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On Multivariate Wilson Bases

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Abstract—A Wilson system is a collection of finite linear combinations of time frequency shifts of a square integrable function. In this paper we give an account of the construction of bimodular Wilson bases in higher dimensions from Gabor frames of redundancy two.

I. INTRODUCTION

Daubechies, Jaffard and Journé [4], inspired by work of K. G. Wilson [11], introduced in 1991 the Wilson system of (linear combinations of) time-frequency shifts of a univariate function $g \in L^2(\mathbb{R})$:

$$\begin{aligned} \mathcal{W}_1(g) = & \{T_n g\}_{n \in \mathbb{Z}} \\ & \cup \left\{ \frac{1}{\sqrt{2}} T_n (M_m + (-1)^m M_{-m}) g \right\}_{n \in \mathbb{Z}, m \in \mathbb{N}} \\ & \cup \left\{ \frac{1}{\sqrt{2}} T_{n+1/2} (M_m - (-1)^m M_{-m}) g \right\}_{n \in \mathbb{Z}, m \in \mathbb{N}}, \end{aligned} \quad (1)$$

where the translation operator T_λ and the modulation operator M_γ are given by

$$T_\lambda f(x) = f(x - \lambda), \quad M_\gamma f(x) = e^{2\pi i x \cdot \gamma} f(x),$$

for $f \in L^2(\mathbb{R}^d)$ and $\lambda, \gamma \in \mathbb{R}^d$. The main result in [4] is a construction of Wilson orthonormal bases generated by functions $g \in L^2(\mathbb{R})$ with good time and frequency localization. This should be compared with the Gabor system $\{M_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$, where such basis constructions are impossible for any pair Λ and Γ of full-rank lattices in $\mathbb{R}^d$. Indeed, the Balian-Low Theorem [1], [2], [9] states that if a Gabor system is a orthonormal basis or a Riesz basis for $L^2(\mathbb{R}^d)$, then the generating window function g cannot have rapid decay in time and frequency.

However, it is easy to construct “nice” window functions g generating *redundant* Gabor frames. The main result of [4] illustrates how this fact can be used to construct nice orthonormal Wilson bases for $L^2(\mathbb{R})$:

Theorem I.1 ([4]). *Let $g \in L^2(\mathbb{R})$ be such that $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$ and $\|g\|_2 = 1$. Then the Gabor system $\{M_m T_{n/2} g\}_{m,n \in \mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$ with frame bound $A = 2$ if, and only if, the Wilson system $\mathcal{W}_1(g)$ is an orthonormal basis for $L^2(\mathbb{R})$.*

From the definition (1), it is clear that, except from the pure translations $\{T_n g\}_{n \in \mathbb{Z}}$, the univariate Wilson systems produce a bimodular covering of the frequency line, in the sense that each element of the system has two peaks in its power spectrum $|\hat{g}|^2$, assuming the window function is sufficiently localized in frequency. On the other hand, Gabor systems are unimodular since each element has a single peak in the power spectrum. In many applications a bimodular covering of the frequency domain is as good as a unimodular covering, in particular, if the signals of interest are real-valued.

As an example of an application of Theorem I.1, we mention that the continuous, symmetric function $g(x) = \cos(\pi x) \mathbb{1}_{[-1/2, 1/2]}(x)$ with compact support generates a tight Gabor frame $\{M_m T_{n/2} g\}_{m,n \in \mathbb{Z}}$ with frame bound $A = 2$ and $\|g\|_2 = 1$. Thus, the Wilson system $\mathcal{W}(g)$ is an orthonormal basis for $L^2(\mathbb{R})$.

The construction of Wilson orthonormal bases has recently been generalized to higher dimensions $L^2(\mathbb{R}^d)$ by the authors in [10]. The aim of this note is to give an overview of these results.

II. CONSTRUCTION OF MULTIVARIATE BIMODULAR WILSON BASES

In higher dimensions, it is obviously possible to construct orthonormal Wilson bases for $L^2(\mathbb{R}^d)$ by taking tensor products of univariate window functions from Theorem I.1. However, tensoring Wilson bases to $L^2(\mathbb{R}^d)$ has several undesirable side effects:

- a) the basis functions of tensored Wilson basis are 2^d -modular hence they give rise to a 2^d -modular covering of the frequency domain,
- b) tensored Wilson bases are associated with highly redundant Gabor frames of redundancy 2^d ,
- c) the generating function of a tensored Wilson basis has to be a separable function of the form $g_1(x_1) \dots g_d(x_d)$.

Gabor frames $\{M_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ are unimodular in all dimensions, hence give rise to a unimodular covering of the frequency domain. The 2^d -modular coverings are a curse of dimensionality of tensored Wilson bases since, e.g., symmetric peaks of the power spectrum of real-valued signals will leak out to 2^{d-1} other locations in frequency.

Our goal in this paper is to construct Wilson orthonormal bases in higher dimension that do not suffer from these tensoring artifacts.

A. Setup and Notation

We use the following assumptions throughout the paper: We let N be a subset of $\mathbb{Z}^d$ such that $N \cap (-N) = \emptyset$ and $N \cup (-N) \cup \{0\} = \mathbb{Z}^d$. For $g \in L^2(\mathbb{R}^d)$ we consider the Gabor system¹

$$\mathcal{G}(g) = \{T_\lambda M_\gamma g\}_{\lambda \in \Lambda, \gamma \in \mathbb{Z}^d},$$

where $\Lambda := \mathbb{Z}^d \cup (\mathbf{1}/2 + \mathbb{Z}^d)$ and where we use boldface $\mathbf{1}/2$ to denote the constant vector $(1/2, \dots, 1/2) \in \mathbb{R}^d$, and the Wilson system

$$\begin{aligned} \mathcal{W}(g) = & \{T_\lambda g\}_{\lambda \in \mathbb{Z}^d} \\ & \cup \left\{ \frac{1}{\sqrt{2}} T_\lambda (M_\gamma + (-1)^{|\gamma|} M_{-\gamma}) g \right\}_{\lambda \in \mathbb{Z}^d, \gamma \in N} \\ & \cup \left\{ \frac{1}{\sqrt{2}} T_{\lambda + \frac{1}{2}} (M_\gamma - (-1)^{|\gamma|} M_{-\gamma}) g \right\}_{\lambda \in \mathbb{Z}^d, \gamma \in N}, \end{aligned}$$

where $(-1)^{|\mathbf{n}|}$ means $(-1)^{n_1 + n_2 + \dots + n_d}$ for vectors $\mathbf{n} \in \mathbb{Z}^d$. For any dimension $d \in \mathbb{N}$ the Wilson system $\mathcal{W}(g)$, as opposed to tensored Wilson systems, provides a bimodular covering of the frequency domain for localized windows with the two peaks in frequency being symmetric about the origin. A priori it is not clear that this is the correct definition of Wilson systems in $L^2(\mathbb{R}^d)$, in particular, as there are many more symmetries possible in higher dimensions. The results of the next subsection will justify the definition of $\mathcal{W}(g)$.

B. Main result

We have the following duality principles of Gabor frames and Wilson bases.

¹This convention is a slight abuse of language since the defined system is only unitarily equivalent with the Gabor system $\{M_\gamma T_\lambda g\}$. However, these systems share all frame theoretic properties, and we will not make any distinction between such systems in the remainder of this paper.

Theorem II.1 ([10]). *Suppose that $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$. Then the following holds:*

- (i) *The Gabor system $\mathcal{G}(g)$ is a frame with bounds $2A$ and $2B$ for $L^2(\mathbb{R}^d)$ if and only if the Wilson system $\mathcal{W}(g)$ is a Riesz basis with bounds A and B for $L^2(\mathbb{R}^d)$.*
- (ii) *The Gabor system $\mathcal{G}(g)$ is a tight frame for $L^2(\mathbb{R}^d)$ with frame bound $A = 2$ if and only if the Wilson system $\mathcal{W}(g)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.*

The “only if”-assertion in (i) is Corollary 8.5.6 in [5] for $d = 1$, albeit without bounds. Assertion (ii) of Theorem II.1 generalizes Theorem I.1 to higher dimensions in a non-tensored way.

In the following example we construct bimodular multivariate Wilson bases from Gabor frames of redundancy two. The construction follows the standard construction procedure of “nice” generators g of univariate Wilson bases, see e.g., [4].

Example 1. Let $g \in L^2(\mathbb{R}^d)$ be a window function in the Wiener space $W(\mathbb{R}^d)$ satisfying the symmetry condition $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$ and

$$\text{ess inf}_{x, \omega \in [0, 1]^d} \left(|Zg(x, \omega)|^2 + |Zg(x - \frac{1}{2}, \omega)|^2 \right) > 0,$$

where Z denotes the Zak transform. By [5, Theorem 8.3.1], it follows that the Gabor system $\{T_\lambda M_\gamma g\}_{\lambda \in \mathbb{Z}^d \cup (\mathbf{1}/2 + \mathbb{Z}^d), \gamma \in \mathbb{Z}^d}$ is a frame. Let S denote the Gabor frame operator. Define $h = S^{-1/2}g = Z^{-1}qZg$, where $q = (|Zg|^2 + |ZT_{\mathbf{1}/2}g|^2)^{-1/2} \in L^\infty([0, 1]^2)$. Since the action of the frame operator preserves the symmetry of g , we conclude that $\{T_\lambda M_\gamma h\}_{\lambda \in \mathbb{Z}^d \cup (\mathbf{1}/2 + \mathbb{Z}^d), \gamma \in \mathbb{Z}^d}$ is a Parseval frame whose window function satisfies $\hat{h}(\omega) = \overline{\hat{h}(\omega)}$. An application of Theorem II.1 yields that the Wilson system generated by $\sqrt{2}h \in W(\mathbb{R}^d)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.

C. Auxiliary results

The following simple relationship between frame operators of the Gabor system and the Wilson system in Proposition II.2 seems not to have been noticed before in the literature.

Proposition II.2 ([10]). *Suppose that $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$. Then the Gabor system $\mathcal{G}(g)$ is a Bessel sequence with bound B if and only if the Wilson system $\mathcal{W}(g)$ is a Bessel sequence with bound $B/2$. Furthermore, in either (and hence both cases) the Gabor frame operator S_G and the Wilson frame operator S_W satisfy*

$$S_G = 2S_W. \quad (2)$$

The following density-type theorem for Wilson system is an easy consequence of Theorem II.1 and Proposition II.2.

Corollary II.3. *If $\mathcal{W}(g)$ is a frame for $L^2(\mathbb{R}^d)$ with bounds A and B , then $\mathcal{W}(g)$ is a Riesz basis for $L^2(\mathbb{R}^d)$ with bounds A and B .*

Proof. If $\mathcal{W}(g)$ is a frame for $L^2(\mathbb{R}^d)$ with bounds A and B , then, by Proposition II.2, so is $\mathcal{G}(g)$ with bounds $2A$ and $2B$. The conclusion now follows from Theorem II.1(i). $\square$

III. PROOF TECHNIQUES

The proof of the results in Section II from [10] are mostly based on the frame theory of shift-invariant systems [3], [6], [7]. We illustrate in this section how this theory can be applied in the study of Wilson bases. We refer to [10] for a detailed argumentation and a proof of Theorem II.1.

A. Frame theory of shift-invariant system

Definition III.1. Let Γ be a countable index set and let $\{g_\gamma\}_{\gamma \in \Gamma} \subset L^2(\mathbb{R}^d)$. For a full-rank lattice $\Lambda = P\mathbb{Z}^d$, where $P \in \text{GL}_d(\mathbb{R})$, the dual lattice is given by $\Lambda^\perp = (P^{-1})^\top \mathbb{Z}^d$. Suppose that

$$\sum_{\gamma \in \Gamma} |\hat{g}_\gamma(\omega)|^2 < \infty \quad \text{for a.e. } \omega \in \mathbb{R}^d. \quad (3)$$

For the shift-invariant system $\{T_\lambda g_\gamma\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ we define its autocorrelation functions $\{t_\alpha\}_{\alpha \in \Lambda^\perp}$ by

$$t_\alpha(\omega) := \frac{1}{|\det P|} \sum_{\gamma \in \Gamma} \hat{g}_\gamma(\omega) \overline{\hat{g}_\gamma(\omega - \alpha)} \quad (4)$$

for a.e. $\omega \in \mathbb{R}^d$ and $\alpha \in \Lambda^\perp$.

For a given function $t \in L^\infty(\mathbb{R}^d)$, define the multiplication operator

$$M_t f(x) = t(x)f(x) \quad \text{for } f \in L^2(\mathbb{R}^d).$$

For the special choice of $t(x) = e^{2\pi i \langle x, \gamma \rangle}$ with $\gamma \in \mathbb{R}^d$, this yields the modulation operator M_γ , justifying our notation.

Lemma III.2 ([6], [7]). *Let $K = P\mathbb{Z}^d$, where $P \in \text{GL}_d(\mathbb{R})$, be a full-rank lattice, let Γ be a countable index set, and let $\{g_\gamma\}_{\gamma \in \Gamma} \subset L^2(\mathbb{R}^d)$. For the shift-invariant system $\{T_\lambda g_\gamma\}_{\lambda \in K, \gamma \in \Gamma}$ the followings holds:*

- (i) *$\{T_\lambda g_\gamma\}_{\lambda \in K, \gamma \in \Gamma}$ is a tight frame for $L^2(\mathbb{R}^d)$ with frame bound A if, and only if, for all $\alpha \in K^\perp$,*

$$t_\alpha(\omega) = A\delta_{\alpha,0} \quad (5)$$

for almost all $\omega \in \mathbb{R}^d$. In either case, the series defining t_α are a.e. absolutely convergent and $|t_\alpha(\omega)| \leq A$ for a.e. $\omega \in \mathbb{R}^d$.

- (ii) *If $\{T_\lambda g_\gamma\}_{\lambda \in K, \gamma \in \Gamma}$ is a Bessel sequence with bound B , then the series (4) defining t_α are a.e. absolutely convergent, $|t_\alpha(\omega)| \leq B$ for a.e.*

$\omega \in \mathbb{R}^d$, and the frame operator S has a Walnut representation:

$$\widehat{S}f = \sum_{\alpha \in K^\perp} M_{t_\alpha} T_\alpha f$$

with unconditionally norm convergence for all $\hat{f} \in L^\infty(\mathbb{R}^d)$ with compact support.

B. On autocorrelation functions of Wilson and Gabor systems

Recall that $\Lambda := \mathbb{Z}^d \cup (\mathbf{1}/2 + \mathbb{Z}^d)$. One readily verifies that Λ is a full-rank lattice with its dual lattice given by

$$\Lambda^\perp = \{n \in \mathbb{Z}^d : n_1 + n_2 + \dots + n_d \in 2\mathbb{Z}\}.$$

The usefulness of the frame theory of shift-invariant systems in the construction of Wilson bases is the following key lemma.

Lemma III.3 ([10]). *Let $g \in L^2(\mathbb{R}^d)$ and $\hat{g}(\omega) = \hat{g}(\omega)$. Suppose that*

$$\sum_{\gamma \in \mathbb{Z}^d} |\hat{g}(\omega - \gamma)|^2 < \infty \quad (6)$$

for a.e. $\omega \in \mathbb{R}^d$. Then the following holds:

- (i) *If the Gabor system $\mathcal{G}(g)$ is considered as a shift-invariant system with generators $\{M_\gamma g\}_{\gamma \in \mathbb{Z}^d}$ and with shifts along the lattice $\Lambda = \mathbb{Z}^d \cup (\mathbf{1}/2 + \mathbb{Z}^d)$, then its autocorrelation functions are given by*

$$t_{\alpha, \mathcal{G}}(\omega) = 2 \sum_{\gamma \in \mathbb{Z}^d} \hat{g}(\omega - \gamma) \overline{\hat{g}(\omega - \gamma - \alpha)},$$

for $\alpha \in \Lambda^\perp$, a.e. $\omega \in \mathbb{R}^d$.

- (ii) *If the Wilson system $\mathcal{W}(g)$ is considered as a shift-invariant system with generators*

$$g, \left\{ \frac{1}{\sqrt{2}} (M_\gamma + (-1)^{|\gamma|} M_{-\gamma}) g \right\}_{\gamma \in N}$$

and $\left\{ \frac{1}{\sqrt{2}} T_{\mathbf{1}/2} (M_\gamma - (-1)^{|\gamma|} M_{-\gamma}) g \right\}_{\gamma \in N}$

and with shifts along the lattice $\mathbb{Z}^d$, then its autocorrelation functions are given by, for $\alpha \in \Lambda^\perp$,

$$t_{\alpha, \mathcal{W}}(\omega) = \sum_{\gamma \in \mathbb{Z}^d} \hat{g}(\omega - \gamma) \overline{\hat{g}(\omega - \gamma - \alpha)},$$

and for $\alpha \in \mathbb{Z}^d \setminus \Lambda^\perp$, $t_{\alpha, \mathcal{W}}(\omega) = 0$ for a.e. $\omega \in \mathbb{R}^d$.

Sketch of the proof. Assumption (6) guarantees that generators of $\mathcal{G}(g)$ and $\mathcal{W}(g)$ satisfy condition (3). Hence, their autocorrelation functions are well-defined. It is straightforward to verify (i) from (4).

In the following we sketch the proof of (ii) for $d = 1$. In dimension one, we find $\Lambda = (1/2)\mathbb{Z}$, $\Lambda^\perp = 2\mathbb{Z}$; the canonical choice of the set N is $\mathbb{N}$, but we will allow

a general index set N . By Definition III.1, for $\alpha \in \mathbb{Z}$ we have

$$\begin{aligned} t_{\alpha, \mathcal{W}}(\omega) &= \hat{g}(\omega) \overline{\hat{g}(\omega - \alpha)} \\ &+ \frac{1}{2} \sum_{\gamma \in N} G(--) + (-1)^{|\gamma|} G(-+) \\ &\quad + (-1)^{|\gamma|} G(+-) + G(++) \\ &+ \frac{1}{2} (-1)^\alpha \sum_{\gamma \in N} G(--) - (-1)^{|\gamma|} G(-+) \\ &\quad - (-1)^{|\gamma|} G(+-) + G(++). \end{aligned} \quad (7)$$

where $G(\pm\pm) := \hat{g}(\omega \pm \gamma) \overline{\hat{g}(\omega \pm \gamma - \alpha)}$. Due to the phase factor $(-1)^\alpha$, we will consider two cases: (I) $\alpha \in 2\mathbb{Z}$, and (II) $\alpha \in 2\mathbb{Z} + 1$, corresponding to $\alpha \in \Lambda^\perp$ and $\alpha \in \mathbb{Z}^d \setminus \Lambda^\perp$, respectively. Using that $N \cup (-N) \cup \{0\} = \mathbb{Z}$ and $N, -N$ and $\{0\}$ are mutually disjoint sets, formula (7) yields:

(I) for $\alpha \in \Lambda^\perp = 2\mathbb{Z}$

$$t_{\alpha, \mathcal{W}}(\omega) = \sum_{\gamma \in \mathbb{Z}} G(--) \quad (8)$$

(II) for $\alpha \in \mathbb{Z} \setminus \Lambda^\perp = \mathbb{Z} \setminus 2\mathbb{Z}$

$$t_{\alpha, \mathcal{W}}(\omega) = \sum_{\gamma \in \mathbb{Z}} (-1)^{|\gamma|} G(-+) \quad (9)$$

Note that the symmetry property $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$ was not used in the verification of (8).

It remains to show that (9) is equal to zero. Fix $\alpha \in \mathbb{Z} \setminus 2\mathbb{Z}$. By a change of variables $\gamma \mapsto -\gamma' + \alpha$, we obtain

$$t_{\alpha, \mathcal{W}}(\omega) = \sum_{\gamma' \in \mathbb{Z}} (-1)^{|(-\gamma' + \alpha)|} \hat{g}(\omega + \gamma' - \alpha) \overline{\hat{g}(\omega - \gamma')}$$

for a.e. $\omega \in \mathbb{R}$. For odd $\alpha \in \mathbb{Z}$, we note that

$$(-1)^{|-\gamma' + \alpha|} = -(-1)^{|\gamma'|}.$$

By our assumption $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$, it follows that

$$\hat{g}(\omega + \gamma' - \alpha) \overline{\hat{g}(\omega - \gamma')} = \overline{\hat{g}(\omega + \gamma' - \alpha)} \hat{g}(\omega - \gamma').$$

From the last three displayed equations, we see that $t_{\alpha, \mathcal{W}}(\omega) = -t_{\alpha, \mathcal{W}}(\omega)$, hence $t_{\alpha, \mathcal{W}}(\omega) = 0$. $\square$

Lemma III.3 shows a very simple relationship between the autocorrelation functions of Gabor systems and Wilson systems. In fact:

$$t_{\alpha, \mathcal{W}}(\omega) = \begin{cases} 2^{-1} t_{\alpha, \mathcal{G}}(\omega) & \alpha \in \Lambda^\perp, \\ 0 & \alpha \in \mathbb{Z}^d \setminus (\Lambda^\perp). \end{cases} \quad (10)$$

C. A proof of $S_{\mathcal{G}} = 2S_{\mathcal{W}}$

To illustrate the usefulness of Lemma III.3, we give a simple proof of $S_{\mathcal{G}} = 2S_{\mathcal{W}}$.

Proof of Proposition II.2. We only argue for the proof of the “furthermore”-part, i.e., (2). The Walnut representations of the two frame operators show that

$$\begin{aligned} \widehat{S_{\mathcal{W}}} f &= \sum_{\alpha \in \mathbb{Z}^d} M_{t_{\alpha, \mathcal{W}}} T_{\alpha} \hat{f} \\ &= 2^{-1} \sum_{\alpha \in \Lambda^\perp} M_{t_{\alpha, \mathcal{G}}} T_{\alpha} \hat{f} = 2^{-1} \widehat{S_{\mathcal{G}}} f \end{aligned}$$

for f in a dense subset of $L^2(\mathbb{R}^d)$. By continuity the relation extends to all of $L^2(\mathbb{R}^d)$. $\square$

IV. EXTENSIONS

It is possible to extend the results from Section II on multivariate Wilson bases in several directions. Firstly, it is possible to construct multivariate Wilson bases on symplectic lattices. In dimension one this result generalizes a result by Kutyniok and Strohmer [8]. Secondly, it is possible to define a huge number of distinct intermediate 2^k -modular Wilson systems for $k = 1, \dots, d-1$. We refer the reader to [10] for the precise statements.

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