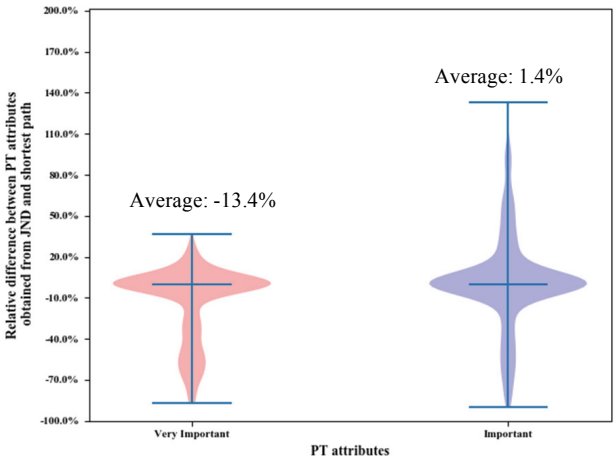
path method, we find that the difference between path Options 2 and 3 is small ($10.16 for Option 2 and $10.10 for Option 3); however, for the JND method path Option 2 is clearly the preferred one from the perspective of the most important preference (fare related). This observation demonstrates some of the differences between the two methods.

## 4. Case Study

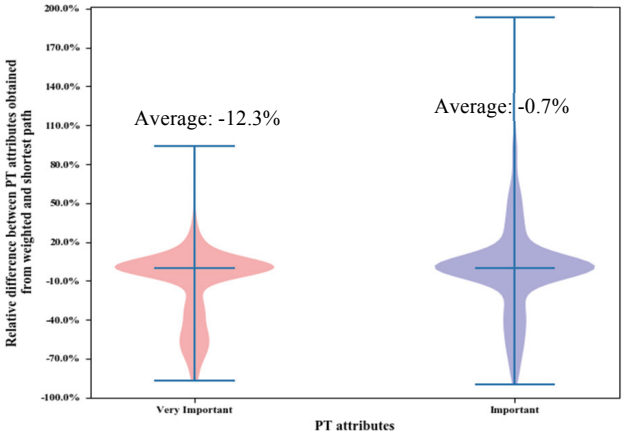
The methodology developed was tested using the Copenhagen network shown in Fig. 4(a) and (b). The Greater Copenhagen Area is divided into 99 zones. The network studied contains Copenhagen's Zones 1, 2, 3, and half of 4,



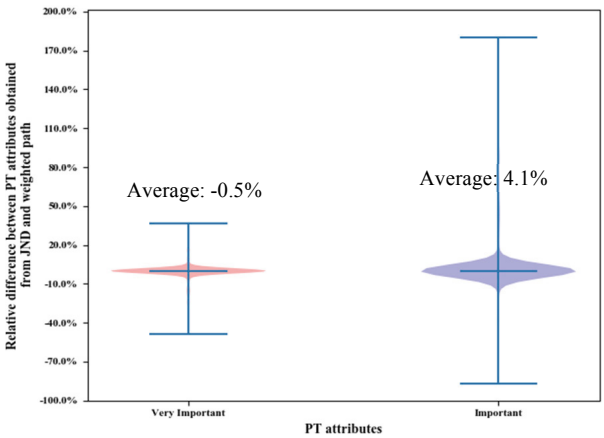

(a) Copenhagen PT lines




(b) Comparison between the best routes of the JND and shortest path methods




(c) Comparison between the best routes of the *k*-weighted and shortest path methods




(d) Comparison between the best routes of the *k*-weighted and JND methods

Fig.4 Case study description and results for Copenhagen Network

covering most of central Copenhagen. In short, there are 278 PT lines and 397 stops. The experiments were conducted based on the following setting and assumptions. A1) Two attributes were considered: travel time (TT) and waiting time (WT). The WT of each line is based on half of the headway, the inverse of the frequency (Ceder 2016). A2) Total of 1000 travelers were simulated. For each traveler the JND values, $\beta_m^{JND}$, were randomly generated between 10% - 30%. Simultaneously, the passenger's weighting parameters for each attribute were also simulated. Passenger preference was derived from the values of weighting parameters, (i.e., the higher the weighting parameter is, the higher

the order is); A3) Because of the variation in the magnitude of PT attributes across the OD pairs, it is not easy to set a constant max acceptable value for all the OD pairs. For example, the TT for one OD could be 90 mins, but only 10 mins for another. In such a case, a constant max acceptable value (i.e., 60 mins) is infeasible for the first OD pair and has no effect on the second one. Therefore, we use the minimum value of each attribute's shorest path, as a benchmark, and set its max acceptable value to be twice the minimum value; A4) For analyzing and comparing the results of the three different methods we assume that a traveler selects only the first recommended path. Then we compare the two PT attributes of the first path recommended by each of the three methods to evaluate whether or not this traveler can benefit by each of the recommended paths and by what magnitude; A5) The shortest path refers only to travel time, not travel cost. Based on the above setting we found the shortest travel time path, *k*-weighted shortest path (we set $k = 1$ according to A4)), and the lexicographical shortest path with JND threshold. The program was coded in C++ and run on a personal laptop with Intel(R) Core(TM) i7-6600U CPU @2.60GHz.

Fig. 4(b), (c) and (d) illustrate how the changes in the two attributes vary. For instance, Fig. 4(b) compares the JND and shortest path methods. A positive percentage means that the value of the attribute in the JND method is larger than that obtained in the shortest path method. That is, for the very important PT attribute, most of the relative difference values are negative, indicating that the JND method is better than the shortest path method in reducing the very important attribute. Moreover, though some relative difference values are positive, meaning that the JND path is worse, the max relative difference is less than the assumed max noticeable difference of 0.3 for the very important attribute. All in all, the JND method prefers the most important PT attribute with the possibility to consider the next important attribute for cases where the path options are perceived same (below the JND threshold) as is shown, for example, in Table 4 for Passenger B between Options 3 and 1 for the JND method. Fig. 4(c) shows that the weighted path method outperforms the shortest path method with a 12.3% reduced average value of the most important attribute. The comparison between the JND and the weighted path methods, in Fig. 4(d), illustrates a superiority of the proposed JND method, but not as significant as in Fig. 4(b), probably because of the use of only two attributes being caused by data limitation. Thus, a future study could incorporate more PT attributes such as fare (if applicable), and comfort.

The reasoning of the distributions shown in Fig. 4 is two-fold. The first reason being is the randomness existed in generating the parameters of weights and the JND threshold. The second and more important reason is the variation of the attributes across different PT services; for example, an express bus with TT = 10 min and WT = 60 min compared with a regular bus with TT = 20 min and WT=30 min. Such differences also warrant the importance of the route guidance tools to provide a special attention to the preferred PT attribute. From a PT operation's perspective these large differences of TT and WT can serve as food for thought for the design of different paths. These considerations, in the upcoming personalized era, may have impact on PT network design models for both ordinary and automated PT services – whether or not to design multiple PT routes such that each route caters to a class of travelers associated with certain preferences.

A note worth mentioning is that the simulation study could also incorporate the selection process of the users, but this will require establishing a transit assignment model to be consistent with the proposed route guidance methodology considering both users' personalised requests and the JND values. This deserves a further study.

Finally the computation time of the Copenhagen case study for the shortest path, *k*-weighted shortest path, and the lexicographical order with JND path methods are 0.10, 0.15, and 0.21 seconds, respectively, implying favorable potential for real-life applications.

## 5. Conclusion

In view of the upcoming era of personalized PT mobility, this work provides, as a background, an adjusted design framework for creating the modeling required for a personalized PT service. The framework integrates the PT operators' planning and operation components with users' experiences gathered using smartphone technologies. This work focuses on the key element of the adjusted design framework, namely the personalized route guidance methodology. Explication of three different route guidance methods is offered and they are compared – the classical shortest path method, a *k*-weighted shortest path method, and a novel lexicographical ordering shortest path method with a just noticeable difference (JND) consideration. The JND based method is based on Ernst Weber's law of the

human perception threshold. This work discovers that a straightforward application of Weber's Law does not satisfy the axiom of transitivity required for an implementable algorithm, and thus a revised method was developed and proved for its correctness. A small network example is illustrated as an explanatory device to demonstrate the differences between the three methods. In addition, a large simulation study of the PT network of Copenhagen is conducted. The results of the case study show that the average reduction of the value of the most important PT traveler's attribute is 12.3% for the *k*-weighted shortest path method, and 13.4% for the lexicographical JND based shortest path method in comparison with the classical shortest path method.

This work opens a new arena for public transport researchers and planners with the use of smartphone apps to acquire real time and readily available journey planning information. It is apparent that through the apps, a traveler seeks a satisfactory path that caters to the traveler's preferences at the time of a requested trip. Future research, for instance, can (i) be connected to PT network design models for both ordinary and automated PT services; (ii) conduct empirical studies for a calibration of the parameters used in this work, especially the human perception parameters; and (iii) continue the theoretical research using the rational choice theory for adjusting travelers' unintentional input.

## Acknowledgements

This work is partly funded by the Innovation Fund Denmark (IFD) under File No. 4109-00005.

## References

Arentze, T.A., 2013. Adaptive personalized travel information systems: A Bayesian method to learn users' personal preferences in multimodal transport networks. IEEE Transactions on intelligent transportation systems, 14(4):1957-1966.

Baird, J. C. and E. Noma. 1978. Fundamentals of scaling and psychophysics. John Wiley & Sons, Inc., New York.

Bouhana, A., Fekih, A., Abed, M., Chabchoub, H., 2013. An integrated case-based reasoning approach for personalized itinerary search in multimodal transportation systems. Transportation Research Part C, 31:30-50.

Calabrese, F., Pereira, F.C., Di Lorenzo, G., Liang, L., Ratti, C., 2010. The geography of taste: Analyzing cell-phone mobility and social events. In: Pervasive Computing, 6030:22-37.

Ceder, A. 2016. Public Transit Planning and Operation: Modeling, Practice and Behavior. Second Edition, CRC Press, Boca Raton, USA

Chorus, C.G., 2012. Travel Information: Time to Drop the Labels?. IEEE Transactions on Intelligent Transportation Systems, 13(3):1235-1242.

Chorus, C.G., Arentze, T.A., Timmermans, H.J., 2009. Traveler compliance with advice: A Bayesian utilitarian perspective. Transportation Research Part E, 45(3):486-500.

Chowdhury, S., Ceder, A., Schwalger, B. 2015. The effects of travel time and cost savings on urban commuters' perception of public transport routes involving transfers. Journal of Transport Geography, 43:151-159.

Eppstein, D., 1998. Finding the k shortest paths. SIAM Journal on computing, 28(2):652-673.

Galdames, C., Tudela, A., Carrasco, J.A., 2011. Exploring the role of psychological factors in mode choice models by a latent variables approach. Transportation Research Record, 2230:68-74.

Global Mass Transit report, 2017. Smartphone Apllications in Transit Services: Growing Popularity. Retrieved from http://www.globalmasstransit.net/archive.php?id+16210, 30 June 2016.

Grison, E., Burkhardt, J.M., Gyselinck, V., 2017. How do users choose their routes in public transport? The effect of individual profile and contextual factors. Transportation Research Part F, 51:24-37.

Handler, G. Y., Zang, I. 1980. A dual algorithm for the constrained shortest path problem. Networks 10(4): 293-309.

Handte, M., Foell, S., Wagner, S., Kortuem, G., Marrón, P.J., 2016. An Internet-of-Things Enabled Connected Navigation System for Urban Bus Riders. IEEE internet of things journal, 3(5):735-744.

Hadas, Y., Ceder, A., 1996. Shortest Path of Emergency Vehciles Under Uncertain Urban Traffic Conditions. Transportation Research Record, 1560:34-39.

Jiang, Y., Szeto, W.Y., 2016. Reliability-based stochastic transit assignment: Formulations and capacity paradox. Transportation Research Part B, 93:181-206.

Laming, D.R.J 2008. Weber's law, in "Inside psychology: A science over 50 years", Rabbit. P. (Ed.), Oxford University Press.

Kingham, S., Dickinson, J., Copsey, S., 2001. Travelling to work: Will people move out of their cars. Transport Policy, 8 (2):151-160.

Lathia, N., Smith, C., Froehlich, J., Capra, L., 2013. Individuals among commuters: Building personalised transport information services from fare collection systems. Pervasive and Mobile Computing, 9(5):643-664.

Lorenz, D. H., Raz, D. 2001. A simple efficient approximation scheme for the restricted shortest path problem. Operations Research Letters, 28(5): 213-219.

Nuzzolo, A., Crisalli, U., Comi, A., Rosati, L. 2014. An Advanced Traveller Advisory Tool based on Individual Preferences. Procedia-Social and Behavioral Sciences, 160:539-547.

Peng, Z.R., Huang, R., 2000. Design and development of interactive trip planning for web-based transit information systems. Transportation Research Part C, 8(1):409-425.

Shaheen, S., Martin, E., Cohen, A., Musunuri, A., Bhattacharyya, A., 2016. Mobile Apps and Transportation: A Review of Smartphone Apps and a Study of User Response to Multimodal Traveler Information. Final report, ITS of UC Berkeley, California Department of Transportation (Caltrans), October.

Vo, H. T., Ni, P., Yan, C., Eswaran, T., Shah, J., Massarczyk, B., 2015. Diversification of route planning results for improved user satisfaction. In: Intelligent Transportation Systems (ITSC), 2015 IEEE 18th International Conference on (pp. 2625-2630). IEEE, September.

Xiao, Y., Lo, H., 2016. Day-to-day Departure Time Modeling under Social Network Influence. Transportation Research Part B, 92:54-72

Xu, W., He, S., Song, R., Chaudhry, S.S., 2012. Finding the K shortest path in a schedule-based transit network, Computers and Operations Research, 39:1812-1826.

Yen, J.Y., 1971. Finding the k shortest loopless paths in a network. Management Science, 17(11), 712-716.

Zografos, K.G., Androutsopoulos, K.N., Spitadakis, V., 2009. Design and assessment of an online passenger information system for integrated multimodal trip planning. IEEE Transactions on Intelligent Transportation Systems, 10(2):311-323.

Hamdouch, Y., Szeto, W.Y., Jiang, Y., 2014. A new schedule-based transit assignment model with travel strategies and supply uncertainties. Transportation Research Part B, 67, 35-67.

## Appendix A

### *A.1. Proof of Theorem 1*

For clarity, we use $A$, $B$, $C$ to denote three options and we omit the superscript *id*. Therefore, proving Theorem 1 is equivalent to proving that if $A \succeq B$ and $B \succeq C$, then $A \succeq C$. Because of space limitations, we only present proof of transitivity with respect to the strict preference relationship, i.e., if $A \succ B$ and $B \succ C$, then $A \succ C$.
First, given $A \succ B$ and $B \succ C$, we have

$$A \succ B \Leftrightarrow \begin{cases} \exists q \in (1,...,m), v_{B,q} > v_{A,q} \text{ and } v_{B,q} - \min\limits_{i\in\{A,B,C\}}\{v_{i,q}\} > \beta_q \min\limits_{i\in\{A,B,C\}}\{v_{i,q}\} \cdots\cdots\cdots\cdots\cdots\cdots (a) \\ \forall k \le q-1, \begin{cases} v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \le \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ \forall j \in \{A,B\}, \text{or} \\ v_{A,k} = v_{B,k}, \text{if } v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} > \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ j \in \{A,B\} \end{cases} \cdots\cdots\cdots (b) \end{cases} \tag{A.1}$$

$$B \succ C \Leftrightarrow \begin{cases} \exists \bar{q} \in (1,...,m), v_{C,\bar{q}} > v_{B,\bar{q}} \text{ and } v_{C,\bar{q}} - \min\limits_{i\in\{A,B,C\}}\{v_{i,\bar{q}}\} > \beta_{\bar{q}} \min\limits_{i\in\{A,B,C\}}\{v_{i,\bar{q}}\} \cdots\cdots\cdots\cdots\cdots\cdots (a) \\ \forall k \le \bar{q}-1, \begin{cases} v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \le \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ \forall j \in \{B,C\}, \text{or} \\ v_{B,k} = v_{C,k}, \text{if } v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} > \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ j \in \{B,C\} \end{cases} \cdots\cdots\cdots (b) \end{cases} \tag{A.2}$$

We consider the following scenarios:

a) $m = 1$ and $q = \bar{q} = 1$

It is easy to deduce from (A.1a) and (A.2a) that $v_{C,1} > v_{A,1}$ and $v_{C,1} - \min\limits_{i\in\{A,B,C\}}\{v_{i,1}\} > \beta_1 \min\limits_{i\in\{A,B,C\}}\{v_{i,1}\}$, thus $A \succ C$.

b) $m > 1$ and $q = \bar{q}$

Equations (A.1a) and (A.2a) imply that

$$v_{C,q} > v_{A,q} \text{ and } v_{C,q} - \min_{i\in\{A,B,C\}}\{v_{i,q}\} > \beta_q \min_{i\in\{A,B,C\}}\{v_{i,q}\}. \tag{A.3}$$

Meanshile, when $q = \overline{q}$ , equation (A.2b) can be written as

$$\forall k \le q-1, \begin{cases} v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \le \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ \ \forall j \in \{B,C\}, \text{or} \\ v_{B,k} = v_{C,k}, \text{if } v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} > \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ j \in \{B,C\} \end{cases} \tag{A.4}$$

Equation (A.4) indicates two cases that we need to consider

1) $$\exists k \in (1,q-1), \text{if } v_{B,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} > \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \Rightarrow v_{A,k} = v_{B,k} = v_{C,k} \tag{A.5}$$

2) $$\begin{aligned} &\exists k \in (1,q-1), \text{if } v_{B,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \le \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \\ &\qquad \Rightarrow v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \le \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\}, \forall j \in \{A,B,C\} \end{aligned} \tag{A.6}$$

Combining equations (A.3), (A.5) and (A.6), we conclude that $A \succ C$ .

c) $m > 1$ and $q > \overline{q}$

In such a case, equation (A.1b) implies

$$\forall k \le \overline{q}, \begin{cases} v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \le \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ \ \forall j \in \{A,B\}, \text{or} \\ v_{A,k} = v_{B,k}, \text{if } v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} > \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ j \in \{A,B\} \end{cases} \tag{A.7}$$

Combining equations (A.2) and (A.7), we can deduce

$$v_{C,\overline{q}} > v_{A,\overline{q}} \text{ and } v_{C,\overline{q}} - \min\limits_{i\in\{A,B,C\}}\{v_{i,\overline{q}}\} > \beta_{\overline{q}} \min\limits_{i\in\{A,B,C\}}\{v_{i,\overline{q}}\} \tag{A.8}$$

Meanwhile, equations (A.5) and (A.6) also hold when we replace $q$ with $\overline{q}$ . Therefore, we can conclude $A \succ C$ .

d) $m > 1$ and $q < \overline{q}$

In such a case, equation (A.2b) indicates,

$$\forall k \le q, \begin{cases} v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \le \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ \ \forall j \in \{B,C\}, \text{or} \\ v_{A,k} = v_{B,k}, \text{if } v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} > \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\}, j \in \{B,C\} \end{cases} \tag{A.9}$$

Similarly, two cases are considered,

1) If $v_{B,q} - \min\limits_{i\in\{A,B,C\}}\{v_{i,q}\} \le \beta_q \min\limits_{i\in\{A,B,C\}}\{v_{i,q}\}$. This contradicts equation (A.1a)

2) If $v_{B,q} - \min\limits_{i\in\{A,B,C\}}\{v_{i,q}\} > \beta_q \min\limits_{i\in\{A,B,C\}}\{v_{i,q}\} \Rightarrow v_{C,q} = v_{B,q} > v_{A,q}$.

Meanwhile, equations (A.5) and (A.6) also hold, and thus, we can conclude $A \succ C$ . Combining scenarios a) - d), we conlucde that if $A \succ B$ and $B \succ C$ , then $A \succ C$ . □

*A.2. Proof of corollary 1*

This corollary is a direct conclusion from scenarios a) - *d*) in the proof of Theorem 1. □

*A.3. Proof of corollary 2*

Given $A \succ C$ and $B \succ C$, equation (A.2) and the following equation (A.10) hold.

$$A \succ C \Leftrightarrow \begin{cases} \exists q \in (1,...,m), v_{C,q} > v_{A,q} \text{ and } v_{C,q} - \min\limits_{i\in\{A,B,C\}}\{v_{i,q}\} > \beta_q \min\limits_{i\in\{A,B,C\}}\{v_{i,q}\} \cdots\cdots\cdots\cdots\cdots\cdots (a) \\ \forall k \le q-1, \begin{cases} v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \le \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ \forall j \in \{A,C\}, \text{or} \\ v_{A,k} = v_{C,k}, \text{if } v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} > \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ j \in \{A,C\} \end{cases} \cdots\cdots\cdots (b) \end{cases} \tag{A.10}$$

When $q < \bar{q}$, equation (A.2b) indicates

$$\forall k \le q, \begin{cases} v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \le \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ \forall j \in \{B,C\}, \text{or} \\ v_{B,k} = v_{C,k}, \text{if } v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} > \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ j \in \{B,C\} \end{cases} \tag{A.11}$$

Meanwhile, equation (A.10a) states that $v_{C,q} - \min\limits_{i\in\{A,B,C\}}\{v_{i,q}\} > \beta_q \min\limits_{i\in\{A,B,C\}}\{v_{i,q}\}$. Thus $v_{B,q} = v_{C,q} > v_{A,q}$.

According to equations (A.10b) and (A.11), we have

$$\forall k \le q-1, \begin{cases} v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \le \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ \forall j \in \{A,B\}, \text{or} \\ v_{A,k} = v_{B,k}, \text{if } v_{j,k} - \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} > \beta_k \min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} \ j \in \{A,B\} \end{cases} \tag{A.12}$$

Therefore, we can prove *corollary 2*(a), i.e., $A \succ B$. Similarly, we can prove *corollary 2*(b). □

*A.4. Proof of corollary 3*

Proof: Due to space limitations, the proof is not presented here. In short, we follow the procedure for the proof of Theorem 1 and consider two cases: $\min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} = v_{A,k}$ and $\min\limits_{i\in\{A,B,C\}}\{v_{i,k}\} = \min\limits_{i\in\{B,C\}}\{v_{i,k}\}$

**Appendix B**

This appendix proves Theorem 2.

1) If $m = 1$: the algorithm reduces to a simple sorting algorithm, which is addressed by line 2.
2) When $m > 1$, we compare the sorted set generated at $i = k$ and $i = k+1$

   When $i = k$, we obtain $\bar{Q}^k = \{l_0^k \succeq l_1^k \succeq ... \succeq l_j^k\}$

   When $i = k+1$, we obtain $\bar{Q}^{k+1} = \{l_0^{k+1} \succeq l_1^{k+1} \succeq ... \succeq l_h^{k+1}\}$

Given Theorem 1, the axiom of transitivity, we only need to prove that $l_h^{k+1} \succeq l_0^k$. In other words, the worst option generated at $i = k+1$ is better than the best option at $i = k$. According to line 2 of the algorithm, we know

$$\forall j \in \bar{Q}^k, \begin{cases} v_{j,k} - \min\limits_{\bar{l}\in S}\{v_{\bar{l},k}\} > \beta_k \min\limits_{\bar{l}\in S}\{v_{\bar{l},k}\} \\ \forall \bar{k} \le k-1, v_{j,\bar{k}} - \min\limits_{\bar{l}\in S}\{v_{\bar{l},\bar{k}}\} \le \beta_{\bar{k}} \min\limits_{\bar{l}\in S}\{v_{\bar{l},\bar{k}}\} \end{cases} \text{and } \forall j \in \bar{Q}^{k+1}, \begin{cases} v_{j,k+1} - \min\limits_{\bar{l}\in S}\{v_{\bar{l},k+1}\} > \beta_{k+1} \min\limits_{\bar{l}\in S}\{v_{\bar{l},k+1}\} \\ \forall \bar{k} \le k, v_{j,\bar{k}} - \min\limits_{\bar{l}\in S}\{v_{\bar{l},\bar{k}}\} \le \beta_{\bar{k}} \min\limits_{\bar{l}\in S}\{v_{\bar{l},\bar{k}}\} \end{cases}$$

Thus, it is concluded that

$$l_h^{k+1} \succ l_0^k \Leftrightarrow \begin{cases} v_{l_0^k,k} > v_{l_h^{k+1},k} \text{ and } v_{l_0^k,k} - \min\limits_{\bar{l} \in S}\left\{v_{\bar{l},k}\right\} > \beta_k \min\limits_{\bar{l} \in S}\left\{v_{\bar{l},k}\right\} \\ \forall \bar{k} \le k-1, v_{j,\bar{k}} - \min\limits_{\bar{l} \in S}\left\{v_{\bar{l},\bar{k}}\right\} \le \beta_{\bar{k}} \min\limits_{\bar{l} \in S}\left\{v_{\bar{l},\bar{k}}\right\} \text{ and } v_{j,\bar{k}} - \min\limits_{\bar{l} \in S}\left\{v_{\bar{l},\bar{k}}\right\} \le \beta_{\bar{k}} \min\limits_{\bar{l} \in S}\left\{v_{\bar{l},\bar{k}}\right\} \end{cases}$$

This completes the proof.