



## On energy-momentum tensors and proton structure

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*Published in:*  
EPL

*Link to article, DOI:*  
[10.1209/0295-5075/128/11004](https://doi.org/10.1209/0295-5075/128/11004)

*Publication date:*  
2019

*Document Version*  
Publisher's PDF, also known as Version of record

[Link back to DTU Orbit](#)

*Citation (APA):*  
Trinhammer, O. L. (2019). On energy-momentum tensors and proton structure. *EPL*, 128(1), Article 11004.  
<https://doi.org/10.1209/0295-5075/128/11004>

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To cite this article: Ole L. Trinhammer 2019 *EPL* **128** 11004

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# On energy-momentum tensors and proton structure

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received 16 October 2019; accepted 28 October 2019  
published online 27 November 2019

PACS 12.40.Yx – Hadron mass models and calculations

**Abstract** – We discuss baryonic energy-momentum tensors (EMT). The quantum nature of the particle is reflected in the square of the components of the energy-momentum tensor being proportional to interior probability densities depending on dynamical intrinsic configuration variables. We exemplify by an approximate protonic state from which we determine density distributions. We find a mass density vanishing at the centre.



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**Introduction.** – The energy-momentum tensor of baryons is currently under discussion [1,2]. Baryons are quantum particles and their interior structure is traditionally thought to be in principle describable from quantum chromodynamics, QCD [3]. Thus, one tries to introduce radial (and more involved) distribution coefficient functions which can be extracted experimentally by scattering experiments analysed in various coordinate frames [2,4].

We derive an energy-momentum tensor based on an intrinsic conception of baryon structure [5]. We discuss how the structure may be related to observations in laboratory scattering experiments both as spherical distributions and as distributions in the transverse plane.

**Gravity and quantum of action.** – Consider Einstein's equation for the metric  $g_{\mu\nu}$  of spacetime interwoven with the local energy-momentum tensor  $\mathcal{T}_{\mu\nu}$  which depends on derivatives of  $g_{\mu\nu}$  through the curvature tensor  $\mathcal{R}_{\mu\nu}$  (cf., e.g., p. 81 in [6] and p. 154 in [7]),

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = -\kappa\mathcal{T}_{\mu\nu}. \quad (1)$$

Here  $\kappa = 8\pi G/c^4$ , where  $G$  is Newton's gravitational constant of universal mutual mass attraction and  $c$  is the speed of light in empty space. Usually the quantization of gravity is thought to be relevant when the energy transferred in the scattering approaches the Planck scale. We introduce a Planck length  $l_{\text{Pl}}$  by

$$\frac{l_{\text{Pl}}^2}{\kappa} \equiv hc \rightarrow l_{\text{Pl}} = 2 \cdot 10^{-19} \text{ fm}, \quad (2)$$

where  $h$  is Planck's constant and we identify  $hc$  as a *quantum of space-action*. With this we can rewrite (1) to get

$$\frac{hc}{l_{\text{Pl}}} \left[ r_{\mu\nu} - \frac{1}{2}g_{\mu\nu}r \right] = -l_{\text{Pl}}^3 \mathcal{T}_{\mu\nu}, \quad r_{\mu\nu} \equiv l_{\text{Pl}}^2 \mathcal{R}_{\mu\nu}, \quad (3)$$

where  $r_{\mu\nu}$  is a dimensionless curvature tensor. As we shall see for an approximate protonic structure the energy-momentum tensor components vary smoothly and show densities so low that the curvature in the interior is not noticeable, because for a baryonic length scale  $a \approx 1$  fm

$$\frac{\hbar c}{a} = \frac{hc}{2\pi a} \ll \frac{hc}{l_{\text{Pl}}}. \quad (4)$$

Note, however, that many scenarios of a minimal length scale exist [8].

**Intrinsic baryon configuration.** – Consider a description of baryons as stationary states on an intrinsic  $U(3)$  configuration space

$$\frac{\hbar c}{a} \left[ -\frac{1}{2}\Delta + \frac{1}{2}d^2(e, u) \right] \Psi(u) = \mathcal{E}\Psi(u). \quad (5)$$

The intrinsic configuration space is non-spatial —there is no gravitation in the intrinsic space. The configuration variable  $u \in U(3)$  may be seen as a kind of generalized spin to be excited kinematically by momentum, spin and Laplace-Runge-Lenz generators from the laboratory space, cf., e.g., [5,9]. Thus,

$$u = e^{i(a\theta_j p_j + \alpha_j S_j + \beta_j M_j)/\hbar}, \quad \theta_j, \alpha_j, \beta_j \in \mathbb{R}, \quad j = 1, 2, 3, \quad (6)$$

where  $a$  is a length scale for mapping the intrinsic dynamics to laboratory space by the projection [5]

$$x_j = a\theta_j. \quad (7)$$

We name the dynamical variables  $\theta_j$  *eigenangles* as they determine the eigenvalues  $e^{i\theta_j}$  of the configuration variable  $u$ . The intrinsic momenta  $p_j$ , conjugate to the eigenangles, generate three toroidal (colour) degrees of freedom in  $U(3)$  by toroidal generators  $iT_j$ ,

$$iT_j = \frac{\partial}{\partial\theta_j}, \quad p_j = \frac{-i\hbar}{a} \frac{\partial}{\partial\theta_j} = \frac{\hbar}{a} T_j. \quad (8)$$

The intrinsic spin generators  $S_j$  in a coordinate representation read (cf. p. 210 in [10])

$$\begin{aligned} S_1 &= a\theta_2 p_3 - a\theta_3 p_2 = \hbar\lambda_7, \\ S_2 &= a\theta_1 p_3 - a\theta_3 p_1 = \hbar\lambda_5, \\ S_3 &= a\theta_1 p_2 - a\theta_2 p_1 = \hbar\lambda_2 \end{aligned} \quad (9)$$

and the Laplace-Runge-Lenz generators  $M_j$  (which mix with spin and take care of flavour degrees of freedom (14) [5]) read

$$\begin{aligned} M_3/\hbar &= \theta_1\theta_2 + \frac{a^2}{\hbar^2} p_1 p_2 = \lambda_1, \\ M_2/\hbar &= \theta_3\theta_1 + \frac{a^2}{\hbar^2} p_3 p_1 = \lambda_4, \\ M_1/\hbar &= \theta_2\theta_3 + \frac{a^2}{\hbar^2} p_2 p_3 = \lambda_6. \end{aligned} \quad (10)$$

These mixing operators “connect” the algebra by commuting into the spin operators

$$[M_k, M_l] = [S_k, S_l] = -i\hbar S_m, \quad \text{cyclic permutations.} \quad (11)$$

Note the reversed sign in the commutators of the intrinsic spin operators  $S_j$  corresponding to body fixed coordinates in nuclear physics (cf., *e.g.*, p. 87 in [11]). The lambdas are the six off-diagonal Gell-Mann matrices, cf., *e.g.*, p. 209 in [10].

We want to map the spatial components  $\mathcal{T}_{ij}$  for  $i, j = 1, 2, 3$  of the energy-momentum tensor  $\mathcal{T}_{\mu\nu}$  to laboratory space from the structure of the wave function  $\Psi$  in (5). Our Hamiltonian in (5) may look too simple to capture the complexity expected from the standard description by quantum chromodynamics. However, the fact that our configuration space is the Lie group  $U(3)$  introduces the complex structure needed and leads to generation of quark and gluon fields transforming properly as the fundamental and the adjoint representation under  $SU(3)$ , respectively [12,13]. Further, the Laplacian  $\Delta$  when written in a polar decomposition [14]

$$\Delta = \sum_{j=1}^3 \frac{1}{J^2} \frac{\partial}{\partial\theta_j} J^2 \frac{\partial}{\partial\theta_j} - \sum_{i<j, k\neq i, j}^3 \frac{(S_k^2 + M_k^2)/\hbar^2}{8 \sin^2 \frac{1}{2}(\theta_i - \theta_j)} \quad (12)$$

even discloses flavour degrees of freedom (14) from a mix of the off-toroidal generators. In (12) the “Jacobian”, the van de Monde-determinant (cf. [15], p. 197), is

$$J = \prod_{i<j}^3 2 \sin \left( \frac{1}{2}(\theta_i - \theta_j) \right). \quad (13)$$

The off-toroidal generators  $\mathbf{S} = (S_1, S_2, S_3)$  and  $\mathbf{M} = (M_1, M_2, M_3)$  contain spin  $s$ , isospin  $i_3$  and hypercharge  $y$ ,

$$\begin{aligned} (\mathbf{S}^2 + \mathbf{M}^2)/\hbar^2 &= s(s+1) + M^2 \\ &= \frac{4}{3} \left( n + \frac{3}{2} \right)^2 - 3 - \frac{1}{3} y^2 - 4i_3^2. \end{aligned} \quad (14)$$

Here  $n \geq 0$  is a natural number [5]. The minimum value of  $(\mathbf{S}^2 + \mathbf{M}^2)/\hbar^2$  is 4 for  $n = 1, y = 1$  and  $i_3 = \pm 1/2$ .

Finally it should be mentioned that our potential in (5) is inspired by the Manton analogue [16] of the Wilson action [17,18]. In the intrinsic conception of the configuration space we apply the (measure-scaled) exterior derivative  $d\Phi$  to generate quark and gluon fields [13]. The  $d\Phi$  is also called the momentum form [19] and is obtained from the wave function  $\Psi$  by scaling with the Jacobian  $J$ , thus  $\Phi = J\Psi$ . The potential in (5) depends on the shortest geodesic distance  $d(e, u)$  in the configuration space from its *origo* (the neutral element  $e$ ) to the configuration variable  $u$ . The potential is

$$\frac{1}{2} d^2(e, u) = \frac{1}{2} \text{Tr} \chi^2, \quad \chi = (a\theta_j p_j + \alpha_j S_j + \beta_j M_j)/\hbar. \quad (15)$$

The potential is periodic and only depends on the eigenangles  $\theta_j$  since the eigenvalues  $e^{i\theta_j}$  of  $u$  are unchanged by conjugation  $u \rightarrow v^{-1} u v, v \in U(3)$  as is the trace  $\text{Tr}$ . The distance  $d$  can be thought of as the Euclidean measure folded onto the (compact) group space manifold [20].

**Energy-momentum components from intrinsic dynamics.** – Using the exterior derivative we can extract information to the laboratory space from the intrinsic dynamics inherent in the wave function and thus derive an energy-momentum distribution for the spatial components  $\mathcal{T}_{ij}$ . We introduce generalized intrinsic momenta by derivations along the  $ij$ -th degree of freedom [21]

$$\frac{-i\hbar}{a} d\Phi_u(iE_{ij}) \quad (16)$$

to be read off from the wave function by the exterior derivative  $d\Phi$ , cf. the next section. In a matrix representation  $E_{ij}$  is the  $3 \times 3$  matrix with the  $ij$ -th element equal to one and all other elements are zero. The generalized momenta are read off in a fixed base in laboratory space and averaged over the off-toroidal degrees of freedom by integrating over these to get the spatial energy-momentum components for the unpolarized case

$$\mathcal{T}_{ij}^2 = \frac{1}{V^2} \int d\alpha^3 d\beta^3 \left( \frac{-i\hbar c}{a} d\Phi_u(iE_{ij}) \right)^2, \quad i, j = 1, 2, 3 \quad (17)$$

normalizing over a volume  $V$  in laboratory space. Further, we take

$$\mathcal{T}_{00} = \frac{1}{V} \int d\alpha^3 d\beta^3 J^2 \Psi^\dagger H \Psi. \quad (18)$$

The wave function in (5) can be factorized into

$$\Psi(u) = \tau(\theta_1, \theta_2, \theta_3) \Upsilon(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \quad (19)$$

and the off-toroidal variables  $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$  can be integrated out [5]. Thus, (17) and (18) reduce to

$$\mathcal{T}_{ij}^2(\theta_1, \theta_2, \theta_3) = \frac{\hbar^2 c^2 (\mathbf{S}^2 + \mathbf{M}^2)/(3\hbar^2)}{a^2 V^2} \frac{R^2(\theta_1, \theta_2, \theta_3)}{4}, \quad (20)$$

$$i \neq j, \quad i, j = 1, 2, 3$$

with  $R = J\tau$  and

$$\mathcal{T}_{jj}^2 = \frac{\hbar^2 c^2}{a^2 V^2} \left( \frac{\partial R}{\partial \theta_j} \right)^2, \quad \mathcal{T}_{00} = \frac{\mathcal{E}}{V} R^2. \quad (21)$$

**Exterior derivative calculations.** – We here show how to reach (20) and (21). In case the reader is unfamiliar with the concept of derivations on Lie groups, we refer to [22,23] and possibly the appendix on “Vector fields, derivations and forms on smooth manifolds” in [21].

First we note the equivalence between derivations and vector fields. A vector field  $Z$  on a smooth manifold  $M$  maps points  $u$  in the manifold into the tangent space  $TM$  of the manifold

$$Z: u \in M \rightarrow Z(u) \in TM. \quad (22)$$

In our case the manifold is the  $U(3)$  Lie group configuration space and the algebra  $u(3)$  is its tangent space. We use left invariant vector fields for our derivations on the wave functions, *i.e.*,

$$Z_u = u Z_e. \quad (23)$$

The derivation  $Z[\Phi]$  at  $u$  along the direction given by  $Z$  becomes operational by differentiation of the function  $\Phi$  along a one-parameter curve through  $u$ , *i.e.*,

$$Z[\Phi](u) = Z_u[\Phi] = d\Phi_u(Z) = \frac{d}{dt} \Phi(ue^{tZ})|_{t=0}. \quad (24)$$

Think of a tangential velocity along a curved surface. In our application of these differential geometry concepts we use the generators  $iT_j, iS_j/\hbar, iM_j/\hbar$  and combinations thereof as derivations. We have

$$iE_{ij} = \frac{1}{2}(-S_k + iM_k)/\hbar, \quad i < j, \quad k \neq i, j, \quad (25)$$

where  $S_k^\dagger = S_k$  and  $M_k^\dagger = M_k$ . For the calculation of  $\mathcal{T}_{ij}^2$  we consider the integrand in (17) and use left invariance of the vector fields and unitarity of the configuration variable

$$\begin{aligned} & \left( \frac{-i\hbar c}{a} d\Phi_u(iE_{ij}) \right)^\dagger \left( \frac{-i\hbar c}{a} d\Phi_u(iE_{ij}) \right) = \\ & \frac{\hbar^2 c^2}{a^2} ([uiE_{ij}]\Phi)^\dagger ([uiE_{ij}]\Phi) = \\ & \frac{\hbar^2 c^2}{a^2} (\Phi^\dagger(iE_{ij})^\dagger u^\dagger uiE_{ij}\Phi) = \\ & \frac{\hbar^2 c^2}{a^2} \frac{(S_k^2 + M_k^2)/\hbar^2}{4} \Phi^2(u), \quad i \neq j, k \neq i, j. \end{aligned} \quad (26)$$

We note immediately that  $\Psi$  cannot be a mutual eigenstate for all three  $S_k^2 + M_k^2$ ,  $k = 1, 2, 3$ . This problem is handled by integrating out the off-toroidal degrees of freedom in (17) by exploiting the arbitrary labelling of the  $\theta_j$ 's as when we solved (5) by a factorization of the wave function [5]. We thus have the expectation value over the off-toroidal degrees of freedom

$$\begin{aligned} \mathcal{T}_{ij}^2(u) &= \frac{\hbar^2 c^2}{a^2 V^2} R^2(\theta_1, \theta_2, \theta_3) \\ &\times \int d\alpha^3 d\beta^3 \Upsilon^\dagger \frac{(S_k^2 + M_k^2)/\hbar^2}{4} \Upsilon, \quad i \neq j, k \neq i, j. \end{aligned} \quad (27)$$

Since the indexing of the eigenangles  $\theta_j$  is arbitrary, we can average in (27) to get

$$\mathcal{T}_{ij}^2(\theta_1, \theta_2, \theta_3) |_{i \neq j} = \frac{\hbar^2 c^2 (\mathbf{S}^2 + \mathbf{M}^2)/(3\hbar^2)}{a^2 V^2} R^2(\theta_1, \theta_2, \theta_3). \quad (28)$$

We here exploited an individual normalization of the off-toroidal wave function  $\Upsilon$  made possible by a factorization of the measure [24]. (It is common to refer to the existence of the Haar measure [25], but the invariant measure on Lie groups and its factorization were described already by Hurwitz [24] and acknowledged by Haar). It can be shown [5,12] that  $\mathbf{S}^2 + \mathbf{M}^2 = 4\hbar^2$  is the minimum value and it yields the  $N$  and  $\Delta$  baryon spectrum from (5) [13].

The diagonal components we obtain similarly ( $iE_{jj} = iT_j$ )

$$\begin{aligned} \mathcal{T}_{jj}^2(\theta_1, \theta_2, \theta_3) &= \frac{1}{V^2} \int \left( \frac{-i\hbar c}{a} d\Phi_u(iT_j) \right)^2 d\alpha^3 d\beta^3 \\ &= \frac{\hbar^2 c^2}{a^2 V^2} (iT_j R)^\dagger iT_j R = \frac{\hbar^2 c^2}{a^2 V^2} \left( \frac{\partial R}{\partial \theta_j} \right)^2. \end{aligned} \quad (29)$$

This concludes the calculation of the general results (20) and (21) for the spatial components of the energy-momentum tensor  $\mathcal{T}_{\mu\nu}$  of an intrinsic configuration.

**Application to the proton.** – We here apply our definitions (17) and (18) to an approximate protonic state [5]

$$R = \frac{1}{N} \begin{vmatrix} 1 & 1 & 1 \\ \sin \frac{1}{2}\theta_1 & \sin \frac{1}{2}\theta_2 & \sin \frac{1}{2}\theta_3 \\ \cos \theta_1 & \cos \theta_2 & \cos \theta_3 \end{vmatrix}, \quad (30)$$

where the normalization constant  $N$  is dimensionless on  $U(3)$ . It is straightforward to use (20) and (21) on  $R$ , but it is not obvious how the results can be interpreted to compare with experimentally possible investigations.

First we refer to Heinz Pagels' work on the energy-momentum structure form factors of particles [26] where

he noted that Lorentz invariance requires in the rest frame of the particle that the only non-zero element of  $\mathcal{T}_{\mu\nu}$  be

$$\mathcal{T}_{00,\text{particle}} = mc^2, \quad \text{in rest frame.} \quad (31)$$

It must be understood here that the talk is about a point-particle. The proton is usually considered as a ‘‘soup’’ of quarks and gluons confined by quantum chromodynamics to a certain interior in laboratory space. In the intrinsic conception the idea is that the intrinsic dynamics is carried by the structure of the wave function from (5). The structure is then mapped to the laboratory space by use of the momentum form as described in the definitions (17) and (18) leading to the results (20) and (21). Consistency with the appearance of the proton as an entire, sole entity in laboratory space is upheld by the fact that

$$\begin{aligned} \int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} R \, d\theta_1 d\theta_2 d\theta_3 &= 0, \\ \sim \int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} \mathcal{T}_{ij} \, d\theta_1 d\theta_2 d\theta_3 &= 0, \quad i \neq j \end{aligned} \quad (32)$$

simply because  $R$  is constructed as a Slater determinant and thus is antisymmetric under interchange of any pair of the three eigenangles  $\theta_j$  corresponding to interchange of columns in the determinant. When integrated over the full period of its periodic components the integral vanishes. The same goes for the diagonal elements. Thus, all spatial elements  $\mathcal{T}_{ij}$  and  $\mathcal{T}_{jj}$  are zero when integrated over the proton ‘‘volume’’ (the unfolded compact intrinsic space). At the same time

$$\int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} R^2 \, d\theta_1 d\theta_2 d\theta_3 = 1, \quad (33)$$

which ensures that the intrinsic dynamics stays inside the protonic volume with probability one —the proton is an integral entity.

What becomes interesting then is to try to look ‘‘inside’’ the proton to see if one can discern the complicated structure that is expected both by QCD and by the structure inherent in the intrinsic description (5) from a Lie group configuration space. We can use the *squares*  $\mathcal{T}_{ij}^2$  and  $\mathcal{T}_{jj}^2$  of the energy-momentum components to map the *density* of protonic appearance, in other words to map the probability density for interacting with the proton inside its laboratory space interior. As an example we indicate in fig. 1 the densities in the  $xy$ -plane for varying  $z$  with the protonic centre located at  $(x, y, z) = (0, 0, 0)$  in the rest frame.

Now, actually fig. 1 is in the parameter space, from where the parametrizations are thought to be scaled by  $a$  when projected to the laboratory space as mentioned in (7), thus

$$\theta_1 = x/a, \quad \theta_2 = y/a, \quad \theta_3 = z/a \quad (34)$$

and the tensor density from (28) becomes

$$\begin{aligned} \mathcal{T}_{ij}^2(\mathbf{x}) &= \frac{\hbar^2 c^2 (\mathbf{S}^2 + \mathbf{M}^2)/(3\hbar^2)}{a^2} \\ &\times \frac{1}{V^2 N^2} \begin{vmatrix} 1 & 1 & 1 \\ \sin\left(\frac{1}{2}\frac{x}{a}\right) & \sin\left(\frac{1}{2}\frac{y}{a}\right) & \sin\left(\frac{1}{2}\frac{z}{a}\right) \\ \cos\left(\frac{x}{a}\right) & \cos\left(\frac{y}{a}\right) & \cos\left(\frac{z}{a}\right) \end{vmatrix}^2. \end{aligned} \quad (35)$$

Equation (35) from the projections (34) implies a periodic structure in laboratory space. This corresponds to periodic boundary conditions on the proton wave function commonly used in lattice gauge theory [27]. It makes sense in the rest frame as well-defined momentum (zero) implies an unlocalized proton, but needs clarification with respect to proton extension in scattering phenomenology. We stress that (34) is only a parametrization in toroidal angles of the intrinsic configuration space. The configuration space is compact and thus maps in periodic parameters. We therefore assume a box normalization over  $V$  in (35).

To try to compare the densities (35) to laboratory distributions, we introduce spherical coordinates

$$\begin{aligned} x &= r \sin \theta \cos \phi = a\theta_1, \\ y &= r \sin \theta \sin \phi = a\theta_2, \\ z &= r \cos \theta = a\theta_3, \end{aligned} \quad (36)$$

where  $r^2 = x^2 + y^2 + z^2$ ,  $\theta \in [0, \pi]$  is the polar angle and  $\phi \in [0, 2\pi]$  the azimuth. We integrate over the polar angle and the azimuth to get a radial off-diagonal energy-momentum density

$$\sigma_{ij}(r) = \frac{1}{4\pi(r/a)^2} \sqrt{\int_0^{2\pi} d\phi \int_0^\pi d\theta \, \mathcal{T}_{ij}^2(r, \theta, \phi)}. \quad (37)$$

Note that the solid angle element on the sphere of surface  $4\pi r^2$  is simply  $d\phi d\theta$ . Likewise we define from (21) diagonal and mass densities

$$\begin{aligned} \tau_{jj}(r) &= \frac{1}{4\pi(r/a)^2} \sqrt{\int_0^{2\pi} d\phi \int_0^\pi d\theta \, \mathcal{T}_{jj}^2(r, \theta, \phi)}, \\ \mu(r) &= \frac{1}{4\pi(r/a)^2} \int_0^{2\pi} d\phi \int_0^\pi d\theta \, \mathcal{T}_{00}(r, \theta, \phi). \end{aligned} \quad (38)$$

Because of the symmetry in  $\mathcal{T}_{ij}^2$  for  $i \neq j$  under interchange of the toroidal angles, we have  $\sigma_{ij} \equiv \sigma$  for  $i \neq j$ .

Now confer, *e.g.*, p. 10 in [28], for stress tensors in continuous matter used, for instance in Polyakov’s decomposition of the static tensor [4],

$$\mathcal{T}_{ij} = s(r) \left( \frac{x_i x_j}{r^2} - \frac{1}{3} \delta_{ij} \right) + p(r) \delta_{ij}. \quad (39)$$

The function  $p(r)$  is interpreted as the radial distribution of ‘‘pressure’’ inside the particle and the function  $s(r)$  is

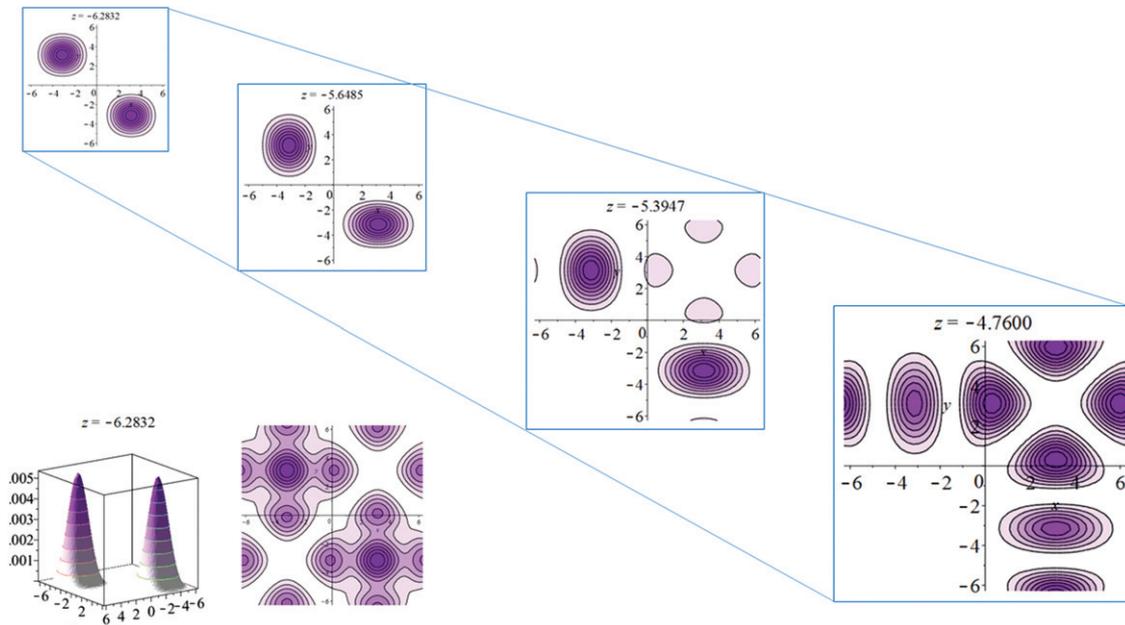


Fig. 1: Commencing a travel through the off-diagonal energy-momentum tensor density for an approximate protonic state. The probe enters at  $z = -2\pi$  (back, top left) and proceeds to increasing  $z$ -values. The pattern will repeat from  $z = 2\pi$ . The pattern is a density in the toroidal part of the intrinsic space unfolded to laboratory space. Therefore the cubic frame should not be interpreted as a cubic volume for the proton in laboratory space! The first insert (bottom, first from left) is to indicate the colour scale of the contour plots. It is a 3D analogue of the 2D contour plot seen at the top left where the travel commences. We have run an animation through the whole range  $z \in [-2\pi, 2\pi]$ . It shows that the density values stay within the values shown in the insert and, as expected from (30) inserted into (20), there are no singularities. The second insert (bottom, second from left) shows the density integrated over  $z$ , *i.e.*, a contraction in the third toroidal dimension.

related to the distribution of shear forces. We slightly change the definitions to read

$$\mathcal{T}_{ij} = s(r) \left( E_{ij} - \frac{1}{3} \delta_{ij} \right) + p_{jj}(r) \delta_{ij} \quad (40)$$

with no summation over  $j$ . We sum up our results from (37) and (38) as

$$\mathcal{T}_{ij} = \sigma(E_{ij} - \delta_{ij}) + \tau_{jj} \delta_{ij}, \quad i, j = 1, 2, 3. \quad (41)$$

By comparison with (40) we infer

$$\sigma(r) \sim s(r), \quad p_{jj}(r) \sim \tau_{jj}(r) - \frac{2}{3} \sigma(r). \quad (42)$$

We show  $p_{33}$  and  $\sigma$  in fig. 2 as obtained by numerical integration over the angular variables in (38) and (37). The qualitative similarity with fig. 6(a) on isotropic pressure and fig. 7(a) on pressure anisotropy in [2] is astonishing—even though a scale issue remains to be understood: In [5,13] we used the classical electron radius  $r_e$  (cf. p. 97 in [28]) to set the length scale  $a$  by

$$\pi a = r_e = 2.82 \text{ fm}, \quad \frac{e^2}{4\pi\epsilon_0 r_e} \equiv m_e c^2 \quad (43)$$

and got fine agreement with spectroscopic phenomena. In fig. 2 we used  $2\pi a = 1 \text{ fm}$  to compare with [2]—no other fitting is involved. The corresponding distributions in [2]

are based on 20 parameters, some of which are extracted from experiments while others are based on theoretical arguments and phenomenological heuristics. Finally some are hypothesized to make the distributions be realistic. In fig. 2 we also show the mass distribution  $\mu$  which seems distinctly different from [2]. We find the mass surface density (not just the accumulated mass) vanishing as the proton centre is approached.

The radial distributions in fig. 2 show a shear stress and mass density increasing towards a proton “surface”. This is in accord with the success of the MIT bag model in the earlier days of parton models, cf. [29] for a contemporary review. Note that the  $R$  we used as an example above is only approximate. It does, however, give quite convincing parton distribution functions for  $u$  and  $d$  valence quarks in the proton [5].

**Spherical or transverse distribution?** – It would be interesting if the spatial energy-momentum distributions of the lumps from the unfolded toroidal dynamics visualized in fig. 1 could be related to spatial gluon distributions obtained from diffractive scattering as discussed in [30,31]. In the QCD Lagrangian [3] the gluon fields are expanded on the  $SU(3)$  generators,

$$t_a = \lambda_a/2, \quad a = 1, 2, \dots, 8. \quad (44)$$

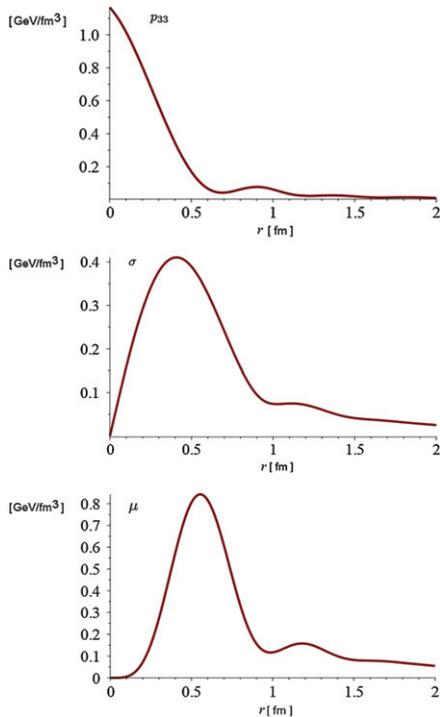


Fig. 2: Radial density of pressure  $p_{33}$  (top) and shear stress  $\sigma$  (middle) in the proton interior from (37). The graphs correspond to the decomposition (40). The plots are for  $2\pi a = 1$  fm, chosen to compare with [2]. At the bottom is shown the mass density  $\mu$  (38) with energy scale  $\Lambda = \hbar c/a$  in (21). Normalization is discussed from eq. (44).

We have

$$t_3 = \frac{1}{2}(E_{11} - E_{22}), \quad t_8 = \frac{1}{2}(E_{11} + E_{22} - 2E_{33})/\sqrt{3} \quad (45)$$

and introduce to complete for  $U(3)$

$$t_9 = (E_{11} + E_{22} + E_{33})/\sqrt{6}. \quad (46)$$

It shares the trace normalization with the others, *i.e.*,

$$\text{Tr } t_a^\dagger t_a = \frac{1}{2}, \quad t_a^\dagger = t_a, \quad a = 1, 2, \dots, 9, \quad (47)$$

where all  $t_a$ 's are Hermitian. To apply the scaled Gell-Mann generators  $t_a$  in (16) one may exploit the linearity of the momentum form

$$d\Phi(\alpha X + \beta Y) = \alpha d\Phi(X) + \beta d\Phi(Y), \quad \alpha, \beta \in \mathbb{C}. \quad (48)$$

This linearity is very practical if one wants to change the base of the algebra, as we did in (25), and still respect the commutation relations from the Lie group structure. We observe that there is a factor 2 between the trace norm of the  $t_a$ 's when compared to  $E_{ij}$  in (25) and to  $T_j$  in (8). This relates to the scaling issue in fig. 2 mentioned in connection with eq. (43). We have not settled the scaling issue in the present work.

Actually it seems more important to settle how the intrinsic structure maps to scattering centres in the

transverse plane with highly contracted structures for relativistic impact momenta. Imagine the “interface” between the intrinsic space as a door with a keyhole into which one looks from the laboratory space. One looks into the intrinsic structure with the scattering particle probing the density of the structure present behind the keyhole. At first thought one may think of an impact parameter description with the keyhole positioned at different places in the door. But that conception would require the intrinsic structure to be like a fixed wallpaper glued on the inside of the door. This would correspond to identifying the *origo* in the laboratory system with the *origo* in the configuration space. However, *the choice of origo is arbitrary* in configuration space. This fact leads to the praised local gauge invariance [32], but causes problems when probing through the keyhole. We suggest that the *origo* for the intrinsic structure at the time of impact be given a random position in the space of toroidal angles with an even probability distribution in these angles. One could then build on the ideas from the energy-dependent hot-spot model which assumes randomly distributed impact parameters and a varying number of hot-spot scattering centres [33]. The energy-dependent hot-spot model assumes Gaussian density distributions around each hot spot. Provided we use the information from the intrinsic structure derived from the intrinsic wave function, the Gaussian distributions should be replaced by contracted distributions like the one shown in fig. 1<sup>1</sup>.

**Conclusion** – We have introduced an analysis of the energy-momentum tensor for baryonic interior based on an intrinsic conception of baryonic dynamics. We used the momentum form on the  $U(3)$  Lie group configuration space to project energy-momentum components to the laboratory space and to do actual calculations on an exemplar protonic state. We indicated that the energy-momentum tensor behaves smoothly all through the spatial interior of the proton with a mass density vanishing at the centre. Further work should discuss the radial distributions and the off-centre structure in the transverse plane expected in scattering experiments.

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I thank JAKOB BOHR and STEEN MARKVORSEN for helpful discussions and the Technical University of Denmark for an inspiring working environment.

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<sup>1</sup>Possibly with axes  $(x', y', z')$  in a random orientation relative to laboratory  $(x, y, z)$ . The  $z$ -contraction is then along the beam axis and the distribution in the laboratory coordinates is (after contraction) to be averaged over the arbitrary angle  $\varphi$  in polar coordinates  $x = r \cos \varphi$  and  $y = r \sin \varphi$ .

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