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Izumi, Shuro; Neergaard-Nielsen, Jonas Schou; Andersen, Ulrik Lund

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Tomography of a single-rail qubit projector

Shuro Izumi, Jonas S. Neergaard-Nielsen, and Ulrik L. Andersen
Center for Macroscopic Quantum States (bigQ), Department of Physics, Technical University of Denmark, Building 307, Fysikvej, 2800 Kgs. Lyngby, Denmark

The projective measurement of a qubit in its computational basis as well as its conjugate basis is an incredibly important tool in both fundamental studies and practical applications of quantum information science. For some qubits (e.g., spin and polarization qubits), these measurements are simple while for other qubits (e.g., single-rail and cat qubits) it is notoriously difficult. Here we experimentally demonstrate a measurement that is known to be, in principle, an ideal projection onto the superposition of vacuum and single photon states – the single-rail qubit. Our measurement consists of a displacement operation combined with a photon counting measurement followed by a real time feedback operation. We characterize the single-rail qubit projector by detector tomography and find that it can discriminate the conjugate basis with a certainty of 96%. Such a feedback controlled photon counter will facilitate the realization of quantum information protocols with single-rail qubits as well as the non-locality test of certain entangled states.

Introduction.— In quantum computing and quantum communication, the information is usually encoded as qubits; examples being single-rail qubits, polarization qubits, cat state qubits and Gottesman-Kitaev-Preskill state qubits [1–5]. Among these qubit realizations, the single-rail encoding – corresponding to the superposition of the vacuum state, |0⟩, and the single photon state, |1⟩ – is particularly interesting due to its natural interconvertibility between different physical systems such as atomic, mechanical and optical systems [6]. Furthermore, the single-rail qubit is often a suitable encoding in both discrete variable (DV) as well as some continuous variable (CV) protocols, which is exemplified by single-rail qubit teleportation using a DV [7] and CV [8] teleporter as well as universal quantum computation using a DV [9] and a CV [10] computer. Finally, the single-rail qubit can be also seamlessly converted into other qubit formats such as cat state qubits and polarization qubits [11, 12].

An important ingredient in many quantum information protocols based on single-rail qubits is a projective measurement that discriminates the basis states, |0⟩ and |1⟩, as well as one that discriminates the conjugate states, |±⟩ = 1/√2(|0⟩ ± |1⟩). While the basis states can be seamlessly discriminated using a high-efficiency photon counter, the discrimination of the conjugated states is non-trivial since they are not associated with simple physical observables. One discrimination strategy for the conjugate states is to apply a Hadamard transform to the state (converting |±⟩ into |0⟩ or |1⟩) prior to the measurement of its photon number, but such a transformation is highly non-trivial requiring a strong non-linearity [13]. However, a much simpler strategy is based on applying a displacement operation onto the state prior to photon-counting and controlling the displacement in real-time based on the photon measurement outcomes using real-time feedback [14–22]. This strategy is asymptotically approaching the ideal conjugate basis state projector for infinitely fast feedback, and furthermore, it may enable the projection onto any arbitrary single-rail qubit [16].

In this paper, we experimentally demonstrate a feedback controlled projective measurement onto the conjugate basis |±⟩ of a single-rail qubit. By combining a photon counter with a displacement transformation that is optimally controlled by the measurement outcome, we discriminate the two conjugate basis states with a low error rate. By changing the parameters of the displacement operations, our measurement can be generalized to the projection onto arbitrary orthogonal states on the Bloch sphere. We adopt quantum detector tomography with coherent states to characterize our measurement and evaluate the discrimination error for the superposition states as the main figure of merit. Although direct evaluation of the discrimination error would be possible by probing with the superposition states, the accessible information regarding the measurement would be rather specific, whereas full characterization of the measurement via detector tomography enables an in-depth analysis of the measurement. Tomography of measurements has been demonstrated for various types of static measurements [23–27], but here we perform it for the first time for a dynamically updated measurement.

Concept.—The original proposal for quantum state discrimination using a displacement based photon counter with feedback operations shown in Fig. 1(a) consists of a displacement operation, a photon counter and real-time feedback of the photon counter’s measurement outcome to the displacement amplitude [14]. An incoming quantum state with full time width T is virtually divided into M temporal mode bins with time widths t_i for the i'th bin. The phase and amplitude of the displacement in each bin, β_i, is dependent on the photon counting history of the earlier time bins. The measurement strategy can be equivalently analyzed using spatial modes, where the measurement consists of beam splitters (BSs) having reflectances r_i^2 and a displacement operation with the amplitude β_i and a photon counter in each of the M spatial modes [15, 16], as depicted in Fig. 1(b).

For concreteness, we will describe the protocol in the
setting of discrimination of the states $|\pm\rangle$. The quantum state to be distinguished is split by the BSs and the first displacement operation is implemented such that the $|\pm\rangle$ state is displaced close to the vacuum state. The displaced state is detected by the photon counter, whose binary outcomes indicate whether the state is more likely to be $|\pm\rangle$ (off) or $|\mp\rangle$ (on). Once an outcome from the photon counter is obtained, one can calculate an \textit{a posteriori} probability $P(|\mp\rangle|\{e_i\}$ for given outcomes $\{e_i\}$, where $e_i \in \{\text{off, on}\}$. The displacement amplitude is controlled dependent on the \textit{a posteriori} probability, i.e., the most probable state is displaced close to the vacuum state. By repeating the operations and detections recursively, we conclude whether the state is $|\pm\rangle$ or $|\mp\rangle$ depending on the \textit{a posteriori} probability. The optimal strategy, it turns out, is to change the sign of the $i + 1$th displacement with respect to the $i$th displacement if the $i$th counter detects a photon and to maintain the phase otherwise. The conclusion of the state discrimination is then $|\pm\rangle$ if the total number of “on” events is even and $|\mp\rangle$ if it is odd [15, 16]. The discrimination error approaches zero if $M \to \infty$. We adopt the spatial mode analysis to investigate the performance of the measurement in the finite number case.

Each stage of the measurement, consisting of beam splitter, displacement, and photon counting with the outcome $e_i$, can be considered as a single quantum operation $\hat{E}_e(i)$ on the state that was output from the previous stage. The post-measurement state of the $i$'th stage is then $\hat{\rho}^{(i+1)} = \hat{E}_e(i)(\hat{\rho}(i))/\text{Tr} \hat{E}_e(i)(\hat{\rho}(i))$, where the normalization factor is the probability of getting the outcome $e_i$. For an “off” detection, the map of the operation is $\hat{E}_e(i)(\hat{\rho}(i)) = \hat{K}_0(i)\hat{\rho}(i)\hat{K}_0(i)^\dagger$, with the Kraus operator corresponding to zero photons at the detector given by $\hat{K}_0(i) = e^{-\frac{1}{2}b^2}e^{-\alpha t_i^2}e^{-\alpha^2 b_i^2/4t_i}e^{\alpha^2 b_i^2 t_i/4}$. Since the photon counter cannot distinguish between one and more photons, the corresponding map for an “on” detection is $\hat{E}_e(i)(\hat{\rho}(i)) = \sum_{n=1}^\infty \hat{K}_n(i)\hat{\rho}(i)\hat{K}_n(i)^\dagger$ with the $n$-photon Kraus operators $\hat{K}_n(i) = \frac{1}{\sqrt{r^n_m}}(\hat{a}^n\hat{a}^n + \hat{a}^n\hat{a}^n)\hat{K}_0(i)^\dagger$. The total probability for obtaining the series of detection events $\{e_1, \ldots, e_M\}$ given an input state $\hat{\rho}$ is then the trace of the composition of the maps for each stage, $P(\{e_i\} | \hat{\rho}) = \text{Tr} \hat{E}(M)^{(i)} \circ \cdots \circ \hat{E}(1)^{(i)}(\hat{\rho})$. Equivalently, the probability can be written in terms of a POVM corresponding to that specific measurement outcome, $P(\{e_i\} | \hat{\rho}) = \text{Tr} [\hat{\rho} \hat{\Pi}_{\{e_1, \ldots, e_M\}}]$. While the former formulation is most natural for understanding and modelling the iterative detection scheme, the latter is more relevant for the process of detector tomography which returns the elements of the POVM. Thus, denoting the sets of all possible outcomes with even (odd) number of “on” events as $E$ ($O$), the error probability for the discrimination of the superposition states is given by,

$$P_e = \frac{1}{2} \left( \sum_{\{e_i\} \in E} P(\{e_i\} | |\mp\rangle) + \sum_{\{e_i\} \in \{\{e_i\}_{e_{-1}}\}} P(\{e_i\} | |\pm\rangle) \right).$$

To illustrate the scheme and the optimizations involved, we first consider the feedback measurement of $|\pm\rangle$ in the case of $M = 2$. Using the decision strategy and the expressions outlined above, we obtain the error probability for the feedback measurement with $M = 2$ as,

$$P_e^{M=2} = \frac{1}{2} - r \text{Re}[\beta_1]e^{-|\beta_1|^2} - (1 - e^{-|\beta_{2,\text{on}}|^2} - e^{-|\beta_{2,\text{off}}|^2})$$
$$- \sqrt{1 - r^2} \text{Re}[\beta_{2,\text{on}}]e^{-|\beta_{2,\text{on}}|^2} - (1 - e^{-|\beta_1|^2})$$
$$+ \sqrt{1 - r^2} \text{Re}[\beta_{2,\text{off}}]e^{-|\beta_{2,\text{off}}|^2} - e^{-|\beta_1|^2}. $$

The “off” and “on” indices on $\beta_2$ indicate that the amplitude of the second displacement depends on the outcome in the first channel. The minimum achievable error probability is obtained to be $P_e^{M=2} = 0.040$ at the optimized parameter values $r^2 = 0.336$, $\beta_1 = -0.643$, $\beta_{2,\text{off}} = -0.514$, $\beta_{2,\text{on}} = 0.390$. For comparison, the displacement photon counting scheme without feedback operation (i.e. $M = 1$) obtains $P_e^{M=1} \approx 0.071$, while homodyne detection would be able to achieve an error probability of 0.101 [27].

Finding the ultimate performance for a given $M$ requires optimization over $M - 1$ parameters for the BS reflection coefficients and $2^{M-1}$ parameters for the displacement magnitudes $|\beta_i|$. This problem becomes intractable for large $M$. In a simplification of the scheme, we may assume that only the displacement phases, i.e. the sign of the $\beta_i$’s should depend on the outcome history, whereas the displacement magnitudes will be kept fixed at each stage, e.g. $|\beta_{2,\text{off}}| = |\beta_{2,\text{on}}|$. It turns out (see later) that there is only a small penalty to pay for the error probability in using this simplified scheme. The results of the minimization are plotted in Fig. 1(c).

Experiment.—Fig. 2 illustrates our experimental setup. Quantum detector tomography requires well characterized probe states that cover the Hilbert space of interest. We use densely spaced coherent states as probes, since they are readily available and tomographically complete [23]. A continuous-wave, fiber-coupled laser at 1550 nm is split in two paths, one for preparation of the probe states and one for the reference field for the displacement operation. The laser intensity is switched between high and low for the purposes of phase calibration/stabilization and measurement, respectively. The intensity and the phase of the probe states are adjusted by a variable attenuator and a phase shifter that consists of a piezo transducer embedded in a circular mount with an optical fiber looped around. We prepare probe states with 4 weak magnitudes $|\alpha|^2 \approx \{0.4, 0.6, 0.8, 1.0\}$ and 8 phase conditions $j\pi/4, j = 0, \ldots, 7$ as well as the vacuum state to characterize our measurement in the two-dimensional Hilbert space spanned by $|0\rangle$ and $|1\rangle$. The displacement operation is physically implemented with a 99:1 fiber coupler. Its magnitude and direction is controlled by a phase and an intensity modulator. When the laser intensity is high (locking mode), a switch directs the displaced probe to a conventional photo detector for stabilization of the relative phase between probe and reference. When the intensity is low (measurement mode), the displaced probe state is detected by a superconducting nanowire single photon detector (SSPD) [28, 29]. A field programmable gate array (FPGA) counts the electrical signal from the SSPD and rapidly changes—dependent on the measurement outcome—the voltages applied to the intensity and phase modulators. The procedure is repeated 10,000 times for each of the probe states. The POVM elements of the measurement are reconstructed following the maximum likelihood procedure using the known density matrices of the probe states and the distribution of the outcomes [30].

Losses in the switch and other optical components and the finite detection efficiency of the SSPD limit the performance of our measurement. The total transmittance from the 99:1 fiber coupler to the fiber right before the SSPD is measured to be 65%. A benefit of the SSPD is that by changing the applied bias voltage, one can tune the trade off between high detection efficiency and low dark count rate. We set the efficiency to $\sim 51\%$ which results in a dark count rate of $\sim 20$ counts per second. As a proof of concept and to highlight the functionality of the measurement, we choose to disregard the finite overall efficiency $\eta$. In practice, we do this by calibrating the magnitude of the probe states and displacements by the actual count rate of the SSPD. This corresponds to a rescaling of the probe coherent state amplitudes, $\sqrt{\eta}\alpha \rightarrow \alpha$.

We first explore the $M = 2$ case in detail, investigating the error probability with variable beam-splitter ratio and optimized displacement amplitudes. The quantum states of the probes are defined within a rectangular temporal mode of length $T = 100 \mu$s. The feedback bandwidth of our experiment is limited by a digital analog converter whose bandwidth is roughly 1 MHz. The delay of the feedback operations degrades the discrimination error. Therefore, we discard counts observed in a $2 \mu$s time interval after the first time bin, which corresponds to 2% loss. The delay analysis is further discussed in Supplemental Material.

Experimental results of the estimated error probability for $M = 2$ with various settings of $t_1$ (corresponding to the beam-splitter ratio in the spatial picture) are shown in Fig. 3(a). Red data points are the experimentally estimated error probability. The mean values and the error bars are evaluated from 5 independent procedures. The blue and red solid curves represent, respectively, the ex-
Next, we investigate the performance of the displacement photon counting for up to five feedback stages. The experimental results agree well with the theoretical prediction and the optimization of the feedback measurements in the ideal case and with experimental imperfections, the non-unit visibility $\xi = 0.98$, the dark count noise $2.38 \times 10^{-3}$ counts/state and 2% loss due to the feedback delay. The slightly degraded performance shown by the red dashed line is what would be obtained by keeping the displacement magnitude fixed ($|\beta_{2,\text{off}}| = |\beta_{2,\text{on}}|$). Measurement without feedback operation with and without the experimental imperfections are shown by red and blue dot lines. Figure 3(b) depicts the experimental conditions as well as the theoretical optimum values of the displacement amplitudes. The experimental results agree well with the theoretical prediction and the optimization of the feedback timing is essential to achieve the minimum error probability for a given $M$.

Next, we investigate the performance of the displacement photon counting for up to five feedback stages. The experimentally obtained error probability is plotted in Fig. 4(a) and expected theoretical values are shown by solid gray lines showing the ideal values for minimum error discrimination. The gaps between time bins are the periods that are discounted to alleviate errors due to the finite feedback time. The red lines indicate the values expected from the model, while the blue lines indicate the ideal performance with no experimental imperfections. Dashed lines are for fixed displacement magnitudes at each stage, while the solid lines are for magnitudes adapted to the count history. (b) Experimental conditions of the displacement magnitudes and time bin divisions. The thickness of the top lines indicate the uncertainty ($\pm 1$ standard deviation) of the magnitudes, while the solid gray lines show the ideal values for minimum error discrimination. The gaps between time bins are the periods that are discounted to alleviate errors due to the finite feedback time. The red lines indicate the values expected from the model, while the blue lines indicate the ideal performance with no experimental imperfections. Dashed lines are for fixed displacement magnitudes at each stage, while the solid lines are for magnitudes adapted to the count history. (b) Experimental conditions of the displacement magnitudes and time bin divisions. The thickness of the top lines indicate the uncertainty ($\pm 1$ standard deviation) of the magnitudes, while the solid gray lines show the ideal values for minimum error discrimination. The gaps between time bins are the periods that are discounted to alleviate errors due to the finite feedback time. The red lines indicate the values expected from the model, while the blue lines indicate the ideal performance with no experimental imperfections. Dashed lines are for fixed displacement magnitudes at each stage, while the solid lines are for magnitudes adapted to the count history. (b) Experimental conditions of the displacement magnitudes and time bin divisions. The thickness of the top lines indicate the uncertainty ($\pm 1$ standard deviation) of the magnitudes, while the solid gray lines show the ideal values for minimum error discrimination. The gaps between time bins are the periods that are discounted to alleviate errors due to the finite feedback time. The red lines indicate the values expected from the model, while the blue lines indicate the ideal performance with no experimental imperfections. Dashed lines are for fixed displacement magnitudes at each stage, while the solid lines are for magnitudes adapted to the count history. (b) Experimental conditions of the displacement magnitudes and time bin divisions. The thickness of the top lines indicate the uncertainty ($\pm 1$ standard deviation) of the magnitudes, while the solid gray lines show the ideal values for minimum error discrimination. The gaps between time bins are the periods that are discounted to alleviate errors due to the finite feedback time.

FIG. 3. (a) Error probability for the discrimination of the $|\pm\rangle$ states using our displacement photon counter with a single feedback operation ($M = 2$) as a function of the relative temporal width of the first time bin, $t_1/T$. (b) Amplitude conditions for the displacement operation. $\beta_1$ is the displacement amplitude in the first time bin, while $\beta_{2,\text{off}}$ and $\beta_{2,\text{on}}$ are for the second time bin if the outcome of the first time bin is “off” or “on”. The curves indicate the theoretical optimum values as a function of $t_1$. Solid curves are for second-stage magnitudes adapted to the first stage’s detector outcome, while dashed curves are for fixed magnitudes ($|\beta_{2,\text{off}}| = |\beta_{2,\text{on}}|$).

FIG. 4. (a) Estimated error probabilities for the discrimination of $|\pm\rangle$ using various number of feedback operations. The points are experimentally obtained values with error bars. The red lines indicate the values expected from the model, while the blue lines indicate the ideal performance with no experimental imperfections. Dashed lines are for fixed displacement magnitudes at each stage, while the solid lines are for magnitudes adapted to the count history. (b) Experimental conditions of the displacement magnitudes and time bin divisions. The thickness of the top lines indicate the uncertainty ($\pm 1$ standard deviation) of the magnitudes, while the solid gray lines show the ideal values for minimum error discrimination. The gaps between time bins are the periods that are discounted to alleviate errors due to the finite feedback time.
ity owing to the feedback operation could be saturated. Therefore, we increase the time width for the probe state as \( T = 100 \times (M - 1) \mu s \) such that the delay loss can be constant 2% and decrease the bias voltage applied to the SSPD to get the constant dark counts \( \sim 2 \times 10^{-3} \) counts/state. Although the actual performance does not reach the ideal one (due to the experimental imperfections), we observe a clear improvement of the error probability by increasing the number of feedback operations.

Summary.—We experimentally demonstrated a measurement system designed for the discrimination of single-rail qubits. It is composed of a displacement operation, a photon counter and feedback operations that depend on the outcome of the photon counter. Our measurement was characterized by quantum detector tomography using coherent state probes. We first investigated the error probability attainable by our measurement with a single feedback operation for varying timing conditions, showing the importance of optimizing this parameter. Secondly, we showed that the expected discrimination error can be improved by increasing the number of feedback operations.

We expect that our projector will pave the way for various applications in quantum information science utilizing the single-rail optical qubit. Moreover, the projector can be used to demonstrate quantum non-locality between many parties using a multi-mode delocalized single photon state (also known as a W state) [13, 31–34], which in turn will provide device-independent security in quantum communication [35, 36], and may lead to remote preparation of the superposition state [37–40]. Furthermore, it is interesting to note that the displacement photon counter with optimized feedback timing can also provide an advantage for coherent state discrimination in practical scenarios with finite feedback bandwidth [19].

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Supplemental material

Shuro Izumi, Jonas S. Neergaard-Nielsen, and Ulrik L. Andersen
Center for Macroscopic Quantum States (bigQ), Department of Physics, Technical University of Denmark, Building 307, Fysikvej, 2800 Kgs. Lyngby, Denmark

I. FUNDAMENTAL BOUND OF THE DISCRIMINATION ERROR WITH LINEAR LOSS

We analyze the fundamental bound of the discrimination error for the superposition states with a linear loss $\eta$. The loss can be discussed by a beam splitter model and the superposition states after the loss can be described as

$$\rho_\pm^\eta = \frac{1}{2} \left[ 2 - \eta \pm \sqrt{\eta} \right]$$

in the photon number basis. Thus, the error probability for the superposition states with an equal prior probability is,

$$P_e = \frac{1}{2} (\text{Tr}[\rho_+^\eta \hat{\Pi}_-] + \text{Tr}[\rho_-^\eta \hat{\Pi}_+])$$

$$= \frac{1}{2} \left( 1 - \text{Tr}[\hat{\Pi}_+ (\rho_+ - \rho_-)] \right),$$

where $\hat{\Pi}_\pm$ are the POVM for concluding the input state as $|\pm\rangle$. The minimum error probability for the discrimination of $\rho_+^\eta$ is obtained by maximizing $\text{Tr}[\hat{\Pi}_+ (\rho_+ - \rho_-)]$ and given by,

$$P_e = \frac{1}{2} (1 - \lambda)$$

$$= \frac{1}{2} (1 - \sqrt{\eta}),$$

where $\lambda = \sqrt{\eta}$ is a positive eigenvalue of the operator $(\rho_+^\eta - \rho_-^\eta)$ [1]. An optimal measurement to accomplish the error probability Eq.(3) is given by a projector onto the superposition basis $|\pm\rangle$ irrespective of the loss.

In addition to the linear loss, various experimental imperfections cause the degradation of the performance and we show a recipe to calculate the error probability for the measurement consisting of the displacement and the photon counter with the experimental imperfections [2, 3]. In order to analyze the imperfection of the displacement operation due to limited visibility in the interference of the input and reference fields, we assume that the state is split into two modes where mode 0 does not interfere with the coherent state for the displacement operation and mode 1 is displaced with $\hat{D}_1(\sqrt{\xi} \beta)$. Both modes are detected by the photon counter with imperfections. By denoting the visibility of the displacement operation as $\xi$, the state is divided into two modes,

$$\hat{B}(\xi)|\pm\rangle_1 |0\rangle_2$$

$$= \{ |0\rangle_0 |0\rangle_1 + (\sqrt{\xi} |0\rangle_0 |1\rangle_1 \pm \sqrt{1-\xi} |1\rangle_0 |0\rangle_1) \}/\sqrt{2}.$$

The probability of having “off” outcome is given by a product of the “off” probability for each mode. Therefore, the POVM for the final outcomes is described as,

$$\hat{\Pi}_\text{off} = e^{-\nu - (1-\xi)\eta^2} \hat{\Pi}_\text{off}^1 \hat{D}_1(\sqrt{\xi} \beta) \hat{D}_1(\sqrt{\xi} \beta),$$

$$\hat{\Pi}_\text{on} = \hat{I} - \hat{\Pi}_\text{off},$$

where $\hat{\Pi}_\text{off}^i = \sum_{n=0}^\infty (1-\eta)^n |n\rangle \langle n|$ is the probability of having “off” from the photon counter for $i$ mode. The dark count rate (defined as the probability per state of getting at least one dark count) and the detection efficiency of the photon counter are represented by $\nu$ and $\eta$ respectively. An additional unwanted count due to the coherent state for the displacement operation that does not interfere with the superposition state induces “on” event irrespective of the states to be discriminated. Hence, the error probability for the discrimination of the superposition states with the non-ideal measurement is,

$$P_e = \frac{1}{2} \left( 1 \langle 0 | \hat{B}(\xi)^\dagger \hat{\Pi}_\text{on} \hat{B}(\xi) |0\rangle \right)$$

$$+ 1 \langle 0 | \hat{B}(\xi)^\dagger \hat{\Pi}_\text{off} \hat{B}(\xi) |0\rangle$$

$$= \frac{1}{2} + \eta \xi \beta e^{-\nu - \eta^2}.$$

Since derivation of the Kraus operator is not as straightforward as the ideal case, finding the analytical expression for the state evolution is not trivial. Nevertheless, we can calculate the state evolution using the method outlined above and obtain the theoretical error probability for the feedback measurement with the imperfections up to 5 stages, which is shown in Fig. 4(a) in the main text.

II. EXPERIMENTAL PROCEDURE

Figure 1 shows the procedure of our experiment for the feedback measurement with $M = 2$. In order to prepare the probe state with the targeted phase condition, the interferometer is stabilized with a strong laser beam and a conventional photo detector. The relative phase can be set to an arbitrary phase condition except close to $\theta = 0, \pi$ by actively controlling a piezo transducer (PZT). The probe state with the condition $\theta = 0, \pi$ is prepared such that the phase is first stabilized to $\theta = \pi/2$ and an offset, that is calibrated to induce an additional $\pi/2$ shift, is added to the PZT after releasing the phase stabilization. In the data acquisition period, we release the phase stabilization and switch the input to the interferometer to be a weak field. The number of data points that can be
acquired after releasing the phase stabilization is limited by the passive stability of the relative phase between the probe state and the displacement beam. We acquire 500 data points in each data acquisition period, which takes $\sim 80$ ms including the switching time ($\sim 3$ ms) and repeat the procedure 20 times to get the total 10,000 points for each probe state. We set 50 $\mu$s time interval between each probe state in order to avoid phase drift due to the thermal effect of the phase and intensity modulators. A program running on an FPGA detects the electrical signal from the SSPD and changes the voltages applied to the phase and the intensity modulators depending on the outcome from the SSPD for the feedback operation.

The output power from the intensity modulator slowly drifts with time since the modulator consists of a small interferometer. To implement reliable displacement operation with well characterized magnitude, while stabilizing the relative phase, we stabilize the intensity modulator by monitoring the output from the 99:1 fiber coupler by a photo detector and feedback-controlling the offset voltage applied to the intensity modulator.

FIG. 2. (a) Estimated error probability for the experimental result (red point) and the theoretical prediction (blue line) as a function of discarding time $\Delta t$. (b) Electric signals from SSPD (blue), to the phase (grey) and intensity modulator (yellow). Second time bin starts at $t = 0$. 

III. DELAY ANALYSIS OF FEEDBACK OPERATION

The finite time required to implement the feedback operation is a critical imperfection in practice. While the intensity and phase modulators settle on their updated values, the displacement amplitude is not correct, thereby causing a possibility of erroneous detection outcomes. We compensate the degradation of the performance because of the feedback delay by discarding counts observed in a time interval $\Delta t$ between each time bin. For the case of the single feedback operation $M = 2$, we set the time width of the probe state to be $T = 100 \mu$s. Figure 2(a) depicts the error probability for $M = 2$ with the first time length $t_1/T = 0.31$ and the displacement magnitudes $\{|\beta_1|, |\beta_2|\} = \{0.71, 0.49\}$. The discarding time is taken into account as linear loss for the theoretical analysis, $\Delta t/T = 1 - \eta$, and the effect of the feedback delay is not considered because of the complexity of its analysis. The outcome distributions for different $\Delta t$ are obtained simultaneously and the experimentally obtained error probabilities (red points) are evaluated from the same experimental procedure. The gap between the
FIG. 3. Estimated error probability for the displacement photon counter with the feedback operation $M = 2$.

Theoretical prediction (blue line) and the experimental result for $\Delta t = 0$ would be due to the delay of the feedback operation. The error probability can be improved by discarding the counts detected in $\Delta t$ if $\Delta t \leq 1$ and degrades as $\Delta t$ further increases because the loss becomes dominant.

The electrical signals applied to the modulators are shown in Fig. 2(b). The delay of the signal for the phase modulator can be estimated to be $1 \mu s$, which can explain the improvement of the error probability by discarding the counts right after the first time bin. The delay is mostly due to the setting time of the digital to analog converter employed for our experiment. In our experiment, the time interval $\Delta t$ is set to $2 \mu s$ which corresponds to $2\%$ of the total time length of the probe state. A degradation of the error probability because of the $2\%$ loss is smaller than the improvement thanks to the feedback operation.

Figure 3 shows the error probability for $M = 2$. Blue and red data points are obtained from the distribution of the outcomes including the counts observed in the time interval $\Delta t$ and neglecting the counts respectively. The experimental results agree well with a theoretical prediction that takes into account the visibility, the dark count and the $2\%$ loss due to the delay compensation (black solid line). The dark dashed line represents the theoretical prediction for the ideal condition. The error probability for the displacement photon counting measurement with and without the experimental imperfections are respectively shown by green solid line and green dashed line.

For larger $M$, the degradation becomes comparable with the gain obtained from the added feedback operations, so the error probability could be saturated. Therefore, we change the total time length as $T = 100 \times (M - 1) \mu s$ and the voltage applied to the SSPD in order to achieve a similar dark count noise level. The dark count rate $\nu_M$ for each experimental condition is measured to be $\{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5\} = \{2.38, 2.38, 2.04, 1.91, 2.00\} \times 10^{-3}$ counts/state. Though the dark count noise is not exactly constant, the dependence of the error probability on the dark count noise is enough small compared with the improvement owing to the feedback operation. A detection efficiency of the SSPD $\eta_M$ also varies depending on the voltage condition and is measured to be, $\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\} = \{51.4, 51.4, 41.1, 30.7, 23.6\} \%$.

IV. MAXIMUM LIKELIHOOD RECONSTRUCTION OF MEASUREMENT OPERATORS

The most likely POVMs for the experimentally obtained data set can be estimated via a maximum likelihood reconstruction method (ML) [4]. The log-likelihood functional defined as,

$$L[\{\hat{\Pi}_l\}] = \sum_{l=1}^{L} \sum_{k=1}^{K} f_{kl} \ln \text{Tr} [\hat{\rho}_k \hat{\Pi}_l],$$

is maximized for the most likely POVMs, where $L, K, f_{kl}$ indicate the number of POVMs to be estimated, the num-

FIG. 4. (a) Schematic of the tomography of the displacement operation with feedback operations. A maximum likelihood reconstruction (ML) is performed using the outcome statistics $\{f^+_k, f^-_k\}$ and the characteristics of the probe states, illustrated in the phase space diagram on the left. (b) Experimentally reconstructed POVM elements for $M = 2$. 

ber of probe states used for the tomography and the experimentally obtained frequency of the outcome $l$ for the state $\hat{\rho}_k$. The POVMs maximizing Eq. (7), under the constraints for the POVMs to be physically valid $\{\hat{\Pi}_l \geq 0, \sum_{l=1}^{L} \hat{\Pi}_l = \hat{I}\}$, can be found by recursively applying the following transformation,

$$\hat{\Pi}_l = \hat{\lambda}^{-1} \hat{R}_l \hat{\Pi}_l \hat{R}_l \hat{\lambda}^{-1},$$

where

$$\hat{R}_l = \sum_{k=1}^{K} \frac{f_{kl}}{\text{Tr}[\hat{\rho}_k \hat{\Pi}_l]} \hat{\rho}_k,$$

$$\hat{\lambda} = (\sum_{l=1}^{L} \hat{R}_l \hat{\Pi}_l \hat{R}_l)^{1/2}.$$

A simple schematic of the tomography is depicted in Fig. 4(a). We prepare 33 different probe coherent states ($K = 33$) with 4 weak amplitude conditions $|\alpha| \approx \{0.4, 0.6, 0.8, 1.0\}$ and 8 phase conditions $j \pi/4$, $j = 0, \ldots, 7$ in addition to the vacuum state. Experimentally reconstructed POVM elements for $M = 2$ with the displacement amplitudes and the feedback timing $\{\beta_1, \beta_2\} = \{-0.63, -0.51, 0.39\}$ and $t_1/T = 0.30$ are shown in Fig. 4(b). They are first reconstructed in 5-dimensional space in the photon number basis and then truncated to two dimensions to evaluate the discrimination error. Our measurement provides $2^M$ outcomes for a given number of $M$ and the error probability is finally calculated using the POVMs for the parity of the total number of “on” events.