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Edge-Weightings of Graphs and Applications of Non-Separating Cycles

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Edge-Weightings of Graphs and Applications of Non-Separating Cycles

Research in the AlgoLoG section at DTU

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Summary

English

This PhD thesis consists of two parts. The first part is about edge-weightings of graphs and the second part is about applying the existence of non-separating cycles to prove properties of graphs containing such cycles.

Given a graph G and a set of numbers $S \subset \mathbb{R}$, we say that G has the *S-property*, if there there exists a function $w : E(G) \to S$ assigning a number from S to each edge of G such that for any two neighbouring vertices u, v in G, the sum of the numbers assigned to the edges incident with u is distinct from the sum of the numbers assigned to the edges incident with v, i.e. $\sum_{e \in E(u)} w(e) \neq \sum_{e \in E(v)} w(e)$ for any two adjacent vertices u, v in G. Such a function w is called a *proper edge-weighting* of G. A conjecture from 2004 by Karoński, Łuczak, and Thomason, known as the *1-2-3 Conjecture*, states that any connected graph with at least 3 vertices has the $\{1, 2, 3\}$-property. In the first part of this thesis we study problems related to this 1-2-3 Conjecture.
Firstly, we characterise all 2-edge-connected bipartite graphs and all trees without the $\{0, 1\}$-property. We also characterise all 2-connected bipartite graphs and all trees without the $\{a, a+2\}$-property, where a is any odd integer. After having considered bipartite graphs we move on to consider general graphs. A *list-variant* of the 1-2-3 Conjecture called *weight-choosability* asks the following question: Given a graph G, an integer k and for each edge e in G a set $S_e \subset \mathbb{R}$ of k numbers, does there exist a proper edge-weighting w of G such that $w(e) \in S_e$ for each edge e? If such an edge-weighting w exists for any assignment of k-element sets (also called lists) to the edges of G, then G is said to be *k-weight-choosable*. We prove that any connected graph G with at least 3 vertices is $(2\lceil \log_2(\Delta(G)) \rceil + 1)$-weight-chooseable, where $\Delta(G)$ denotes the maximum degree in G, providing the first upper bound on the weight-choosability of general graphs which is logarithmic in $\Delta(G)$.
We finish the chapter concerning edge-weightings by considering a variant of the *Antimagic Labelling Conjecture* formulated by Hartsfield and Ringel. The Antimagic Labelling Conjecture states that for any connected graph G with at least three vertices, there exists a bijection $w : E(G) \to \{1, \ldots, |E(G)|\}$ such that for any two vertices u, v in G, the sum of the numbers assigned to the edges incident with u is distinct from the sum of the numbers assigned to the edges incident with v. We prove the following result which is a list-variant of the Antimagic Labelling Conjecture where

only adjacent vertices are required to have distinct sums of incident edge-weights: For any connected graph G which is not a star and any set $S \subset \mathbb{R}$ of $|E(G)|$ distinct numbers, there exists a bijection $w : E(G) \to S$ such that for any two adjacent vertices u, v in G, the sum of the numbers assigned to the edges incident with u is distinct from the sum of the numbers assigned to the edges incident with v.

The second part of this thesis starts by considering a relaxation of the well-studied notion of *homeomorphically irreducible* spanning trees which are spanning trees with no vertices of degree 2. By using the existence of non-separating cycles in graphs of minimum degree at least 3, we prove that any graph G with minimum degree at least 3 contains a spanning tree T without a path with 3 vertices all of degree precisely 2 in T (we also say that there are no 3 consecutive vertices of degree 2 in T).
After considering this relaxation of homeomorphically irreducible spanning trees we consider what is known as the *3-Decomposition Conjecture* formulated by Hoffmann-Ostenhof. This conjecture states that the edge set of any connected cubic graph G can be partitioned into three parts, where one part induces a spanning tree in G, another part induces a collection of pairwise disjoint cycles in G and a third part induces a matching in G. Inspired by this conjecture we formulate and prove the result that the edges of any connected graph can be partitioned into three parts, such that one part induces a spanning tree in G, another part induces a graph where all vertices have even degree in G and a third part induces a star forest in G.
Finally, in the last section of this thesis, we use non-separating cycles to prove that if G is a 2-connected subcubic graph where all vertices except possibly three x, y, z have degree 3, then there are two $x - y$ paths in G whose lengths differ by 1 or 2. We also show that under some mild additional assumptions we can allow there to be four vertices of degree 2. These results are some of the main tools in a manuscript by the author of this thesis and Martin Merker which proves that for any two integers m, k where k is odd, there exists a number $N(k)$ such that any 3-connected cubic graph with at least $N(k)$ vertices contains a cycle whose length is congruent to m modulo k.

Danish

Denne PhD-afhandling består overordnet set af to dele. Den første del handler om kant-vægtninger af grafer og den anden del viser eksempler på hvordan eksistensen af ikke-separerende kredse kan bruges til at bevise særlige egenskaber for grafer som indeholder sådanne kredse.

Givet en graf G og et sæt af tal $S \subset \mathbb{R}$, så siges G at have *S-egenskaben* hvis der findes en funktion $w : E(G) \to S$, som til enhver kant i G tildeler et tal fra S, således at der for ethvert par af forbundne punkter u, v i G gælder at summen af de tal tildelt til kanterne som støder op til u er forskellig fra summen af de tal tildelt til kanterne som støder op til v, altså, $\sum_{e \in E(u)} w(e) \neq \sum_{e \in E(v)} w(e)$. En sådan funk-

tion w kaldes en *ordentlig kantvægtning* af G. En formodning fra 2004 formuleret af Karoński, Łuczak og Thomason, kendt som *1-2-3 Formodningen* siger at enhver sammenhængende graf med mindst tre punkter har $\{1, 2, 3\}$-egenskaben. I første del af denne afhandling studerer vi problemer som relaterer sig til denne 1-2-3 Formodning. Først karakteriserer vi alle 2-kant-sammenhængende to-delte grafer og alle træer uden $\{0, 1\}$-egenskaben. Vi karakteriserer også all 2-sammenhængende to-delte grafer og alle træer uden $\{a, a+2\}$-egenskaben hvor a er et hvilket som helst ulige heltal. Efter at have betragtet to-delte grafer ser vi på generelle grafer. En *liste-variant* af 1-2-3 Formodningen kendt som vægt-udvælgning stiller følgende spørgsmål: Givet en graf G, et heltal k og for enhver kant e i G et sæt $S_e \subset \mathbb{R}$ af k forskellige tal findes der så en ordentlig kant-vægtning w således at $w(e) \in S_e$ for enhver kant e i G? Hvis der findes sådan en kantvægtning w for alle tildelinger af k-element sæt (også kaldet lister) til kanterne så siges grafen G at være *k-vægt-udvælgelig*. Vi beviser at enhver sammenhængende graf G er $(2\lceil \log_2(\Delta(G)) \rceil + 1)$-vægt-udvælgelig, hvor $\Delta(G)$ betegner maksimumsvalensen i G. Således udleder vi den første øvre grænse for kant-udvælgighed for generelle grafer som er logaritmisk i maksimumsvalensen.
Vi afslutter kapitlet om kant-vægtninger med at betragte en variant af den *Antimagiske Mærkningsformodning* (”The Antimagic Labelling Conjecture” på engelsk) formuleret af Hartsfield og Ringel. Den Antimagiske Mærkningsformodning siger at for enhver sammenhængende graf G med mindst tre punkter findes der en bijektion $w : E(G) \to \{1, \dots, |E(G)|\}$ således at der for ethvert par af punkter u, v gælder at $\sum_{e \in E(u)} w(e) \neq \sum_{e \in E(v)} w(e)$. Vi beviser følgende resultat som er en liste-variant af den Antimagiske Mærkningsformodning hvor kun forbundne punkter kræves at have forskellige summmer af tilstødende kantvægte: For enhver sammenhængende graf G som ikke er en stjerne og ethvert sæt $S \subset \mathbb{R}$ af $|E(G)|$ tal findes der en bijektion $w : E(G) \to \{1, \dots, |E(G)|\}$ således at der for ethvert par af forbundne punkter u, v gælder at $\sum_{e \in E(u)} w(e) \neq \sum_{e \in E(v)} w(e)$.

Den anden del af denne afhandling starter med at undersøge en opblødning af det velstuderede fænomen, *homeomorfiske ireducible* udspændende træer, som er udspændende træer helt uden punkter af valens 2. Ved at bruge ikke-separerende kredse beviser vi at enhver graf G med minimumsvalens mindst 3 har et udspændende træ T uden en vej med tre punkter som har valens præcis 2 i T (vi siger også at der ikke er 3 punkter af valens 2 i T som er på hinanden følgende).
Efter at have studeret denne opblødning af homeomorfiske ireducible udspændende træer fokuserer vi på en formodning kaldet *3-Dekompositions-Formodningen* formuleret af Hoffmann-Ostenhof. Denne formodning siger at kanterne i enhver sammenhængende kubisk graf G kan partitioneres i tre dele E_1, E_2, E_3 således at E_1 inducerer et udspændende træ i G, E_2 inducerer en mængde af disjunkte kredse i G og E_3 inducerer en parring i G. Inspireret af denne formodning formulerer og beviser vi et resultat der siger at kanterne i enhver sammenhængende graf G kan partitioneres i tre dele E_1, E_2, E_3 således at E_1 inducerer et udspændende træ i G, E_2 inducerer en graf i G hvor alle punkter har lige valens og E_3 inducerer en stjerneskov i G.
Slutteligt, i afhandlingens sidste sektion, bruger vi ikke-separerende kredse til at be-

vise at hvis G er en 2-sammenhængende subkubisk graf hvor all punkter på nær op til tre x, y, z har valens 3, så findes der to veje i G fra x til y hvis længder varierer med 1 eller 2. Vi beviser også at vi med et par ekstra antagelser kan tillade at der er op til fire punkter med valens 2. Disse resultater er nogle af de centrale redskaber i et manuskript af forfatteren af denne afhandling og Martin Merker, som viser at for to vilkårlige heltal m, k hvor k er ulige findes der et tal $N(k)$ således at alle 3-sammenhængende kubiske grafer med mindst $N(k)$ punkter indeholder en kreds hvis længde er kongruent med m modulo k.

Preface

This thesis was prepared at the Department of Applied Mathematics and Computer Science at the Technical University of Denmark in fulfilment of the requirements for acquiring a PhD degree in mathematics. The research was financed by a DTU stipend. The PhD studies included a 3 months stay as a visiting graduate student at the University of Waterloo, Canada.
The results presented in this thesis have been submitted to various journals and some of them can also be found on arXiv.org:

- Lyngsie, K. S. On neighbour sum-distinguishing $\{0, 1\}$-edge-weightings of bipartite graphs. Discrete Mathematics and Theoretical Computer Science, 2018, vol. 20.
- Lyngsie, K. S and Zhong, L. A Generalized Version of a Local Antimagic Labelling Conjecture. Graphs and Combinatorics, 2018, vol. 34, 1363-1369.
- Lyngsie, K. S and Zhong, L. Vertex colouring edge weightings: A logarithmic upper bound on weight-choosability. Manuscript.
- Bensmail, J. and Inerney, F. and Lyngsie, K. S. On $\{a, b\}$-edge-weightings of bipartite graphs with odd a, b. Manuscript.
- Lyngsie, K. S. and Merker, M. Spanning trees without adjacent vertices of degree 2. arXiv: 1801.07025.
- Lyngsie, K. S. and Merker, M. Decomposing Graphs into a Spanning tree, an Even Graph, and a Star Forest. Electronic Journal of Combinatorics, 2019, vol. 26.
- Lyngsie, K. S. and Merker, M. Cycle lengths modulo k in large 3-connected cubic graphs. arXiv: 1904.05076.

Kongens Lyngby, June 14, 2019

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Also Carsten's wife, Lis, deserves a special thank for baking the many home made cakes our research group has enjoyed during many coffee breaks. I even got a birthday cake for my $(26+\frac{1}{2})$-birthday. During these coffee breaks we would talk about anything we had on our minds and listen to Carsten's many stories and anecdotes. Carsten's group is very international so here you could meet mathematicians from all over the world. I am glad to have met André Kündgen, John Gimble and Bruce Richter who all visited our group for several months as well as Binlong Li, Daniel Harvey, and Karl Heuer, who worked in our group as postdocs and Seongmin Ok, Alan Arroyo and Robert (Tank) Aldred who also paid our group a visit. They all contributed to the great atmosphere.

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Martin later on returned to our group as a postdoc and we worked together the last one and a half year of my PhD studies. During this time he was effectively working as my second advisor and I owe a great deal of my personal growth over the last year

to him. I have learned a lot about mathematics from him and we have shared many great experiences. He has also been a great help in the writing process of this thesis. I am also glad to have gotten to know Rikke Langhede and David Roberson who worked in our group the last one and half year of my PhD studies.
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Contents

CHAPTER 1

Introduction

1.1 Overview of The Thesis

The thesis is split into two chapters. The first chapter is about so-called *neighbour sum-distinguishing edge-weightings* and the second chapter is about applications of *non-separating cycles*.
A neighbour sum-distinguishing edge-weighting of a graph G is a function $w : E(G) \to \mathbb{R}$ assigning a number to each of the edges of G such that for any two neighbouring vertices u, v in G it holds that the sum of the numbers assigned to the edges incident with u is distinct from the sum of the numbers assigned to the edges incident with v. If $S \subset \mathbb{R}$ is a set of numbers and G is a graph, then G is said to have the *S-property* if there is a neighbour sum-distinguishing edge-weighting $w : E(G) \to S$ of G only using numbers in S.
The work regarding neighbour sum-distinguishing edge-weightings presented in this thesis is mainly motivated by the *1-2-3 Conjecture* from 2004 formulated by Karoński, Łuczak, and Thomason [Kar04]. This conjecture states that if G is a connected graph with at least 3 vertices, then G has the $\{1, 2, 3\}$-property.
Chapter 2 concerning neighbour sum-distinguishing edge-weightings is split into four sections. The first section gives an introduction to the history of the 1-2-3 Conjecture, related problems, and to the basic terminology we will use when working with edge-weightings of graphs. The second section concerns neighbour sum-distinguishing edge-weightings of bipartite graphs using only two edge-weights. This is motivated by the work of Thomassen, Wu and Zhang [Tho16] who characterised all the bipartite graphs without the $\{1, 2\}$-property. Any such graph is a so-called *odd multi-cactus*. Thomassen et al. mentioned that their result and their proof generalise to the S-property whenever S consists of two positive integers of distinct parity. However, they also mentioned that the result does not generalise to the $\{0, 1\}$-property. We show that if G is a 2-edge-connected bipartite graph without the $\{0, 1\}$-property, then G is an odd multi-cactus. Thus, we show that their characterisation also holds for the $\{0, 1\}$-property if we restrict ourselves to 2-edge-connected bipartite graphs. We also show a recursive way to construct all trees without the $\{0, 1\}$-property. After having considered the $\{0, 1\}$-property for bipartite graphs we move on to consider the $\{a, a + 2\}$-property for bipartite graphs, where a is any odd integer. We show that if G is a 2-connected bipartite graph without the $\{a, a + 2\}$-property, then G is an odd multi-cactus and we also characterise all trees without the $\{a, a + 2\}$-property.

The third section concerning neighbour sum-distinguishing edge-weightings is about *weight-choosability* of graphs which can be defined as follows: if k is a natural number, then a graph G is said to be *k-weight-choosable* if for any assignment of k-element lists L_e, $e \in E(G)$ to the edges of G there exists a neighbour sum-distinguishing edge-weighting w of G such that $w(e) \in L_e$ for all $e \in E(G)$. In particular, any graph which is 3-weight-choosable satisfies the 1-2-3 Conjecture. In this thesis we show that any connected graph with at least 3 vertices is $(2\lceil \log_2(\Delta(G)) \rceil + 1)$-weight-choosable, where $\Delta(G)$ denotes the maximum degree in G. This is the first upper bound for weight-choosability which is logarithmic in the maximum degree of G. The earlier best known upper bound was a linear function in $\Delta(G)$.
The fourth and last section concerning neighbour sum-distinguishing edge-weightings is about a result related to the *Antimagic Labelling Conjecture* which states that for any connected graph G with at least 3 vertices there exists a bijection $w : E(G) \to \{1, \ldots, |E(G)|\}$ such that for any two distinct vertices u, v in G, the sum of the numbers assigned to the edges incident with u by w is distinct from the sum of the numbers assigned to the edges incident with v by w. This conjecture was formulated by Hartsfield and Ringel [Har90]. We show that if G is a connected graph with at least 3 vertices which is not a star and S is any set of real numbers with $|S| = |E(G)|$, then there exists a neighbour sum-distinguishing edge-weighting $w : E(G) \to S$ which is a bijection.

A cycle C in a connected graph G is called *non-separating* if $G - V(C)$ is connected. In Chapter 3 about applications of these non-separating cycles we prove three theorems whose proofs all use the existence of non-separating cycles. These three theorems are related to *homeomorphically irreducible spanning trees*, *The 3-Decomposition Conjecture*, and the existence of certain paths in cubic graphs whose lengths differ by 1 or 2.
A *homeomorphically irreducible* tree, also called a *HIT*, is a tree with no vertices of degree 2. A homeomorphically irreducible spanning tree in a graph is called a *HIST*. Since Hill [Hil74] in 1974 studied the existence of HISTs in certain graphs the following question has served as inspiration for several research papers: what properties of a graph G implies the existence of a HIST in G? In general this question is hard as pointed out by Douglas [Dou92] who showed that it is **NP**-complete to decide whether a given planar subcubic graph contains a HIST and Lemke [Lem88] showed that it is also **NP**-complete for general cubic graphs. Hill conjectured that any triangulation of the plane with at least four vertices contains a HIST. This conjecture was strengthened by Malkevitch [Mal79] who conjectured that any near-triangulation of the plane contains a HIST. In 1990 Malkevitch's conjecture was proven by Albertson, Berman, Hutchinson and Thomassen [Alb90]. Albertson et al. also proved that there exists a constant c such that any graph with n vertices and minimum degree at least $c\sqrt{n}$ contains a HISTs and that this lower bound is best possible up to the constant c. In contrast to this result they also provided a construction showing that for each natural number k there exists a k-connected graph with no HIST.
Many other results regarding the existence of HISTs can be found in the literature,

see for example [Che13], [Fur13] and [Die15]. We will study a relaxation of the notion of HISTs and prove that any connected graph with minimum degree at least 3 has a spanning tree in which there is no path of length at least 2 only containing vertices having degree 2 in the spanning tree. We also say that there are no three *consecutive* vertices of degree 2.
A *decomposition* of a graph is a partition of its edge set. The 3-Decomposition Conjecture formulated by Hoffmann-Ostenhof [Cam11] states that there is a decomposition of any connected cubic graph G into three parts, one which induces a spanning tree in G, one which induces a collection of cycles in G, and one which induces a matching in G. Note that a HIST in a cubic graph G is a spanning tree T such that $G - E(T)$ is a collection of pairwise disjoint cycles and isolated vertices. In this way one can look at the 3-Decomposition Conjecture as a statement about the existence of a spanning tree T which is close to being homeomorphically irreducible in the sense that the components of $G - E(T)$ which are not cycles or isolated vertices (one can think of this part as an error term") form a matching. Inspired by the 3-Decomposition Conjecture we formulate and prove a 3-decomposition theorem for all connected graphs: we prove that the edges of any connected graph G can be partitioned into three sets E_1, E_2, E_3 such that E_1 induces a spanning tree in G, the set E_2 induces a star forest in G and finally E_3 induces a graph where all vertices have even degree graph in G.
In the last section of chapter 3 we study the existence of certain paths in subcubic graphs. A result by Fan [Fan02] implies that if x, y are vertices in a 2-connected graph G where $d_G(z) = 3$ for all $z \in V(G) \setminus \{x, y\}$, then there exist two $x - y$ paths in G whose lengths differ by 1 or 2. This result by Fan was in part motivated by the work of Bondy and Vince [Bon98] who answered a question of Erdös by proving that any graph with minimum degree at least 3 contains two cycles whose lengths differ by 1 or 2. We will show that under some mild additional conditions we can allow up to two additional vertices besides x and y to possibly have degree 2 and still have two $x - y$ paths whose lengths differ by 1 or 2. These results we prove are some of the key tools in [Lyna] where the author of this thesis and Merker prove that for any odd natural number k, there exists a number $N(k)$ such that any 3-connected cubic graph with at least $N(k)$ vertices contains a cycle of length congruent to m modulo k for every integer m.

1.2 Notation and Definitions

For basic graph theory terminology and definitions not explained in this thesis the reader is referred to the book by Diestel [Die16].

Given a graph G we let $V(G)$ and $E(G)$ denote its vertex- and edge set, respectively. If H is a subgraph of G, then we sometimes write $G - H$ instead of $G - V(H)$ (the graph obtained from G by removing all vertices in H and removing all edges having an end in a vertex in H). If v is a vertex in $G - H$, then $H + v$ is the graph obtained from the disjoint union of H and v by adding all edges incident to v in G having an end in H. Similarly, if H' is a subgraph of $G - H$, then $H + H'$ is the

graph obtained from the disjoint union of H and H' by adding all edges in G which have an end in H and H'.
The number of edges in G, that is $|E(G)|$, is also referred to as the *size* of G and $|V(G)|$ is called the *order* of G.
If v is a vertex in a graph G, then $d_G(v)$, or simply $d(v)$, denotes the degree of v in G. We let $E_G(v)$ or simply $E(v)$ denote the set of edges in G incident to v and we let $N_G(v)$ or simply $N(v)$ denote the set of vertices which are adjacent to v in G. The vertices in $N(v)$ are also called the *neighbours* of v. The maximum degree in G is denoted $\Delta(G)$.
All graphs in this thesis are undirected and finite. A graph G is said to be *regular* if all vertices in G have the same degree and G is called *irregular* if there are no two vertices in G with the same degree. A graph G is called *simple* if all edges in G join two distinct vertices and if no two vertices in G are joined by more than one edge. If u and v are vertices in a graph G and are joined by more than one edge, then any edge joining u and v is called a *multiple* edge. If e is a multiple edge in a graph G joining two vertices u and v, then the *multiplicity* of e, denoted $M(uv)$ or $M(e)$, is the number of edges in G joining u and v. Graphs with multiple edges only play a role in the second section ("Edge-Weightings of Bipartite Graphs Using Two Weights") in Chapter 2. In all other places in the thesis, unless stated otherwise, we assume that graphs are simple. We will sometimes use the term *multigraph* about a graph which contains multiple edges.
By a *bipartition* of a bipartite graph G we mean a partition of $V(G)$ into two independent sets and these two sets are called the *bipartition sets* of G.
A *leaf* is a vertex of degree 1 in a graph. An edge incident to a leaf in a graph is called a *pendant* edge. A *star* is a tree with at most one vertex which is not a leaf. This vertex is called the *center* of the star (if it is not unique, then the star must be an isolated edge and any of the two vertices can be referred to as the center). A *bistar* is a tree obtained from the disjoint union of two stars by adding an edge between their centers. The centers of the two stars are also called the centers of the bistar. A *star forest* is a forest where each component is a star. A *matching* is a star forest where each component has exactly one edge.
If G is a graph and A, B are disjoint subset of $V(G)$, then an $A - B$ *path* in G is a path whose endvertices are in A and B, respectively, and whose internal vertices are disjoint from $A \cup B$. Similarly if $x, y \in V(G)$, then an $x - y$ *path* in G is a path in G with endvertices x and y. If P is a path in G and u, v are vertices on P, then uPv denotes the $u - v$ subpath of P.

CHAPTER 2

Neighbour Sum-Distinguishing Edge-Weightings

2.1 Introduction

Given a graph G and a set of numbers $S \subset \mathbb{R}$ an *S-edge-weighting* of G is a function $w : E(G) \to S$. Questions regarding edge-weightings of graphs are often phrased in the following way:

Which properties of a graph G and a set $S \subset \mathbb{R}$ are sufficient to guarantee the existence of an S-edge-weighting of G with property $(\star)$?

The property $(\star)$ asked for can vary depending on the motivation behind the problem investigated. One such property that has been studied a lot in recent years was motivated by the so-called *irregularity strength* of graphs. The *irregularity strength* of a simple graph G is a number associated to G which in some sense measurse how far away G is from being irregular. More precisely, it is the smallest number k such that G can be turned into an irregular multigraph by replacing each edge of G by an edge of multiplicity at most k. Instead of thinking of an edge of multiplicity k as a collection of k distinct edges we can simply think of it as an edge of *weight* k and we can define the weighted degree of a vertex accordingly: If $w : E(G) \to \mathbb{R}$ is an edge weighting of a graph G and $v \in V(G)$, then the *weighted degree* of v induced by w, denoted by $d_w(v)$ or $w(v)$ or $C_w(v)$, is the sum of the weights of the edges incident to v, i.e. $d_w(v) = w(v) = C_w(v) = \sum_{e \in E(v)} w(e)$. If $e = uv$ is an edge in G with $w(e) = a$, then we call e an *a-edge* and if $C_w(u) = C_w(v)$, then e is called a *conflict*. Now we can formulate the irregularity strength of a graph in terms of the existence of edge-weightings with certain properties: the irregularity strength of a graph G is the smallest number k such that there exists an edge-weighting $w : E(G) \to \{1, \ldots, k\}$ such that all vertices of G have distinct weighted degrees induced by w. Such an edge-weighting w is called *sum-distinguishing*. Note that the irregularity strength of

a graph having a component with exactly two vertices (also called an *isolated edge*) is not well-defined. On the other hand, it is easy to check that the irregularity strength of a graph with no isolated edges is indeed well-defined. For more on the irregularity strength of graphs we refer to the survey by Bača et al. [Bač15].
For a graph to be irregular we require all vertices to have distinct degrees. A natural relaxation of this requirement is to only require neighbouring vertices to have distinct degrees. A graph G is said to be *locally irregular* if there are no two neighbouring vertices in G which have the same degree. Like we defined irregularity strength of graphs we can naturally define local irregularity strength of graphs: The *local irregularity strength* of a graph G is the smallest number k such that there exists an edge-weighting $w : E(G) \to \{1, \dots, k\}$ such that any two adjacent vertices in G have distinct weighted degrees induced by w. Such an edge-weighting w is called *neighbour sum-distinguishing* or *proper*. If S is a set of numbers and G is a graph, we say that G has the *S-property* if there exists a neighbour sum-distinguishing edge-weighting of G only using weights in S.
Note that also the local irregularity strength is not defined for a connected graph having exactly two vertices and is well-defined for all graphs without any isolated edges. For a graph G without any isolated edges we let $\chi_l(G)$ denote the local irregularity strength of G. The local irregularity of graphs was first studied in 2004 by Karoński, Łuczak, and Thomason [Kar04]. They formulated the now well-known 1-2-3 Conjecture which conjectures that actually the weights 1, 2, and 3 always suffice when constructing a neighbour sum-distinguishing edge-weighting of a graph without isolated edges:

Conjecture 2.1.1 (The 1-2-3 Conjecture, Karoński, Łuczak, Thomason, 2004)**.** *If G is a graph with no isolated edges, then $\chi_l(G) \leq 3$.*

When studying the local irregularity strength of a graph G we can, by studying one component at a time, assume that G is connected and has at least three vertices. For convenience we call such graphs *nice*. It is easy to find infinite families of nice graphs with local irregularity strength at least 3 (all complete graphs for example), so the value 3 in the conjecture is indeed smallest possible. Karoński et al. also proved that if G is a nice k-colourable graph and k is odd, then $\chi_l(G) \leq k$. This result reflects some of the early approaches to the 1-2-3 Conjecture, namely to relate the local irregularity strength of a given graph to the chromatic number of the graph, see for example [Dua12], [Lu 09].
The first constant upper bound for the local irregularity strength of nice graphs was obtained in 2007 by Addario-Berry, Dalal, McDiarmid, Reed and Thomason [Add07] who showed that if G is a nice graph, then $\chi_l(G) \leq 30$. This bound was improved to $\chi_l(G) \leq 16$ by Addario-Berry, Dalal and Reed [Add08], then to $\chi_l(G) \leq 13$ by Wang and Yu [Wan08], then to $\chi_l(G) \leq 6$ by Kalkowski, Karoński and Pfender [Kal09], and finally in 2011 this bound was improved to the currently best known $\chi_l(G) \leq 5$ by Kalkowski, Karoński and Pfender [Kal10].

The 1-2-3 Conjecture gave rise to many related edge-weighting problems, see for example the survey by Seamone [Sea]. One related problem involving so-called *to-*

tal weightings of graphs was introduced by Przybyło and Woźniak [Prz10]. A total weighting of a graph G is a mapping $w : V(G) \cup E(G) \to \mathbb{R}$. One can think of it as a weight function assigning weights to both the edges and the vertices. A total weighting w of a graph G is called neighbour sum-distinguishing or proper if for any edge $uv \in E(G)$ we have $w(u) + \sum_{e \in E(u)} w(e) \neq w(v) + \sum_{e \in E(v)} w(e)$. Przybyło and Woźniak conjectured that any connected graph has a neighbour sum-distinguishing total weighting only using weights 1 and 2. This conjecture is now known as the *1-2 Conjecture*. Note that in this conjecture we do not need exclude isolated edges.

As for many colouring problems in graph theory there are associated *list-versions* of the 1-2-3 Conjecture and the 1-2 Conjecture. This concept is known as *weight-choosability* and was introduced in 2009 by Bartnicki, Grytczuk and Niwczyk [Bar09]. Let k be a natural number. Recall that a graph G is called *k-weight-choosable* if for any assignment of k-element lists L_e, $e \in E(G)$ to the edges of G there exists a neighbour sum-distinguishing edge-weighting w of G such that $w(e) \in L_e$ for all $e \in E(G)$. In particular, any graph which is 3-weight-choosable must satisfy the 1-2-3 Conjecture. The list-version generalising the concept of total weightings is defined as follows. Let l, k be two natural numbers. A graph G is called *(l, k)-weight-choosable* if for any assignment of l-element lists L_v, $v \in V(G)$ to the vertices of G and any assignment of k-element lists L_e, $e \in E(G)$ to the edges of G there exists a neighbour sum-distinguishing total weighting w of G such that $w(v) \in L_v$ for all $v \in V(G)$ and $w(e) \in L_e$ for all $e \in E(G)$. In particular, any graph which is $(2, 2)$-weight-choosable must satisfy the 1-2 Conjecture. An interesting special case of (l, k)-weight-choosability is when $l = 1$. This case corresponds to having a prescribed weight on each vertex and in the special subcase where all prescribed vertex-weights are 0 this is the same as k-weight-choosability.

When having defined k-weight-choosability it is natural to wonder whether the 1-2-3 Conjecture can be generalised and whether every nice graph is 3-weight-choosable. This was indeed conjectured by Bartnicki, Grytczuk and Niwczyk [Bar09] and is still an open problem. However, there is a marked difference in the progress towards this conjecture compared to the progress towards the 1-2-3 Conjecture. As mentioned above, we know that there exists a neighbour sum-distinguishing edge-weighting of any nice graph only using weights in the set $\{1, 2, 3, 4, 5\}$. For the list-version no constant upper bound has yet been proved: We don't know any constant c such that any nice graph is c-weight-choosable. Bartnicki, Grytczuk and Niwczyk [Bar09] used the Combinatorial Nullstellensatz and the permanent of matrices to show that any complete graph and any tree is 3-weight-choosable. Instead of providing a constant upper bound for the weight-choosability of nice graphs many results provide a linear function in the maximum degree as an upper bound. A result by Ding et al. [Din+19] mentioned by Wong and Zhu [Won] for example says that any nice graph G is $(1, \Delta(G) + 1)$-weight-choosable.

Similar to conjecturing that any nice graph is 3-weight-choosable it is, given the 1-2 Conjecture, natural to Conjecture that any graph is $(2, 2)$-weight-choosable. This was conjectured by Wong and Zhu [Won11] in 2011 who also conjectured that any nice graph is $(1, 3)$-weight-choosable. But unlike for the problem of k-weight-choosability we do have significant progress towards this conjecture: In 2016 Wong and Zhu [Won16] used the Combinatorial Nullstellensatz and the permanent of matrices to show that any graph is $(2, 3)$-weight-choosable.

We will start our study about neighbour sum-distinguishing edge-weightings by only looking at bipartite graphs.

2.2 Edge-Weightings of Bipartite Graphs Using Two Weights

The material presented in this section essentially consists of one research article [Lyn18a] and one manuscript [Ben]. In this section we allow multiple edges unless stated otherwise.

Let a and b be two integers. Note that if c is any non-zero real number, then any proper $\{ca, cb\}$-edge-weighting of a graph G also yields a proper $\{a, b\}$-edge-weighting of G, by simply replacing each weight ca by a and replacing each weight cb by b. Thus, when investigating the $\{a, b\}$-property of graphs where a and b are integers we can assume that a and b are co-prime. In particular, we can assume that one of a, b is odd.
One way to approach the 1-2-3 Conjecture could be to start by characterising all graphs without the $\{1, 2\}$-property. However, Dudek and Wajc [Dud11] showed that deciding the S-property for general graphs is **NP**-complete for $S = \{0, 1\}$ and $S = \{1, 2\}$. So if one wants to find a useful characterisation of the graphs without the $\{1, 2\}$-property one has to restrict the class of graphs considered. This motivated the investigation of the $\{1, 2\}$-property for bipartite graphs. Recall from the introduction above that if G is a nice k-colourable graph and k is odd, then $\chi_l(G) \leq k$. So all nice bipartite graphs have the $\{1, 2, 3\}$-property. In 2016 Thomassen, Wu and Zhang [Tho16] gave a complete characterisation of all bipartite graphs without the $\{1, 2\}$-property. Any such graph is a so-called *odd multi-cactus* which we will define later in Section 2.2.2 - ”Odd Multi-Cacti”.

An essential tool used by Thomassen, Wu and Zhang [Tho16] in the characterisation of bipartite graphs without the $\{1, 2\}$-property is the following lemma.

Lemma 2.2.1. *[Tho16] Let G be a connected graph. If $f : V(G) \to \mathbb{Z}_2$ is a mapping such that*

$$\sum_{v \in V(G)} f(v) \equiv 0 \mod 2,$$

then G contains a subgraph H such that $d_H(v) \equiv f(v) \mod 2$ for every $v \in V(G)$.

Proof. Let $A \subset V(G)$ be the set of vertices such that $f(a) = 1$ if and only if $a \in A$. Since $\sum_{v \in V(G)} f(v) \equiv 0 \mod 2$, the set A has even size, say $A = \{a_1, \ldots, a_{2m}\}$. For each $i \in \{1, 3, 5, \ldots, 2m-1\}$ let P_i be a $v_i - v_{i+1}$ path in G. Then the symmetric difference of all the sets $E(P_i)$, $i \in \{1, 3, 5, \ldots 2m-1\}$ induces a desired subgraph of G. $\square$

Let a, b be two distinct integers, let G be a connected graph and let $X \subset V(G)$ have even size. Lemma 2.2.1 implies that there is a subgraph H of G such that $d_H(v)$ is odd if and only if $v \in X$. By assigning weight a to all the edges in H and weight b

to all the edges in $E(G) \setminus E(H)$ we obtain an edge-weighting where all vertices in X are incident to an odd number of a-edges and all vertices in $V(G) \setminus X$ are incident to an even number a-edges. Thus we have derived the following lemma.

Lemma 2.2.2. *Let a, b be distinct integers and let G be a connected graph. If $X \subset V(G)$ is such that $|X|$ is even, then there is an $\{a, b\}$-edge-weighting of G such that all vertices in X are incident to an odd number of a-edges and all vertices in $V(G) \setminus X$ are incident to an even number of a-edges.*

If a and b are integers of distinct parity, say a is odd and b is even, then Lemma 2.2.2 immediately reduces the problem of determining the $\{a, b\}$-property for bipartite graphs to the case where both bipartition sets have odd size: for suppose one of the bipartition sets X of a bipartite connected graph G has even size. Then there exists an $\{a, b\}$-edge-weighting of G such that all vertices in X are incident to an odd number of a-edges and all vertices in $V(G) \setminus X$ are incident to an even number a-edges. Since b is even this means that all vertices in X have odd weighted degree and all vertices in $V(G) \setminus X$ have even weighted degree and hence the edge-weighting is proper. This shows why Lemma 2.2.1 is an essential tool.

If a and b are natural numbers, G is a bipartite graph, (X, Y) is a partition of $V(G)$ and w is an $\{a, b\}$-edge-weighting of G, then w is called an *(X, Y)-a-parity* edge-weighting of G if all vertices in X are incident to an odd number of a-edges and all vertices in Y are incident to an even number of a-edges. An edge $xy \in E(G)$ is called a *parity conflict* if $C_w(x)$ and $C_w(y)$ have the same parity. Recall that the edge xy is called a conflict if $C_w(x) = C_w(y)$, so a conflict is always a parity conflict, but a parity conflict is not necessarily a conflict. If we want to prove that G has the $\{a, b\}$-property and X, Y are the bipartition sets of G, then we can, as mentioned above, assume that $|X|$ and $|Y|$ are both odd. Thus, for any vertex $v \in X$, Lemma 2.2.2 implies that there is a $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting w_v of G. Note that in this case the only parity conflicts are between v and its neighbours and since we can choose v freely we can basically choose where the potential conflicts should be. This observation leads to an important idea behind some of the proofs in this section: We will pick $v \in V(G)$ and the corresponding edge-weighting w_v of G such that the local structure in G around v allows us to get rid of the potential conflicts involving v. One way to do this is to "swap" weights on so-called "changing" cycles. By *swapping* the weight of an edge, we mean changing its weight to a if it is a b-edge, or changing its weight to b if it is an a-edge. By swapping the weights on a path or a cycle, we mean swapping the weights of all of its edges. A cycle C containing two edges incident to a vertex $u \in V(G)$ having the same weight and avoiding another vertex $u' \in V(G)$ is called *u-changing* and *u'-avoiding.*
Let us again focus on the edge-weighting w_v of G where all parity conflicts involve v and suppose that vv' is an actual conflict. If we can find a v'-changing and v-avoiding cycle C in G, then if we swap all the weights on C, the weighted degree of v' changes while the weighted degree of v does not change. Furthermore, note that swapping the

weights on a cycle does not change the parity of the number of incident a-edges of any vertex so we still have an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting of G, but we have now removed the conflict vv'. This demonstrates why these v-changing cycles are useful.

Now we will present what we will prove about neighbour sum-distinguishing edge-weightings of bipartite graphs in this thesis.

2.2.1 Results

Thomassen, Wu and Zhang [Tho16] mentioned that their proof of the characterisation of bipartite graphs without the S-property for $S = \{1, 2\}$ works whenever S consists of two positive integers of distinct parity. They also explicitly mentioned that the characterisation does not hold when $S = \{0, 1\}$. For example Lu [Lu 16] gave the following example of a graph with the $\{1, 2\}$-property but without the $\{0, 1\}$-property: Two 6-cycles joined by a path of length 3. Actually, joining any two graphs G_1, G_2 without the $\{0, 1\}$-property by a path $P = v_1xyv_2$ where $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ gives a graph G' without the $\{0, 1\}$-property: for in any proper $\{0, 1\}$-edge-weighting of G' one of the edge v_1x or v_2y, say v_1x, must receive the weight 0 and hence such a proper $\{0, 1\}$-edge-weighting of G' would yield a proper $\{0, 1\}$-edge-weighting of G_1. We will prove that if we restrict ourselves to 2-edge-connected bipartite graphs, then the $\{0, 1\}$- and the $\{1, 2\}$-property are actually equivalent:

Theorem 2.2.3. *G is a 2-edge-connected bipartite graph without the $\{0, 1\}$-property if and only if G is an odd multi-cactus.*

The construction of bipartite graphs without the $\{0, 1\}$-property mentioned above is based on the following observation. If e is a cut-edge in a connected graph G and one of the components of $G - e$ does not have the $\{0, 1\}$-property, then the edge e must receive weight 1 in any proper $\{0, 1\}$-edge-weighting of G. This is the motivation for restricting ourselves to 2-edge-connected bipartite graphs in Theorem 2.2.3. Indeed this observation yields several ways one can construct bipartite graphs without the $\{0, 1\}$-property, for example the following. Let $s \geq 0$ be an integer and let P be a path with $n \equiv 2 \mod 4$ vertices. Join each internal vertex on P to s graphs without the $\{0, 1\}$-property by s edges and join the endvertices of P to $s + 1$ graphs without the $\{0, 1\}$-property by $s + 1$ edges, see Figure 2.1. In any proper $\{0, 1\}$-edge-weighting of this resulting graph all edges in $E(v_i) \setminus E(P)$ must have weight 1 for all $i \in \{1, \dots n\}$. To avoid the conflict v_1v_2 the edge v_2v_3, must receive weight 0. Then to avoid the conflict v_3v_4 the edge v_4v_5 must receive weight 1. We continue arguing like this and see that the weights of the edges $v_2v_3, v_4v_5, \dots$ must alternate starting with the weight 0. Since $n \equiv 2 \mod 4$ this means that $v_{n-2}v_{n-1}$ must receive weight 1 which means that $v_{n-1}v_n$ is a conflict.

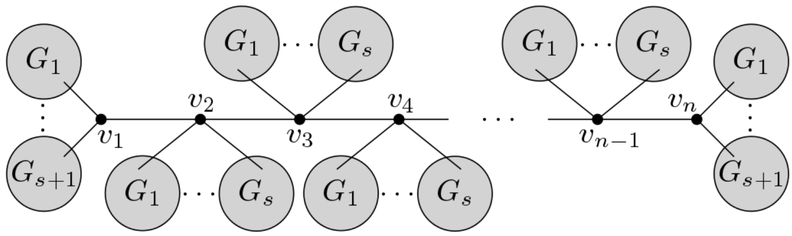


Figure 2.1: A graph G. For any natural numbers s and n with $n \equiv 2 \mod 4$, if all the graphs $G_1, G_2, \dots G_{s+1}$ do not have the $\{0,1\}$-property, then G does also not have the $\{0,1\}$-property.

As the construction above also points out, there are many trees without the $\{0,1\}$-property whereas all trees distinct from K_2 have the $\{1,2\}$-property. After proving Theorem 2.2.3 we turn our attention to trees and prove a characterisation of all trees without the $\{0,1\}$-property. We will recursively define a class of trees $\mathcal{B}$ and prove that this class of trees is exactly the trees without the $\{0,1\}$-property.

Theorem 2.2.4. *A tree T has the {0,1}-property unless T is a member of $\mathcal{B}$.*

How $\mathcal{B}$ exactly is defined will be explained later. It contains more trees than the above construction gives, since that construction is indeed not sufficient to get all trees without the $\{0,1\}$-property: the tree in Figure 2.2 can for example not be obtained by the construction described in Figure 2.1.

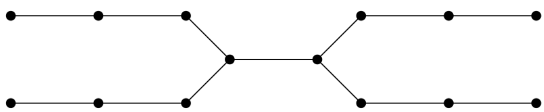


Figure 2.2: A tree without the $\{0,1\}$-property.

In the proofs of Theorem 2.2.3 and Theorem 2.2.4 it is important that the edge-weights we use have distinct parity. So these proofs do not provide any results for the case where S consists of two odd numbers which are co-prime (recall that we can always assume that these two numbers are co-prime). After having considered the $\{0,1\}$-property we will turn our attention to the case where $S = \{a, a+2\}$ for any odd number a. We will provide a tool called a *mod 4 vertex-colouring* which will allow us to examine the $\{a, a+2\}$-property in a similar way as the $\{0,1\}$-property. Using this tool we will prove the following theorem.

Theorem 2.2.5. *Let a be an odd integer. A 2-connected bipartite graph G does not have the $\{a, a+2\}$-property if and only if G is an odd multi-cactus.*

The restriction to 2-connected bipartite graphs in Theorem 2.2.5 is particularly motivated by the special case where $S = \{-1, 1\}$. In this case one can construct graphs without the $\{-1,1\}$-property having cut-vertices in the following way:

- Let G_1, G_2, G_3, G_4 be four bipartite graphs without the $\{-1,1\}$-property, and let v_1, v_2, v_3, v_4 be four degree-1 vertices in G_1, G_2, G_3, G_4, respectively. Let G denote the graph obtained from the disjoint union $G_1 \cup G_2 \cup G_3 \cup G_4$ by identifying the vertices v_1 and v_2, identifying the vertices v_3 and v_4, and adding an edge joining the two vertices resulting from these identifications, see Figure 2.3, (a)-(b). If all of G_1, G_2, G_3, G_4 do not have the $\{-1,1\}$-property, then neither has G. To see this, note that the two edges incident to $v_1 \sim v_2$ not going to G_1 cannot have distinct weights since G_1 does not have the $\{-1,1\}$-property. Similarly, the two edges incident to $v_1 \sim v_2$ not going to G_2 must also have the same weight and hence all edges incident to $v_1 \sim v_2$ must have the same weight. By symmetry also all edges incident to $v_3 \sim v_4$ must have the same weight so $v_1 \sim v_2$ and $v_3 \sim v_4$ must have the same weighted degree.

- Let G_1, G_2 be two bipartite graphs without the $\{-1,1\}$-property, and let v_1, v_2 be any two vertices in G_1 and G_2, respectively. Let G denote the graph obtained from the disjoint union $G_1 \cup G_2$ by adding the edge v_1v_2, and joining v_1 and v_2 by a path P whose length is congruent to 3 modulo 4, see Figure 2.3, (c)-(d). If G_1 and G_2 do not have the $\{-1,1\}$-property, then neither has G. To see this, note that the two edges in P going to G_1 and G_2, respectively must have distinct weights. So the sum of the two edge-weights incident to v_1 outside G_1 or the sum of the two edge-weights incident to v_2 outside G_2 must be 0. This means that any proper $\{-1,1\}$-edge-weighting of G would yield a proper $\{-1,1\}$-edge-weighting of one of G_1, G_2.

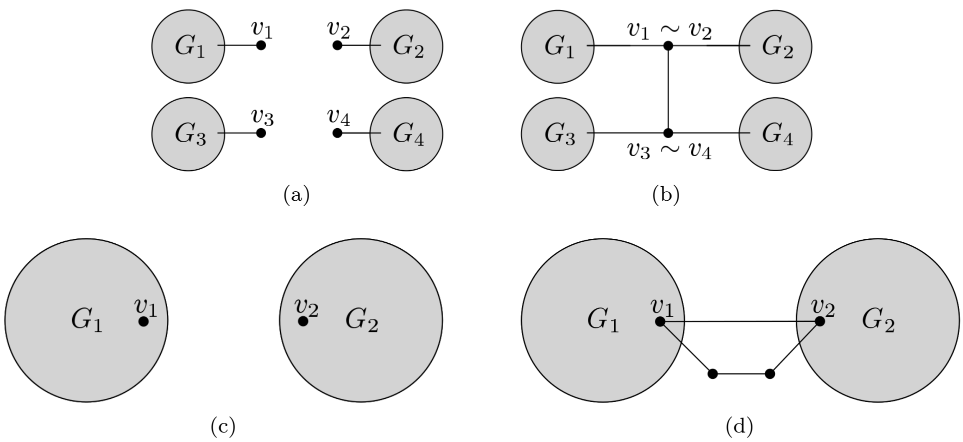


Figure 2.3: Construction of graphs without the $\{-1,1\}$-property.

Aligned with the analysis for the $\{0,1\}$-property we also characterise all trees without the $\{a, a+2\}$-property when a is an odd integer. When $a \neq -1$ it is easy to see that all trees distinct from K_2 have the $\{a, a+2\}$-property by the following induction argument: consider a vertex v all of whose neighbours $u_1, \ldots, u_{d-1}$ but one u_d are leaves. Remove $u_1, \ldots, u_{d-1}$, apply induction (it is easy to see that all stars except K_2 have the $\{a, a+2\}$-property, so we can assume the resulting tree is not an isolated edge) to obtain a neighbour sum-distinguishing $\{a, a+2\}$-edge-weighting, and extend the weighting to the edges $vu_1, \ldots, vu_{d-1}$ so that the conflict vu_d is avoided. By this observation only the case $a = -1$ is potentially interesting. We solve this case by proving the following theorem.

Theorem 2.2.6. *A tree does not have the $\{-1, 1\}$-property if and only if it can be constructed from a disjoint union of graphs isomorphic to K_2 by repeated applications of the operation in Figure 2.3 (a)-(b).*

As can be seen above, odd multi-cacti play an important role when investigating the S-property of bipartite graphs, where S is a set of two integers. Therefore we will start out with a section concerning these graphs.

2.2.2 Odd Multi-Cacti

An *odd multi-cactus* is a connected bipartite graph defined as follows. Take a collection of simple cycles whose lengths are congruent to 2 modulo 4, each of which has edges coloured alternately red and green. Then form a connected simple graph by pasting the cycles together, one by one, in a tree-like fashion along green edges. Finally replace every green edge by a multiple edge of any multiplicity. The graph with one edge and two vertices is also called an odd multi-cactus. See Figure 2.4 for an example of odd multi-cacti. Note that the red/green edge colouring of an odd multi-cactus M is unique if and only if M has at least 6 vertices and is not a simple cycle. If M has a vertex with at least three distinct neighbours, then M is called a *proper* odd multi-cactus.

If M is a proper odd multi-cactus then, by definition, M contains at least two cycles C_1, C_2 whose lengths are congruent to 2 modulo 4 each containing two adjacent vertices which have at least three distinct neighbours in M while the other vertices in C_1 and C_2 all have exactly two neighbours in M. Cycles of this type are called *end-cycles* in M. We will now show that odd multi-cacti do not have the S-property whenever S is a set of 2 integers.

Lemma 2.2.7. *If M is an odd multi-cactus, then M does not have the $\{a, b\}$-property for every $a, b \in \mathbb{Z}$.*

Proof. Suppose the lemma is false and let M be an odd multi-cactus of minimum size which has the $\{a, b\}$-property for some $a, b \in \mathbb{Z}$. First suppose $M = v_0v_1 \cdots v_{n-1}v_0$ is just a cycle (possibly with some multiple edges) of length $n \equiv 2 \mod 4$. We can assume that the edge $e_i = v_iv_{i+1}$ (indices taken modulo n) is green whenever i is even. Thus, the edge e_i is red and hence simple whenever i is odd. Now fix

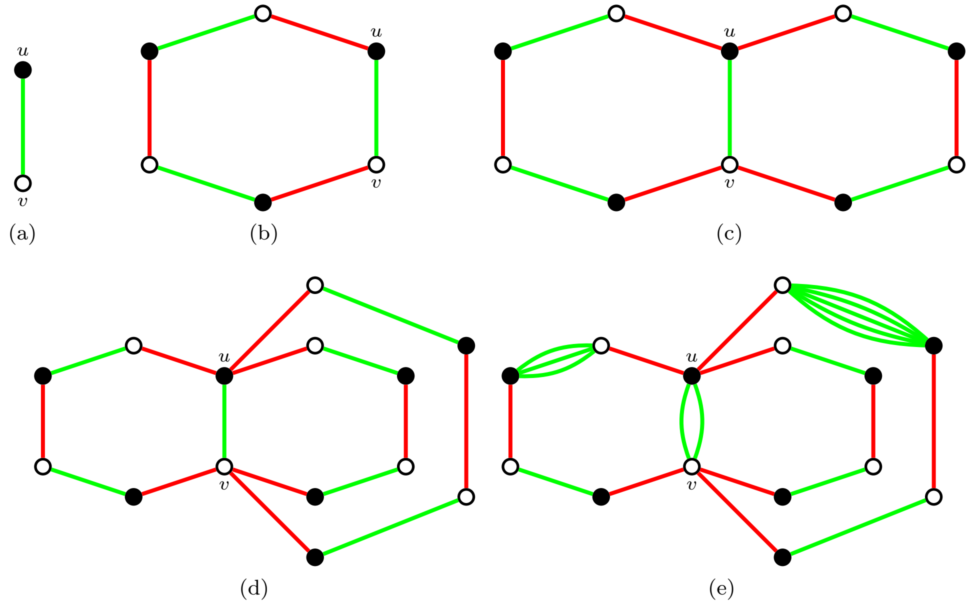


Figure 2.4: Stepwise constructing of an odd multi-cactus starting from K_2 (a). Simple cycles with edges alternating red and green of length congruent to 2 modulo 4 are being glued along a green edge to the green edge uv in the steps from (b) to (d). In step (e) some green edges are replaced by edges of larger multiplicity.

$i \in \{0, 2, \ldots, n-2\}$. Note that the vertices v_i and v_{i+1} have distinct weighted degrees if and only if the weights of the two red edges e_{i-1} and e_{i+1} are different. Thus, in any proper $\{a, b\}$-edge-weighting of M the weights of $e_1, e_3, e_5, \ldots, e_{n-1}$ must alternate between a and b. Since $n \equiv 2 \mod 4$ this means that e_1 and e_{n-1} must receive the same weight and hence v_0 and v_{n-1} will have the same weighted degree. So we can assume that M is a proper odd multi-cactus.

Let $C = v_0 \cdots v_{m-1} v_0$ (indices taken modulo m) be an end-cycle in M where each of v_0, v_1 has at least three distinct neighbours in M. Note that since $v_0 v_1$ must be a green edge, all the edges $v_1v_2, v_3v_4, v_5v_6, \ldots, v_{m-1}v_0$ are red and thus simple. As before, in any proper $\{a, b\}$-edge-weighting of M the weights of these edges $v_1v_2, v_3v_4, v_5v_6, \ldots, v_{m-1}v_0$ must alternate and since $m \equiv 2 \mod 4$, this means that v_1v_2 and $v_{m-1}v_0$ must receive the same weight. Now let w be a proper $\{a, b\}$-edge-weighting of M and let M' be the odd multi-cactus obtained from M by replacing the path $P = v_1 v_2 \cdots v_0$ with an edge e. Since $w(v_1v_2) = w(v_{m-1}v_0)$ the edge-weighting obtained by the restriction of w to M' and assigning the edge e weight $w(v_0v_1) = w(v_{m-2}v_{m-1})$ is a proper edge-weighting of M'. Thus, M' has the $\{a, b\}$-property contradicting the minimality of M. $\square$

Next we prove the following slightly technical lemma which shows that modifying an odd multi-cactus slightly will give a graph having the S-property for any set S of 2 integers. This fact will be useful later on. Note that if G is a not an odd multi-cactus but is obtained from an odd multi-cactus M by replacing a green edge e by a path whose length is congruent to 1 modulo 4, then e must have multiplicity 1 in M and be an edge in an endcycle of M such that the ends of e both have at least three neighbours in M, for example e could be the edge uv in the odd multi-cactus in Figure 2.4 (c) or (d).

Lemma 2.2.8. *If G is not an odd multi-cactus and is obtained from an odd multi-cactus M by either replacing a red edge with an edge of multiplicity at least 2, or by replacing a green edge by a path of length $k \geq 5$ with $k \equiv 1 \mod 4$, then G has the $\{a, b\}$-property for any two distinct integers $a, b \in \mathbb{Z}$.*

Proof. Let two distinct integers a, b be given. The proof is by induction on the order of M. Let e denote the edge of M which is replaced, as described in the lemma, to obtain G.
First suppose M is not proper. Then, since G is not an odd multi-cactus, the graph G was obtained from M by replacing a red edge with an edge of multiplicity at least 2. In this case $M = v_0v_1 \cdots v_{n-1}v_0$ (indices taken modulo n) must be a cycle (possibly with multiple edges) of length $n \equiv 2 \mod 4$. Since G is not an odd multi-cactus, the green/red edge-colouring of M must be unique, so at least one green edge of M, say v_0v_1 has multiplicity at least 2. This implies that whenever i is even the edge v_iv_{i+1} is green and whenever i is odd the edge v_iv_{i+1} is red. So for some odd number j we have $e = v_jv_{j+1}$. Now we weight the red edges $v_{j+2}v_{j+3}, v_{j+4}v_{j+5}, \ldots, v_{j-2}v_{j-1}$ except $e = v_jv_{j+1}$ alternately a and b, and we weight the green edges of G except v_0v_1 such that for $i \in \{3, 5, \ldots, n-1\}$ the sum of the weights of the edges joining v_{i-1} and v_i is distinct from the sum of the weights of the edges joining v_{i+1} and v_{i+2}. Now only the edges joining v_0 and v_1 and the edges joining v_j and v_{j+1} are missing a weight. When assigning weights to the edges joining v_0 and v_1 we only have to worry about the two potential conflicts $v_{n-1}v_0$ and v_1v_2, and when assigning weights to the edges joining v_j and v_{j+1} we only have to worry about the two potential conflicts $v_{j-1}v_j$ and $v_{j+1}v_{j+2}$. Since there are at least two edges joining v_0 and v_1 and at least two edges joining v_j and v_{j+1}, we have at least three choices for the sum of the weights of the edges joining v_0 and v_1 and at least three choices for the sum of the weights of the edges joining v_j and v_{j+1}. This means that we can avoid the potential conflicts and obtain a proper $\{a, b\}$-edge-weighting of G. Thus, we can assume that M is proper.
Suppose there is an end-cycle $C = v_0v_1 \cdots v_{n-1}v_0$ in M which does not contain e. By possibly relabelling the vertices of C we can assume that all vertices of C except v_0 and v_1 only have two distinct neighbours in M. Let G' denote the graph obtained from G by replacing the path $v_1v_2 \cdots v_0$ with an edge e'. Note that G' is obtained from an odd multi-cactus M' in the same way as G was obtained from M and that M' has smaller order than M. By induction G' has the $\{a, b\}$-property. Let w' be a

proper $\{a, b\}$-edge-weighting of G'. We will now define a proper $\{a, b\}$-edge-weighting of G. Any edge in G which is also in G' receives the weight assigned to it by w' in G'. The edges v_1v_2 and $v_{n-1}v_0$ receive the weight $w(e')$ and we assign weights to the remaining edges $v_2v_3, \ldots, v_{n-2}v_{n-1}$ in a way avoiding conflicts inside C. This gives a proper $\{a, b\}$-edge-weighting of G. So we can assume that all end-cycles contain the edge e.
Since all end-cycles in M contain e the odd multi-cactus M was constructed by only pasting cycles together along e as in Figure 2.4 (c), (d) and (e) where there have only been pasted cycles together along the edge uv. Since G is not an odd multi-cactus, the edge e must be simple in M and in this case it is easy to check that G has the $\{a, b\}$-property. $\square$

In the proof of Theorem 2.2.3 we will need some facts about $\{0, 1\}$-edge-weightings of odd multi-cacti. We can formulate these in the two following lemmas.

Lemma 2.2.9. *Let M be an odd multi-cactus. For any vertex $v \in V(M)$ there is a $\{0, 1\}$-edge-weighting of M such that v and all vertices in the opposite bipartition set to v have weighted degree 1 and all other vertices have weighted degree 0 or 2.*

Proof. The proof is by induction on the order of M. It is easy to check that the statement is true if M is not proper, so we can proceed to the induction step assuming M is proper. We can also assume that there are no multiple edges in M, since otherwise we can remove an edge e, use induction, put the edge e back and assign it weight 0. Let C be an end-cycle in M such that v is not a vertex in C with only two neighbours. It is easy to check that subdividing edges with four vertices preserves the conclusion of the lemma so we can assume that C is a 6-cycle, say $C = v_0v_1 \cdots v_5v_0$, where v_0 and v_1 have at least three neighbours in M. Since v is in $M' = M - \{v_2, v_3, v_4, v_5\}$ and M' is an odd multi-cactus we can use the induction hypothesis on M'. So there is a desired edge-weighting of M' and it is easy to check this can be extended to a desired $\{0, 1\}$-edge-weighting of M. $\square$

If w is an edge-weighting of a graph G, b is a natural number and v is a vertex in G, then we let $C_w(v, b)$ denote the vertex-colouring of G obtained from C_w by replacing the colour of v with $C_w(v) + b$. If $C_w(v, b)$ is a proper vertex-colouring we say that w *is a proper $\{0, 1\}$-edge-weighting of G when the weighted degree of v is increased by b.* This may be thought of as a neighbour sum-distinguishing edge-weighting where the vertex v has some pre-assigned weight.

Lemma 2.2.10. *Let M be an odd multi-cactus and let u, v be any two vertices in M belonging to the same bipartition set (possibly $u = v$). If $u \neq v$, then there is a proper $\{0, 1\}$-weighting of M when the weighted degrees of both u and v are increased by 1 and if $u = v$, then there is a proper $\{0, 1\}$-weighting of G when the weighted degree of u is increased by 2.*

Proof. First note that the case $u = v$ follows from Lemma 2.2.9, so we may assume that $u \neq v$. The proof is by induction on the order of M. It is easy to check that the statement holds if M is not proper, so we may proceed to the induction step assuming M is proper. As in the proof of Lemma 2.2.9 we can assume that M is simple and we choose an end-cycle C such that one of v, u, say u, is not a vertex in C with only two neighbours in M, and we may assume $C = v_0v_1 \cdots v_5v_0$, where v_0 and v_1 have at least three neighbours in M. If v and u are both in $M' = M - \{v_2, v_3, v_4, v_5\}$, then we use the induction hypothesis on M' to obtain a $\{0,1\}$-edge-weighting of M' which is proper when the weighted degree of both u and v are increased by 1. We can easily extend this $\{0,1\}$-edge-weighting to a desired edge-weighting of M, so we can assume that u is in M' and v is one of v_2, v_4 (the other cases are similar). We split the proof into three cases:

Case 1: *u is one of v_0, v_1, say, $u = v_0$ and $v = v_4$.*
By Lemma 2.2.9 there is a $\{0,1\}$-edge-weighting w of M' where v_0 and all vertices in the opposite bipartition set to v_0 have weighted degree 1 and all other vertices have weighted degree 0 or 2. We extend this $\{0,1\}$-edge-weighting to M by defining $w(v_0v_5) = w(v_3v_4) = 1$ and $w(v_1v_2) = w(v_2v_3) = w(v_4v_5) = 0$ and obtain a desired edge-weighting of M.

Case 2: *$u = v_0$ and v is v_2.*
As in Case 1 there is a $\{0,1\}$-edge-weighting w of M' where v_0 and all vertices in the opposite bipartition set to v_0 have weighted degree 1 and all other vertices have weighted degree 0 or 2, and we can easily extend this edge-weighting to a desired edge-weighting of M.

Case 3: *$u \in V(M - C)$ and $v \in \{v_2, v_4\}$.*
Note that v_0 is in the same bipartition set as u. We start by considering the subcase where $v = v_2$. In this subcase we use the induction hypothesis on M' choosing u and v_0 as our special vertices. We extend this $\{0,1\}$-weighting, letting the edge v_0v_5 play the role as the extra weight on v_0 by defining $w(v_0v_5) = 1$ and $w(v_1v_2) = 0$. Now v_0 and v_1 have different weighted degrees by the induction hypothesis so we can choose the weights on v_2v_3 and v_4v_5 to be different such that we avoid conflicts between v_5 and v_0, between v_2 and v_1 and between v_3 and v_4. Finally we define $w(v_3v_4) = 0$ to avoid conflicts between v_4 and v_5, and between v_2 and v_3.
The subcase where $v = v_4$ remains. Here we use Lemma 2.2.9 on M' choosing u as our special vertex and extend this $\{0,1\}$-weighting to M by defining $w(v_1v_2) = w(v_0v_5) = w(v_2v_3) = 0$ and $w(v_4v_5) = w(v_3v_4) = 1$. □

Now we have established the necessary facts about odd multi-cacti in order to start our investigation of the $\{0,1\}$-property of bipartite graphs.

2.2.3 $\{0,1\}$-Edge-Weightings of Bipartite Graphs

Let G be a connected bipartite graph with bipartition sets X and Y. When investigating whether or not G has the $\{0,1\}$-property, Lemma 2.2.2, as mentioned above, allows us to assume that both X and Y have odd size. Recall that for any vertex $v \in X$, Lemma 2.2.2 thus implies that there is an $(X \setminus \{v\}, Y \cup \{v\})$-1-parity $\{0,1\}$-edge-weighting w_v of G. In such an edge-weighting of G the only parity conflicts are between v and the neighbours of v. Since such an edge-weighing w_v exists for all vertices $v \in X$ we can more or less choose where in the graph the potential conflicts are. This observation leads to the main idea behind the proof of Theorem 2.2.3 which can be roughly formulated as follows. We will find a local structure in G around a vertex $v \in V(G)$ such that when we find a $\{0,1\}$-edge-weighting w_v of G where the only parity conflicts are between v and the neighbours of v, then the local structure around v allows us to modify w_v so that there are no actual conflicts. One example of a useful local structure around a vertex is explained in the following lemma which will be useful in the proof of Theorem 2.2.3.

Lemma 2.2.11. *Let G be a connected bipartite graph and let $v \in V(G)$ be a vertex such that $G - v$ is connected. If $|N(v)| \geq 3$ and $G - v - N(v)$ is connected, then G has the $\{0,1\}$-property.*

Proof. Suppose $|N(v)| \geq 3$, the graph $H = G - v - N(v)$ is connected and let X, Y denote the two bipartition sets of G with $v \in X$. Furthermore, let $S = (E(G) \setminus E(H)) \setminus E(v)$ and let $N'(v) \subset N(v)$ denote the set of vertices which are incident to an odd number of edges having v as an end. We can assume that both X and Y have odd size. For each $a \in N(v)$ let e_a denote an edge in $E(a) \setminus E(v)$ and define $G' = G - v - \cup_{a \in N(v)}(E(a) \setminus e_a)$ and note that G' is connected. First suppose $|N'(v)|$ is even. Lemma 2.2.2 implies that there is an $((X \setminus \{v\}) \cup N'(v), Y \setminus N'(v))$-1-parity $\{0,1\}$-edge-weighting w' of G'. We now extend w' to the whole of G by assigning weight 0 to all edges in $(E(G) \setminus E(v)) \setminus E(G')$ and weight 1 to all edges in $E(v)$. This gives an $(X \setminus \{v\}, Y \cup \{v\})$-1-parity $\{0,1\}$-edge-weighting w of G. The only potential conflicts are between v and its neighbours, but the weighted degree of any vertex v' in $N(v)$ is at most one more than the number of edges joining v and v'. So the weighted degree of v is greater than that of any of its neighbours and there can therefore be no conflicts.
Now suppose $|N'(v)|$ is odd. Lemma 2.2.2 implies that there is an $(Y \setminus N'(v), (X \setminus \{v\}) \cup N'(v))$-1-parity $\{0,1\}$-edge-weighting w' of G'. Again, we extend w' to the whole of G by assigning weight 0 to all edges in $(E(G) \setminus E(v)) \setminus E(G')$ and weight 1 to all edges in $E(v)$. As before we obtain an edge-weighting of G where the only parity conflicts are between v and its neighbours and where the weighted degree of v is strictly greater than that of any of its neighbours. □

Before we move on to the proof of Theorem 2.2.3 we need one more technical lemma.

Lemma 2.2.12. *Let G be a connected bipartite graph with bipartition sets X, Y. Let G' be an induced subgraph of G where both $|X \cap V(G')|$ and $|Y \cap V(G')|$ are odd and $G - G'$ is connected. Furthermore, assume that there are only two edges joining G' and $G - G'$ in G, let $v_1, v_2 \in V(G')$ denote the ends of these two edges in G' and let $u_1, u_2 \in V(G - G')$ denote the ends of these two edges in $G - G'$. Possibly $v_1 = v_2$ or $u_1 = u_2$. Assume that $v_1, v_2 \in X$ and that both u_1 and u_2 have degree at least 2 in $G - G'$ and none of u_1, u_2 are cut-vertices in $G - G'$. If G' has the $\{0,1\}$-property or if there is a $\{0,1\}$-edge-weighting of G' which is proper when the weighted degrees of both v_1 and v_2 are increased by 1 (if $v_1 = v_2$, then the weighted degree of $v_1 = v_2$ is increased by 2), then G has the $\{0,1\}$-property.*

Proof. We can assume that X and Y both have odd size and hence $X_1 = X \cap V(G - G')$ and $Y_1 = Y \cap V(G - G')$ both have even size. We distinguish two cases:

Case 1: G' has the $\{0,1\}$-property.
Let w' be a proper $\{0,1\}$-edge-weighting of G'. By Lemma 2.2.2 there is an (Y_1, X_1)-1-parity $\{0,1\}$-edge-weighting w_1 of $G - G'$. We assign weight 0 to the edges u_1v_1 and u_2v_2 and combine w' and w_1 to obtain an edge-weighting of G where the only potential conflicts are u_1v_1 and u_2v_2. So we can assume that one of v_1, v_2, say v_1, has odd weighted degree. By Lemma 2.2.2 there is also an (X_1, Y_1)-1-parity $\{0,1\}$-edge-weighting w_2 of $G - G'$. Again we assign weight 0 to the edges u_1v_1 and u_2v_2 and now we combine w' and w_2 to obtain an edge-weighting of G where the only potential conflict is u_2v_2 (since v_1 has odd weighted degree) and where there are no parity conflicts inside $G - G'$. Since u_2 has degree at least 2 in $G - G'$ and is not a cut-vertex in $G - G'$, and since u_2 has even weighted degree there is a u_2-changing cycle in $G - G'$ and swapping the weights on such a cycle yields a proper edge-weighting of G.

Case 2: There is a $\{0,1\}$-edge-weighting of G' which is proper when the weighted degrees of v_1 and v_2 are increased by 1 (if $v_1 = v_2$, then the weighted degree of $v_1 = v_2$ is increased by 2).
If $v_1 \neq v_2$, let w' be a $\{0,1\}$-edge-weighting of G' which is proper when the weighted degrees of v_1 and v_2 are increased by 1, and if $v_1 = v_2$ let w' be a $\{0,1\}$-edge-weighting of G' which is proper when the weighted degree of $v_1 = v_2$ is increased by 2. First suppose $u_1 \neq u_2$. By Lemma 2.2.2 there is a $(Y_1 \setminus \{u_1, u_2\}, X_1 \cup \{u_1, u_2\})$-1-parity $\{0,1\}$-edge-weighting w_1 of $G - G'$. We assign weight 1 to the edges u_1v_1 and u_2v_2 and combine w' and w_1 to obtain an edge-weighting w_1^* of G where the only potential conflicts are u_1v_1 and u_2v_2. So we can assume that one of v_1, v_2, say v_1, has odd weighted degree. By Lemma 2.2.2 there is an $(X_1 \cup \{u_1, u_2\}, Y_1 \setminus \{u_1, u_2\})$-1-parity $\{0,1\}$-edge-weighting w_2 of $G - G'$. Again we assign weight 1 to the edges u_1v_1 and u_2v_2 and now we combine w' and w_2 to obtain an edge-weighting w_2^* of G where the only potential conflict is u_2v_2 (since v_1 has odd weighted degree) and where there are no parity conflicts inside $G - G'$. So we can assume that v_2 has even weighted degree. Let us now return to the edge-weighting w_1^* of G. The only possible conflict is v_1u_1, but since u_1 has even weighted degree inside $G - G'$, we can now, as in Case 1, avoid

this conflict by swapping the weights on a u_1-changing cycle in $G - G'$. Thus, we can assume $u_1 = u_2$. This case is completely analogous to Case 1; we just assign weight 1 to the edges u_1v_1 and u_2v_2 when combining the edge-weightings of $G - G'$ and G'. □

Now we have established all the tools necessary for the proof of Theorem 2.2.3 which characterises all 2-edge-connected bipartite graphs without the $\{0, 1\}$-property. The proof is rather long and technical and contains many special cases that have to be dealt with separately. Therefore the proof is moved to its own subsection ”2-Edge-Connected Bipartite Graphs” below.

2.2.3.1 2-Edge-Connected Bipartite Graphs

Lemma 2.2.7 implies that in order to prove Theorem 2.2.3 it suffices to show that if G is a 2-edge-connected bipartite graph which is not an odd multi-cactus, then G has the $\{0, 1\}$-property. To prove this we will use the overall strategy roughly outlined in the beginning of Section 2.2.3 above. By a *suspended* path in a graph G we mean a path in G all of whose internal vertices have degree 2 in G and whose endvertices have degree at least 3 in G. A suspended cycle in a graph G is a cycle where all vertices except one have degree exactly 2 in G and one vertex has degree at least 3.

Proof of Theorem 2.2.3. By Lemma 2.2.7 it suffices to prove that if G is a 2-edge-connected bipartite graph without the $\{0, 1\}$-property, then G is an odd multi-cactus. Suppose this is false and let G be counterexample with smallest size. If $e = uv \in E(G)$ is an edge of multiplicity at least 2, then the graph G' obtained from G by removing one of the parallel edges e_1 between u and v cannot have the $\{0, 1\}$-property, since any proper $\{0, 1\}$-edge-weighting of G' can be extended to G by assigning weight 0 to the edge e_1. Since G does not have the $\{0, 1\}$-property and is not an odd multi-cactus, Lemma 2.2.8 implies that G' is also not an odd multi-cactus and hence, by the minimality of G, the edge e must be a cut-edge in G'. In particular, any multiple edge has multiplicity exactly 2 and there are no multiple edges in any 2-connected block of G.

Let X, Y denote the two bipartition sets of G. By the remark following Lemma 2.2.2 we can assume that both X and Y have odd size.

Claim 1. *$|N(v)| \geq 2$ for every $v \in V(G)$.*

Proof of the claim. Suppose $v \in V(G)$ only has one neighbour u. Since G is 2-edge-connected the edge uv is a multiple edge so by the above uv has multiplicity exactly 2. We can assume $v \in X$. By Lemma 2.2.2 there is an $(X \setminus \{v\}, Y \cup \{v\})$-1-parity $\{0, 1\}$-edge-weighting of G. The only potential conflict is uv in the case where all edges in $E(u) \setminus E(v)$ have weight 0. Since G is 2-edge-connected the graph $G - v$ contains a u-changing cycle C. By possibly swapping the weights on C we can obtain a proper $\{0, 1\}$-edge-weighting of G. ◇

Let B be an endblock of G. Note that by Claim 1 the block B is 2-connected, so by the above remark B contains no multiple edges.

Claim 2. *B contains no suspended path of length 2.*

Proof of the claim. Suppose the claim is false and let y_1xy_2 be a suspended path of length 2 in B. We can assume $y_1, y_2 \in Y$ and $x \in X$. By Lemma 2.2.2 there is an $(X \setminus \{x\}, Y)$-1-parity $\{0,1\}$-edge-weighting $w_{G'}$ of $G' = G - x$. We now extend this weighting to an $(X \setminus \{x\}, Y \cup \{x\})$-1-parity $\{0,1\}$-edge-weighting w_G of G by assigning weight 0 to the two edges y_1x and y_2x. Note that the only parity conflicts are y_1x and y_2x, so we can assume y_1x is a conflict, which means that all edges incident to y_1 are 0-edges. Since G is 2-edge-connected and since y_1 has degree at least 3, the graph G' contains a y_1-changing cycle C. After swapping the weights on C the edge y_1x is no longer a conflict, so we can now assume y_2x is a conflict. The graph G' also contains a y_2-changing cycle C' and after swapping the weights on C' the edge y_2x is no longer a conflict. The only potential conflict is now y_1x in the case where C' contained two 1-edges incident to y_1, whose weights were changed to 0. Thus, we can assume that all cycles in G' that contain y_2 also contain y_1 and similarly all cycles in G' that contain y_1 also contain y_2. Furthermore, we can assume that all cycles in G' containing y_1 contain the two 1-edges incident to y_2. Note that this also implies that none of y_1, y_2 is the unique cut-vertex in B (this cut-vertex only exists if G is not 2-connected and is unique because B is an endblock). The only possibility is that $G'' = G - y_1 - x - y_2$ consists of exactly two components G_1, G_2 and that both y_1 and y_2 have exactly one neighbour in each of G_1, G_2. Let x_1, x_2 denote the neighbours of y_1 in G_1 and G_2, respectively and let z_1, z_2 denote the neighbours of y_2 in G_1 and G_2, respectively. Possibly $x_1 = z_1$ or $x_2 = z_2$. Let $X_1 = V(G_1) \cap X, Y_1 = V(G_1) \cap Y, X_2 = V(G_2) \cap X, Y_2 = V(G_2) \cap Y$. Let us again focus on the $(X \setminus \{x\}, Y \cup \{x\})$-1-parity $\{0,1\}$-edge-weighting w_G of G where $w_G(y_1x) = w_G(y_2x) = 0$ and where y_1x is a conflict. Both the edges y_1x_1 and y_1x_2 must have weight 0 and both y_2z_1 and y_2z_2 have weight 1. This means that all the vertices in $X_1 \setminus \{z_1\}$ and $X_2 \setminus \{z_2\}$ are incident to an odd number of 1-edges in G_1 and G_2, respectively, and all vertices in $Y_1 \cup \{z_1\}$ and $Y_2 \cup \{z_2\}$ are incident to an even number of 1-edges in G_1 and G_2, respectively. It follows that both X_1 and X_2 have odd size (since otherwise if, say $|X_1|$ is even, the subgraph of G induced by the 1-edges in G_1 has an odd number of vertices of odd degree). By Lemma 2.2.11 we can assume that $G - y_1 - x - x_1 - x_2$ is disconnected so one of x_1, x_2, say x_1, must be a cut-vertex in G. Similarly, since also $G - y_2 - x - z_1 - z_2$ is disconnected, one of z_1, z_2 must be a cut-vertex in G. Since B only contains one cut-vertex of G it must be the case that $x_1 = z_1$. By Lemma 2.2.2 there is an $(X \setminus \{x_1, x, x_2\}, (Y \setminus \{y_1\}) \cup \{x, x_1\})$-1-parity $\{0,1\}$-edge-weighting $w_{G'''}$ of the connected graph $G''' = G - y_1 - x_2$. We extend $w_{G'''}$ to G by assigning weight 1 to the edges in $E(y_1)$ and weight 0 to the edges in $E(x_2) \setminus E(y_1)$ and obtain a $(X \cup \{y_1\}, Y \setminus \{y_1\})$-1-parity $\{0,1\}$-edge-weighting w'_G of G where $w'_G(y_1) = 3$ and $w'_G(x_2) = 1$ and the only potential conflict is y_1x_1. So we can assume that $w'_G(x_1) = 3$. We can also assume that there is no x_1-changing cycle avoiding y_1 and x_2 in G, since swapping the weights on such a cycle would yield a proper $\{0,1\}$-edge-weighting of G. Hence, x_1 must be incident to one 1-edge and one 0-edge going to each component of $G - x_1$ not containing y_1 and x_2. This implies that each component of $G - x_1$ not containing y_1 must contain an odd number of vertices in X (since otherwise if K is a component of $G - x_1$ not containing y_1 which has an even number of vertices in X, then the subgraph of K induced by the 1-edges contains an odd number of vertices with odd degree). Since $|X_1|$ is odd there must be an even number of such components. Moreover, since $w'_G(x_1) = 3$ we can conclude that there are exactly two such components and hence x_1 has degree 6 in G. We distinguish two cases:

Case 1: $|Y_2|$ is odd.
By Lemma 2.2.2 there is an $(X_1 \cup Y_2 \cup \{y_1, y_2\}, Y_1 \cup X_2 \cup \{x\})$-1-parity $\{0,1\}$-edge-weighting of $G - y_1x_2 - x_1y_2$. We extend this edge-weighting to the whole of G by assining weight 0 to the edges y_1x_2, x_1y_2. Now the only parity conflicts are x_1y_1 and x_1y_2, but the weighted degree of x_1 is 3 and the weighted degree of both y_1 and y_2 is 1 so this edge-weighting of G is proper.

Case 2: $|Y_2|$ is even.
Note that since Y has odd size it must be the case that $|Y_1|$ is odd. Note that G_1 is not 2-connected and hence not an odd multi-cactus. By the minimality of G it follows that G_1 has the $\{0,1\}$-property and thus, Lemma 2.2.12 implies that G has the $\{0,1\}$-property. $\diamond$

Claim 3. *B contains no suspended path or cycle of length 4.*

Proof of the claim. First assume that $y_1x_1y_2x_2y_1$ is a suspended cycle of length 4 in B with $d_G(y_1) \geq 3$ and $y_1 \in Y$. By Lemma 2.2.2 there is an $(Y \setminus \{y_2\}, X \setminus \{x_1, x_2\})$-1-parity $\{0,1\}$-edge-weighting of $G - x_1 - y_2 - x_2$. We extend this edge-weighting to the whole of G by assigning weight 1 to y_1x_1 and y_1x_2 and weight 0 to x_1y_2 and x_2y_2. Now the only possible conflicts are y_1x_2 and y_1x_2, but y_1 has weighted degree at least 3 and both x_1 and x_2 have weighted degree 1.

Now suppose $y_1x_1y_2x_2y_3$ is a suspended path of length 4. Define $G' = G - x_1 - y_2 - x_2$. By Lemma 2.2.2 there is an $(Y \setminus \{y_1, y_2, y_3\}, (X \setminus \{x_1, x_2\}) \cup \{y_1, y_3\})$-1-parity $\{0,1\}$-edge-weighting of G'. We extend this weighting to the whole of G by assigning weight 0 to the edges in $E(y_2)$ and weight 1 to the edges y_1x_2 and y_3x_2 and obtain an $((Y \setminus \{y_2\}) \cup \{x_1, x_2\}, (X \setminus \{x_1, x_2\}) \cup \{y_2\})$-1-parity $\{0,1\}$-edge-weighting of G. The only parity conflicts are y_1x_1 and y_3x_2. As in the proof of Claim 2 we can conclude that $G' - y_1 - y_3$ consists of two components G_1, G_2 and both y_1 and y_3 have exactly one neighbour in each of G_1, G_2. Let x'_1, x'_2 denote the neighbours of y_1 in G_1 and G_2, respectively and let z_1, z_2 denote the neighbours of y_3 in G_1 and G_2, respectively. Possibly $x'_1 = z_1$ or $x'_2 = z_2$. Let $X_1 = V(G_1) \cap X, Y_1 = V(G_1) \cap Y, X_2 = V(G_2) \cap X, Y_2 = V(G_2) \cap Y$. As in the proof of Claim 2 we can conclude that both Y_1 and Y_2 have odd size and that one of x'_1, x'_2, say $x'_1 = z_1$, is a cut-vertex in G. Furthermore, as in the proof of Claim 2 we can argue that there are at most two components of $G - x_1'$ not containing y_1, and that each such component is incident with exactly two edges incident to x_1 and contains an odd number of vertices in X. We distinguish two cases:

Case 1: $|X_2|$ is even.
In this case $|X_1|$ must be odd and therefore there is an even number of components in $G - x_1'$ not containing y_1. Since there are at most two such components there must then be exactly two such components K_1, K_2. The minimality of G implies that G_1 has the $\{0,1\}$-property and thus, Lemma 2.2.12 implies that also G has the $\{0,1\}$-property.

Case 2: $|X_2|$ is odd.
In this case $|X_1|$ must be even and therefore there is exactly one component K in $G - x_1'$ not containing y_1. Note that K must contain an odd number of vertices in Y. Let G'' denote the bipartite graph obtained from G' by identifying y_1 and y_3 and let $y' = y_1 \sim y_3 \in V(G'')$ denote this new resulting vertex. By the minimality of G, there is a proper $\{0,1\}$-edge-weighting of G'' and in such an edge-weighting either the two edges going out of G_2 have the same weight, or the two edges going out of K have the same weight (otherwise $x_1'y'$ is a conflict). Now Lemma 2.2.12 implies that G has the $\{0,1\}$-property. $\diamond$

Claim 4. *G contains no suspended path or cycle of length at least 5.*

Proof of the claim. Suppose $y_1x_1y_2x_2y_3x_3$ is a path in G where all internal vertices have degree 2. Let G' be the graph obtained from $G - x_1 - y_2 - x_2 - y_3$ by adding the edge $e = y_1x_3$ (if this edge is already there we just add a new parallel edge). By Lemma 2.2.8 we can assume that G is not obtained from an odd multi-cactus by replacing a green edge with a path of length 5 or by adding an edge parallel to a red edge (see the definition of odd multi-cactus for an explanation of red and green edges). Since G is not an odd multi-cactus, also G' is not an odd multi-cactus. The minimality of G implies that there is a proper $\{0,1\}$-weighting of G'. This $\{0,1\}$-edge-weighting can now be used to find a proper $\{0,1\}$-edge-weighting of G: we put back the vertices x_1, y_2, x_2, y_3 and give y_1x_1 and y_3x_3 the same weight as $e = y_1x_3$ and delete that edge. We give y_2x_2 the opposite weight. Then we give x_1y_2 and x_2y_3 distinct weights. There are two ways to do this and since y_1 and x_3 have different weighted degrees, one way will give a proper $\{0,1\}$-edge-weighting. $\diamond$

By Claims 1, 2, 3 and 4 any endblock B of G is simple, 2-connected, and all vertices of degree 2 in B lie on a suspended path of length 3. In G we replace all suspended paths of length 3 with an edge to form a new bipartite graph G^*. Edges arising from suspended paths will be called *blue edges* and the edges of G are called *white edges*. Note that G^* is 2-edge-connected and the minimum degree in any endblock is at least 3. Let B^* be an endblock of G^* and let x_0 denote the unique cut-vertex of G^* in B^* if it exists.

Claim 5. *Any vertex in B^* has at least two distinct neighbours.*

Proof of the claim. Suppose $v \in V(B^*)$ only has one neighbour u. Since v has degree at least 3 the edge uv must have multiplicity at least 3 and since there are no multiple edges in any endblock of G, at most one of these parallel edges is white. By Lemma 2.2.11 we can assume that $G - v - N(v)$ is disconnected, so exactly one of the edges joining u and v is white. Thus B^* consists of two vertices joined by 1 white edge and $s \geq 2$ blue edges. In G this corresponds to an endblock B_G consisting of two adjacent vertices joined by s paths of length 3. It is easy to check that B_G has the $\{0,1\}$-property, so we can assume that G is not 2-connected and that one of u, v, say u, is the cut-vertex x_0. Let G' be obtained from G by removing all vertices except u in B_G. Note that G' has an odd number of vertices so by the remark following Lemma 2.2.2 it has a $\{0,1\}$-edge-weighting w' with no parity conflicts. By possibly swapping the weights on a u-changing cycle we can assume that u has weighted degree at least 1 in G'. If s is even, we extend w' to the whole of G by assigning weight 0 to the edge uv, and for each path of length 3 $uxyv$ joining u and v assigning weight 1 to ux and xy and weight 0 to yv. This will yield a proper $\{0,1\}$-edge-weighting of G. If s is odd, then we extend w' to the whole of G by assigning weight 1 to the edge uv, and for each path of length 3 $uxyv$ joining u and v assigning weight 1 to ux and weight 0 to the edges xy and yv. This will yield a proper $\{0,1\}$-edge-weighting of G. $\diamond$

By Claim 5 all endblocks of G^* are 2-connected.

Claim 6. *If $v \in V(B^*)$, then $G^* - v - N_{G^*}(v)$ is disconnected.*

Proof of the claim. Suppose the claim is false and let $v \in V(B^*)$ be a vertex such that $H^* = G^* - v - N_{G^*}(v)$ is connected. Since $v \in V(B^*)$ the degree of v is at least 3. We can assume $v \in X$. By Lemma 2.2.11 we can assume that $G - v - N_G(v)$ is disconnected, so there must be at least one pair of parallel edges e_1, e_2 incident to v in G^* such that e_1 is blue and e_2 is white. For each neighbour v' of v in G of degree at least 3 let $S_{v'}$ denote the set of edges incident to v' in G which are contained in a suspended path of length 3 ending in v. Let G' denote the graph obtained from $G - v$ by for each $v' \in N_G(v)$ of degree at least 3 removing all edges except one in $E_{G-v}(v') \setminus S_{v'}$. Since H^* is connected, G' is also connected. We distinguish two cases.

Case 1: $d_G(v)$ is odd.
By Lemma 2.2.2 there is an $(Y \setminus N_G(v), (X \setminus \{v\}) \cup N_G(v))$-1-parity $\{0,1\}$-edge-weighting of G'. We extend this to an $(Y \cup \{v\}, X \setminus \{v\})$-1-parity $\{0,1\}$-edge-weighting of G by assigning weight 1 to all edges in $E_G(v)$ and weight 0 to all edges in $(E(G) \setminus E_G(v)) \setminus E(G')$. The only parity conflicts are between v and its neighbours. Note that v has weighted degree at least 3. Furthermore, note that if $v' \in N_G(v)$ has degree at least 3, then each edge $e \in S_{v'}$ has weight 0, since the edge incident to v on the suspended path of length 3 containing e has weight 1. Thus, each neighbour of v has weighted degree at most 2 and hence the edge-weighting is proper.

Case 2: $d_G(v)$ is even.
By Lemma 2.2.2 there is an $((X \setminus \{v\}) \cup N_G(v), Y \setminus N_G(v))$-1-parity $\{0,1\}$-edge-weighting of G'. We extend this to an $(X \setminus \{v\}, Y \cup \{v\})$-1-parity $\{0,1\}$-edge-weighting of G by assigning weight 1 to all edges in $E_G(v)$ and weight 0 to all edges in $(E(G) \setminus E_G(v)) \setminus E(G')$. The only parity conflicts are between v and its neighbours. Note that v has weighted degree at least 4. Furthermore, note that, as above, if $v' \in N_G(v)$ has degree at least 3, then each edge $e \in S_{v'}$ has weight 0. Thus, each neighbour of v has weighted degree at most 2 and hence the edge-weighting is proper. ◇

By Claim 6 we can assume that whenever we pick a vertex $v \in V(B^*)$, then $G^* - v - N_{G^*}(v)$ is disconnected. By Claim 5 there is a vertex in $V(B^*)$ not adjacent to x_0 (if x_0 exists), so there is a vertex $v \in V(B^*)$ such that x_0 (if it exists) is in $G^* - v - N_{G^*}(v)$. Now we choose $v \in V(B^*)$ such that the component K of $G^* - v - N_{G^*}(v)$ containing x_0 has maximum size (if x_0 does not exists we just maximize the size of some component). By the maximality of K each vertex in $N_{G^*}(v)$ must have a neighbour in K. Also, each other component of $G^* - v - N_{G^*}(v)$ must be an isolated vertex with the same neighbourhood as v. Let U denote the set of isolated vertices in $G^* - N_{G^*}(v)$ making up the components distinct from K. We can assume that $|U| \geq 2$ and that $U \subset X$.

Claim 7. *If $u \in U$, then $d(u) \leq 4$.*

Proof of the claim. Suppose the claim is false and let $u \in U$ be a vertex of degree at least 5. By Lemma 2.2.11 the graph $G - u - N(u)$ is disconnected, so u must be incident to at least one white edge in B^*. Let $u' \in N_G(u)$ be a neighbour of u in $N_{G^*}(v)$. For each vertex $a \in N_{G^*}(v) \setminus \{u'\}$ let e_a denote an edge incident to a going to K and let S_a denote the set of edges in G incident to a which are contained in a suspended path of length 3 containing u. Let G' be obtained from $G - u$ by for each $a \in N_{G^*}(v) \setminus \{u'\}$ removing all edges in $E(a) \setminus (S_a \cup \{e_a\})$. Note that G' is connected. We look at two cases.

Case 1: $d(u)$ is odd.
By Lemma 2.2.2 there is an $(Y \setminus N_G(u), (X \setminus \{u\}) \cup N_G(u))$-1-parity $\{0,1\}$-edge-weighting of G'. We now extend this weighting to the whole of G by assigning weight 1 to all edges in $E(u)$ and weight 0 to all edges in $(E(G) \setminus E(G')) \setminus E(u)$, and obtain a $(Y \cup \{u\}, X \setminus \{u\})$-1-parity $\{0,1\}$-edge-weighting of G, where u has weighted degree $d(u) \geq 5$ and each neighbour of u distinct from u' has weighted degree 1. The only possible conflict is now uu', so we can assume this is indeed a conflict. Thus, u' has weighted degree at least 5 and therefore there is a u'-changing cycle in $G - u$. After swapping the weights on such a cycle we avoid the conflict uu'. The weighted degree of a neighbour of u distinct from u' might now have increased by 2, but since the weighted degree of u is at least 5 this gives a proper edge-weighting of G.

Case 2: $d(u)$ is even.
In this case $d(u)$ is at least 6. By Lemma 2.2.2 there is an $(X \setminus \{u\} \cup N_G(u), Y \setminus N_G(u))$-1-parity $\{0,1\}$-edge-weighting of G'. We now extend this weighting to the whole of G by assigning weight 1 to all edges in $E(u)$ and weight 0 to all edges in $(E(G) \setminus E(G')) \setminus E(u)$, and obtain a $(X \setminus \{u\}, Y \cup \{u\})$-1-parity $\{0,1\}$-edge-weighting of G, where u has weighted degree $d(u)$ and each neighbour of u distinct from u' has weighted degree 2. As in Case 1 the only possible conflict is now uu' and we can avoid this conflict by swapping the weights on a u'-changing cycle avoiding u without creating any new conflicts. $\diamond$

By Claim 5 and Claim 7 we have that $|N_{G^*}(v)| \in \{2,3,4\}$. Now we show that all vertices in U actually have degree 3. and hence $|N_{G^*}(v)| \in \{2,3\}$:

Claim 8. *If $u \in U$, then $d(u) = 3$.*

Proof of the claim. Suppose the claim is false and let $u \in U$ be a vertex of degree 4. Fix a vertex $u' \in N_{G^*}(v)$. For each vertex $a \in N_{G^*}(v) \setminus \{u'\}$ let e_a denote an edge incident to a going to K and let S_a denote the set of edges in G incident to a which are contained in a suspended path of length 3 containing u. Let G' be obtained from $G-u$ by for each $a \in N_{G^*}(v) \setminus \{u'\}$ removing all edges in $E(a) \setminus (S_a \cup \{e_a\})$. Note that G' is connected. By Lemma 2.2.2 there is an $((X \setminus \{u\}) \cup N_G(u), Y \setminus N_G(u))$-1-parity $\{0,1\}$-edge-weighting of G'. We now extend this edge-weighting to the whole of G by assigning weight 1 to all edges in $E(u)$ and weight 0 to all edges in $(E(G) \setminus E(G')) \setminus E(u)$, and obtain an $(X \setminus \{u\}, Y \cup \{u\})$-1-parity $\{0,1\}$-edge-weighting of G, where u has weighted degree $d(u) = 4$ and each neighbour of u in G distinct from u' has weighted degree exactly 2 and each vertex in $N_{G^*}(v) \setminus \{u'\}$ has weighted degree at most 2. The only possible conflict is now uu', so we can assume that u' has weighted degree 4. Furthermore, note that any vertex u'' in $N_G(u) \setminus \{u'\}$ of degree at least 3 is incident to a 1-edge going to K and a 1-edge incident to u and all other edges incident to u'' are 0-edges. The vertex u' is incident to three 1-edges not incident to u. If two of these 1-edges go to K we can find a u'-changing cycle avoiding $N_{G^*}(v) \setminus \{u'\}$ and after swapping the weights on such a cycle we obtain a proper edge-weighting of G. Hence we can assume that u'' is incident to at most one 1-edge going to K. So u' is incident to at least two 1-edges e_1, e_2 such that e_i is either incident to a vertex $u_i \in U$, or is contained in a suspended path of length 3 ending in a vertex $u_i \in U$ for $i = 1, 2$. We can assume $u_1 \neq u_2$ since otherwise there is a u'-changing cycle in $G - u$ avoiding $N_{G^*}(v) \setminus \{u'\}$. Recall that in G^* both u_1 and u_2 are adjacent to all vertices in $N_{G^*}(v)$ and the weight of any edge incident to a vertex in $N_{G^*}(v) \setminus \{u'\}$ which is either incident to u_1 or u_2, or is contained in a suspended path of length 3 containing u_1 or u_2 is 0.
First suppose there are two distinct vertices $x_1, x_2 \in N_{G^*}(v) \setminus \{u'\}$. Then let P^* be the path in G^* consisting of the edges e_1, e_2, u_1x_1, u_2x_2 and let P be the corresponding path in G. Let P' be a path from x_1 to x_2 in $K+x_1+x_2$ using the edges e_{x_1}, e_{x_2}. After swapping the weights on the cycle $P \cup P'$ we obtain a proper $\{0,1\}$-edge-weighting of G. Thus, we can assume that $N_{G^*}(v) \setminus \{u'\}$ consists of a single vertex x and hence

$|N_{G^*}(v)| = 2$. If u' is incident to a 1-edge-going to K we can find a u'-changing cycle in $G - u$ containing e_x and swapping the weights on this cycle will yield a proper edge-weighting of G, so we can assume that there is no such 1-edge. Since u' has weighted degree 4, this implies that there is a third 1-edge e_3 incident to u' such that e_3 is either incident to a vertex $u_3 \in U$, or is contained in a suspended path of length 3 containing a vertex $u_3 \in U$. As before, we can assume that $u_3 \neq u_1$ and $u_3 \neq u_2$. Since each u_i has degree at least 3 in G, there must be at least two parallel edges between u_i and u' or between u_i and x in G^*. First suppose there are at least two edges between u' and u_1 in G^*. The corresponding edges incident to u' in G must have different weights since otherwise we find a u'-changing cycle in $G - u$ which is not x-changing. Now define a u'-changing cycle in $G - u$ as follows. We take the union of the path of length 1 or 3 from u' to u_1 using an edge incident to u' with weight 0 and the path of length 1 or 3 from u_1 to x and together with a path from x to u' in $K + x + u'$ containing e_x. Swapping the weights on this cycle yields a proper edge-weighting of G. Hence we can assume that for all $i \in \{1, 2, 3\}$ the edge $u_i x \in E(G^*)$ has multiplicity at least 2. Now we first swap the weights on a cycle in $G - u$ avoiding u_3 which is both u'-changing and x-changing to get rid of the conflict uu'. Next we swap the weights on an x-changing cycle in $G - u$ avoiding u', u_1, u_2 and K (the one containing u_3) to avoid the conflict xu. ◇

Claim 9. $|N_{G^*}(v)| = 2$.

Proof of the claim. Suppose the claim is false and $|N_{G^*}(v)| = 3$. By Claim 8 all vertices in U have degree 3 and since $G - u - N(u)$ is disconnected for all $u \in U$ no vertices in U are incident to a blue edge in G^*. Let $u_1, u_2 \in U$ be distinct vertices. By Lemma 2.2.2 there is an $((X \setminus \{u_1, u_2\}) \cup N_{G^*}(v), Y \setminus N_{G^*}(v))$-1-parity $\{0, 1\}$-edge-weighting of $G - u_1 - u_2$. We can now extend this edge-weighting to a proper edge-weighting of G by assigning weight 0 to all edges in $E(u_1)$ and weight 1 to all edges in $E(u_2)$. ◇

By Claim 8 and Claim 9 and since for each vertex $u \in U$ we must have that $G - u - N(u)$ is disconnected (by Lemma 2.2.11), each vertex in U is incident to at most 2 blue edges. Now suppose that there is a vertex $u \in U$ which is incident to two blue edges in G^*. Let $v' \in N_G(u)$ denote the vertex joined to u by a white edge in G^*. Since $G - u - N_G(u)$ is disconnected v' is also joined to u by a blue edge. Now let H denote the graph obtained from $G - u$ by deleting all edges incident to v' except one going to K and the one contained in a suspended path of length 3 containing u (corresponding to the blue edge joining u and v' in G^*). The graph H is connected so by Lemma 2.2.2 there is a $(Y \setminus N_G(u), (X \setminus \{u\}) \cup N_G(u))$-1-parity $\{0, 1\}$-edge-weighting of H. We now extend this edge-weighting to G by assigning weight 1 to all edges in $E(u)$ and weight 0 to all remaining edges. The only parity conflict are between u and its neighbours, but the weighted degree of u is 3 and the weighted degree of all its neighbours is 1, so this edge-weighting is proper. Thus we can assume that all vertices in U are incident to exactly one blue edge.
For a vertex $u \in U$ let b_u denote the suspended path of length 3 containing u in G

(this corresponds to the blue edge in G^*), and let e_u, f_u denote the two white edges incident to u. Since u is not a cut-vertex the two white edges e_u, f_u are not parallel. Let v_1, v_2 denote the two vertices in $N_{G^*}(v)$.

Claim 10. *Both v_1 and v_2 are only incident to one edge going to K.*

Proof of the claim. Suppose for a contradiction that v_1 is incident to at least two edges e_1, e_2 going to K and let $u \in U$. We distinguish two cases:

Case 1: $|Y \cap V(K)|$ is even.
Let G' denote the graph obtained from $G - u$ by removing all edges incident to v_2 except one edge going to K and one edge contained in b_u if b_u contains v_2. Note that G' is connected. By Lemma 2.2.2 there is an $(Y \setminus N_G(u), (X \setminus \{u\}) \cup N_G(u))$-1-parity $\{0,1\}$-edge-weighting of G'. We now obtain a $(Y \cup \{u\}, X \setminus \{u\})$-1-parity $\{0,1\}$-edge-weighting of G by assigning weight 1 to all edges in $E(u)$ and weight 0 to all remaining edges. Note the all edges incident to v_2 except uv_2 has weight 0 so the only possible conflict is uv_1. Since $|Y \cap V(K)|$ is even there must be an even number of 1-edges incident to v_1 going to K. But this means that there is a v_1-changing cycle avoiding v_2 and u. Swapping the weights on such a cycle yields a proper $\{0,1\}$-edge-weighting of G.

Case 2: $|Y \cap V(K)|$ is odd.
Since $|Y|$ is odd, $|Y \setminus V(K)|$ is even and hence there is an even number of vertices in U (since each such vertex u' adds one vertex to Y from the suspended path of length 3 containing u'). This also implies that $|X \setminus V(K)|$ is even and hence $|X \cap V(K)|$ is odd. Let K' be obtained from $K + v_1 + v_2$ by deleting all edges between K and v_1, v_2 except e_1, e_2 and one edge incident to v_2. By Lemma 2.2.2 there is an $((Y \cap V(K)) \cup \{v_2\}, (X \cap V(K)) \cup \{v_1\})$-1-parity $\{0,1\}$-edge-weighting $w_{K'}$ of K' and by possibly swapping the weights on a v_1-changing cycle we can assume that v_1 has weighted degree 2. Let G' be obtained from $G - K$ by removing the edge uv_1 and the edge incident to u in b_u. Note that G' is connected and both $X' = X \cap V(G')$ and $Y' = Y \cap V(G')$ have even size. Let $y \in Y$ and $x \in X$ denote the vertices of degree 2 in b_u. By Lemma 2.2.2 there is an $((Y' \setminus \{v_2, y\}) \cup \{u, x\}, ((X' \setminus \{u, x\}) \cup \{v_2, y\})$-1-parity $\{0,1\}$-edge-weighting $w_{G'}$ of G'. We now combine $w_{K'}$ and $w_{G'}$ and assign weight 0 to all edges which are not in K' or G' and obtain a $((Y \setminus \{y\}) \cup \{u, x\}, (X \setminus \{u, x\}) \cup \{y\})$-1-parity $\{0,1\}$-edge-weighting of G, where u has weighted degree 1 and v_1 and v_2 have weighted degree at least 3. This is a proper edge-weighting of G. $\diamond$

Claim 11. $d(v_1), d(v_2) \leq 5$

Proof of the claim. Suppose for a contradiction that $d(v_1) \geq 6$. As in the proof of Claim 7, since $G - v_1$, is connected we can use Lemma 2.2.2 to find a $\{0,1\}$-edge-weighting of G where v_1 has weighted degree $d(v_1) \geq 6$ and the only parity conflicts are between v_1 and its neighbours. By Claim 10 the vertex v_1 has only one neighbour v_k in K. By possibly swapping the weights on a v_1-changing cycle in $G - K$ we can assume that v_1v_k is not a conflict and that v_1 has weighted degree at least 4. Since all neighbours of v_1 in $G - K$ has degree at most 3, this yields a proper edge-weighting of G. $\diamond$

Note that Claim 11 and Claim 9 implies that $|U| = 2$ and hence both $|Y \cap V(K)|$ and $|X \cap V(K)|$ are odd. One of v_1, v_2, say v_1, has degree at least 4. As in the proof of Claim 11 since $G - v_1$ is connected we can use Lemma 2.2.2 to find a $\{0,1\}$-edge-weighting of G where v_1 has weighted degree $d(v_1) \geq 4$ and the only parity conflicts are between v_1 and its neighbours. By Claim 10 the vertex v_1 has only one neighbour v_k in K and as above v_1v_k is the only possible conflict. So we can assume that v_k also has weighted degree $d(v_1) \geq 4$. The conflict v_1v_k can be avoided by swapping the weights on a v_k-changing cycle in K and such a cycle exists unless v_k is a cut-vertex in K. So we can assume that v_k is a cut-vertex in K. If K is 2-edge-connected, then the minimality of G implies that K has the $\{0,1\}$-property, and by Lemma 2.2.12, so does G. So we can assume that K is not 2-edge-connected. Since B is an endblock this implies that the neighbour of v_2 in K is not a cut-vertex in K. By symmetry of v_1 and v_2 we can assume that v_2 does not have degree at least 4. Hence the degree of v_2 is 3. Now $G - v_2 - N_G(v_2)$ is connected so by Lemma 2.2.11, the graph G has the $\{0,1\}$-property.

□

2.2.3.2 Trees

In this section we will characterise all trees without the $\{0,1\}$-property by providing a recursive way to construct all these trees. This recursion will include four different types of trees, one of them being trees without the $\{0,1\}$-property. A tree which is of one of the other three types will have a special vertex surrounded by a certain local structure. The first new type of trees is defined as follows. If v is a vertex in a tree T such that there is no proper $\{0,1\}$-edge-weighting of T when the weighted degree of v is increased by 1, then we say that T is *v-sensitive*. Note that if xyz is a path of length 2 in a tree T without the $\{0,1\}$-property where x is a leaf in T and y has degree 2 in T, then $T' = T - x - y$ must be z-sensitive (otherwise there is an $\{0,1\}$-edge-weighting of $T - x$ where yz has weight 1 and the only potential conflict is yz, and such an edge-weighting can be extended to a proper $\{0,1\}$-edge-weighting of T by assigning a weight to xy such that yz is not a conflict). Conversely, note that if z is a vertex in a tree T' which is z-sensitive, then the tree T obtained from the disjoint union of T' and an isolated edge xy by adding the edge yz does not have the $\{0,1\}$-property (in any proper $\{0,1\}$-edge-weighting of T the edge yz must have weight 1, so this would yield an edge-weighting of T' which is proper when the weighted degree of z is increased by 1). Thus, we have derived the following lemma.

Lemma 2.2.13. *Let T' be a tree and let v be a vertex in T'. Let T be the tree obtained from T' by adding two vertices v_1 and v_2 and the edges vv_1 and v_1v_2. The tree T' is v-sensitive if and only if T does not have the $\{0,1\}$-property.*

Lemma 2.2.13 and Lemma 2.2.2 imply that a tree which is z-sensitive for some vertex z must have an even number of vertices in both bipartition sets.
Before introducing the remaining two types of trees we will need in our recursive construction of all trees without the $\{0,1\}$-property, we will prove two lemmas describing some of the local structure around a vertex of degree 1 in a simple connected bipartite graph without the $\{0,1\}$-property. Later on we will only use these lemmas when dealing with trees, however, since they hold more generally, we do not restrict ourselves to trees here.

Lemma 2.2.14. *Let G be a simple connected bipartite graph without the $\{0,1\}$-property. If v is a vertex of degree 1, and v' is the unique neighbour of v, then all edges incident to v' are cut-edges in G.*

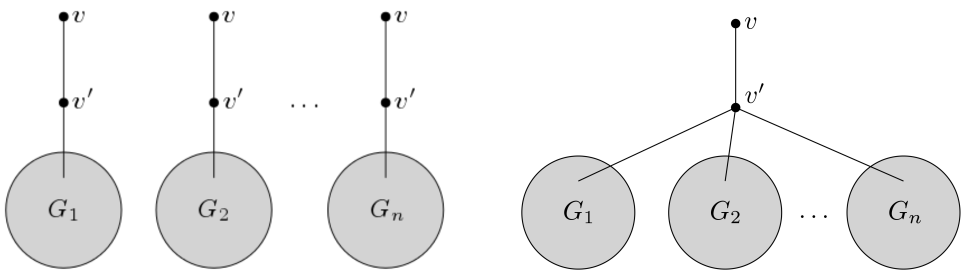
Proof. By Lemma 2.2.2 there is a $\{0,1\}$-weighting of $G-v$ with no parity conflicts. The only problem we can have in extending this $\{0,1\}$-weighting to G is that the weighted degree of v' might be 0. If v' is contained in a cycle in $G-v$ we can avoid this by swapping the weights on a v'-changing cycle. Thus, v' is not contained in a cycle in $G-v$ and hence all edges incident to v' are cut-edges in G. □

Lemma 2.2.14 is fairly trivial and of course redundant when only working with trees. It is included here since it is needed in the proof of the following lemma which describes a reduction of degree 1-vertices being adjacent to a vertex of degree at least 3.

Lemma 2.2.15. *Let G be a simple connected bipartite graph and let $v \in V(G)$ be a vertex of degree 1. Let v' denote the neighbour of v and let $e_0, e_1, \ldots, e_n$ be the edges incident to v' in G with $e_0 = vv'$. Assume that all edges incident to v' are cut-edges in G. For $i \in \{1,\ldots,n\}$ let G_i be the component of $G - e_i$ not containing v and let G_i' denote the connected graph obtained from G_i by adding the vertices v, v' and the edges e_0, e_i. Then the graph G does not have the $\{0,1\}$-property if and only if none of the graphs $G_1', \ldots, G_n'$ have the $\{0,1\}$-property.*

Proof. See Figure 2.5 for an illustration of the graphs $G_1', \ldots, G_n'$ and G. For $i \in \{1,\ldots,n\}$ let v_i be the vertex in G_i incident to v'. First assume that all the graphs $G_1', \ldots, G_n'$ do not have the $\{0,1\}$-property. By Lemma 2.2.13 each G_i is v_i-sensitive. It follows that in any proper $\{0,1\}$-edge-weighting of G all the edges $e_1, \ldots e_n$ must receive weight 0. But then v and v' have the same weighted degree, so no proper $\{0,1\}$-edge-weighting of G exists, which means that G does not have the $\{0,1\}$-property.
Now assume that G does not have the $\{0,1\}$-property. Let X, Y denote the bipartition sets of G such that $v \in X, v' \in Y$ and for each $i \in \{1,\ldots,n\}$ let $X_i = X \cap V(G_i)$ and $Y_i = Y \cap V(G_i)$. Recall that X and Y must have odd size. By Lemma 2.2.2 there is an $(X \setminus \{v\}, Y \cup \{v\})$-1-parity $\{0,1\}$-edge-weighting w_1 of G. We can assume that vv' is a conflict so both v and v' have weighted degree 0 induced by w_1. Note that this implies that all the sets X_i have even size. By Lemma 2.2.2 there is also a $(Y \cup \{v\}, X \setminus \{v\})$-1-parity $\{0,1\}$-edge-weighting w_2 of G. Again we can assume that

vv' is a conflict which means that both v and v' have weighted degree 1 induced by w_2. Note that this implies that all the sets Y_i have even size.
For a contradiction assume that for some i the graph G'_i has the $\{0,1\}$-property. By Lemma 2.2.13 there is a $\{0,1\}$-edge-weighting w_i of G_i which is proper when the weighted degree of v_i is increased by 1. By Lemma 2.2.2 there is an $((X \setminus X_i) \cup \{v'\}, (Y \setminus \{v'\}) \setminus Y_i)$-1-parity $\{0,1\}$-edge-weighting w_3 of $G - G_i$. We now let w_i and w_3 together with assigning weight 1 to the edge $v'v_i$ form an edge-weighting of G. The only possible conflict is $v'v_i$ in the case where v_i have odd weighted degree induced by w_i in G_i. In this case we use Lemma 2.2.2 again to find an $((Y \setminus Y_i) \setminus \{v'\}, (X \setminus X_i) \cup \{v'\})$-1-parity $\{0,1\}$-edge-weighting w_4 of $G - G_i$. Now the edge-weighting of G formed by w_i and w_4 together with assigning weight 1 to the edge $v'v_i$ is a proper edge-weighting of G. Hence none of the graphs G'_i for $i \in \{1, \ldots, n\}$ have the $\{0,1\}$-property. $\square$



(a) The graphs $G'_1, \ldots, G'_n$. (b) The graph G.

Figure 2.5: Lemma 2.2.15.

We now describe the second and third type of trees we will use to construct all trees without the $\{0,1\}$-property. They are special cases of the graphs defined as follows. Let v be a vertex in a connected bipartite graph G and let a and b be integers. We say that G is $v(a,b)$*-strict* if G has the $\{0,1\}$-property and v has weighted degree a in all proper $\{0,1\}$-edge-weightings of G, and v has weighted degree b in all proper $\{0,1\}$-weightings of G where the weighted degree of v is increased by 1. The two special cases we will use in our construction of all trees without the $\{0,1\}$-property are trees which are $v(s,s+1)$-strict and trees which are $v(s,s+3)$-strict for some integer s and some vertex v. Note that a $v(a,b)$-strict tree T cannot have an even number of vertices in both bipartition sets, since in this case Lemma 2.2.2 implies that there are two proper $\{0,1\}$-edge-weightings w_1, w_2 of G where v have even weighted degree induced by w_1 and odd weighted degree induced by w_2 (so v cannot have weighted degree a in both cases). Lemma 2.2.2 also implies that T cannot have an odd number of vertices in both bipartition sets. To see this, suppose T have an odd number of vertices in both bipartition sets and let T' be obtained from T by adding two vertices x, y and the edges xy and yv. Then T' has an even number of vertices in

both bipartition sets so as above there are proper $\{0,1\}$-edge-weightings of T' where v have even and odd weighted degree. But in each of these edge-weightings the edge yv must have weight 1, so these yield proper edge-weightings of T where v have even and odd weighted degree after the weighted degree of v is increased by 1 (so v cannot have weighted degree b in both cases).
The above shows that for any two integers a, b it holds that any tree which is $v(a,b)$-strict for some vertex v has an odd number of vertices.
We will need the following two lemmas describing the local structure around a vertex v in a tree which is $v(s,s+1)$-strict or $v(s,s+3)$-strict.

Lemma 2.2.16. *If s is an integer and v is a vertex in a tree T which is $v(s,s+1)$-strict, then either*

(a) (See Figure 2.6(a)) T is obtained from the disjoint union of a tree T_1 which is $v_1(s-1,s+2)$-strict for some $v_1 \in V(T_1)$, a tree T_2 which is $v_2(s,s+1)$-strict for some $v_2 \in V(T_2)$, together with some trees $T_3,\dots,T_m$ where for each $i \in \{3,\dots,m\}$ the tree T_i is v_i-sensitive for some $v_i \in V(T_i)$ and $s-1$ trees $T_{m+1},\dots,T_{m+s-1}$ without the $\{0,1\}$-property, by adding the vertex v and all the edges $vv_1, vv_2, \dots, vv_m$ and also an edge from v to each of the trees $T_{m+1},\dots,T_{m+s-1}$, or

(b) (See Figure 2.6(b)) T is obtained from the disjoint union of s trees without the $\{0,1\}$-property $T_1,\dots,T_s$ and some trees $T_{s+1},\dots,T_{s+n}$ where for each $i \in \{s+1,\dots,s+n\}$ the tree T_i is v_i-sensitive for some $v_i \in V(T_i)$, by adding the vertex v and all the edges $vv_{s+1}, vv_{s+2},\dots,vv_{s+n}$ and also an edge from v to each of the trees $T_1,\dots,T_s$.

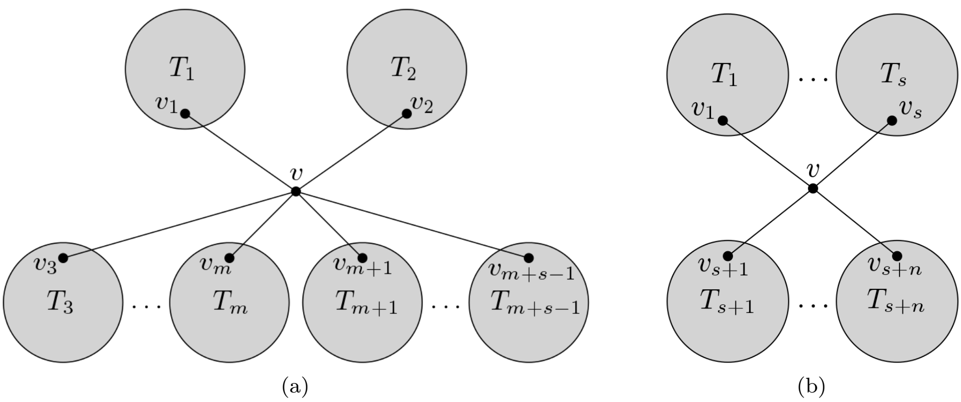


Figure 2.6: The two possible situations explained in Lemma 2.2.16.

Proof. Assume s is even (the case where s is odd is similar). Let X, Y denote the bipartition sets of T such that $v \in Y$, let $d = d(v)$, and let $e_1 = vv_1, \dots, e_d = vv_d$

denote the edges incident to v. For $i \in \{1, \dots, d\}$ let T_i denote the component of $T - v$ containing v_i and let $X_i = X \cap V(T_i)$ and $Y_i = Y \cap V(T_i)$. By possibly adjusting the order of $T_1, \dots, T_d$ we may assume that for three natural numbers s', n_1, n_2 we have that

1. if $i \leq n_1$, then $|X_i|$ is even and $|Y_i|$ is odd, and
2. if $n_i < i \leq n_1 + n_2$, then $|X_i|$ is odd and $|Y_i|$ is even, and
3. if $n_1 + n_2 < i \leq n_1 + n_2 + s'$, then both $|X_i|$ and $|Y_i|$ are odd, and
4. if $i > n_1 + n_2 + s'$ then both $|X_i|$ and $|Y_i|$ are even.

Since T is $v(s, s+1)$-strict T has an odd number of vertices so one of $|X|$, $|Y|$ is even. However, if $|Y|$ is even, then by Lemma 2.2.2, the tree T has an (Y, X)-1-parity $\{0, 1\}$-edge-weighting and since we assumed that s is even this contradicts T being $v(s, s+1)$-strict. So $|Y|$ is odd and $|X|$ is even. By Lemma 2.2.2 the tree T has an (X, Y)-1-parity $\{0, 1\}$-edge-weighting. In such a $\{0, 1\}$-edge-weighting all the edges $vv_1, \dots, vv_{n_1}$ must have weight 0, since otherwise if say vv_1 is weighted 1, then the subgraph of T_1 induced by the 1-edges has an odd number of vertices of odd degree. By a similar argument, all the edges $vv_{n_1+1}, \dots, vv_{n_1+n_2}$ have weight 1, all the edges $vv_{n_1+n_2+1}, \dots, vv_{n_1+n_2+s'}$ also have weight 1 and all the edges $vv_{n_1+n_2+s'}, \dots, vv_d$ have weight 0. Since T is $v(s, s+1)$-strict it follows that $n_2 + s' = s$.
Lemma 2.2.2 also implies that T has a $(Y \setminus \{v\}, X \cup \{v\})$-1-parity $\{0, 1\}$-edge-weighting. We can argue as before and see that all the edges $vv_1, \dots, vv_{n_1}$ must have weight 1, all the edges $vv_{n_1+1}, \dots, vv_{n_1+n_2}$ have weight 0, all the edges $vv_{n_1+n_2+1}, \dots, vv_{n_1+n_2+s'}$ have weight 1 and all the edges $vv_{n_1+n_2+s'}, \dots, vv_d$ have weight 0. Since we can obtain a proper edge-weighting of T by increasing the weighted degree of v by 1, then, since T is $v(s, s+1)$-strict, we must have that $n_1 + s' = s$ and hence $n_1 = n_2$.
Now we show that if $i > n_1 + n_2 + s'$, then T_i is v_i-sensitive. Suppose this is not the case and let $j > n_1 + n_2 + s'$ be such that T_j has a $\{0, 1\}$-edge-weighting which is proper when the weighted degree of v_j is increased by 1. This implies that there is a $\{0, 1\}$-edge-weighting w' of $T_j + v$, where the weight of vv_j is 1 and the only possible conflict is between v and v_j. By Lemma 2.2.2 there is a $(Y \setminus (Y_j \cup \{v\}), (X \cup \{v\}) \setminus X_j)$-1-parity $\{0, 1\}$-edge-weighting w_1 of $T - T_j$. Note that, as before, the weighted degree of v induced by w_1 is s. Now we combine w' and w_1 to obtain an edge-weighting of T where the only potential conflict is vv_j and where the weighted degree of v is $s + 1$. Since T is $v(s, s+1)$-strict the edge vv_j must be a conflict, so v_j also has weighted degree $s + 1$. By Lemma 2.2.2 there is also an $(X \setminus X_j, Y \setminus Y_j)$-1-parity $\{0, 1\}$-edge-weighting w_2 of $T - T_j$. As before the weighted degree of v induced by w_2 is s. Now combining w' and w_2 gives an edge-weighting of T where v has weighted degree $s + 1$ which is proper when the weighted degree of v is increased by 1. This contradicts T being $v(s, s+1)$-strict. We conclude that if $i > n_1 + n_2 + s'$, then T_i is v_i-sensitive.
Next we show that if $n_1 + n_2 < i \leq n_1 + n_2 + s'$, then T_i does not have the $\{0, 1\}$-property. Suppose this is not the case and let $j \in \{n_1 + n_2 + 1, \dots, n_1 + n_2 + s'\}$

be such that T_j has a the $\{0,1\}$-property. This implies that there is a $\{0,1\}$-edge-weighting w' of $T_j + v$ where the only conflict is vv_j and where the weight of vv_j is 0. By Lemma 2.2.2 there is an $(Y \setminus Y_j, X \setminus X_j)$-1-parity $\{0,1\}$-edge-weighting w_1 of $T - T_j$. Note that the weighted degree of v induced by w_1 is $s-1$. As before we combine w' and w_1 and obtain an edge-weighting of T where the only conflict is vv_j and where the weighted degree of v and v_j is $s-1$. By Lemma 2.2.2 there is an $((X \setminus X_j) \cup \{v\}, (Y \setminus Y_j) \setminus \{v\})$-1-parity $\{0,1\}$-edge-weighting w_2 of $T - T_j$. Note that the weighted degree of v induced by w_2 is $s-1$. We now combine w' and w_2 and increase the weighted degree of v by 1 which gives a proper edge-weighting of T, contradicting T being $v(s, s+1)$-strict. We conclude that if $n_1 + n_2 < i \leq n_1 + n_2 + s'$, then T_i does not have the $\{0,1\}$-property.
Note that if $n_1 = n_2 = 0$, then we have showed (b), so we may assume that $n_1 = n_2 > 0$. We will now prove that if $i \leq n_1$, then T_i is $v_i(s-1, s+2)$-strict. So let $i \leq n_1$ and $T' = T - T_i$. By Lemma 2.2.2 there is an $(Y \setminus Y_i, X \setminus X_i)$-1-parity $\{0,1\}$-edge-weighting w_1 of T'. Note that v has weighted degree $s-1$ induced by w_1. If there is a proper edge-weighting of T_i where v_i does not have weighted degree $s-1$, then we can find a proper edge-weighting of T where v has degree $s-1$ contradicting T being $v(s, s+1)$-strict. By Lemma 2.2.2 there is also an $(X \setminus X_i, Y \setminus Y_i)$-1-parity $\{0,1\}$-edge-weighting w_2 of T'. Note that v has weighted degree s induced by w_2. If there is an edge-weighting of $T_i + v$ where vv_i has weight 1 and where v_i does not have weighted degree $s+2$, then we can find an edge-weighting of T where v has weighted degree $s+1$ and which is proper when the weighted degree of v is increased by 1 contradicting T being $v(s, s+1)$-strict. Hence T_i is $v_i(s-1, s+2)$-strict.
Next we show that if $n_1 < i \leq n_2$, then T_i is $v_i(s, s+1)$-strict. So let $n_1 < i \leq n_2$ and $T' = T - T_i$. By Lemma 2.2.2 there is an $(Y \setminus (Y_i \cup \{v\}), (X \cup \{v\}) \setminus X_i)$-1-parity $\{0,1\}$-edge-weighting w_1 of T'. Note that v has weighted degree s induced by w_1. If there is a proper edge-weighting of $T_i + v$ where vv_i has weight 1 and where v_i does not have weighted degree $s+1$, then we can find a proper edge-weighting of T where v has weighted degree $s+1$ contradicting T being $v(s, s+1)$-strict. By Lemma 2.2.2 there is also an $((X \cup \{v\}) \setminus X_i, (Y_i \setminus \{v\}))$-1-parity $\{0,1\}$-edge-weighting w_2 of T'. Note that v has weighted degree $s-1$ induced by w_2. If there is an edge-weighting of $T_i + v$ where vv_i has weight 0 and where v_i does not have weighted degree s, then we can find a $\{0,1\}$-edge-weighting of T where v has weighted degree $s-1$ and which is proper when the weighted degree of v is increased by 1 contradicting T being $v(s, s+1)$-strict. Hence T_i is $v_i(s, s+1)$-strict.
It remains to show that $n_1 = n_2 = 1$. Suppose this is false and let $T' = T - T_{n_1+1} - T_{n_1+2}$. By the above there is a $\{0,1\}$-edge-weighting w of $v + T_{n_1+1} + T_{n_1+2}$, where the two edges incident to v have weight 0 and where v_{n_1+1} and v_{n_1+2} have weighted degree s and where there are no conflicts in T_{n_1+1} or T_{n_1+2}. By Lemma 2.2.2 there is an $(X \cap V(T'), Y \cap V(T'))$-1-parity $\{0,1\}$-edge-weighting w' of T'. Note that the weighted degree of v induced by w' is $s-2$. By combining w and w' we obtain a proper $\{0,1\}$-edge-weighting of T where v has weighted degree $s-2$ contradicting T being $v(s, s+1)$-strict. $\square$

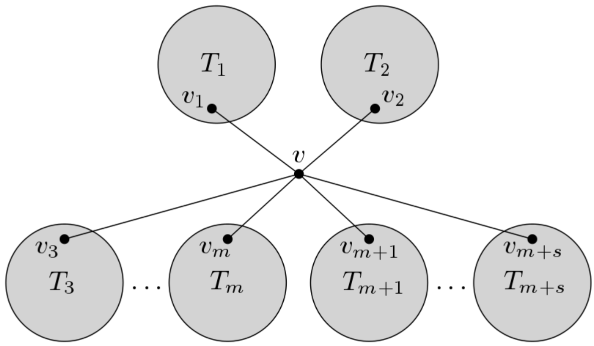


Figure 2.7: Lemma 2.2.17.

Similarly to what we did in the proof of Lemma 2.2.16 we can describe the local structure around a vertex v in tree which is $v(s, s+3)$-strict (some parts of this proof is very similar to the proof of Lemma 2.2.16 and we thus refer to that for some of the details).

Lemma 2.2.17. *Let s be a non-negative integer and let v be a vertex in a tree T. If T is $v(s, s+3)$-strict, then T is obtained from the disjoint union of a tree T_1 which is $v_1(s+1, s+2)$-strict for some $v_1 \in V(T_1)$, a tree T_2 which is $v_2(s+1, s+2)$-strict for some $v_2 \in V(T_1)$ together with some trees $T_3, \dots, T_m$ where for each $i \in \{3, \dots, m\}$ the tree T_i is v_i-sensitive for some $v_i \in V(T_i)$ and s trees $T_{m+1}, \dots, T_{m+s}$ without the $\{0,1\}$-property, by adding the vertex v, all the edges $vv_1, vv_2, \dots, vv_m$ and also an edge from v to each of the trees $T_{m+1}, \dots, T_{m+s}$. See Figure 2.7 for an illustration.*

Proof. Assume s is even (the case where s is odd is similar). We let $d = d(v)$ and define $X, Y \subset V(T)$, n_1, n_2, s' and e_i, T_i, X_i, Y_i for $i \in \{1, \dots, d\}$ as in the proof of Lemma 2.2.16. Since $|V(T)|$ is odd, one of $|X|$, $|Y|$ is even and as in the proof of Lemma 2.2.16 we conclude that $|X|$ is even and $|Y|$ is odd. By Lemma 2.2.2 the tree T has an (X, Y)-1-parity $\{0,1\}$-edge-weighting. In such a $\{0,1\}$-edge-weighting all the edges $vv_1, \dots, vv_{n_1}$ must have weight 0, all the edges $vv_{n_1+1}, \dots, vv_{n_1+n_2}$ have weight 1, all the edges $vv_{n_1+n_2+1}, \dots, vv_{n_1+n_2+s'}$ also have weight 1 and all the edges $vv_{n_1+n_2+s'}, \dots, vv_n$ have weight 0. It follows that $n_2 + s' = s$.
By Lemma 2.2.2, there is also an $(Y \setminus \{v\}, X \cup \{v\})$-1-parity $\{0,1\}$-edge-weighting of T. This means that there is a $\{0,1\}$-weighting of T where all vertices in Y have odd weighted degree and all vertices in X have even weighted degree when the weighted degree of v is increased by 1. In such a $\{0,1\}$-weighting all the edges $vv_1, \dots, vv_{n_1}$ must have weight 1, all the edges $vv_{n_1+1}, \dots, vv_{n_1+n_2}$ have weight 0, all the edges $vv_{n_1+n_2+1}, \dots, vv_{n_1+n_2+s'}$ have weight 1 and all the edges $vv_{n_1+n_2+s'}, \dots, vv_n$ have weight 0. Since T is $v(s, s+3)$-strict it follows that $n_1 + s' + 1 = s + 3$ and hence $n_1 = n_2 + 2$.
As in the proof of Lemma 2.2.16 we can argue that if $i > n_1 + n_2 + s'$, then T_i is v_i-sensitive and if $n_1 + n_2 < i \le n_1 + n_2 + s'$, then T_i does not have the $\{0,1\}$-property.

By the same methods as in the proof of Lemma 2.2.16 we can also show that if $i \leq n_1$, then T_i is $v_i(s+1, s+2)$-strict. So it remains to show that $n_1 = 2$. Clearly $n_1 \geq 2$ so suppose $n_1 \geq 3$ and let $T' = T - T_1 - T_2 - T_3$. By Lemma 2.2.2 there is an $(Y \setminus (Y_1 \cup Y_2 \cup Y_3), X \setminus (X_1 \cup X_2 \cup X_3))$-1-parity $\{0,1\}$-edge-weighting w_1 of T'. Note that v has weighted degree $s-1$ induced by w_1. We can now find a proper edge-weighting of each of T_1, T_2, T_3 such that v_1, v_2 and v_3 have weighted degree $s+1$ and combine these weightings with w_1 to find a proper edge-weighting of T where v has weighted degree $s-1$ and this contradicts T being $v(s, s+3)$-strict. Hence $n_1 = 2$ and $n_2 = 0$. □

The following lemma shows how trees without the $\{0,1\}$-property are constructed from graphs isomorphic to K_2 and trees which are $v(s, s+1)$-strict.

Lemma 2.2.18. *Any tree without the $\{0,1\}$-property distinct from K_2 is obtained from the disjoint union of a tree which is $v(s, s+1)$-strict for some vertex v where $s > 0$ and s graphs isomorphic to K_2 by adding a vertex v', the edge vv' and an edge from v to each of the graphs isomorphic to K_2.*

Proof. Suppose the lemma is false and let T be a counterexample of smallest size. It is easy to check that the statement holds for all trees of diameter at most 3, so we can assume that the diameter of T is at least 4. Lemma 2.2.15 implies that there cannot be two leaves in T having the same neighbour. Let v be the fourth last vertex on a longest path P in T and let v' be the third last vertex. Since P is a longest path in T and since there cannot be two leaves in T having the same neighbour, the components of $T - v'$ not containing v contain at most two vertices. Note that Lemma 2.2.15 actually implies that all these components contain exactly two vertices. Thus, all components of $T - v'$ not containing v are isomorphic to K_2. Since T does not have the $\{0,1\}$-property the subtree of $T - vv'$ containing v must be $v(s, s+1)$-strict where s is the number of components of $T - v'$ not containing v. Hence T is constructed as claimed. □

We list a recursive way to construct trees without the $\{0,1\}$-property below. The construction of $v(s, s+3)$-strict trees is explained by the operation (a)-(b) in Figure 2.9. The construction of v-sensitive trees is explained by the operation (a)-(b) in Figure 2.10. The construction of $v(s, s+1)$-strict trees is illustrated in Figure 2.8 by the operations (a)-(b) and (c)-(d). Finally, the construction of trees without the $\{0,1\}$-property is explained in Figure 2.11. The class of trees without the $\{0,1\}$-property which can be obtained in this way starting with K_2 as the smallest tree without the $\{0,1\}$-property is denoted $\mathcal{B}$.

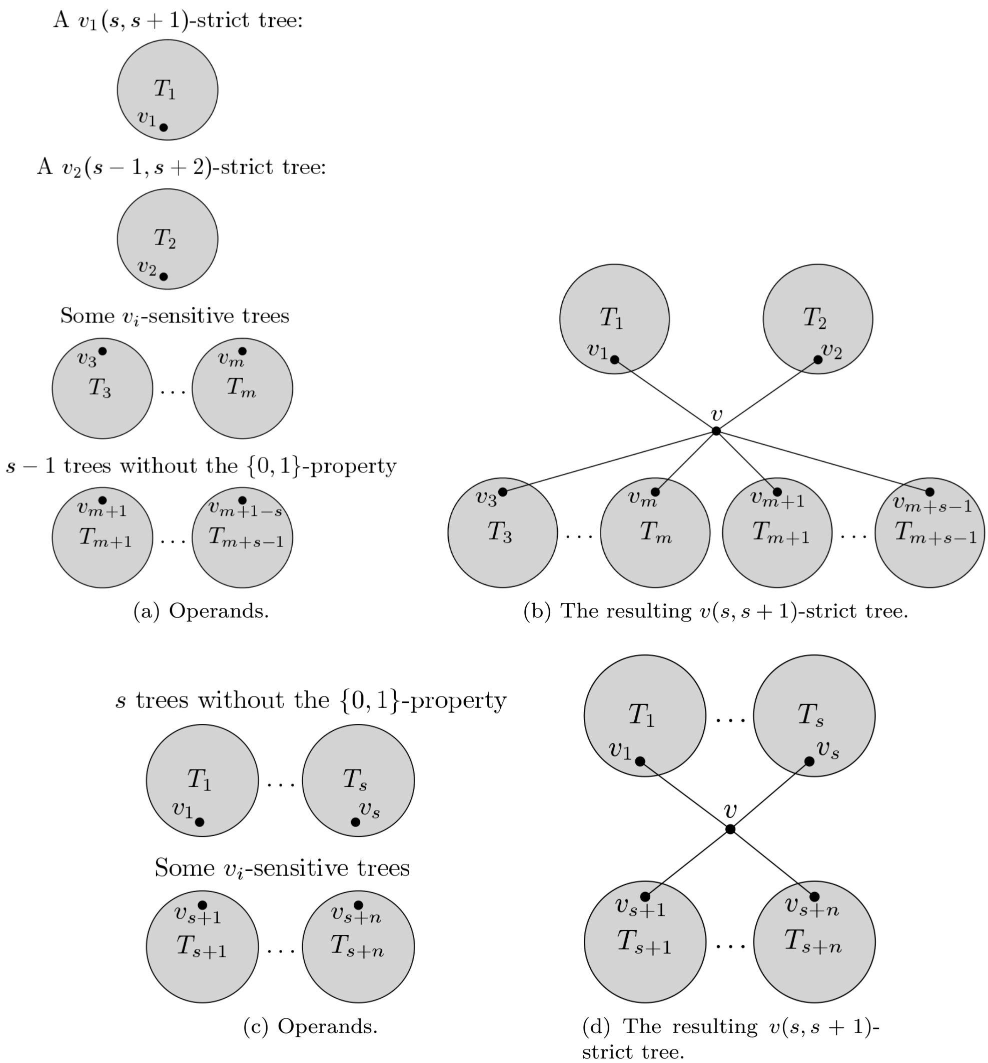
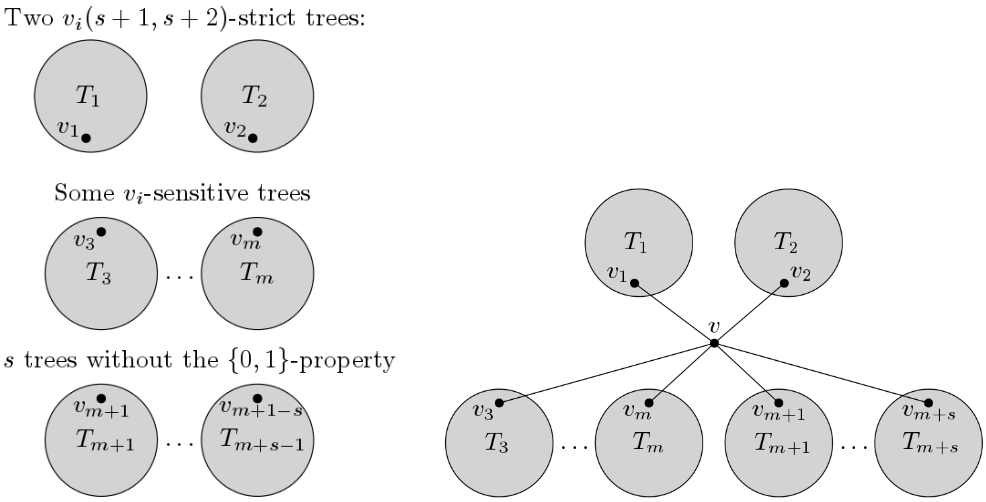
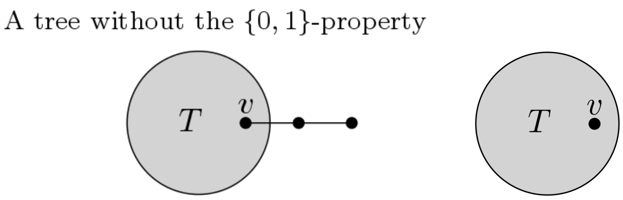


Figure 2.8: Construction of $v(s, s+1)$-strict trees.



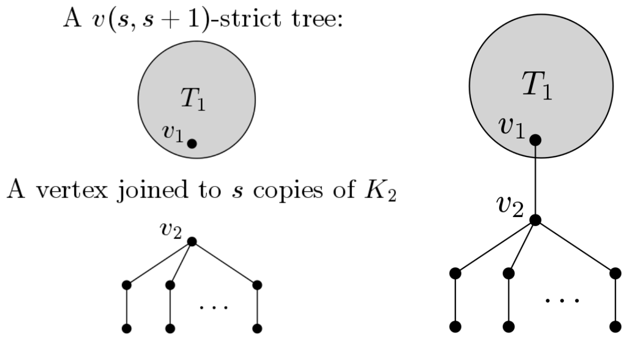
(a) Operands. (b) The resulting $v(s, s+3)$-strict tree.

Figure 2.9: Construction of $v(s, s+3)$-strict trees.



(a) Operand. (b) The resulting v-sensitive tree.

Figure 2.10: Construction of v-sensitive trees.



(a) Operands. (b) The resulting tree without the $\{0,1\}$-property.

Figure 2.11: Construction of trees without the $\{0,1\}$-property.

The above constructions do indeed construct all trees without the $\{0,1\}$-property as we will show now:

Proof of Theorem 2.2.4. Suppose the theorem is false and let T be a tree of smallest size without the $\{0,1\}$-property which cannot be constructed by the above recursion. It is easy to check that the diameter of T must be at least 4. Let n be the number of vertices in T. By Lemma 2.2.13 and the minimality of T we can assume that all trees which have at most $n-4$ vertices and which are v-sensitive for some vertex v can be constructed using the above recursion. By Lemma 2.2.17 and Lemma 2.2.16 we can also assume that all trees which have at most $n-3$ vertices and which are $v(s,s+1)$-strict or $v(s,s+3)$-strict for some vertex v and some integer s can be constructed using the above recursion. By Lemma 2.2.18 our counterexample T is obtained from a tree T' which is $v(s,s+1)$-strict where $s>0$ and v is some vertex in T', and a vertex joined to s distinct graphs isomorphic to K_2. But T' has at most $n-3$ vertices so T' can be constructed by the recursion, and then so can T. $\square$

This completes the section about $\{0,1\}$-edge-weightings of bipartite graphs. We will now turn our attention to $\{a,a+2\}$-edge-weightings of bipartite graphs where a is an odd number.

2.2.4 $\{a,a+2\}$-Edge-Weightings of Bipartite Graphs

In this section we will consider $\{a,a+2\}$-edge-weightings of bipartite graphs where a is an odd number. Since both a and $a+2$ are odd, an edge $uv \in E(G)$ in a graph G is a parity conflict in an $\{a,a+2\}$-edge-weighting of G if and only if $d_G(u)$ and $d_G(v)$ have the same parity. Because of this, we cannot apply Lemma 2.2.2 in the same way as for the $\{0,1\}$-property in the previous sections when trying to construct a proper

$\{a, a+2\}$-edge-weighting of G. To use a parity-argument it is therefore necessary to modify the technique a bit. For this the following observation is useful.

Observation 2.2.19. *Let a be an odd integer and let uv be an edge in a graph G whose edges are weighted with a and $a+2$. If*

1. *$d(u)$ and $d(v)$ have distinct parity, or*
2. *$d(u) \equiv d(v) \mod 4$ and v is incident to an odd number of a-edges while u is incident to an even number of a-edges, or*
3. *$d(u) \not\equiv d(v) \mod 4$ and both v and u are incident to an odd number of a-edges or both v and u are incident to even number of a-edges,*

then u and v have distinct weighted degrees. This is also true if one considers the parity of the number of incident $(a+2)$-edges instead of the parity of the number of incident a-edges.

Proof of Observation. We will consider the case where the parity of the number of a-edges is considered (the case with the $(a+2)$-edges is similar). Clearly we can assume that $d(u)$ and $d(v)$ have the same parity.

Case 1: $d(u) \equiv d(v) \mod 4$.
Since v is incident to an odd number of a-edges the weighted degree of v is congruent to $d(v)\cdot(a+2)-2$ modulo 4 while the weighted degree of u is congruent to $d(u)\cdot(a+2)$ modulo 4. Thus, since $d(u) \equiv d(v) \mod 4$ and $\gcd(a+2, 4) = 1$ it follows that v and u have different weighted degrees.

Case 2: $d(u) \not\equiv d(v) \mod 4$.
If both v and u are incident to an odd number of a-edges, the weighted degree of v is congruent to $d(v)\cdot(a+2)-2$ modulo 4 while the weighted degree of u is congruent to $d(u)\cdot(a+2)-2$ modulo 4. Since $d(u) \not\equiv d(v) \mod 4$ and $\gcd(a+2, 4) = 1$ it follows that v and u have different weighted degrees. If both v and u are incident to an even number of a-edges, the weighted degree of v is congruent to $d(v)\cdot(a+2)$ modulo 4 while the weighted degree of u is congruent to $d(u)\cdot(a+2)$ modulo 4. Again, since $d(u) \not\equiv d(v) \mod 4$ and $\gcd(a, 4) = 1$ it follows that v and u have different weighted degrees. □

Let a be an odd integer and let G be a connected bipartite graph. Assume that we have a vertex-colouring $c : V(G) \to \{1, 2\}$ such that for any $uv \in E(G)$ where $d(u)$ and $d(v)$ have the same parity it holds that if $d(u) \equiv d(v) \mod 4$, then $c(u) \neq c(v)$ and if $d(u) \not\equiv d(v) \mod 4$, then $c(u) = c(v)$. Let X, Y denote the two colour classes of $V(G)$ and assume that one of them, say X, has even size. Lemma 2.2.2 implies that there is an (X, Y)-a-parity $\{a, a+2\}$-edge-weighting of G and Observation 2.2.19 implies that this edge-weighting is proper. This is how Observation 2.2.19 allows us to use techniques similar to the ones in the previous section. One simply has to work with a vertex-colouring which is different from the standard proper 2-vertex-colouring of a bipartite graph. Such a vertex-colouring c is called a *mod-4 vertex-colouring.*

Definition 2.2.20 (mod-4 vertex-colouring)**.** *A mod-4 vertex-colouring of a graph G is a mapping $c : V(G) \to \{1, 2\}$ of G satisfying the following conditions for any $uv \in E(G)$ where $d(u)$ and $d(v)$ have the same parity:*

1. *If $d(u) \equiv d(v) \mod 4$, then $c(u) \neq c(v)$.*
2. *If $d(u) \not\equiv d(v) \mod 4$, then $c(u) = c(v)$.*

As pointed out above both colour classes induced by a mod-4 vertex-colouring of a graph without the $\{a, a + 2\}$-property, where a is odd, must have odd size:

Lemma 2.2.21. *Let G be a connected graph and let a be an odd integer. If G has a mod-4 vertex-colouring where at least one of the two colour classes has even size, then G has the $\{a, a + 2\}$-property.*

We will now show that any bipartite graph has a mod-4 vertex-colouring. This fact together with Lemma 2.2.21 reduces the problem of deciding the $\{a, a + 2\}$-property of bipartite graphs for a odd significantly. The fact that any bipartite graph has a mod-4 vertex-colouring will be implied by the following more general lemma.

Lemma 2.2.22. *Let G be a bipartite graph and let $c : V(G) \to \{1, 2\}$ be any mapping. Then there exists a mapping $c' : V(G) \to \{1, 2\}$ such that for any $uv \in E(G)$ it holds that*

1. *if $c(u) = c(v)$, then $c'(u) \neq c'(v)$ and*
2. *if $c(u) \neq c(v)$, then $c'(u) = c'(v)$.*

Proof. Let X, Y denote the bipartition sets of G. Define $c' : V(G) \to \{1, 2\}$ as $c'(v) = c(v)$ if $v \in X$, and $c'(v) = 1$ if $v \in Y$ and $c(v) = 2$, and $c'(v) = 2$ if $v \in Y$ and $c(v) = 1$. It is easy to check that c' is as desired. □

We can now use Lemma 2.2.22 to show that any bipartite graph has a mod-4 vertex-colouring.

Lemma 2.2.23. *Every bipartite graph has a mod-4 vertex-colouring.*

Proof. Let G_1 and G_2 denote the subgraphs of G induced by the vertices of odd degree and the vertices of even degree, respectively. Let $c_1 : V(G_1) \to \{1, 2\}$ be defined as $c_1(v) = 1$ if $d_G(v) \equiv 1 \mod 4$ and $c_1(v) = 2$ if $d_G(v) \equiv 3 \mod 4$. Let $c_2 : V(G_2) \to \{1, 2\}$ be defined as $c_2(v) = 1$ if $d_G(v) \equiv 0 \mod 4$ and $c_2(v) = 2$ if $d_G(v) \equiv 2 \mod 4$. For $i \in \{1, 2\}$ let c'_i be the mapping we get by applying Lemma 2.2.22 to G_i and c_i and let $c' : V(G) \to \{1, 2\}$ denote the mapping whose restriction to $V(G_i)$ is c'_i for $i \in \{1, 2\}$. It is easy to check that c' is a mod-4 vertex-colouring of G. □

Note that because of Observation 2.2.19 the same ideas as in the previous section with swapping weights on cycles can also be used in the case of $\{a, a+2\}$-edge-weightings where a is odd: Suppose X, Y are the two colour classes induced by a mod-4 vertex-colouring of a graph G, $u \in X$ and that w is an $(X \setminus \{u\}, Y \cup \{u\})$-$a$-parity $\{a, a+2\}$-edge-weighting of G. Observation 2.2.19 implies that all potential conflicts involve u. So suppose uv is conflict. If we now swap the weights on an u-changing cycle avoiding v, then we get rid of the conflict uv and, by Observation 2.2.19, all the potential conflicts still involve u.
With these preliminary tools in hand we are ready to proceed to the section dealing with 2-connected bipartite graphs.

2.2.4.1 2-Connected Bipartite Graphs

The goal of this section is to prove Theorem 2.2.5. Before we proceed to this, it will be convenient to prove the following two lemmas which will simplify the proof of Theorem 2.2.5. Recall that if e is an edge in a multigraph G, then $M(e)$ denotes the multiplicity of e.

Lemma 2.2.24. *Let G be a 2-connected bipartite graph let X, Y be the two colour classes of a mod-4 vertex-colouring of G, and let a be an odd integer. If both X and Y have odd size and $v \in X$ is such that $G - v - N(v)$ is connected, then there is an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, a+2\}$-edge-weighting of G where all edges incident to v have weight $a+2$ and every vertex $u \in N(v)$ is incident to at most $1 + M(uv)$ $(a+2)$-edges.*

Proof. Assume that $v \in X$ and that $G' = G - v - N(v)$ is connected. Let G'' be obtained from $G - v$ by for each vertex $u \in N(v)$ removing all edges but one incident to u in $G - v$. For each $u \in N(v)$ let e_u be the unique edge incident to u in G'' and let $n(u)$ denote the unique neighbour of u in G''. Note that since G' is connected, then so is G''. Let S denote the set of edges in G not incident to v and not in G''. That is, S is the set of edges removed from $G - v$ to obtain G''. Let $G[S]$ denote the subgraph of G induced by the edges in S and let Z denote the vertices of odd degree in $G[S]$. Clearly $|Z|$ is even and since $X \setminus \{v\}$ has even size, this implies that the set $X' = (X \setminus (Z \cup \{v\})) \cup Z \cap Y$ also has even size. Thus, Lemma 2.2.2 implies that there is an $(X', V(G'') \setminus X')$-$a$-parity $\{a, a+2\}$-edge-weighting of G''. We now extend this weighting to G by assigning weight a to all edges in S and weight $a+2$ to all edges in $E(v)$. This results in a desired $\{a, a+2\}$-edge-weighting of G. $\square$

The special case where $a = -1$ (when we are considering the $\{-1, 1\}$-property) will play a special role in some cases in the proof of Theorem 2.2.5. To simplify these cases we prove the following lemma.

Lemma 2.2.25. *Let G be a 2-connected bipartite graph. If there is a vertex $v \in V(G)$ of degree at least 4 and with $|N(v)| \geq 3$ such that $G - v - N(v)$ is connected, then G has the $\{-1, 1\}$-property.*

Proof. Let X, Y be the colour classed induced by a mod-4 vertex colouring of G. By Lemma 2.2.21 we can assume that both X and Y have odd size. Lemma 2.2.24 implies that there is an $(X \setminus \{v\}, Y \cup \{v\})$-$(-1)$-parity $\{-1, 1\}$-edge-weighting of G, where all edges incident to v have weight 1 and any vertex $u \in N(v)$ is incident to at most $1 + M(uv)$ 1-edges. Observation 2.2.19 implies that the only potential conflicts are between v and its neighbours. The weighted degree of v is $d(v)$ and since $|N(v)| \geq 3$, the multiplicity of any edge incident to v is strictly less than $d(v) - 1$. Thus, the weighted degree of any $u \in N(v)$ is less than $d(v)$ and therefore there can be no conflicts. $\square$

We need one more tool before starting the proof of Theorem 2.2.5. This tool is Lemma 2.2.26 below which was proved by Thomassen et al. [Tho16] who used it in their characterisation of bipartite graphs without the $\{1, 2\}$-property. Immediately following the statement of the lemma we will explain how it is useful for our purposes.

Lemma 2.2.26. *[Tho16] Let q be a natural number such that $q \geq 4$. Let G be a connected graph and let A be an independent set of at most q vertices such that each vertex in A has degree at least $q - 1$, or, each vertex in A, except possibly one has degree at least q. Assume that no vertex in A is adjacent to a cut-edge in G. Then, for each vertex a of A, there is an edge e_a incident with a such that the deletion of all e_a, $a \in A$, results in a connected graph unless $|A| = q = 4$, all vertices of A have degree* 3 *and $G - A$ has six components each of which is joined to two distinct vertices of A.*

Let a be an odd integer and let $b = a + 2$. In some cases Lemma 2.2.2 and Lemma 2.2.26 can work well together when trying to construct a proper $\{a, b\}$-edge-weighting of a connected bipartite graph G. This can be seen thorugh the following example. Suppose $c : V(G) \to \{1, 2\}$ is a mod-4 vertex-colouring of a 2-connected bipartite graph G with at least 3 vertices and let X and Y denote the two colour classes induced by c. Recall that we can assume that both X and Y have odd size. Furthermore, suppose that the degree of a vertex $v \in X$ is at least 4 and no vertex in $N(v)$ has degree strictly larger than $d(v)$. Let A be the vertices in $N(v)$ with the same degree as v and suppose that no vertex in A is incident to a cut-edge in $G' = G - v$ and we are not in the exceptional case of Lemma 2.2.26. That is, for each $u \in A$ there is an edge e_u such that $G' - \cup_{u \in A}\{e_u\}$ is connected. Define $S = \cup_{u \in A}\{e_u\}$ and let Z denote the set of vertices in G' which have odd degree in the subgraph of G induced by S (note that $A \subset Z$). Since $X \setminus \{v\}$ and Z have even size, the set $X' = (X \setminus (Z \cup \{v\})) \cup (Z \cap Y)$ also has even size. Thus, Lemma 2.2.2 implies that there is an $(X', V(G') \setminus X')$-a-parity $\{a, b\}$-edge-weighting of $G' - S$. We can extend this edge-weighting to G by assigning weight a to all edges in S and weight b to all edges incident to v to obtain an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting of G where all edges incident to v have weight b and every vertex $u \in N(v)$ which has the same degree as v is incident to at least one a-edge. The weighted degree of v is greater than that of its neighbours, so Observation 2.2.19 implies that the edge-weighting is proper. This is why Lemma 2.2.26 can be a useful tool: in some cases it allows us to

remove edges from a set of independent vertices while maintaining connectivity.
We have now collected all the tools necessary for the proof of Theorem 2.2.5.

Proof of Theorem 2.2.5. By Lemma 2.2.7 it suffices to show that if G is a 2-connected bipartite graph without the $\{a, a+2\}$-property for some odd integer a, then G is an odd multi-cactus. Suppose this is false and, for some odd integer a and $b = a+2$, let G be a counterexample which has smallest size possible. By possibly multiplying the weights by -1 we can assume $b > 0$. Let c be a mod-4 vertex-colouring of G (such a colouring exists by Lemma 2.2.20), and let X denote the set of vertices with colour 1 and let Y denote the set of vertices with colour 2. By Lemma 2.2.21, we can assume that both X and Y have odd size. The proof is split into several claims.

Claim 1. *If $M(uv) > 1$, then $|N(u)| \geq 3$ or $|N(v)| \geq 3$.*

Proof of the claim. Suppose uv is a multiple edge and both u and v have only two distinct neighbours. By the minimality of G, Lemma 2.2.8, and the fact that G is not an odd multi-cactus, the graph obtained from G by replacing uv with one non-multiple edge has a proper $\{a, b\}$-edge-weighting w. But since the multiplicity of uv in G is at least 2 and since u and v can each be in only one conflict distinct from uv, we can obtain a proper $\{a, b\}$-edge-weighting of G from w by weighting the edges joining u and v in a way that avoids the potential conflicts involving u and v (we can do this because there are at least three distinct possible sums for the weights of the edges joining u and v). $\diamond$

Claim 2. *G has no suspended path of length 2.*

Proof of the claim. Suppose v_1xv_2 is a suspended path in G, where $d(x) = 2$ and $d(v_1), d(v_2) \geq 3$. We can assume $x \in X$. Define $G' = G - x$. Since G is 2-connected the graph G' is connected. Lemma 2.2.2 implies that there is an $(X \setminus \{x\}, Y)$-a-parity $\{a, b\}$-edge-weighting of G'. By putting back x and assigning weight b to the edges v_1x, v_2x and swapping the weights on an x-changing cycle we obtain a $(X \setminus \{x\}, Y \cup \{x\})$-$a$-parity $\{a, b\}$-edge-weighting w of G, where $w(v_1x) = w(v_2x) = a$. Observation 2.2.19 implies that the only conflicts that can arise are xv_1 and xv_2, so we can assume that xv_1 is a conflict, that is, both x and v_1 have weighted degree $2a$. This implies that v_1 has even degree at least 4 and $a < 0 < b$. Hence $a = -1$ and $b = 1$. Let u_1, u_2 be the ends of two distinct (-1)-edges incident to v_1 in G', possibly $u_1 = u_2$. Since G is 2-connected, there exists a cycle C in G' containing the edges u_1v_1 and v_1u_2. The cycle C is v_1-changing and x-avoiding, so if we swap the weights on C we do not create new conflicts in G' and we lose the conflict xv_1. In particular, x still has weighted degree -2 while v_1 now has weighted degree 2. We can now assume that xv_2 is a conflict. This implies that v_2 also has even degree at least 4. We can get rid of the conflict xv_2 in the same way as we got rid of the conflict v_1x. The only problem in doing this is that we might recreate the conflict v_1x if the v_2-changing cycle contains two 1-edges incident to v_1. Thus, we can assume that all v_2-changing cycles in G' contain two 1-edges incident to v_1. Since v_2 has weighted degree -2, the

vertex v_2 must be incident to at least two (-1)-edges in G' and at least one 1-edge. First, assume that v_1 is incident to a (-1)-edge e in G'. Since G is 2-connected there is a path P in G' from v_1 to v_2 using e. If the weight on the last edge e' of P (the one incident to v_2) is 1, then swapping the weights on the cycle $P \cup v_1x \cup xv_2$ yields a proper edge-weighting of G, so we can assume e' has weight -1. The vertex v_2 must be incident to a (-1)-edge $e'' \neq e'$ in G'. Because G is 2-connected, the graph $G - v_2$ has a path P' from the end of e'' different from v_2 to the end of e' different from v_2. Note that if P' does not contain v_1, then there is a v_2-changing cycle in G' which is not v_1-changing, which contradicts the above. The same conclusion holds if when walking from v_2 along P' we intersect P in a vertex different from v_1. So we can assume that v_1 is the first intersection between P and P'. Now there is a cycle C in G containing v_2 and the edges e, e', e'', (first go from v_2 to v_1 along P', before going back to v_2 along P). Swapping the weights on C yields a proper edge-weighting of G. Thus we can assume that v_1 is not incident to a (-1)-edge e in G' and therefore must have degree exactly 4. By symmetry, v_2 also has degree exactly 4. Note that this implies that $v_1, v_2 \in X$ and that all four edges incident to v_1 except v_1x have weight 1. Furthermore recall that v_1 has weighted degree 2 and v_2 has weighted degree -2, so two edges incident to v_2 in G' have weight -1 while the last edge incident to v_2 in G' has weight 1.
We now consider the graph $G'' = G' - v_1 - v_2$. If G'' is connected, then we can find a v_2-changing and v_1 avoiding cycle in G', so we can assume that G'' is disconnected. We may also assume that the two (-1)-edges incident to v_2 in G' go to two distinct components of G''. This leaves us with the following three cases to consider:

Case 1: G'' has two components K_1, K_2, such that v_1 is incident to two edges going to K_1 and one edge going to K_2, and v_2 is incident to two edges going to K_2 and one going to K_1.
Let $e_{1,a}$ and $e_{1,b}$ denote the two edges incident to v_1 going to K_1. Recall that $e_{1,a}$ and $e_{1,b}$ have weight 1. Since K_1 is connected, there is a path in K_1 from the end of $e_{1,a}$ different from v_1 to the end of $e_{1,b}$ different from v_1. We now swap all weights along the cycle formed by this path and $e_{1,a}, e_{1,b}$ to get an edge-weighting of G where v_1 has weighted degree -2. Note that now both xv_1 and xv_2 are conflicts.
Let $e_{1,c}$ denote the edge incident to v_1 going to K_2 and let $e_{2,a}$ denote the 1-edge incident to v_2 going to K_2. Both these edges are weighted 1. Since K_2 is connected, there is a path P in K_2 from the end of $e_{1,c}$ different from v_1 to the end of $e_{2,a}$ different from v_2. Now consider the cycle $C = xv_2 \cup e_{2,a} \cup e_{1,c} \cup v_1x \cup P$. When swapping all weights on C, the vertices v_1, v_2 remain of weighted degree -2, while x gets weighted degree 2, so this yields a proper $\{-1, 1\}$-edge-weighting of G.

Case 2: G'' has two components K_1, K_2, such that both v_1 and v_2 are incident to two edges going to K_1 and one going to K_2.
Let $v_{1,a}, v_{1,b}$ denote the ends of the edges incident to v_1 in K_1, and let $v_{1,c}$ denote the neighbour of v_1 in K_2 (possibly $v_{1,a} = v_{1,b}$). Note that for one of $v_{1,a}, v_{1,b}$, say $v_{1,a}$, the graph $G''' = G - v_1 - v_{1,a} - v_{1,c}$ is connected: to see this suppose that G'''

is disconnected. Then it must be the case that $v_{1,a}$ is a cut-vertex in the connected graph $G - v_1$ and in this case it is easy to see that $G - v_1 - v_{1,b} - v_{1,c}$ is connected. By Lemma 2.2.25 we can assume that $G''' - v_{1,b}$ is disconnected. Let $L_1, \dots, L_n$ denote the components of $G''' - v_{1,b}$ such that $v_2 \in V(L_1)$. Since $v_{1,b}$ is not a cut-vertex in G, the vertex $v_{1,a}$ has a neighbour in each of the components L_i for $i \geq 2$. Let us now consider the graph H obtained from $G - v_1$ by removing all edges but one incident to $v_{1,a}$ and removing all edges but one incident to $v_{1,c}$. Note that H is connected. Now define the following two sets X', Y':

$$X' = (X \setminus \{v_1, v_{1,a}, v_{1,b}, v_{1,c}, x\}) \cup (Y \cap \{v_{1,a}, v_{1,b}, v_{1,c}\})$$

and

$$Y' = (Y \setminus \{v_{1,a}, v_{1,b}, v_{1,c}\}) \cup (X \cap \{v_{1,a}, v_{1,b}, v_{1,c}, x\}).$$

Note that $|X'|$ is even. Lemma 2.2.2 implies that H has an (X', Y')-(-1)-parity $\{-1, 1\}$-edge-weighting. We now extend this edge-weighting to the whole of G by assigning weight -1 to all removed edges incident to v_1 and weight 1 to all remaining edges (incident to one of $v_{1,a}, v_{1,c}$). We obtain an $(X \setminus \{v_1\}, Y \cup \{v_1\})$-$(-1)$-parity $\{-1, 1\}$-edge-weighting w of G where all four edges incident to v_1 are weighted -1, and each of $v_{1,a}, v_{1,c}$ is incident to at most two (-1)-edges. The only possible conflict is $v_1 v_{1,b}$, so we can assume that v_1 and $v_{1,b}$ have the same weighted degree and that must be -4. We can also assume that we cannot swap the weights on a $v_{1,b}$-changing cycle in G''', so $v_{1,b}$ is incident to at most two edges going to each of L_i for $i = 1, \dots, n$. Since the weighted degree of $v_{1,b}$ is -4, there must be some components in $G''' - v_{1,b}$ which are incident to strictly more (-1)-edges in $E(v_{1,b})$ than 1-edges in $E(v_{1,b})$. Again, since there is no $v_{1,b}$-changing cycle in G''' there are no two edges incident to $v_{1,b}$ with the same weight that go to the same component of $G''' - v_{1,b}$. Thus, there are at least three components L'_1, L'_2, L'_3 in $G''' - v_{1,b}$, each of which is incident to only one edge in $E(v_{1,b})$ and each of these edges has weight -1. We can assume that L'_1 and L'_2 are distinct from L_1. See Figure 2.12. Recall that $v_{1,a}$ has a neighbour u'_i in each L'_i for $i = 1, 2$. Now there is a cycle C in $G''' + v_{1,a}$ containing two edges incident to $v_{1,b}$ having weight -1 and containing the two edges $v_{1,a}u'_1$ and $v_{1,a}u'_2$. If we swap the weights on C, then the only possible conflict is $v_1 v_{1,a}$ in the case where $v_{1,a}$ is a vertex of degree 4, both $v_{1,a}u'_1$ and $v_{1,a}u'_2$ have weight 1, and $v_{1,a}$ is incident to some fourth (-1)-edge $v_{1,a}z$. We can assume that the component L' to which z belongs in $G''' - v_{1,b}$ is not incident to a (-1)-edge in $E(v_{1,b})$, since otherwise, we could have modified C to contain the edge $v_{1,a}z$. This also implies that $L'_3 = L_1$. Since $v_{1,b}$ had weighted degree -4 induced by w, this implies that there is another component L'_4 distinct from all of L'_1, L'_2, L'_3, which is incident to a (-1)-edge in $E(v_{1,b})$. The vertex $v_{1,a}$ must have a neighbour z' in this component L'_4. But we must have $z \neq z'$ which contradicts $v_{1,a}$ having degree 4.

Case 3: G'' has three components K_1, K_2, K_3, such that both v_1 and v_2 are incident to one edge going to each of these three components.
In this case it is easy to check that $G - v_1 - N_G(v_1)$ is connected and hence G has the $\{-1, 1\}$-property according to Lemma 2.2.25. $\diamond$

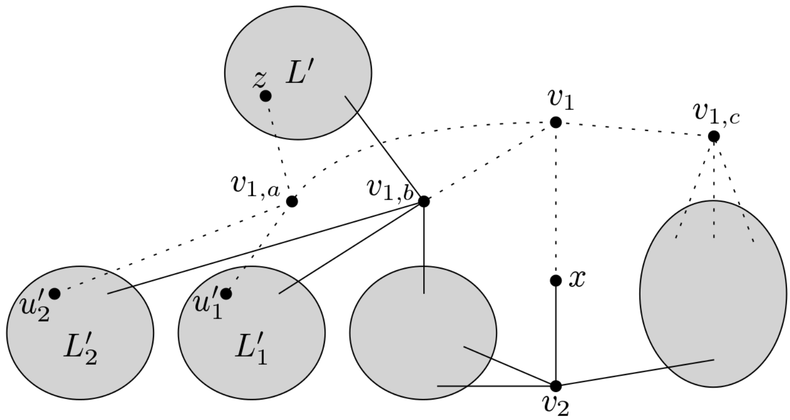


Figure 2.12: the graph G''' in Case 2 of Claim 2.

Claim 3. *G has no suspended path of length 4.*

Proof of the claim. Suppose the claim is false and let $v_1y_1xy_2v_2$ be a suspended path in G, where $d(y_1) = d(x) = d(y_2) = 2$ and $d(v_1), d(v_2) \geq 3$. We can assume $x \in X$, which implies that $y_1, y_2 \in Y$. Define $G' = G - y_1 - x - y_2$. By Lemma 2.2.2 there is an $(X \setminus \{x\}, Y \setminus \{y_1, y_2\})$-$a$-parity $\{a, b\}$-edge-weighting of G'. Now we assign weight b to the edges v_1y_1 and v_2y_2 and weight a to the edges y_1x and xy_2 and obtain an $((X \setminus \{x\}) \cup \{y_1, y_2\}, (Y \setminus \{y_1, y_2\}) \cup \{x\})$-$a$-parity $\{a, b\}$-edge-weighting of G, where the only potential conflicts are v_1y_1 and v_2y_2.
One can check that by slightly modifying the exact same arguments used in the proof of Claim 2, we can eventually remove all conflicts and obtain a proper $\{a, b\}$-edge-weighting of G. The approach is exactly the same, simply treat the path $v_1y_1xy_2v_2$ as the path v_1xv_2 in the proof of Claim 2.

$\diamond$

Claim 4. *G has no suspended path of length at least 5.*

Proof of the claim. Suppose the claim is false and let $v_1x_1x_2x_3x_4v_2$ be a path in G, where x_1, x_2, x_3, x_4 all have degree 2. The vertices v_1, v_2 might also have degree 2. Let G' be obtained from G by replacing $v_1x_1x_2x_3x_4v_2$ by an edge $e = v_1v_2$, also, if that edge is already there. First suppose there is a proper $\{a, b\}$-edge-weighting of G' where the weight of e is, say a. We can now remove e and extend the weighting to G by assigning weight a to v_1x_1 and x_4v_2 (so that v_1 and v_2 keep the same weighted

degree as in G'). We assign weights to the remaining edges in the following way. Since v_1v_2 is an edge in G', the vertices v_1 and v_2 have different weighted degrees. Thus we can assume that v_1 has weighted degree different from $2a$ and v_2 has weighted degree different from $a+b$. We can now obtain a proper edge-weighting of G by assigning weight a to x_1x_2 and weight b to x_2x_3 and x_3x_4.
By the above G' does not have the $\{a,b\}$-property. By the minimality of G the graph G' must be an odd multi-cactus. The edge e cannot be red in G', since then G would also be an odd multi-cactus. Thus, e is green and Lemma 2.2.8 implies that G has the $\{a,b\}$-property. $\diamond$

By Claims 2, 3, 4, all vertices of degree 2 in G lie on suspended paths of length 3. As in the proof of Theorem 2.2.3 we now replace all suspended paths of length 3 in G by edges to form a bipartite multigraph G^*. Edges arising from suspended paths of length 3 are called *blue edges* and the other edges of G^* are called *white edges*.
Note that G^* is bipartite, 2-connected and has minimum degree at least 3. Also, note that for every vertex v in G^*, we have $d_{G^*}(v) = d_G(v)$.
If the deletion of some pair of adjacent vertices disconnects G^*, then let $z_0y_0 \in E(G^*)$ be such that $G^* - z_0 - y_0$ is disconnected and such that some component H of $G^* - z_0 - y_0$ has smallest possible order. The union of that component H and z_0 and y_0 together with all edges connecting them is denoted B. If G^* has no pair of adjacent vertices whose removal disconnects G^* we define $H = B = G^*$ and y_0 and z_0 do not exist.
Define d^* to be the maximum of $d_{G^*}(v)$ for $v \in V(H)$. For a vertex $v \in V(H)$ let $A(v)$ denote the vertices in $N_H(v)$ with the same degree as v which are not joined to v by a blue edge in G^*.

Claim 5. *If $d^* \geq 4$, then there exists a vertex $v_0 \in V(H)$ such that $d_{G^*}(v_0) = d^*$ and for each $x \in A(v_0)$ there is an edge $e_x \in E(x) \setminus E(v_0)$ such that $G - v_0 - \cup_{x \in A(v_0)} e_x$ is connected.*

Proof of the claim. Assume $d^* \geq 4$ and let $v_0 \in V(H)$ be such that $d_{G^*}(v_0) = d^*$ and subject to that, such that $|N_G(v_0)|$ is maximum. Note that for any $v' \in A(v_0)$ it holds that $d_{G-v_0}(v') \geq |A(v_0)| - 1$, since $d_{G^*}(v') = d_{G^*}(v_0)$. Furthermore, note that the minimality of H and Claim 1 implies that no vertex in $A(v_0)$ is incident to a cut-edge in the connected graph $G - v_0$. For each vertex $v' \in A(v_0)$ we want to remove an edge incident to v' in the graph $G - v_0$ and maintain connectivity, so we can assume that no vertex in $A(v_0)$ is incident to a multiple edge in $G^* - v_0$. Lemma 2.2.26 implies that the statement of the claim holds if $|N_G(v_0)| \geq 5$, so we can assume $|N_G(v_0)| \leq 4$. Now suppose $|N_G(v_0)| = 4$. Lemma 2.2.26 implies that $G - v_0 - A(v_0)$ has exactly 6 components and that $d^* = 4$. Furthermore we can assume that this is the case whenever $v_0 \in V(H)$ is a vertex of degree $d^* = 4$ having four neighbours in G^*. But we can avoid this case by choosing v_0 such that the component containing z_0 in $G - v_0 - A(v_0)$ has maximum size (if z_0 do not exist we just maximise some component). Thus we may assume that $|N_G(v_0)| \leq 3$. Since G is 2-connected we also have $|N_G(v_0)| \geq 2$.

If all vertices have degree at least 3 in $G - v_0$, then again Lemma 2.2.26 implies that the statement of the claim holds. So there must be some vertex $u \in A(v_0)$ which is incident to only two edges in $G - v_0$. The choice of v_0 implies that $|N_G(v_0)| \geq |N_G(u)| = 3$. Since $d_{G^*}(v_0) = d_{G^*}(u)$ and u only have two neighbours distinct from v_0, we must have $|N_G(v_0)| = 3$. Now let $v_0' \in A(v_0)$ be distinct from u. Since v_0u is a multiple edge, v_0' must be incident to at least 3 edges in $G - v_0$, which implies $|N_G(v_0')| > 3 = |N_G(v_0)|$, contradicting the choice of v_0. $\diamond$

Claim 6. $d^* = 3$.

Proof of the claim. Suppose the claim is false, let v_0 be a vertex in H satisfying the statement of Claim 5, and let $G' = G - v_0 - \cup_{x \in A(v_0)} e_x$. We may assume $v_0 \in X$. Define $S = \cup_{x \in A(v_0)} e_x$ and let Z denote the set of vertices in $G - v_0$ which are incident to an odd number of edges in S. Clearly $|Z|$ is even so also the set

$$X' = ((X \setminus \{v_0\}) \setminus Z) \cup (Y \cap Z)$$

has even size. Lemma 2.2.2 implies that there is an $(X', V(G') \setminus X')$-a-parity $\{a, b\}$-edge-weighting of G'. We now extend this edge-weighting to the whole of G by assigning weight b to all edges in $E(v_0)$ and weight a to all edges in S. In this way we obtain an $(X \setminus \{v_0\}, Y \cup \{v_0\})$-$a$-parity $\{a, b\}$-edge-weighting of G where all edges incident to v_0 are weighted b and every neighbour of v_0 in H with degree d^* is incident to at least one a-edge (note that the vertices in $N_H(v_0) \setminus A(v_0)$ with the same degree as v_0 are incident to an a-edge contained in a suspended path of length 3 containing v_0). Now v_0 has strictly larger weighted degree than any of its neighbours in H. Thus if v_0 is not incident to any of z_0, y_0 this edge-weighting is proper. So we may assume $z_0v_0 \in E(G)$ is a conflict.
We can assume that we cannot swap the weights on a z_0-changing cycle in $G - V(H)$ since this would yield a proper edge-weighting of G. This implies that $G^* - z_0 - y_0$ has exactly two components (one of them being H) and that z_0 is only incident to two edges in $G^* - H$ whose two corresponding edges in G incident to z_0 must have distinct weights. Let K denote the component of $G^* - z_0 - y_0$ distinct from H and let e_1, e_2 denote the two edges incident to z_0 in G corresponding to the two edges incident to z_0 in $G^* - H$ such that e_1 corresponds to the edge going to K. Since e_1 and e_2 have distinct weights, since all edges incident to v_0 have weight b, and since z_0v_0 is a conflict the vertex z_0 must have larger degree than v_0 and hence $d_G(z_0) \geq d_G(v_0) + 2 \geq 6$.
By possibly relabelling X, Y we can assume $z_0 \in X$. By Lemma 2.2.2, there is an $(X \setminus \{z_0\}, Y)$-a-parity $\{a, b\}$-edge-weighting of $G - z_0$. We can now obtain an $(X \setminus \{z_0\}, Y \cup \{z_0\})$-$a$-parity $\{a, b\}$-edge-weighting of G by putting back z_0 and assigning weight b to all edges in $E(z_0)$. Since $d_G(z_0) > d^*$ the only possible conflicts are z_0y_0 and z_0z_K where z_K is the neighbour of z_0 in K. Note that $G - z_0 - z_K$ is connected so by possibly swapping the weights on a z_K-changing cycle in $G - z_0$ we can assume that z_0z_K is not a conflict. Thus, we may assume z_0y_0 is a conflict and hence $d_G(y_0) \geq d_G(z_0)$. We can also assume that all y_0-changing cycles in $G - z_0$ are also z_K-changing, since otherwise swapping the weights on such a cycle would yield

a proper edge-weighting of G. This implies that y_0 is incident to at most two edges going to H and if it is exactly two, then these two edges must have different weights. Therefore y_0 must be incident to at least 4 edges going to K. We can assume that any y_0-changing cycle in $K + y_0$ is also z_K-changing and that swapping the weights on any such cycle results in $z_K z_0$ being a conflict. This implies that all edges incident to z_K which are contained in some y_0-changing cycle in $K + y_0$ have the same weight and that $d_G(z_K) \geq d_G(z_0)$, see Figure 2.13 for an illustration of a possible configuration. If there are four edges incident to y_0 going to K which have the same weight, say a, then there are two edge-disjoint y_0-changing cycles in $K + y_0$ both containing a-edges incident with y_0 and swapping the weights on both of these cycles will give a proper edge-weighting of G. Thus we can assume that there are at most three edges of the same weight incident to y_0 going to K.
Now consider the connected graph $K' = K + y_0 - z_K$. There must be exactly three edges e_1', e_2', e_3' incident to y_0 in K' having the same weight and these edges must go to three distinct components K_1', K_2', K_3' of $K' - y_0$. We can also assume that z_K is only incident to one edge going to each of K_1', K_2', K_3' and that these three edges f_1, f_2, f_3 all have the same weight (otherwise we can get a proper edge-weighting of G by swapping weights on cycles in $K_1' \cup K_2' \cup K_3' + z_K + y_0$).
Since $d_G(z_K) \geq 6$, there must be at least two more edges f_4, f_5 incident to z_K in $K' + z_K$. Let e_4' and e_5' be two edges incident to y_0 contained in two $z_K - y_0$ paths P_1, P_2 in $K' + z_K$ containing f_4 and f_5, respectively (possibly $e_4' = e_5'$). The weights of e_4' and e_5' must be distinct from the weight assigned to e_1', e_2' and e_3', so e_4' and e_5' have the same weight and thus must go to different components of $K' - y_0$. If the weight of f_4 or f_5, say of f_4, is the same as the weight of f_1, then we swap the weights on a cycle containing e_1' and e_2' (this cycle is both z_K- and y_0-changing) to avoid the conflict $y_0 z_0$ and then afterwards we swap the weights on a cycle containing e_3' and e_4' (this cycle is z_K-changing but not y_0-changing) to avoid the conflict $z_0 z_K$. Thus, we may assume that f_4 and f_5 have the same weight and that this weight is distinct from the weight of f_1, f_2, f_3. If P_1 and P_2 are internally disjoint, then swapping the weights on the cycle $P_1 \cup P_2$ yields a proper edge-weighting of G. If P_1 and P_2 are not internally disjoint, then we swap the weights on a cycle containing e_1' and e_2' and then swap the weights on the z_K-changing and y_0-avoiding cycle contained in $P_1 \cup P_2$.
$\diamond$

Note that by Claim 6 all vertices in H have degree 3 in G.

Claim 7. *There is no vertex $v \in V(H)$ such that $G - v - N(v)$ is connected.*

Proof of the claim. Suppose $v \in V(H)$ is such that $G' = G - v - N(v)$ is connected. We can assume $v \in X$. By Lemma 2.2.24, there is an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting of G, where all edges incident to v have weight b and any vertex $u \in N(v)$ is incident to at most $1 + M(uv)$ b-edges. Note that Claim 1 implies that if vv' is a multiple edge in G, then $v' \in \{z_0, y_0\}$. Thus, the only potential conflict is between v and one of z_0, y_0, say y_0. In this case y_0 must have odd degree at least

5. Since G' is connected there is a y_0-changing cycle in $G - v - (N(v) \setminus \{y_0\})$ and swapping the weights on this cycle yields a proper $\{a, b\}$-edge-weighting of G. $\diamond$

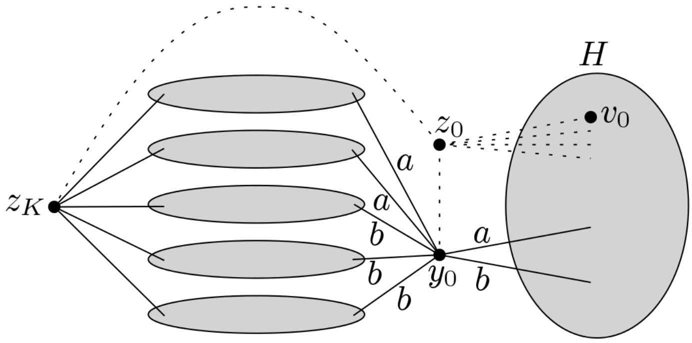


Figure 2.13: Claim 6.

Claim 8. *There are no multiple edges between two vertices in H.*

Proof of the claim. Suppose uv is a multiple edge in H. We can assume $v \in X$. Since u and v have degree 3 in G^* (by Claim 6), the multiplicity of uv is exactly 2. Let e and e' be the two edges between u and v. By Claim 1, the edges e, e' are not both white. Thus, at least one of e, e', say e, is a blue edge in H. Let v' denote neighbour of v in G which is distinct from u and not contained in a suspended path of length 3 containing u.
Let G' be obtained from $G - v$ by removing all edges incident to v' except one edge e''. Clearly G' is connected since $G^* - v - v'$ is connected by the minimality of H. Let $S = E(v') \setminus \{vv', e''\}$. Let Z denote the set of vertices in $G - v$ which are incident to an odd number of edges in S. Note that Z has even size. Thus, $X' = ((X \setminus \{v\}) \setminus Z) \cup (Z \cap Y)$ has even size and Lemma 2.2.2 implies that there is an $(X', V(G') \setminus X')$-a-parity $\{a, b\}$-edge-weighting of G'. We now extend this weighting to G by assigning weight a to all edges in S and weight b to all edges incident to v. This gives an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting of G where all vertices in $N(v)$ are incident to at most two b-edges (the edge in the suspended path of length 3 joining u and v incident to u must have weight a). Observation 2.2.19 implies that the only conflict can be vv' in the case where $v' \in \{z_0, y_0\}$, say $v' = y_0$, and y_0 has degree at least 5. But since $G - v - y_0$ is connected there must then be a y_0-changing cycle avoiding v and u. Swapping the weights on such a cycle yields a proper $\{a, b\}$-edge-weighting of G. $\diamond$

Claim 9. *Every vertex of H is incident to at most one blue edge.*

Proof of the claim. By Claim 6 every vertex v of H has degree 3. If $v \in V(H)$ is incident to at least 2 blue edges, then, by the choice of y_0, z_0 the graph $G - v - N(v)$ is connected, which contradicts Claim 7. $\diamond$

We now have all the tools at hand for completing the proof. Two cases are considered:

Case 1: *There is a vertex $v \in V(H)$ not adjacent to any of z_0, y_0 in G.*
By Claim 6 the graph $G - v - N_G(v)$ is disconnected for all $v \in V(H)$. Let v be a vertex not adjacent to z_0 or y_0 such that the component K of $G' = G - v - N(v)$ containing z_0 and y_0 has maximum order. We can assume that $v \in X$. Note that the choice of v and Claim 8 implies that there is a vertex $v' \in V(H)$ distinct from v with $N_G(v') = N_G(v)$ such that the components of G' are exactly K and the isolated vertex v'.
Let $e_1 = vv_1, e_2 = vv_2, e_3 = vv_3$ denote the three edges incident to v. Since v, v', v_1, v_2, v_3 all belong to H, Claim 8 implies that all these vertices are distinct. Furthermore, since $N_G(v) = N_G(v')$, none of the edges vv_1, vv_2, vv_3 are blue. It follows that $v, v' \in X$ and $v_1, v_2, v_3 \in Y$. Both $G - v$ and $G'' = G - v - v'v_2 - v'v_3$ are connected. Lemma 2.2.2 implies that there is an $(X \setminus \{v\} \cup \{v_2, v_3\}, Y \setminus \{v_2, v_3\})$-$a$-parity $\{a, b\}$-edge-weighting of G''. In particular, the only a-edge incident to v' in G'' is $v'v_1$. We extend this weighting to the whole of G by assigning weight a to $v'v_2, v'v_3$, and weight b to all three edges incident to v. In this way we obtain an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting of G where all three edges incident to v have weight b and all three edges incident to v' have weight a. Observation 2.2.19 implies that the only potential conflicts are between v and its neighbours. All vertices in $N(v)$ are incident to at least one a-edge (the one incident to v') so this edge-weighting is proper.

Case 2: *All vertices in H are adjacent to z_0 or y_0 in G.*
First suppose that in G^* the vertex z_0 is joined to some vertex $v \in V(H)$ by an edge of multiplicity 2. Let e' and e'' be two edges joining z_0 and v in G^*. Claim 7 implies that not both of e', e'' are blue, say e' is white. If e'' is blue, then by Claim 9, the third edge e''' incident to v in G^* must be white. If e'' is white, then Claim 7 implies that e''' is white. Thus, the edge $e''' = vu$ is white.
We can assume $v \in X$ and hence $u \in Y$. Let Z denote the set of vertices in $G - v$ which are incident to exactly one edge incident to u. Note that either Z is empty or Z has size 2. The set $X' = (X \setminus (Z \cup \{v\})) \cup (Y \cap Z)$ has even size, so by Lemma 2.2.2, there is an $(X', V(G - v - u) \setminus X')$-$a$-parity $\{a, b\}$-edge-weighting of $G - v - u$. We now extend this edge-weighting to the whole of G by assigning weight b to all edges in $E(v)$ and weight a to the two edges incident to u distinct from uv. In this way we obtain an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting of G where all edges incident to v have weight b and u is incident to exactly one b-edge. Observation 2.2.19 implies that the only potential conflict is vz_0 in the case where z_0 has degree at least 5. We can also assume that there is no z_0-changing cycle in G avoiding v and u. Hence, z_0 must have degree 2 in $G^* - H$ and be incident to an edge in G^* going to a vertex v' in H distinct from v. We can now find a z_0-changing cycle avoiding v and u unless the only neighbours of v' in B are z_0 and u. But in this case $G - u - N(u)$ is connected, contradicting Claim 7.
By the above we can assume that z_0 is not joined to any vertex in H by a multiple

edge in G^*. By symmetry y_0 is also not joined to a vertex in H by a multiple edge in G^*.
Claim 8 now implies that any vertex in H has three distinct neighbours in G^*. Let $v \in X$ be any vertex in H incident to z_0. The graph $G - v - N(v)$ is disconnected by Claim 7, so there must be a vertex v' in H which in G^* has the same neighbourhood as v. Since all vertices in H have degree 3 this implies that H only has four vertices: two joined to z_0 and two joined to y_0. The graph $G - v - (N(v) \setminus \{z_0\})$ is connected, so as above, there is an $(X \setminus \{v\}, Y \cup \{v\})$-$a$-parity $\{a, b\}$-edge-weighting of G where all edges incident to v have weight b, and the neighbours of v distinct from z_0 which have degree 3 are incident to exactly one b-edge. Observation 2.2.19 implies that the only possible conflict is vz_0 in the case where z_0 has degree at least 5. In this case, it is easy to see that there is an z_0-changing cycle avoiding H. By swapping the weights on this cycle we can get rid of this conflict and obtain a proper $\{a, b\}$-edge-weighting of G.

□

We will now move on to investigating trees without the $\{a, a+2\}$-property for any odd integer a.

2.2.4.2 Trees

As already pointed out earlier it is easy to see that a tree not isomorphic to K_2 has the $\{a, b\}$-property whenever a and b are distinct and both positive or both negative. Therefore, when investigating the $\{a, a+2\}$-property for trees where a is odd, the only interesting case is $a = -1$. In this section we will characterise the trees without the $\{-1, 1\}$-property by proving Theorem 2.2.6. Before we proceed to the proof of Theorem 2.2.6 we will prove two lemmas which will be convenient to have.

Lemma 2.2.27. *If G is a simple connected bipartite graph without the $\{-1, 1\}$-property and e is a cut-edge in G, then the deletion of e results in two components each containing an odd number of vertices.*

Proof. Suppose the lemma is false, let G be a connected bipartite graph without the $\{-1, 1\}$-property and let $e = uv \in E(G)$ be a cut-edge in G such that one of the two components of $G - e$ has an even number of vertices. Let $c : V(G) \to \{1, 2\}$ be a mod-4 vertex-colouring of G and let X, Y denote the sets of vertices coloured 1 and 2, respectively with $u \in X$. By Lemma 2.2.21 both $|X|$ and $|Y|$ are odd so G has an even number of vertices. Let C_1, C_2 denote the two components of $G - e$ with $u \in V(C_1)$ and $v \in V(C_2)$. We may assume that C_1 has an even number of vertices. Since G has an even number of vertices it follows that also C_2 has an even number of vertices. Since both $|X|$ and $|Y|$ are odd we may also assume that both $|V(C_1) \cap X|$ and $|V(C_1) \cap Y|$ are odd and both $|V(C_2) \cap X|$ and $|V(C_2) \cap Y|$ are even. In what follows we can assume that the degrees of u and v have the same parity so there are four cases to be considered. In all four cases we start with an $((X \cap V(C_1)) \setminus \{u\}, (Y \cap V(C_1)) \cup \{u\})$-$(-1)$-parity $\{-1, 1\}$-edge-weighting w_1 of C_1

(which exists by Lemma 2.2.2) and find a way to extend this edge-weighting to a proper edge-weighting of the whole of G.

Case 1: *Both u and v have odd degree and colour 1.*
By Lemma 2.2.2 there is an $(Y \cap V(C_2), X \cap V(C_2))$-1-parity $\{-1,1\}$-edge-weighting of C_2. It follows from Observation 2.2.19 that this edge-weighting together with w_1 and assigning weight -1 to e, forms a proper edge-weighting of the whole G.

Case 2: *Both u and v have even degree and colour 1.*
By Lemma 2.2.2 there is an $(X \cap V(C_2), Y \cap V(C_2))$-1-parity $\{-1,1\}$-edge-weighting of C_2. It follows from Observation 2.2.19 that this edge-weighting together with w_1 and assigning weight -1 to e, forms a proper edge-weighting of the whole G.

Case 3: *Both u and v have odd degree and v has colour 2.*
By Lemma 2.2.2 there is an $(Y \cap V(C_2), X \cap V(C_2))$-1-parity $\{-1,1\}$-edge-weighting of C_2. It follows from Observation 2.2.19 that this edge-weighting together with w_1 and assigning weight -1 to e, forms a proper edge-weighting of the whole G.

Case 4: *Both u and v have even degree and v has colour 2.*
By Lemma 2.2.2, there is an $(X \cap V(C_2), Y \cap V(C_2))$-1-parity $\{-1,1\}$-edge-weighting of C_2. It follows from Observation 2.2.19 that this edge-weighting together with w_1 and assigning weight -1 to e, forms a proper edge-weighting of the whole G.

□

Lemma 2.2.28. *If G is a connected bipartite graph without the $\{-1,1\}$-property and e is a cut-edge in G, then there is a $\{-1,1\}$-edge-weighting of G such that e is the only conflict.*

Proof. Let G be a connected bipartite graph without the $\{-1,1\}$-property containing a cut-edge e and let C_1, C_2 be the two components of $G-e$. Let c be a mod-4 vertex-colouring of G and let X, Y denote the sets of vertices coloured 1 and 2, respectively. By Lemma 2.2.21 both X and Y have odd size and by Lemma 2.2.27 we can assume that both $|X \cap V(C_1)|$ and $|Y \cap V(C_2)|$ are even and both $|Y \cap V(C_1)|$ and $|X \cap V(C_2)|$ are odd. Now Lemma 2.2.2 implies that there is an $(X \cap V(C_1), Y \cap V(C_1))$-1-parity $\{-1,1\}$-edge-weighting of C_1 and an $(Y \cap V(C_2), X \cap V(C_2),)$-1-parity $\{-1,1\}$-edge-weighting of C_2. Observation 2.2.19 implies that these two edge-weightings, together with assigning weight -1 to the edge e, is a $\{-1,1\}$-edge-weighting of G where e is the only potential conflict. □

With the above lemmas in hand we are ready for the characterisation of trees without the $\{-1,1\}$-property. Recall that Theorem 2.2.6 states that a tree does not have the $\{-1,1\}$-property if and only if it can be constructed from a disjoint union of graphs isomorphic to K_2 by repeated applications of the operation (a)-(b) in Figure 2.3.

Proof of Theorem 2.2.6. We refer to operation (a)-(b) in Figure 2.3 as Operation 1. It is straightforward to check that a graph G constructed by Operation 1 from four graphs without the $\{-1,1\}$-property does not have the $\{-1,1\}$-property itself: In any proper $\{-1,1\}$-edge-weighting of G all five edges incident to the vertices $v_1 \sim v_2$ and $v_3 \sim v_4$ must have the same weight, since otherwise the proper $\{-1,1\}$-edge-weighting of G would yield one of at least one of the four graphs used in the construction. Thus v_1v_3 will be a conflict.
By the above it suffices to prove that any tree without the $\{-1,1\}$-property is constructed from a disjoint union of K_2's by repeated applications of Operation 1. Suppose this is false and let T be a counterexample with minimum size. Note that Lemma 2.2.27 implies that, for any vertex $v \in V(T)$ and any edge $e \in E(v)$, the component C_e not containing v in $T-e$ has an odd number of vertices. We can write $|V(T)| = 1 + \sum_{e\in E(v)} |V(C_e)|$ for any vertex $v \in V(T)$ and since $|V(T)|$ is even, this implies that all vertices in T have odd degree.
Let $P = v_1 \cdots v_m$ be a longest path in T. Clearly, all neighbours of v_{m-1} except v_{m-2} are leaves and since all vertices have odd degree the vertex v_{m-1} is incident to an even number n of leaves. First suppose $n \geq 4$ and let $u_1, \dots, u_n$ be the leaves incident to v_{m-1}, with $u_1 = v_m$. Since $T' = T - \{u_1, \dots, u_{n-1}\}$ has an odd number of vertices Lemma 2.2.21 implies that T' has a proper $\{-1,1\}$-edge-weighting. We can now obtain a proper $\{-1,1\}$-edge-weighting of T by possibly changing the weight of $v_{m-1}u_n$ and assigning weights to the edges in $E(v_{m-1}) \setminus \{v_{m-1}v_{m-2}, v_{m-1}u_n\}$ such that all the edges $v_{m-1}u_1, \dots, v_{m-1}u_n$ have the same weight (we choose whether this weight is 1 or -1 so that we avoid the conflict $v_{m-2}v_{m-1}$).
By the above we can assume that v_{m-1} has degree exactly 3. It follows from this and the maximality of P that any neighbour of v_{m-2} distinct from v_{m-3} and v_{m-1} is either a leaf or a vertex of degree 3 adjacent to two leaves. Let $U' = \{u'_1, \dots, u'_p\}$ be the set of leaves adjacent to v_{m-2} and let $U'' = \{u''_1, \dots, u''_q\}$ be the set of neighbours of v_{m-2} distinct from v_{m-1} and v_{m-3} which have degree 3. Possibly $p = 0$ or $q = 0$, but $p + q$ is odd since v_{m-2} has odd degree. Let T_1 and T_2 be the two components of $T - v_{m-3}v_{m-2}$ such that $v_{m-2} \in V(T_2)$. By Lemma 2.2.28, there is a $\{-1,1\}$-edge-weighting w of T such that the only potential conflict is $v_{m-3}v_{m-2}$. By possibly multiplying all edge-weights by -1, we can assume that the weight of $v_{m-3}v_{m-2}$ is 1. We look at three separate cases:

Case 1: $p + q \geq 5$.
By possibly modifying the weights of the edges in $E(T_2)$ such that they all have weight 1 or -1, the vertex v_{m-2} can obtain weighted degree $2 + p + q$ or $-p - q$. Now we simply pick the one of these two options such that $v_{m-3}v_{m-2}$ is not a conflict. Since all vertices in T_2 except v_{m-2} have degree at most 3, this gives a proper $\{-1,1\}$-edge-weighting of T.

Case 2: $p + q = 3$.
As in Case 1, we can modify the edge-weights such that the vertex v_{m-2} can obtain weighted degree $2+p+q = 5$ or $-p-q = -3$. Since all vertices in T_2 except v_{m-2}

have degree at most 3, we can in this way find a proper $\{-1,1\}$-edge-weighting of T, unless v_{m-3} has weighted degree 5. So we can assume that v_{m-3} has weighted degree 5. If $p \in \{1,3\}$, then we modify the weights in T_2 such that all edges incident with $u'_1, \ldots, u'_p$ have weight 1 and all other edges in T_2 have weight -1. If $p = 2$, then we modify the weights in T_2 such that all edges incident with $u'_1, \ldots, u'_p$ have weight -1 and all other edges in T_2 have weight 1. This yields a proper $\{-1,1\}$-edge-weighting of T, so we can assume $p = 0$ and $q = 3$. In this case, we modify the weights in T_2 such that all edges incident to v_{m-1} and u''_1 have weight 1 and all other edges in T_2 have weight -1. This yields a proper $\{-1,1\}$-edge-weighting of T.

Case 3: $p + q = 1$.

First suppose $q = 1$ and $p = 0$. If we modify the edge weights in T_2 such that they all have weight -1, then we obtain a proper $\{-1,1\}$-edge-weighting of T, unless v_{m-3} has weighted degree -1. In this case, we change the weights of the three edges incident to v_{m-1} to 1 to obtain a proper $\{-1,1\}$-edge-weighting of T. Thus, we can assume $p = 1$ and $q = 0$. We can assume that $T''' = T - v_m - v_{m-1} - u_2 - u'_1$ has a proper $\{-1,1\}$-edge-weighting w, since otherwise, the minimality of T implies that T''' is constructed from a disjoint union of K_2's by repeated (possibly none) applications of Operation 1, and then so is T. By possibly multiplying all edge weights of w by -1, we can assume that $v_{m-3}v_{m-2}$ has weight 1. Now assigning weight 1 to all edges incident to v_{m-1} and weight -1 to $v_{m-2}u'_1$ yields a proper $\{-1,1\}$-edge-weighting of T. □

This concludes the section about neighbour sum-distinguishing edge-weightings of bipartite graphs. We will now move on to consider weight-choosability of general graphs.

2.3 Weight-Choosability of Graphs

The material presented in this section essentially consists of one research article [Lync].

The goal of this section is to provide an upper bound on weight-choosability which is logarithmic in the maximum degree. More precisely, we will prove that any graph G which has no components isomorphic to K_2 is $(1, 2\lceil\log_2(\Delta(G))\rceil + 1)$-choosable. Thus, we will also allow a list of size 1 assigned to each vertex of G, which means that each vertex has a prescribed weight. This result will be implied by a slightly stronger and more technical result: Given a graph G and an assignment of 1-element lists to the vertices $L_v \subset \mathbb{R}$, $v \in V(G)$ and an assignment of lists to the edges $L_e \subset \mathbb{R}$, $e \in E(G)$, we will show that if G has no components isomorphic to K_2 and if $|L_e| \geq \lceil\log_2(d(u))\rceil + \lceil\log_2(d(v))\rceil + 1$ for each $e = uv \in E(G)$, then there exists a proper total weighting $w : V(G) \cup E(G) \to \mathbb{R}$ such that $w(v) \in L_v$ for each $v \in V(G)$ and $w(e) \in L_e$ for each $e = uv \in E(G)$.
Given a graph G and a function $\phi : E(G) \to \mathbb{N}$ we say that G is $(1, \phi)$-*choosable* if for any assignment of 1-element lists to the vertices $L_v \subset \mathbb{R}$, $v \in V(G)$ and any assignment of lists to the edges $L_e \subset \mathbb{R}$, $e \in E(G)$ satisfying $|L_e| \geq \phi(e)$, there exists a proper total weighting $w : V(G) \cup E(G) \to \mathbb{R}$ such that $w(v) \in L_v$ for each $v \in V(G)$ and $w(e) \in L_e$ for each $e = uv \in E(G)$. With this definition we can state the main theorem of this section as follows.

Theorem 2.3.1. *Any graph G without a component isomorphic to K_2 is $(1, \phi)$-choosable when $\phi : E(G) \to \mathbb{N}$ is defined by $\phi(uv) = \lceil\log_2(d(u))\rceil + \lceil\log_2(d(v))\rceil + 1$ for $uv \in E(G)$.*

The proof of Theorem 2.3.1 consists of formulating an algorithm which will find appropriate edge-weights. The algorithm will consist of two parts. The first part of the algorithm assigns some edge-weights in a greedy way while keeping track of potential conflicts. The second part of the algorithm repairs conflicts which may be present after the first part. Since the algorithm consists of a number of iterations and there are technical details and several case-distinctions in each iteration, the two parts of the algorithm are written in pseudo-code. Before presenting the algorithm in pseudo-code the proof will start with an overview of the algorithm and the notation used in the procedures of the algorithm.

Proof of Theorem 2.3.1. Let G be a graph with no components isomorphic to K_2, with n vertices, and with m edges $e_1, \ldots, e_m$. Let an assignment of 1-element lists to the vertices $L_v \subset \mathbb{R}$, $v \in V(G)$ and an assignment of lists to the edges $L_e \subset \mathbb{R}$, $e \in E(G)$ satisfying $|L_e| \geq \phi(e)$ be given. Clearly we can assume that $|L_e| = \phi(e)$ for each $e \in E(G)$.
For any vertex v let s_v denote the weight making up the list of size 1 assigned to v and for $j = 1, \ldots, m$ let $L_j = \{t_{j,1}, \ldots, t_{j,\phi(e_j)}\}$ be the list assigned to e_j. We can assume that the ordering of each L_j is such that $t_{j,1} < \cdots < t_{j,\phi(e_j)}$.
Through some number of steps $i = 0, \ldots, k+1 \leq n+1$ we will recursively construct

a sequence of edge-weightings $w_i : E(G) \to \mathbb{R}$ where each w_{i+1} will be a modification of w_i and where w_{k+1} will be our final edge-weighting. All the edge-weightings w_i will satisfy $w_i(e) \in L_e$ for each $e \in E(G)$. For each $i \in \{0, \dots, k+1\}$ and for each vertex $v \in V(G)$ let $C_{w_i}(v) = s_v + \sum_{e \in E(v)} w_i(e)$.
A *step* or an *iteration* in the algorithm is when we go from considering w_i to considering w_{i+1}, so the algorithm will consist of $k+1$ steps/iterations. In each step we will define a set of edges whose weights will never be changed again. This will define a sequence of edge sets $\emptyset = E_0 \subset E_1 \subset \cdots \subset E_{k+1} = E(G)$.
For each edge $e_j = uv$ and each step i of the algorithm we define three values

$$f_{u,i}(e_j) \in [0, \lceil \log_2(d(u)) \rceil],$$

$$f_{v,i}(e_j) \in [0, \lceil \log_2(d(v)) \rceil],$$

$$f_i(e_j) = f_{u,i}(e_j) + f_{v,i}(e_j) + 1.$$

If nothing else is explicitly stated it will always be the case that

$$f_{u,i}(e_j) = f_{u,i-1}(e_j),$$

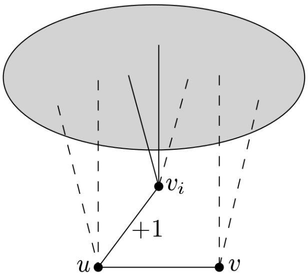
$$f_{v,i}(e_j) = f_{v,i-1}(e_j),$$

$$f_i(e_j) = f_{u,i}(e_j) + f_{v,i}(e_j) + 1.$$

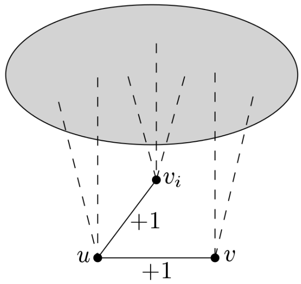
During the first k steps of the algorithm we will also define a sequence of subsets of $V(G) \times E(G)$: $\emptyset = T_0 \subset T_1 \subset \cdots \subset T_k$. Each element (v', uv) of T_k will represent a triangle $v'uv$ in the graph. These triangles will be defined such that the only possible conflicts in the whole graph after the k first steps of the algorithm are $v'u$ and $v'v$ whenever $(v', uv) \in T_k$. The potential conflicts will then be disposed of in the last part of the algorithm.
In each of the first k steps of the algorithm we will also define a sequence of vertex sets $\emptyset = V_0 \subset V_1 \subset \cdots \subset V_k$. We will do this by, in each step i, extending V_{i-1} to V_i by adding at most four vertices to V_{i-1}.
As mentioned the algorithm consists of two parts. These parts are explained in details below. The first part is Procedure 1 and the second part is Procedure 2. Procedure 1 is a greedy way to assign edge-weights and allows us to keep track of potential conflicts by "saving" them in triangles. When we are in a step i of Procedure 1, then for a vertex $v \in V(G) \setminus V_{i-1}$, the number $C_{w_{i-1}}(v)$ is called the *potential* of v.
The conflicts remaining after Procedure 1 will be disposed of in Procedure 2.

Procedure 1 Greedy weight-choosing

Define $i = 1$, $E_0 = \emptyset$, $V_0 = \emptyset$, $T_0 = \emptyset$, $f_{u,0}(e_j) = f_{v,0}(e_j) = 0$, and $w_0(e_j) = t_{j,f_0(e_j)}$ for all $e_j \in E(G)$.
while $E_i \neq E(G)$ **do**
  Choose a vertex v_i in the set $V(G) \setminus V_{i-1}$ minimizing $C_{w_{i-1}}(v_i)$ and subject to that, incident to the fewest number of edges in $E(G) \setminus E_{i-1}$.
  if $G - (E_{i-1} \cup E(v_i))$ contains no isolated edge uv where $C_{w_{i-1}}(u) = C_{w_{i-1}}(v)$ **then**
    Define $V_i = V_{i-1} \cup \{v_i\}$ and $E_i = E_{i-1} \cup E(v_i)$ and $T_i = T_{i-1}$.
    for each edge $v_i v$ in $E(v_i) \setminus E_{i-1}$ **do**
      if $E(v) \setminus E_i \neq \emptyset$ **then**
        Choose an edge e in $E(v) \setminus E_i$ minimizing $f_{v,i-1}(e)$ and define
        $f_{v,i}(e) = f_{v,i-1}(e) + 1$.
    for any edge $e_j \in E(G)$ **do**
      Define $w_i(e_j) = t_{j,f_i(e_j)}$.
  if $G - (E_{i-1} \cup E(v_i))$ contains an isolated edge uv where $C_{w_{i-1}}(u) = C_{w_{i-1}}(v)$ **then**
    if u is adjacent to v_i and v is not adjacent to v_i (as in Figure 2.14) **then**
      Define $V_i = V_{i-1} \cup \{v\}$ and $E_i = E_{i-1} \cup E(v)$ and $T_i = T_{i-1}$.
      Define $f_{u,i}(v_i u) = f_{u,i-1}(v_i u) + 1$.
      for any edge $e_j \in E(G)$ **do**
        Define $w_i(e_j) = t_{j,f_i(e_j)}$.
      if $C_{w_i}(v_i) = C_{w_i}(u)$ and uv_i is an isolated edge in $G - E_i$ **then**
        Define $f_{u,i}(uv) = f_{u,i-1}(uv) + 1$.
    if both u and v are adjacent to v_i (as in Figure 2.15) **then**
      if v_i is not incident to an isolated edge $v_i v'$ in $G-(E_{i-1} \cup \{uv, v_i u, v_i v\})$ **then**
        $V_i = V_{i-1} \cup \{u, v\}$, $E_i = E_{i-1} \cup \{uv, v_i u, v_i v\}$, $T_i = T_{i-1} \cup \{(v_i, uv)\}$.
        Define $f_{u,i}(v_i u) = f_{u,i-1}(v_i u) + 1$.
        for any edge $e_j \in E(G)$ **do**
          Define $w_i(e_j) = t_{j,f_i(e_j)}$.
      if v_i is incident to an isolated edge $v_i v'$ in $G - (E_{i-1} \cup \{uv, v_i u, v_i v\})$ **then**
        Define $V_i = V_{i-1} \cup \{u, v, v_i, v'\}$, $E_i = E_{i-1} \cup \{uv, v_i u, v_i v, v_i v'\}$, and $T_i = T_{i-1} \cup \{(v_i, uv)\}$.
        Define $f_{u,i}(v_i u) = f_{u,i-1}(v_i u) + 1$.
        if now $C_{w_i}(v_i) = C_{w_i}(v')$ **then**
          Redefine $f_{u,i}(v_i u) = f_{u,i-1}(v_i u) + 2$.
        for any edge $e_j \in E(G)$ **do**
          Define $w_i(e_j) = t_{j,f_i(e_j)}$.
  Replace i with $i + 1$.

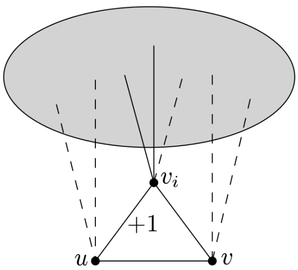


(a) The case in line 13 in Procedure 1.

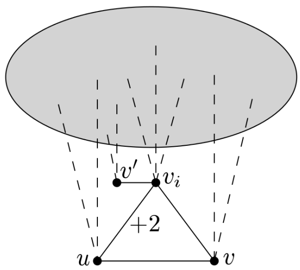


(b) The case in line 18 in Procedure 1.

Figure 2.14: Two special cases in Procedure 1. Dashed edges indicates edges in E_{i-1}



(a) The case in line 21 in Procedure 1.



(b) The case in line 26 in Procedure 1.

Figure 2.15: Two special cases in Procedure 1. Dashed edges indicates edges in E_{i-1}

When Procedure 1 terminates we have a well-defined edge-weighting $w_k : E(G) \to \mathbb{R}$ and a set $T_k \subset V(G) \times E(G)$ representing some triangles in G.
Let $(u_1, e'_1), \ldots, (u_{|T_k|}, e'_{|T_k|})$ denote the elements of T_k enumerated in the order they appeared in Procedure 1. Note that when we repair conflicts in Procedure 2 below, we consider the triangles in T_k in reverse order starting with $(u_{|T_k|}, e'_{|T_k|})$.
When Procedure 2 terminates we have an edge-weighting w_{k+1} of G and it remains to show that for any pair of adjacent vertices u, v we have $C_{w_{k+1}}(u) \neq C_{w_{k+1}}(v)$ and that $f_{k+1}(e) \leq \phi(e)$ holds for any edge $e \in E(G)$.

Procedure 2 Finalisation (see Figure 2.18).

for $i = |T_k| \dots 1$ **do**
  Define $(v', uv) = (u_i, e'_i)$.
  if one of u, v, say, v has the same colour as v' **then**
    Define $f_{v,k+1}(uv) = f_{v,k}(uv) + 1$.
  if now u has the same colour as v' **then**
    Define $f_{v,k+1}(uv) = f_{v,k}(uv) + 2$.
  for any edge $e_j \in E(G)$ **do**
    Define $w_{k+1}(e'_j) = t_{j,f_{k+1}(e'_j)}$.

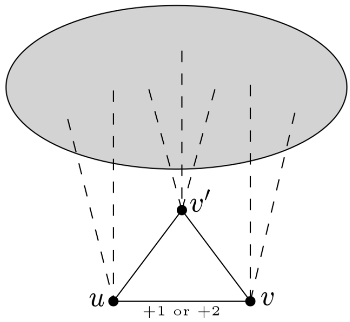


Figure 2.16: An illustration of Procedure 2.

First we prove that for any edge uv we have $C_{w_{k+1}}(u) \neq C_{w_{k+1}}(v)$. To do this we look at three different cases:

1. $(v', uv) \notin T_k$ for all $v' \in V(G)$ and $(u, e') \notin T_k$ and $(v, e') \notin T_k$ for all $e' \in E(u) \cup E(v)$.
2. $(v', uv) \in T_k$ for some $v' \in V(G)$.
3. $(u, e') \in T_k$ or $(v, e') \in T_k$ for some $e' \in E(u) \cup E(v)$.

Case 1:
We split the analysis into two separate subcases.

Case 1.1: For some $i \leq k$ the edge uv is isolated in $G - E_i$.
Let $i \leq k$ be the smallest index such that uv is an isolated edge in $G - E_i$. In a later step of Procedure 1 one of u, v, say u, is chosen as the vertex with minimum potential. That is, for some smallest $i' > i$ we have $u = v_{i'}$, $v \notin V_{i'}$ and $u \notin V_{i'-1}$. Since uv is an isolated edge in $G - E_i$ and hence also in $G - E_{i'-1}$ it follows from lines 4-11 in Procedure 1 that in the i''th step of Procedure 1 no edge-weights changed and $E_{i'} = E_{i'-1} \cup \{uv\}$. Also the weight of uv does not change during Procedure 2. Thus,

$C_{w_i}(u) = C_{w_k}(u) = C_{w_{k+1}}(u)$ and $C_{w_i}(v) = C_{w_k}(v) = C_{w_{k+1}}(v)$, so it suffices to show that $C_{w_i}(u) \neq C_{w_i}(v)$. If the if-statement in line 4 of Procedure 1 was satisfied in the i'th step, then $C_{w_i}(u) \neq C_{w_i}(v)$ follows immediately, so we can assume that the if-statement in line 12 was satisfied in the i'th step. Furthermore, if the if-statement in line 20 was satisfied, then it follows from the lines 20-32, that any isolated edge in $G - E_i$ is also an isolated edge in $G - E_{i-1}$ and this contradicts the choice of i. Thus, we can assume that the if-statement in line 13 was satisfied in the i'th step of Procedure 1. Now it follows from lines 13-19 in Procedure 1 that $C_{w_i}(u) \neq C_{w_i}(v)$.

Case 1.2: For all $i \leq k$ the edge uv is not isolated in $G - E_i$.
Let $i \leq k$ be the smallest index such that $uv \in E_i$. We can assume that $v \notin V_{i-1}$, $v \in V_i$ and $u \notin V_{i-1}$. If also $u \in V_i$, then since $(v', uv) \notin T_k$ for all $v' \in V(G)$, it follows from Procedure 1 that the if-statements in lines 12, 20, and 26 were satisfied in the i'th loop of Procedure 1 and that uv is a pendant edge in a component of $G - E_{i-1}$ which is isomorphic to a triangle with a pendant edge added, see Figure 2.15. In this case it follows from lines 26-32 in Procedure 1 that $C_{w_i}(u) \neq C_{w_i}(v)$. Since $E(u) \cup E(v) \subset E_i$ this implies that $C_{w_k}(u) \neq C_{w_k}(v)$. Furthermore, since $(v', uv) \notin T_k$ for all $v' \in V(G)$ and $(u, e') \notin T_k$ and $(v, e') \notin T_k$ for all $e' \in E(u) \cup E(v)$, the weighted degrees of u or v do not change in Procedure 2 and hence $C_{w_{k+1}}(u) \neq C_{w_{k+1}}(v)$.
By the above we can assume $u \notin V_i$ and since $(v', uv) \notin T_k$ for all $v' \in V(G)$ and $(u, e') \notin T_k$ and $(v, e') \notin T_k$ for all $e' \in E(u) \cup E(v)$ we can assume that either the if-statement in line 4 or both the if-statements in lines 12 and 13 in Procedure 1 were satisfied in the i'th step of Procedure 1. If the if-statement in line 4 was satisfied then $C_{w_i}(v) < C_{w_i}(u)$ follows from lines 4-11 since uv is not an isolated edge in $G - E_{i-1}$. Also, if the if-statements in lines 12 and 13 were satisfied $C_{w_i}(v) < C_{w_i}(u)$ follows from lines 12-17 (note that the if-statement in line 18 was not satisfied in the i'th step as uv is not isolated in $G - E_i$ for any $i \leq k$). Now we have that $C_{w_i}(v) < C_{w_i}(u)$ and hence $C_{w_{k+1}}(v) = C_{w_i}(v) < C_{w_i}(u) \leq C_{w_{k+1}}(u)$.

Case 2:
Let i be the smallest index such that $(v', uv) \in T_i$ for some $v' \in V(G)$. Since we added (v', uv) to T_{i-1} we have $C_{w_{i-1}}(u) = C_{w_{i-1}}(v)$. By lines 20-32 in Procedure 1 we increased the value of $C_{w_{i-1}}(u)$ to make sure that $C_{w_i}(u) \neq C_{w_i}(v)$ and never changed these two values before Procedure 2. It follows from the lines 2-6 in Procedure 2 that we can only change the value of $w_k(uv)$, but not $w_k(uv')$ or $w_k(vv')$ in step $k+1$. Thus, we have that

$$C_{w_{k+1}}(u) = C_{w_i}(u) - w_i(uv) + w_{k+1}(uv) \neq C_{w_i}(v) - w_i(uv) + w_{k+1}(uv) = C_{w_{k+1}}(v).$$

Case 3:
Assume that $(u, e') \in T_k$ and $e' = vv'$. At some point in Procedure 2 the triangle (u, e') is considered. Note that there might exist a vertex u' and an edge e'' incident to u such that $(u', e'') \in T_k$. If this is the case then that triangle (u', e'') appeared later than (u, e') in Procedure 1 and is therefore considered earlier than (u, e') in

Procedure 2 (see Figure 2.17). This implies that at the time Procedure 2 reaches (u, e') and throughout the rest of Procedure 2 the weighted degree of u does not change. By lines 2-6 in Procedure 2 we changed the value of $w_k(e')$ ensuring $C_{w_{k+1}}(u) \neq C_{w_{k+1}}(v)$ as well as $C_{w_{k+1}}(u) \neq C_{w_{k+1}}(v')$. So $C_{w_{k+1}}(u) \neq C_{w_{k+1}}(v)$.

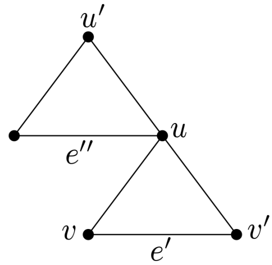


Figure 2.17: How two triangles (u'', e'') and (u, e') in T_k can appear in G. In this case (u'', e'') will be considered before (u, e') in Procedure 2.

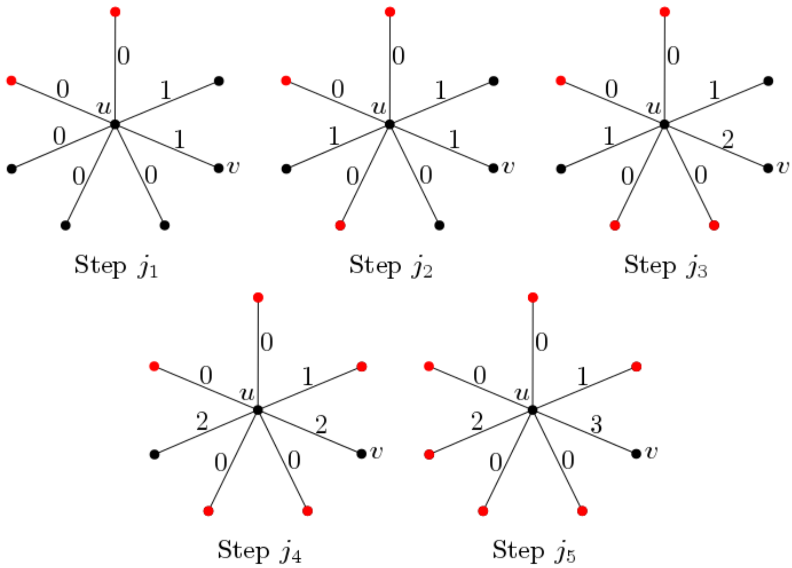


Figure 2.18: An illustration of how edge-weights can increase during Procedure 1. The five graphs illustrate the same vertices in five different steps $j_1, \ldots, j_5$ of the algorithm. A number on an edge e indicates how many times $f_u(e)$ has been increased and the red colour indicates vertices belonging to $V_{j_1}, \ldots, V_{j_5}$. The five shown steps illustrate how the neighbours of u are, one by one, added into $V_{j_1}, \ldots, V_{j_5}$ in such a way that $f_u(uv)$ is increased as many times as possible. This can be thought of as a worst case scenario for $f_u(uv)$.

It remains to show that $f_{k+1}(e) \leq \phi(e) = \lceil \log_2(d(u)) \rceil + \lceil \log_2(d(v)) \rceil + 1$ holds for any edge $e = uv$ in G. This time we also look at the three different cases mentioned above.

Case 1:
Let ℓ be the smallest index such that $e = uv \in E_\ell$. We may without loss of generality assume $v \notin V_{\ell-1}$, $v \in V_\ell$ and $u \notin V_{\ell-1}$. We start by looking at how large $f_{u,\ell-1}(e)$ can possibly be. This is the number of times $f_{u,i}(e)$ (for $i = 0, \dots, \ell - 1$) has increased during Procedure 1 before step ℓ. Suppose we increase $f_{u,i-1}(e)$ in the steps $i_1, i_2, \dots, i_{f_{u,\ell-1}(e)}$. Since we are interested in an upper bound for $f_{u,\ell-1}(e)$ we may assume that in any step j' where Procedure 1 chose a vertex in $N(u)$ as $v_{j'}$ and e minimised $f_{u,j'-1}(x)$ for $x \in E(u) \setminus E_{j'}$, the edge e was chosen (even if there where multiple minimizers) in line 8 in Procedure 1. Note that this implies that in each of the steps i_j for $j \in \{1, \dots, f_{u,\ell-1}(e)\}$ the term $f_{u,i_j-1}(x)$ is constant for $x \in E(u) \setminus E_{i_j}$.
In step i_1 a vertex in $N(u)$ was picked as v_{i_1} and put into V_{i_1} and $f_{u,i_1-1}(e)$ was increased by 1. Note that by the above we can assume that $V_{i_1} \cap N(u) = \{v_{i_1}\}$. In step i_2 another vertex in $N(u)$ was picked as v_{i_2} and $f_{u,i_2-1}(e)$ was increased because $f_{u,i_2-1}(x)$ was constant for $x \in E(u) \setminus E_{i_2}$. Since $f_{u,i_2-1}(e) = 1$ it follows that at least $\left\lfloor \frac{d(u)}{2} \right\rfloor$ of the edges incident to u were in E_{i_2-1}, see Figure 2.18. Similarly, for step i_3 we have $|(E(u) \setminus E_{i_2}) \cap E_{i_3-1}| \geq \left\lfloor \frac{|E(u) \setminus E_{i_2}|}{2} \right\rfloor$. Hence

$$\begin{aligned}
|E(u) \cap E_{i_3-1}| &= |E(u) \cap E_{i_2}| + |(E(u) \setminus E_{i_2}) \cap E_{i_3-1}| \\
&\geq |E(u) \cap E_{i_2}| + \left\lfloor \frac{|E(u) \setminus E_{i_2}|}{2} \right\rfloor \\
&= |E(u) \cap E_{i_2-1}| + 1 + \left\lfloor \frac{|E(u) \setminus E_{i_2-1}| + 1}{2} \right\rfloor \\
&\geq \left\lfloor \frac{d(u)}{2} \right\rfloor + 1 + \left\lfloor \frac{\left\lfloor \frac{d(u)}{2} \right\rfloor + 1}{2} \right\rfloor \\
&\geq \left\lfloor \frac{d(u)}{2} \right\rfloor + \left\lfloor \frac{\frac{d(u)}{2}}{2} \right\rfloor \\
&= \sum_{r=1}^{2} \left\lfloor \frac{d(u)}{2^r} \right\rfloor .
\end{aligned}$$

We continue counting in this way and we get the following for all $j \in \{1, \dots, f_{u,\ell-1}(e)\}$:

$$|E(u) \cap E_{i_j-1}| \geq \sum_{r=1}^{j-1} \left\lfloor \frac{d(u)}{2^r} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{d(u)}{2^{j-1}} \right\rfloor > 0.$$

Furthermore, note that for all $j \in \{1, \ldots, f_{u,\ell-1}(e)\}$ we have $|E(u) \cap E_{i_j-1}| < d(u) - 1$ since $uv \notin E_{i_j-1}$ and $uw \notin E_{i_j-1}$ for some $w \in N(u) \setminus \{v\}$ (where $w \in N(u)$ is the vertex we choose to put into V_{i_j} in step i_j). Thus we have

$$\sum_{r=1}^{f_{u,\ell-1}(e)-1} \left\lfloor \frac{d(u)}{2^r} \right\rfloor < d(u) - 1,$$

which together with $\left\lfloor \frac{d(u)}{2^{f_{u,\ell-1}(e)-1}} \right\rfloor > 0$ implies $f_{u,\ell-1}(e) \leq \lceil \log_2(d(u)) \rceil$. We can repeat the above analysis for $f_{v,\ell-1}(e)$ and get $f_{v,\ell-1}(e) \leq \lceil \log_2(d(v)) \rceil$. If none of $f_{u,\ell-1}(e)$, $f_{v,\ell-1}(e)$ increase in step ℓ of Procedure 1 we now get

$$f_{k+1}(e) = f_{\ell-1}(e) = f_{u,\ell}(e) + f_{v,\ell-1}(e) + 1 \leq \lceil \log_2(d(u)) \rceil + \lceil \log_2(d(v)) \rceil + 1 = \phi(e).$$

Thus, we may assume that one of $f_{u,\ell-1}(e)$, $f_{v,\ell-1}(e)$, say $f_{u,\ell-1}(e)$ increases in step ℓ of Procedure 1. Since $(u, e') \notin T_k$ and $(v, e') \notin T_k$ for all $e' \in E(u) \cup E(v)$ it must be that the if-statement in lines 12, 13 and 18 were satisfied in the ℓ'th step of Procedure 1 and u is a vertex of degree 2 in $G - E_{\ell-1}$ and v is a vertex of degree 1 in $G - E_{\ell-1}$. In this case we have $|E(u) \cap E_{i_j-1}| < d(u) - 2$ for all $j \in \{1, \ldots, f_{u,\ell-1}(e)\}$ and so we get:

$$\sum_{r=1}^{f_{u,\ell-1}(e)-1} \left\lfloor \frac{d(u)}{2^r} \right\rfloor < d(u) - 2,$$

which together with $\left\lfloor \frac{d(u)}{2^{f_{u,\ell-1}(e)-1}} \right\rfloor > 0$ implies $f_{u,\ell-1}(e) \leq \lceil \log_2(d(u)) \rceil - 1$. Hence

$$\begin{aligned} f_{k+1}(e) = f_\ell(e) &= f_{u,\ell}(e) + f_{v,\ell}(e) + 1 \\ &= f_{u,\ell-1}(e) + 1 + f_{v,\ell-1}(e) + 1 \\ &\leq \lceil \log_2(d(u)) \rceil - 1 + 1 + \lceil \log_2(d(v)) \rceil + 1 \\ &= \phi(e). \end{aligned}$$

Case 2:
Let i be the smallest index such that $(v', uv) \in T_i$ for some $v' \in V(G)$. As in Case 1, since $|E(u) - E_{i-1}| = 2$ we have $f_{u,k}(e) = f_{u,i-1}(e) \leq \lceil \log_2(d(u)) \rceil - 1$. Similarly $f_{v,k}(e) = f_{v,i-1}(e) \leq \lceil \log_2(d(v)) \rceil - 1$, thus $f_k(e) \leq \lceil \log_2(d(u)) \rceil + \lceil \log_2(d(v)) \rceil - 1$. Within Procedure 2, we increase the weight of uv at most twice, so we have that $f_{k+1}(e) \leq \lceil \log_2(d(u)) \rceil + \lceil \log_2(d(v)) \rceil + 1 \leq \phi(e)$.

Case 3:
In this case we may assume there is a vertex v' and an edge $e' = vv'$ such that $(u, e') \in T_k$. Let i be the step in Procedure 1 where we put v and v' together into V_i. At this step in Procedure 1 it follows from the same arguments as in Case 1 that $f_{u,i-1}(e) \leq \lceil \log_2(d(u)) \rceil - 1$ as well as $f_{v,i-1}(e) \leq \lceil \log_2(d(v)) \rceil - 1$, which means $f_{i-1}(e) \leq \lceil \log_2(d(u)) \rceil + \lceil \log_2(d(v)) \rceil - 1$. Furthermore, in step i we increase $f_{i-1}(e)$ at most twice and never change its value afterwards, thus $f_{k+1}(e) \leq \phi(e)$. □

2.4 A Local Antimagic Theorem

The material presented in this section essentially consists of one research article [Lyn18b].

The Antimagic Labelling Conjecture which was formulated in 1998 by Hartsfield and Ringel [Har90] states that there is a sum-distinguishing edge-weighting of any nice graph G using weights in the set $\{1, \ldots, |E(G)|\}$ and only using each weight once. The Local Antimagic Labelling Conjecture is a weaker version of this conjecture and was formulated by Bensmail et al. [Ben17] and independently by Arumugam et al. [Aru17]. The Local Antimagic Labelling Conjecture states that there is a neighbour sum-distinguishing edge-weighting of any nice graph G using weights in the set $\{1, \ldots, |E(G)|\}$ and only using each weight once. So in this local version of the Antimagic Labelling Conjecture we only want neighbouring vertices to have distinct weighted degrees whereas in the original conjecture all vertices must have different weighted degrees. The Local Antimagic Labelling Conjecture was proved to be true by Haslegrave [Has18] using a probabilistic argument. In this section we will prove a generalised list-version of the Local Antimagic Labelling Conjecture where instead of using the weights $\{1, \ldots |E(G)|\}$ when weighting the edges of a graph G we use the weights in an arbitrary set of $|E(G)|$ distinct real numbers. In addition we allow each vertex to also have a prescribed weight.

Let G be a graph, let $L \subset \mathbb{R}$ be a set of $|E(G)|$ real numbers and let $s : V(G) \to \mathbb{R}$ be a function. The triple (G, L, s) is called *locally antimagic* if there exists a bijection $w : E(G) \to L$ such that for any two adjacent vertices u, v we have that $s(u) + \sum_{e \in E(u)} w(e) \neq s(v) + \sum_{e \in E(v)} w(e)$. Such an edge-weighting w is also called an (L, s)*-locally antimagic edge-weighting of* G. The tuple (G, s) is called *locally list antimagic* if the for any list $L \subset \mathbb{R}$ of $|E(G)|$ real numbers the triple (G, L, s) is locally antimagic. A graph G is called *locally list antimagic* if for any list $L \subset \mathbb{R}$ of $|E(G)|$ real numbers and any function $s : V(G) \to \mathbb{R}$, the triple (G, L, s) is locally antimagic. For convenience we define $C_{w,s}(v) = s(v) + \sum_{e \in E(v)} w(e)$ for any vertex v in a graph G when w is an edge-weighting of G and $s : V(G) \to \mathbb{R}$ is a function. We also call $C_{w,s}(v)$ the colour of v.
It can easily be checked that stars are not locally list antimagic so these graphs must be omitted when we want to prove that some family of graphs is locally list antimagic. The goal of this section is to prove the following theorem stating that stars are actually the only connected graphs which are not locally list antimagic.

Theorem 2.4.1. *If G is a connected graph which is not locally list-antimagic, then G is a star.*

Since any star which is not isomorphic to K_2 clearly satisfies the Local Antimagic Labelling Conjecture, Theorem 2.4.1 also implies that the Local Antimagic Labelling Conjecture is true.
Since stars play a special role in Theorem 2.4.1 we will start out with a lemma about

stars not being locally list antimagic. If G is a star with leaves $v_1, \ldots, v_m$ and center v_0, L is a set of m real numbers, and $s : V(G) \to \mathbb{R}$ is a function such that s is constant on the leaves of G and $\sum_{x \in L} x + s(v_0) - s(v_1) = l \in L$, then we say that the triple (G, L, s) forms a *bad* star.

Lemma 2.4.2. *If G is a star, L is a set of $|E(G)|$ real numbers and $s : V(G) \to \mathbb{R}$ is a function, then the triple (G, L, s) is not locally antimagic if and only if (G, L, s) forms a bad star. Furthermore if (G, L, s) forms a bad star and $\sum_{x \in L} x + s(v_0) - s(v_1) = l \in L$, then for any injective edge-weighting $w : E(G) \to L$ the only conflict which arises in G is the edge with weight l.*

Proof. Let v_0 be the center of a star G and let $v_1, \ldots, v_m$ denote the leaves. Furthermore, let $e_j = v_0 v_j$ for $j \in \{1, \ldots, m\}$, let $L = \{x_1, \ldots, x_m\} \subset \mathbb{R}$ be an assigned list of real numbers and let $s : V(G) \to \mathbb{R}$ a function. If (G, L, s) forms a bad star, then for any injection $w : E(G) \to L$, there is an $i \in \{1, \ldots, m\}$ so that $w(e_i) = \sum_{x \in L} x + s(v_0) - s(v_1)$. Now we have

$$C_{w,s}(v_0) = s(v_0) + \sum_{x \in L} x = w(e_i) + s(v_1) = w(e_i) + s(v_i) = C_{w,s}(v_i),$$

which means that v_i and v_0 have the same colour and it is easy to see that this conflict is indeed the only conflict.
By the above we can assume that (G, L, s) is not a bad star. Now suppose (G, L, s) is not locally antimagic. It suffices to show that s is constant on the leaves of G, since then it is easy to see that (G, L, s) must form a bad star. So suppose $s(v_1) \neq s(v_2)$. If $m = 2$ it is easy to check that (G, L, s) is locally antimagic so we can assume $m \geq 3$ and proceed by induction on m.
Define $G' = G - v_m$ and let $z \in L$ be such that $\sum_{x \in L} x + s(v_0) \neq z + s(v_m)$. By the induction hypothesis the triple $(G', L \setminus \{z\}, s')$, where s' is $s\big|_{V(G')}$ except $s'(v_0) = s(v_0) + z$, is locally antimagic. So there is an $(L \setminus \{z\}, s')$-locally antimagic edge-weighting w' of G'. To get an (L, s)-locally antimagic edge-weighting w of G we simply extend w' to w defining $w(e_m) = z$. □

The proof of Theorem 2.4.1 is by induction and in order to make that induction work smoothly it is convenient to deal with bistars in a separate lemma. Recall that a bistar is a graph obtained from the disjoint union of two stars each having at least two vertices by adding an edge between the centers of the stars. These two centers are also called the centers of the bistar.

Lemma 2.4.3. *Any bistar is locally list-antimagic.*

Proof. Let G be a bistar with centers v_0, u_0 and leaves $v_1, \ldots, v_{m_1}$ adjacent to v_0 and leaves $u_1, \ldots, u_{m_2}$ adjacent to u_0. Suppose $L \subset \mathbb{R}$ is a set of $|E(G)|$ real numbers and that $s : V(G) \to \mathbb{R}$ is a function such that (G, L, s) is not locally antimagic. Define $C_1 = s(v_1) - s(v_0)$, $C_2 = s(u_1) - s(u_0)$ and $G_1 = G[v_0, \ldots, v_{m_1}]$, $G_2 = G[u_0, \ldots, u_{m_2}]$.

We may assume that $L = \{l_1, \ldots, l_m\}$ is in strictly increasing order, and that we have $s(v_1) \leq \cdots \leq s(v_{m_1})$ and $s(u_1) \leq \cdots \leq s(u_{m_2})$.
First we prove $s(v_1) = s(v_{m_1})$ and $s(u_1) = s(u_{m_2})$. Suppose this is not true and that $s(v_1) \neq s(v_{m_1})$. If we also have $s(u_1) \neq s(u_{m_2})$, then let $L_1, L_2, \{l\}$ be a partition of L such that $|L_1| = m_1$, $|L_2| = m_2$, $L_1 \cup L_2 \cup \{l\} = L$ and $\sum_{x \in L_1} x + s(v_0) \neq \sum_{y \in L_2} y + s(u_0)$. By Lemma 2.4.2 the triples (G_1, L_1, s_1) and (G_2, L_2, s_2) are both locally antimagic, where s_1 is $s\big|_{V(G_1)}$ except $s_1(v_0) = s(v_0)+l$ and s_2 is $s\big|_{V(G_2)}$ except $s_2(u_0) = s(u_0) + l$. But an (L_1, s_1)-locally antimagic edge-weighting of G_1 together with an (L_2, s_2)-locally antimagic edge-weighting of G_2 can be extended to an (L, s)-locally antimagic edge-weighting of G by assigning weight l to the edge v_0u_0. Thus, we can assume that (G_2, L_2, s_2) is a bad star which implies that $s(u_1) = s(u_{m_2})$.
Let $L_2' \subset L$ be such that $|L_2'| = m_2$ and $\sum_{x \in L_2'} x = C_2$ if such a set L_2' exists. If such a set does not exist, then let $L_2' \subset L$ be an arbitrary subset of L with $|L_2'| = m_2$. Let l be an element in $L \setminus L_2'$. Consider the triple (G_2, L_2', s') where s' is $s\big|_{V(G_2)}$ except $s'(u_0) = s(u_0) + l$. If there is no subset of L of size m_2 whose numbers sum to C_2, then Lemma 2.4.2 implies that (G_2, L_2', s') is locally antimagic. On the other hand, if $\sum_{x \in L_2'} x = C_2$, then we have

$$\sum_{x \in L_2'} x + s'(u_0) - s(u_1) = C_2 + s'(u_0) - s(u_1) = s(u_1) + s'(u_0) - s(u_0) - s(u_1) = l \notin L_2',$$

in which case Lemma 2.4.2 also implies that the triple (G_2, L_2', s') is locally antimagic. So (G_2, L_2', s') is locally antimagic and, as before, we can now find an (L, s)-locally antimagic edge-weighting of G assigning weight l to the edge v_0u_0.

The above shows that $s(v_1) = s(v_{m_1})$ and $s(u_1) = s(u_{m_2})$. We say that $l \in L$ is C_1-*representing* if there is a subset $L_1 \subset L$ with $|L_1| = m_1$ such that $l \in L_1$ and $\sum_{x \in L_1} x = C_1$. Similarly, we say that l is C_2-*representing* if there is a subset $L_2 \subset L$ with $|L_2| = m_2$ such that $l \in L_2$ and $\sum_{x \in L_2} x = C_2$.
If some $l \in L$ is neither C_1-representing nor C_2-representing, then we assign weight l to the edge u_0v_0 and choose L_1 and L_2 such that $|L_1| = m_1$, $|L_2| = m_2$, $L_1 \cup L_2 \cup \{l\} = L$, and $\sum_{x \in L_1} x + s(v_0) \neq \sum_{y \in L_2} y + s(u_0)$. Furthermore we distribute the weights in L_1 to the edges $v_0v_1, \ldots, v_0v_{m_1}$ and the weights in L_2 to the edges $u_0u_1, \ldots, u_0u_{m_2}$ arbitrarily. By Lemma 2.4.2 applied to both (G_1, L_1, s_1) and (G_2, L_2, s_2) where s_1 is $s\big|_{G_1}$ except $s_1(v_0) = s(v_0) + l$ and s_2 is $s\big|_{G_2}$ except $s_2(u_0) = s(u_0) + l$, this yields an (L, s)-locally antimagic edge-weighting of G.
Thus, we can assume that any $l \in L$ is C_1-representing or C_2-representing. Define $t = \sum_{x \in L} x - C_1 - C_2$. If $t \in L$, then let $l = t$, otherwise let l be an arbitrary number in L. Without loss of generality assume that l is C_1-representing. Let $L_1 \subset L$ be such that $|L_1| = m_1$, $l \in L_1$ and $\sum_{x \in L_1} x = C_1$. Let $L_2 \subset L \setminus L_1$ be such that $|L_2| = m_2$ and $\sum_{x \in L_1} x + s(v_0) \neq \sum_{y \in L_2} y + s(u_0)$ and define $(L \setminus L_1) \setminus L_2 = \{z\}$.
We must have $\sum_{x \in L_2} x \neq C_2$, since otherwise

$$C_1 + C_2 + z = \sum_{x \in L_1} x + \sum_{x \in L_2} x + z = \sum_{x \in L} x,$$

which implies $z = t$, but $t \notin (L \setminus L_1) \setminus L_2$, a contradiction.
Now we distribute the weights in L_1 to the edges $v_0v_1, \ldots, v_0v_{m_1}$ and the weights in L_2 to the edges $u_0u_1, \ldots, u_0u_{m_2}$ arbitrarily and assign weight z to the edge v_0u_0. The vertices v_0 and u_0 will now have different colours. Since

$$\sum_{x \in L_1} x + s(v_0) + z - s(v_1) = C_1 + s(v_0) + z - s(v_1) = z \notin L_1,$$

Lemma 2.4.2 implies that there are no conflicts between v_0 and $v_1, \ldots, v_{m_1}$. For $i \in \{1, \ldots, m_2\}$ let $l_i \in L_2$ denote the weight assigned to u_0u_i. Note that the difference between the colour of u_0 and the colour of u_i is

$$\sum_{x \in L_2} x + z + s(u_0) - l_i - s(u_i) = \sum_{x \in L} x - C_1 - C_2 - l_i = t - l_i \neq 0,$$

which implies that there are no conflicts between u_0 and $u_1, \ldots, u_{m_2}$. Hence there are no conflicts which means that (G, L, s) is locally antimagic. □

Let G be a graph, let $L = \{l_1, \ldots, l_m\}$ be a list of $|E(G)|$ real numbers in strictly increasing order and let $s : V(G) \to \mathbb{R}$ be a function. For a vertex $v \in V(G)$ we define its *minimum potential* to be the number $s(v) + l_1 + \cdots + l_{d_G(v)}$, that is, the smallest colour it can receive from any injective mapping $w : E(G) \to L$. We are now ready for the proof of Theorem 2.4.1.

Proof of Theorem 2.4.1. Suppose the theorem is false and let G be a counterexample of minimum order. Let $L = \{l_1, l_2, \ldots, l_m\}$ be a list of $|E(G)|$ real numbers in strictly increasing order and let $s : V(G) \to \mathbb{R}$ be a function such that (G, L, s) is not locally antimagic.

Claim: If $v \in V(G)$ and $d_G(v) \geq 2$, then v is not adjacent to $d_G(v) - 1$ leaves in G.
Proof of the claim. Suppose $v \in V(G)$ is a vertex with degree $d_G(v) = k > 1$ adjacent to $k-1$ leaves $v_1, \ldots, v_{k-1}$ and let v_k be the neighbour of v with $d_G(v_k) > 1$. First we prove that $s(v_1) = \ldots = s(v_{k-1})$. Suppose this is not true and assume that $s(v_1) \neq s(v_2)$. Let $L' \subset L$ be such that $|L'| = k-1$ and define $G' = G - \{v_1, \ldots, v_{k-1}\}$. By Lemma 2.4.3 the graph G is not a bistar so G' is not a star and the minimality of G implies that G' is locally list-antimagic. This means that $(G', L \setminus L', s')$ is locally antimagic where s' is $s\big|_{V(G')}$ except that $s'(v) = s(v) + \sum_{x \in L'} x$. But we can extend any $(L \setminus L', s')$-locally antimagic edge-weighting of G' to an (L, s)-antimagic edge-weighting of G by distributing the weights in L' to the edges $vv_1, \ldots, vv_{k-1}$ in a way that yields no conflicts between v and any of $v_1, \ldots, v_{k-1}$ (this can be done by Lemma 2.4.2 since $s(v_1) \neq s(v_2)$).
The above shows that $s(v_1) = \ldots = s(v_{k-1})$. Define $C_1 = s(v_1) - s(v)$. Again, let $L' \subset L$ be such that $|L'| = k - 1$ and $\sum_{x \in L'} x = C_1$ (if such L' does not exist, then just choose an arbitrary set $L' \subset L$ with $|L'| = k-1$). Again, by the minimality of G,

the triple $(G', L \setminus L', s')$ is locally antimagic where $G' = G - \{v_1, \ldots, v_{k-1}\}$ and s' is $s\big|_{V(G')}$ except that $s'(v) = s(v) + \sum_{x \in L'} x$. But we can extend any $(L \setminus L', s')$-locally antimagic edge-weighting w' of G' to an (L, s)-antimagic edge-weighting of G by distributing the weights in L' to the edges $vv_1, \ldots, vv_{k-1}$ arbitrarily: by Lemma 2.4.2 there are no conflicts between v and $v_1, \ldots, v_{k-1}$ if $\sum_{x \in L'} x = C_1$, since

$$\sum_{x \in L'} x + w'(vv_k) + s(v) - s(v_1) = C_1 + w'(vv_k) + s(v) - s(v_1) = w'(vv_k) \notin L',$$

and in the case where there are no subset of L of size $k-1$ whose weights sum to C_2, then Lemma 2.4.2 also implies that there are no conflicts between v and $v_1, \ldots, v_{k-1}$. $\diamond$

Let $v \in V(G)$ be a vertex with smallest minimum potential and subject to that with largest degree. First suppose v has degree 1. Then the neighbour v' of v must have degree at least two and hence has strictly larger minimum potential than v. Also note that by Lemma 2.4.3 and the fact that G is not a star we have that $G - v$ is not a star. So by the minimality of G the triple $(G - v, L \setminus \{l_1\}, s')$, where s' is $s\big|_{V(G-v)}$ except $s'(v') = s(v') + l_1$, is locally antimagic. But any $(L \setminus \{l_1\}, s')$-locally antimagic edge-weighting of $G - v$ can be extended to an (L, s)-locally antimagic edge-weighting of G by assigning weight l_1 to the edge vv'. Thus we can assume that $d_G(v) > 1$. Define $G' = G - v$ and let $C_1, \ldots, C_n$ denote the components of G'. Let $e_1, \ldots, e_{d_G(v)}$ denote the edges incident to v and let $v_1, \ldots, v_{d_G(v)}$ be the corresponding neighbours of v. We can assume that v_1 is not a leaf in G. Define $L' = L \setminus \{l_1, \ldots, l_{d_G(v)}\}$ and $s' : V(G') \to \mathbb{R}$ by $s'(u) = s(u)$ if $u \notin N_G(v)$ and $s'(v_i) = s(v_i) + l_i$ if $v_i \in N_G(v)$. Since v has smallest minimum potential in G and since v_1 is not a leaf in G, any (L', s')-locally antimagic edge-weighting of G' can be extended to an (L, s)-locally antimagic edge-weighting of G by assigning weight l_i to each of the edges e_i (the colour of v will be strictly less than the colour of any other vertex in G). So we can assume that there is no (L', s')-locally antimagic edge-weighting of G' and by the minimality of G this means that some of the components among $C_1, \ldots, C_n$ are stars. If C_j is a star, then, by the above Claim, the vertex v must be incident with at least two edges vv_{j_1}, vv_{j_2} going to C_j. If the tuple $(C_j, s'\big|_{V(C_j)})$ is not locally list-antimagic, then, by Lemma 2.4.2, the function s' must have the same value on all leaves of C_j. If we now modify s' to another function s'' by replacing $s'(v_{j_1}) = s(v_{j_1}) + l_{j_1}$ by $s''(v_{j_1}) = s(v_{j_1}) + l_{j_2}$ and replacing $s'(v_{j_2}) = s(v_{j_2}) + l_{j_2}$ by $s''(v_{j_2}) = s(v_{j_2}) + l_{j_1}$, then the tuple $(C_j, s''\big|_{V(C_j)})$ will be locally list-antimagic since s'' does not have the same value on all leaves of C_j. In this way we can modify s' and obtain s'' such that all the tuples $(C_1, s''\big|_{V(C_1)}), \ldots, (C_n, s''\big|_{V(C_n)})$ are locally list antimagic which means that (G', s'') is locally list antimagic. But as before any (L', s'')-locally antimagic edge-weighting of G' can be extended to an (L, s)-locally antimagic edge-weighting of G, this time by assigning weight $s''(v_j) - s(v_j) \in \{l_1, \ldots, l_{d_G(v)}\}$ to each of the edges vv_j. $\square$

CHAPTER 3

Three Applications of Non-Separating Cycles

3.1 Introduction

Recall that a cycle C in a graph G is called *non-separating* if $G - V(C)$ is connected. In 1981 Thomassen and Toft [Tho81] studied the existence of non-separating induced cycles in graphs. They proved several results about the existence of such cycles for example that any graph with minimum degree at least 3 contains such a cycle. They also proved that under some mild conditions there is a non-separating induced cycle avoiding a given connected subgraph. A classical result of this type is the result by Tutte [Tut63] from 1963 stating that any edge e in a 3-connected graph is contained in at least two induced non-separating cycles having only e in common.
The existence of non-separating cycles is an interesting topic on its own, however, in this thesis we will focus on using these cycles rather than proving theorems about their existence. We will show examples of how such cycles can be useful by using them to prove some structural theorems. Another example of such use of non-separating cycles is by Bondy and Vince [Bon98] who used them to show that any graph of minimum degree at least 3 contains two cycles whose lengths differ by 1 or 2, which answered a question of Erdös. In this thesis we use non-separating cycles to prove three different theorems. In each of the three cases we use at least one of the following results by Thomassen and Toft or Tutte to find these non-separating cycles. A *k-rail* for an integer $k \geq 3$ is a graph consisting of two vertices joined by k internally disjoint paths of which at most one has length 1. A k-rail in a graph G is a subgraph isomorphic to a k-rail where the vertices of degree 2 also have degree two in G.

Lemma 3.1.1. *[Tho81] Let G be a 2-connected graph and let G' be a connected subgraph of G such that $G - V(G')$ contains at least one cycle. Then either $G - V(G')$ contains an induced cycle C such that $G - V(C)$ is connected or there is a connected subgraph G^* of G with $G' \subset G^*$ such that $G - V(G^*)$ is a k-rail in G with $k \geq 3$.*

Lemma 3.1.2. *[Tho81] Let G be a connected graph. If G has minimum degree at least 3, then G contains a non-separating induced cycle.*

Theorem 3.1.3. *[Tut63] Let G be a graph, let $st \in E(G)$ and $r \in V(G) \setminus \{s,t\}$. If G is 3-connected, then G contains a non-separating induced cycle C such that $st \in E(C)$ and $r \notin V(C)$.*

In Sections 3.2 and 3.3 we are sometimes interested in the existence of a cycle C in a graph G such that $G - E(C)$ is connected (instead of $G - V(C)$). Nevertheless, we will still use the above results as tools to find such a cycle. It will be clear from the context exactly which of the two properties we want the cycle to have.

3.2 Spanning Trees with no Three Consecutive Vertices of Degree 2

The material presented in this section essentially consists of one research article [Lynb].

This section is related to homeomorphically irreducible trees, in particular homeomorphically irreducible spanning trees. Recall that a homeomorphically irreducible tree, also called a HIT, is a tree with no vertices of degree 2. These trees were enumerated by Harary and Prince [Har59]. In this section we are not interested in studying these trees themselves, but rather the existence of such spanning trees in connected graphs with certain properties. Actually, we are interested in a relaxation of the notion of HISTs where we want to find spanning trees without many consecutive vertices of degree 2 in connected graphs where we only put restrictions on the minimum degree.
When we only put restrictions on the minimum degree of a connected graph G and look for spanning trees in G without many consecutive vertices of degree 2, then clearly we need to assume that the minimum degree is at least 3. We will prove that this is actually sufficient to guarantee the existence of a spanning tree in which there are no three consecutive vertices of degree 2.

Theorem 3.2.1. *Every connected graph with minimum degree at least 3 contains a spanning tree T without three consecutive vertices of degree 2.*

Figure 3.1 shows a cubic graph in which every spanning tree has two adjacent vertices of degree 2, so in Theorem 3.2.1 we need to allow the spanning tree to contain two consecutive vertices of degree 2.

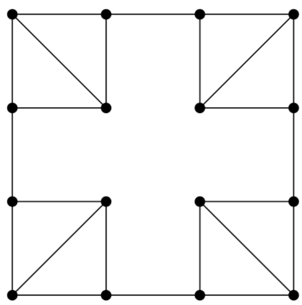


Figure 3.1: A cubic graph where any spanning tree contains two adjacent vertices of degree 2.

In order to use induction it turns out to be more convenient to prove the following theorem which immediately implies Theorem 3.2.1.

Theorem 3.2.2. *Every connected graph G has a spanning tree T, such that there is no path of length 2 in T all of whose vertices have degree 2 in T and degree at least 3 in G.*

To simplify the notation in the following proofs, we introduce the following definition.

Definition 3.2.3 (G-bad, G-good)**.** *Let H be a subgraph of G. We say a path of length 2 in H is G-bad if all its vertices have degree at least 3 in G and degree 2 in H. We say a subgraph H is G-bad if it contains a G-bad path, otherwise we call it G-good.*

Now the statement of Theorem 3.2.2 is simply that every connected graph G has a G-good spanning tree.

We will prove Theorem 3.2.2 by considering a smallest counterexample. First we show that such a minimum counterexample has minimum degree at least 3.

Lemma 3.2.4. *A counterexample to Theorem 3.2.2 with minimum order has minimum degree at least 3.*

Proof. Let G be a connected graph which has no G-good spanning tree and for which $|V(G)|$ is minimal. Clearly $|V(G)| \geq 4$.

Claim 1: G has no vertices of degree 1.
Proof of the claim. Suppose $v \in V(G)$ has degree 1 and let $G' = G - v$. By minimality of G, we can find a spanning tree T' in G' which is G'-good. Let $T_1 = T' + v$. Clearly T_1 is a spanning tree of G. The only way how T_1 could be G-bad is that v is adjacent to an endvertex x of a G-bad path in T_1, say xyz. In this case, let u denote a neighbour of x different from v and y. Now consider the tree $T_2 = T_1 - xy + xu$, which is another spanning tree of G. If T_2 is G-bad, then there must be a G-bad path xuw in T_2. In particular, the vertex w has degree 2 in T_2. Finally, set $T_3 = T_2 - uw + xy$. It is easy to see that T_3 is a G-good spanning tree of G. $\diamond$

Claim 2: G has no vertices of degree 2.
Proof of the claim. Suppose the claim is false, let $v \in V(G)$ be a vertex of degree 2 and let x and y denote the neighbours of v. First suppose x and y are not adjacent. By minimality of G, the graph $G' = G - v + xy$ has a G'-good spanning tree T'. If $xy \in E(T')$, then $T_1 = T' - xy + xv + yv$ is a G-good spanning tree, so we can assume that $xy \notin E(T')$. In this case we can assume that $T_2 = T' + xv$ is a G-bad spanning tree. Hence, x is an endvertex in a G-bad path xwz in T_2. Let u be a neighbour of x in G different from v and w. We can assume that $T_3 = T_2 - xw + xu$ contains a G-bad path xuu'. Notice that u and u' have degree 2 in T_3. Now $T_4 = T_3 - uu' + xw$ is a G-good spanning tree. Thus we may assume that x and y are adjacent and hence every vertex of degree 2 is contained in a triangle in G.
If one of x, y, say x, does not have degree 3 in G, then any $(G - vx)$-good spanning

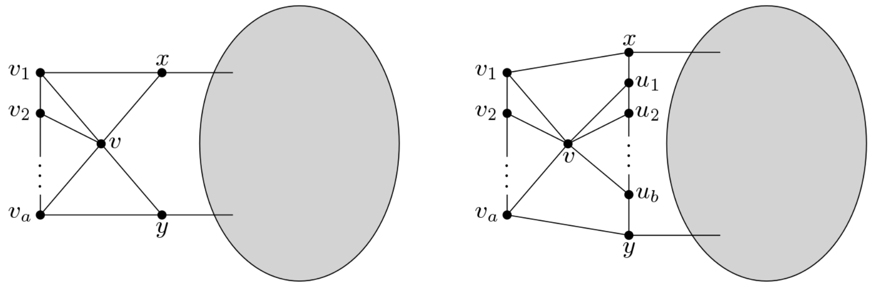
tree of $G-vx$ is also a G-good spanning tree, so by the minimality of G we can assume that both x and y have degree 3 in G. Let x' and y' denote the neighbours of x and y which are different from x, y and v. If $G' = G - v - xy$ is connected, then let T' be a G'-good spanning tree of G'. If both $T_1 = T' + vx$ and $T_2 = T' + vy$ are G-bad, then x' and y' have degree 2 in T' and $T_3 = T_1 + xy - yy'$ is a G-good spanning tree. Thus we may assume that G' is disconnected. In particular $x' \neq y'$ and both x' and y' have degree at least 3 since xx' and yy' are not contained in triangles.
Let $G'' = G - v - x - y + x'y'$ and let T'' be a G''-good spanning tree of G''. Since G' is disconnected we have that $x'y' \in E(T'')$. If both x', y' have degree 2 in T'', then $T_4 = T'' - x'y' + x'x + xv + vy + yy'$ is a G-good spanning tree of G. So one of x', y' does not have degree 2 in T'', say x'. Now $T_5 = T'' - x'y' + x'x + xy + yv + yy'$ is a G-good spanning tree. ◇

Claims 1 and 2 immediately imply that G has minimum degree at least 3. □

By Lemma 3.1.2 any graph with minimum degree at least 3 contains a non-separating cycle. Let G be a counterexample to Theorem 3.2.2 with minimum order. By Lemma 3.2.4 the graph G has minimum degree at least 3 so in G there exists an induced cycle C for which $G - V(C)$ is connected. This implies that also $G' = G - E(C)$ is connected. Note that if C does not contain any vertices of degree 4 in G, then any G'-good spanning tree of G' is also a G-good spanning tree of G, contradicting our choice of G. In particular, every non-separating induced cycle in G contains a vertex of degree 4. This already proves Theorem 3.2.2 for subcubic graphs and Theorem 3.2.1 for cubic graphs.
The proof of Theorem 3.2.2 essentially consists of finding an induced non-separating subgraph H with the property that we can extend every $(G-H)$-good spanning tree of $G - H$ to a G-good spanning tree of G. One of these reducible structures we use is an induced non-separating cycle containing no vertices of degree 4. Two other reducible structures we use are so-called W_a- and $W_{a,b}$-configurations which are defined as follows, see Figure 3.2.

Definition 3.2.5 (W_a-configuration)**.** *A W_a-configuration in G is an induced subgraph H consisting of a path $P = v_1 \cdots v_a$ and three distinct vertices v, x, y not contained in P, such that v is adjacent to all vertices in $V(P) \cup \{x, y\}$, $xv_1, yv_a \in E(H)$, and every vertex in $V(H) \setminus \{v\}$ has degree 3 in G. Moreover, $G-H$ is connected, both x and y have precisely one neighbour in $G-H$ and no other vertex of H has a neighbour in $G - H$. We call v the centre and x, y the connectors of the W_a-configuration.*

Definition 3.2.6 ($W_{a,b}$-configuration)**.** *A $W_{a,b}$-configuration in G is an induced subgraph H consisting of two disjoint paths $P = v_1 \cdots v_a$, $Q = u_1 \cdots u_b$, and three distinct vertices v, x, y not contained in the paths such that v is adjacent to all vertices in $V(P) \cup V(Q)$, $xv_1, xu_1, yv_a, yu_b \in E(H)$, and every vertex in $V(H) \setminus \{v\}$ has degree 3 in G. Moreover, $G - H$ is connected, both x and y have precisely one neighbour in $G - H$ and no other vertex of H has a neighbour in $G - H$. We call v the centre and x, y the connectors of the $W_{a,b}$-configuration.*



(a) A W_a-configuration. (b) A $W_{a,b}$-configuration.

Figure 3.2: Two graph configurations.

Lemma 3.2.7 below will allow us to find the reducible structures we need to finish the proof of Theorem 3.2.2. The proof of Lemma 3.2.7 is rather technical and is postponed to the end of this section.

Lemma 3.2.7. *Let G be a connected graph of minimum degree at least 3. Let S be a set of vertices in G containing all vertices of degree greater than 3 and possibly some vertices of degree 3. Then at least one of the following three conditions is satisfied:*

(C) *There exists an induced cycle C containing no vertex of S such that $G - E(C)$ is connected.*

(P) *There exists an induced path P with endvertices in S such that $G - E(P)$ is connected.*

(W) *There exists a W_a-configuration or a $W_{a,b}$-configuration in G where the center is contained in S.*

Notice that all three conditions in Lemma 3.2.7 are indeed necessary. To see that the statement is not true if we omit condition (W), we can consider the following construction. Let T be any homeomorphically irreducible tree. Now for every leaf t in T, we add a W_a- or $W_{a,b}$-configuration with both connectors joined to t. Let G denote the resulting graph, and let S denote the set of vertices which have degree at least 4 or are centres of the configurations. Now every non-trivial block in G consists of a W_a- or $W_{a,b}$-configuration together with a vertex of degree 2. It is easy to see that every non-separating cycle in G contains precisely one vertex of S. Moreover, any path containing two vertices of S also contains a cut-edge.

Now we will show how to finish the proof of Theorem 3.2.2 having Lemma 3.2.7 available.

Proof of Theorem 3.2.2. Let G be a counterexample with smallest order. By Lemma 3.2.4 the minimum degree of G is at least 3. Let $S \subset V(G)$ be the set of vertices in G

with degree at least 4. By Lemma 3.2.7 it suffices to consider the following three cases.

Case 1: There exists an induced cycle C containing no vertex of S such that $G - E(C)$ is connected.
By the minimality of G there exists a spanning tree T of $G' = G - E(C)$ which is G'-good, but then T is also a G-good spanning tree of G.

Case 2: There exists an induced path P with endvertices in S for which $G - E(P)$ is connected.
We may assume that no interior vertex of P is contained in S by considering a shortest such path. As in Case 1, by minimality of G there exists a spanning tree T of $G' = G - E(P)$ which is G'-good, but then T is also a G-good spanning tree of G.

Case 3: There exists a W_a-configuration or a $W_{a,b}$-configuration in G.
Let H denote such a configuration with centre v and connectors x and y, and let v_1 denote a common neighbour of x and v. By minimality of G, the graph $G' = G - (H - x - y)$ has a G'-good spanning tree T. We can obtain a G-good spanning tree of G by adding all edges incident with x and all edges incident with v apart from vv_1 and vy. □

It remains to prove Lemma 3.2.7. This is what we will do in the rest of this section.

In order to prove Lemma 3.2.7 we will use the following lemma which is an easy corollary of Lemma 3.1.1.

Lemma 3.2.8. *Let G be a 2-connected graph and let G' be a non-empty connected subgraph of G such that G' contains all vertices of degree at least 4 and $G - V(G')$ contains at least one cycle. Then $G - V(G')$ contains an induced cycle C such that $G - V(C)$ is connected.*

Proof. By Lemma 3.1.1 it suffices to show that $G - V(G')$ cannot contain a k-rail for $k \geq 3$. So suppose R is such a k-rail and x and y are the two vertices in R of degree at leat 3. Since G' contains all vertices of degree at least 4 in G and since $k \geq 3$, there can be no edges between R and $G - R$, contradicting that G is connected. □

Tutte [Tut63] showed that any pair of vertices in a 3-connected graph G can be connected by an induced path P such that $G - V(P)$ is connected. We will need the following edge-version which is an easy application of this theorem. We here give a short self-contained proof which we will also refer to in the proof of Lemma 3.2.7.

Lemma 3.2.9. *For any two vertices v_1, v_2 in a 3-edge-connected graph G, there exists an induced path P from v_1 to v_2 such that $G - E(P)$ is connected.*

Proof. Let P be a path from v_1 to v_2 which maximises the size of the largest connected component of $G - E(P)$. Clearly, we may assume that P is an induced path in G. Let K denote the largest component of $G - E(P)$. Notice that

(*) for any vertices z_1, z_2 on P belonging to the same component $L \neq K$ of $G - E(P)$, the path $z_1 P z_2$ does not contain any vertices of K,

since otherwise we could replace $z_1 P z_2$ by a path in L to obtain a new $v_1 v_2$-path P' for which the component of $G - E(P')$ containing K is strictly larger than before, contradicting our choice of P. Let k_1 and k_2 denote the first and last vertex on P, respectively, which is contained in K. If $k_1 \neq v_1$, then let e denote the edge of $v_1 P k_1$ incident to k_1. By (*), the edge e is a cut-edge in G which contradicts 3-edge-connectivity.
Thus, we may assume that $k_1 = v_1$ and similarly $k_2 = v_2$. If $G - E(P)$ is not connected, then there exists a vertex on P which is not in K. Let w be the first such vertex on the path from v_1 to v_2. Let k denote the first vertex on wPv_2 which is contained in K, see Figure 3.3. Let e_w and e_k denote the last edge of $v_1 P w$ and $v_1 P k$, respectively. By (*), the edges e_w, e_k form a 2-edge-cut in G, contradicting 3-edge-connectivity. Thus, every vertex of $G - E(P)$ is contained in K and hence $G - E(P)$ is connected. □

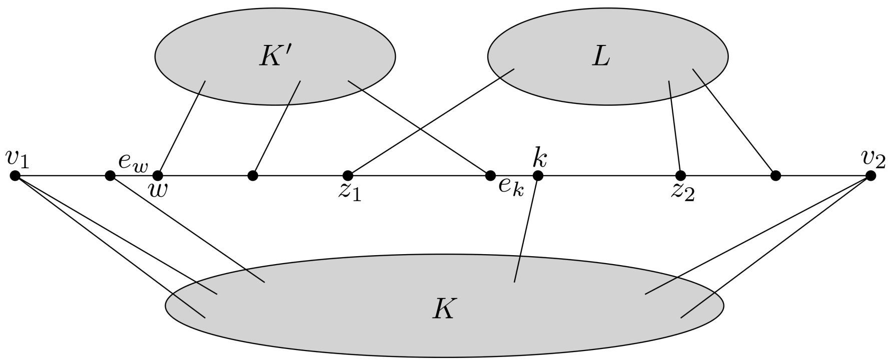


Figure 3.3: Proof of Lemma 3.2.9.

With the above tools at hand we are ready for the proof of Lemma 3.2.7.

Proof of Lemma 3.2.7. Let B be an endblock of G. First suppose that B is 3-edge-connected. Since S contains all vertices of degree at least 4 in G, this implies that if there is a cut-vertex of G in B then that cut-vertex belongs to S. If B contains at least two vertices of S, then we can use Lemma 3.2.9 to find a path between them which satisfies (P). Thus, we can assume that B contains at most one vertex of S, say v. If B contains no vertex of S, let v denote an arbitrary vertex of B. The block B is 2-connected and $B - v$ has minimum degree 2, so $B - v$ is connected and contains a cycle. Now we can use Lemma 3.2.8 on $B - v$ to find a non-separating induced cycle in B not containing v. This cycle is also non-separating in G and satisfies (C).
By the above we may assume that B is not 3-edge connected. If G is not 2-connected

let b denote the unique cut-vertex of G in B. We now choose a 2-edge cut in B minimising the size of the component H not containing b (if b does not exist we just minimise the size of some component H). Note that the choice of H implies that H is 2-edge-connected and contained in the endblock B. The rest of the proof consists of investigating three cases.

Case 1: H contains no vertex of S.
Since every vertex in H has degree 3 in G, every cut-vertex of H would give rise to a connected subgraph $H' \subset H$ which can be separated from $G - H'$ by at most 2 edges, contradicting our choice of H. Hence, we can assume that H is 2-connected. Let $x \in V(H)$ be a vertex joined to $G - H$. Notice that $H - x$ has minimum degree 2 and thus contains a cycle. By Lemma 3.2.8, there exists a non-separating induced cycle in H not containing x. This cycle is also non-separating in G and thus satisfies (C).

Case 2: H contains at least two vertices of S.
Let u_1 and u_2 denote two vertices in H contained in S. Let P be an induced path from u_1 to u_2 in H which maximises the size of the connected component K of $G - E(P)$ containing $G - H$.

Claim: $G - E(P)$ is connected.
Proof of the claim. Suppose $G-E(P)$ is not connected. As in the proof of Lemma 3.2.9 we have that for any vertices z_1, z_2 on P belonging to the same component $L \neq K$ of $G - E(P)$, the subpath z_1Pz_2 of P does not contain any vertices of K. Let k_1 and k_2 denote the first and last vertex on P, respectively, which is contained in K. If $k_1 \neq u_1$, then the last edge of u_1Pk_1 is a cut-edge in G which contradicts H being 2-edge-connected.
By the above we may assume $u_1 = k_1$ and similarly $u_2 = k_2$. Let w be the first vertex on P which is not contained in K and let k denote the first vertex on wPu_2 which is contained in K. Let e_w and e_k denote the last edge of u_1Pw and u_1Pk, respectively. As in the proof of Lemma 3.2.9, the edges e_w, e_k form a 2-edge-cut in G, contradicting the choice of H. Thus, every vertex of $G - E(P)$ is contained in K and hence $G - E(P)$ is connected. $\diamond$

By the claim we have that $G - E(P)$ is connected and thus P satisfies (P).

Case 3: H contains precisely one vertex v of S.
We distinguish three cases depending on the structure of $H - v$. Notice that if H contains a cutvertex w, then by 2-edge-connectivity of H, the vertex w has at least two neighbours in every block of H it is contained in. In particular, w has at least degree 4 and thus $w = v$ is the only possible cutvertex of H. Thus, Case 3.1 is identical to the case where H is not 2-connected.

Case 3.1: $H - v$ is disconnected.
Let B_H be a block of H which contains at most one vertex with a neighbour in $G-H$. The connected graph $B_H - v$ contains at most one vertex of degree 1, so there exists a cycle in $B_H - v$. By Lemma 3.2.8, the block B_H contains a non-separating induced cycle C avoiding v, see Figure 3.4. The cycle C is also non-separating in G and thus satisfies (C).

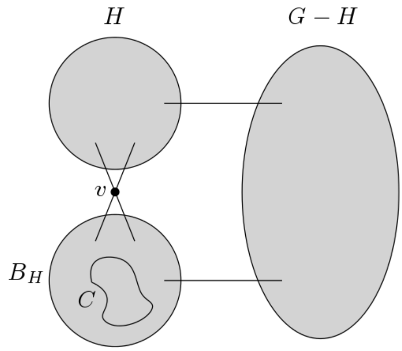


Figure 3.4: $H - v$ is disconnected

Case 3.2: $H - v$ is a tree.
Notice that every vertex of degree 1 in $H-v$ must have a neighbour in $G-H$. Hence, there are at most two vertices of degree 1 in $H - v$. In particular, $H - v$ is a path $P = xv_1 \cdots v_a y$ where both x and y have a neighbour in $G - H$. Thus, there exists a W_a-configuration in G and (W) is satisfied.

Case 3.3: $H - v$ is connected and contains a cycle.
By Lemma 3.2.8, there exists a non-separating induced cycle C in H avoiding v. If C does not satisfy (C), then C contains every vertex of H with neighbours in $G - H$. Since all vertices in H other than v have degree 3 in G and since H is 2-connected, there must be two such vertices x and y which have neighbours in $G - H$, see Figure 3.5. Notice that x and y are not adjacent since otherwise the graph $H - x - y$ would contradict the minimality of H.

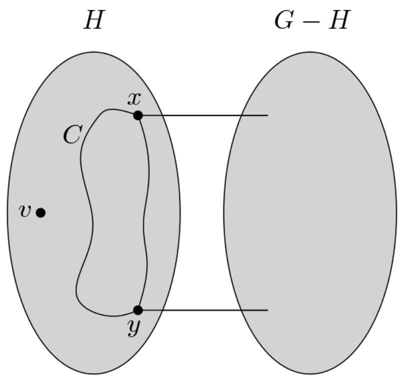
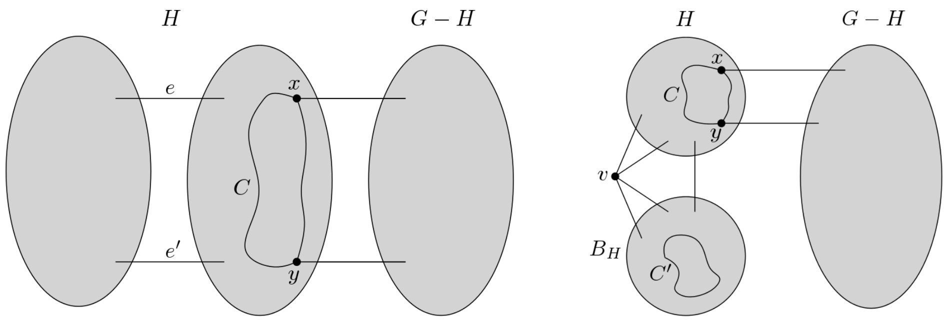


Figure 3.5: Case 3.3

Case 3.3.1: $H - v$ has a cut-vertex w.
Since w has degree at most 3 in $H - v$, there exists an edge e incident with w such that $H' = H - v - e$ is disconnected. Notice that since C does not contain v, and since e is a cut-edge in $H - v$, the cycle C also exists in H'. In particular, x and y are contained in the same block of $H - e$. By definition, v is a cut-vertex in $H - e$. Now suppose $H - e$ contains a cut-vertex $v' \neq v$. Since v' has degree at most 3 in $H - e$, there exists an edge e' which is a cut-edge in $H - e$. Now e, e' form a 2-edge-cut which contradicts our choice of H, see Figure 3.6(a). Thus, v is the only cut-vertex in $H - e$ and $H - e$ has exactly two blocks. Let B_H be the block of $H - e$ containing neither x nor y. Notice that B_H contains an end of e. There exists at most one vertex of degree 1 in $B_H - v$, so $B_H - v$ contains a cycle. By Lemma 3.2.8, there exists a non-separating induced cycle C' in B_H avoiding v. The cycle C' is also non-separating in G and thus satisfies (C), see Figure 3.6(b).



(a) The edges e and e'.

(b) The cycle C in B_H.

Figure 3.6: Case 3.3.1.

Case 3.3.2: $H - v$ is 2-connected.
We begin this case by proving the following claim.

Claim: There exists an induced path P in H from v to x such that $G - V(P)$ is connected.
Proof of the claim. Let P be an induced path in H from v to x, such that the size of the component K of $G - V(P)$ containing $G - H$ is maximum. Notice that since v is the only vertex of degree greater than 3 in H, the graph $G - V(P)$ is connected if and only if $G' = G - E(P)$ is connected and v is not a cut-vertex in G'. First, suppose that G' is disconnected and let K' be the component of G' containing $G-H$. As in the proof of the claim in Case 2 and the proof of Lemma 3.2.9, we have that for any vertices z_1, z_2 on P belonging to the same component $L \neq K'$ of G', the subpath z_1Pz_2 of P does not contain any vertices of K'. Note that $x \in V(K')$ since x has a neighbour in $G - H$ and as in the proof of Lemma 3.2.9 we can also assume $v \in V(K')$. Let w be the first vertex on P which is not contained in K' and let k denote the first vertex on wPx which is contained in K'. As in the proof of the claim in Case 2 we find two edges incident to w and k, respectively, which form a 2-edge-cut in G contradicting the choice of H. Thus G' is connected.
Now suppose that v is a cut-vertex in G'. Let z denote the first vertex on P after v which has a neighbour in K. Let L denote a component different from K in $G-V(P)$, in particular L is adjacent to v. Similar to before, no component $M \neq K$ of $G-V(P)$ has two neighbours z_1, z_2 on P such that z is contained in z_1Pz_2. Thus, the graph $H - \{v, z\}$ is disconnected, which contradicts 2-connectivity of $H - v$. ◇

Let P be the path from the claim above. Clearly we must have $y \notin V(P)$. If $H - V(P)$ contains a cycle, then we can use Lemma 3.2.8 to find an induced cycle C' avoiding P for which $H - E(C')$ is connected. Since C' does not contain x, it satisfies (C). Therefore, we may assume that $T = H - V(P)$ is a tree. We distinguish two cases depending on the length of P.

Case 3.3.2.1 $|E(P)| \leq 2$.
Since x is contained in a cycle not containing v, we have $|E(P)| = 2$. Let u denote the middle vertex of P. Since y is not adjacent to v, each leaf of T is adjacent to at least one of x and u. The vertices x and u are each only adjacent to one vertex outside of P, thus the tree T can contain at most two leaves and is therefore a path. Every vertex in T of degree 2 and different from y is adjacent to v. This shows that there exists a $W_{a,b}$-configuration (with v as its centre) in G and (W) is satisfied.

Case 3.3.2.2 $|E(P)| \geq 3$.
Let u denote the vertex at distance 2 of x on P, and w the neighbour of x on P. Since P is induced and u and w have degree 3 in H, there exist vertices u', w' adjacent to u and w, respectively and not contained in P. First suppose $u' = w'$. Since u' has degree 3 and T is connected, the vertex u' is not adjacent to x. Now $uu'w$ is a non-separating induced cycle in H which also satisfies (C). Thus, we may assume

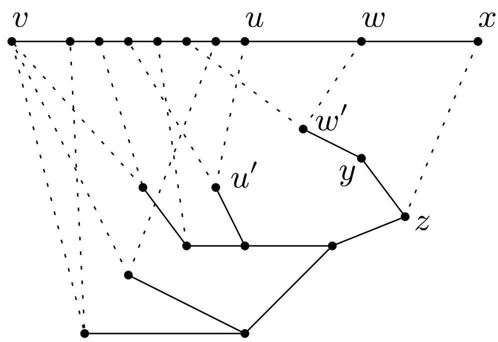


Figure 3.7: An example of how T in Case 3.3.2.2 could look like.

that $u' \neq w'$. Let P' denote the (unique) path in T connecting u' and w'. Let C_P denote the induced cycle consisting of P' and the edges $w'w, wu, uu'$. Notice that every component of $T - V(P')$ contains at least one leaf of T and that every such leaf apart from y is adjacent to two vertices in $P - u - w$. Since x is not adjacent to y, it follows that every component of $T - V(P')$ has a neighbour in $V(P) \setminus \{u, w, x\}$. Thus, $H - x - V(C_P)$ is connected. If y is not contained in C_P or x has a neighbour in $H - V(C_P)$, then C_P satisfies (C). See Figure 3.7 for a specific example of H where C_p does not satisfy (C).

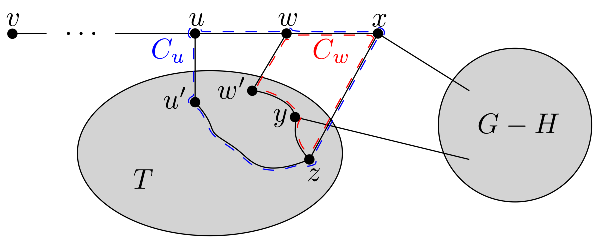


Figure 3.8: The cycles C_u and C_w in Case 3.3.2.2.

If C_P does not satisfy (C), let z denote the neighbour of x in T and define the cycles C_w and C_u as follows: the cycle C_w consists of $zP'w'$ together with the edges $w'w, wx, xz$, while C_u consists of $zP'u'$ together with the edges $u'u, uw, wx, xz$, see Figure 3.8. It is easy to see that C_w satisfies (C) unless y is contained in $zP'w'$. Thus, we may assume that y is contained in $zP'w'$. Recall that x and y are not adjacent so we also have $y \neq z$. Notice that this implies $w' \neq z$ and hence C_u is induced. Now suppose $G - V(C_u)$ is not connected. Clearly $G - V(C_u)$ has at most two components: one containing w' and one containing v. Let $K_{w'}$ denote the connected component of $G - V(C_u)$ containing w'. Let ℓ be a leaf in T contained in $K_{w'}$. If $\ell \neq y$, then ℓ has a neighbour on $P - V(C_u)$ and therefore $K_{w'} = G - V(C_u)$. Thus, we may assume that y is a leaf in T and no other leaf of T is contained in $K_{w'}$. Since y is contained in C_w we have $w' = y$. If y is not adjacent to z, then there exists a vertex k distinct from z, w' on the zw'-path in T. Since $K_{w'}$ contains only one leaf of T, the vertex k has degree 2 in T. Therefore, k has a neighbour on $P - V(C_u)$ and we again get the contradiction $K_{w'} = G - V(C_u)$. Finally, suppose that y is adjacent to z, see Figure 3.9. In particular, y is the only vertex in $K_{w'}$. Now the graph $H' = H - x - y - w - z$ can be separated from $G - H'$ by a 2-edge-cut, contradicting our choice of H. $\square$

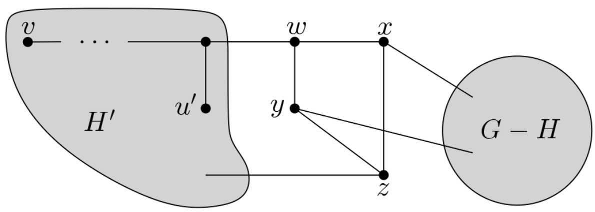


Figure 3.9: Case 3.3.2.2 where $y = w'$ and y is adjacent to z.

3.3 A 3-Decomposition Theorem

The material presented in this section essentially consists of one research article [Lyn19].

In this section we will look at decompositions of graphs. Decomposition of graphs is a well-studied area in graph theory where the typical question of interest is whether some given graph G has a decomposition $E(G) = E_1 \cup \cdots \cup E_k$ where each edge set E_i in the decomposition induces a subgraph of G with a certain property. It is, for example, not hard to see that any graph has a decomposition into an even graph (a graph where all vertices have even degree) and a forest: given a graph G we construct a sequence of subgraphs of $G = G_0 \supset G_1 \supset \ldots \supset G_n$ such that each G_i is obtained from G_{i-1} by removing the edges of a cycle and such that G_n is a forest. The union of the edges we removed $\cup_{i=1}^{n}(E(G_{i-1}) \setminus E(G_i))$ then induces an even subgraph of G and together with $E(G_n)$ it forms a decomposition of G into an even graph and a forest. In the following we will prove that any connected graph G has a decomposition $E(G) = E_1 \cup E_2 \cup E_3$, where the subgraph of G induced by E_1 is a spanning tree in G, the subgraph of G induced by E_2 is an even graph, and the subgraph of G induced by E_3 is a star forest. This result is motivated by the study of HISTs in cubic graphs and the so-called 3-Decomposition Conjecture.

3.3.1 The 3-Decomposition Conjecture and HISTs in Cubic Graphs

If G is a cubic graph, then removing the edges of a spanning tree T in G results in a graph whose components are isolated vertices, cycles, and paths. A HIST T' in a cubic graph G is exactly a spanning tree such that the components of $G - E(T')$ are only isolated vertices and cycles. While there are many connected cubic graphs with no HIST it is an open problem whether every connected cubic graph contains a spanning tree T such that the paths in $G - E(T)$ has length exactly 1, i.e. they form a matching. This problem was formulated by Hoffmann-Ostenhof [Cam11] as a conjecture:

Conjecture 3.3.1 (3-Decomposition Conjecture). *Every connected cubic graph can be decomposed into a spanning tree, a collection of cycles, and a matching.*

Since a HIST in a cubic graph G is a spanning tree T such that $G - E(T)$ contains no paths, one can think of the paths in $G - E(T')$ for a spanning tree T' in G as a kind of "error term". So a connected cubic graph might not have a HIST, but if Conjecture 3.3.1 is true it always has a spanning tree such that the error term is just a matching.
Ozeki and Ye [Oze16] verified Conjecture 3.3.1 for 3-connected planar cubic graphs and this was extended to all planar cubic graphs by Hoffmann-Ostenhof, Kaiser, and Ozeki [Hof18]. Akbari, Jensen and Siggers [Akb15] took a slightly different approach and proved that any cubic graph has a decomposition into a spanning forest, a collection of cycles and a matching.

Inspired by the 3-Decomposition Conjecture we will in the next section formulate and prove a 3-decomposition theorem for all connected graphs.

3.3.2 A General 3-Decomposition Theorem

Above we considered cubic graphs in which case any even subgraph is a collection of disjoint isolated vertices and cycles. As a first step towards a 3-decomposition statement for general connected graphs we will thus replace the collection of cycles in the decomposition by an even subgraph. One might now be tempted to think that every connected graph admits a decomposition into a spanning tree, an even graph, and a matching. However, this is easily seen to be false since the complete bipartite graph $K_{2,n}$ for $n \geq 4$ has no such decomposition. Even if we restrict our attention to regular graphs there are graphs with no such decomposition:

Theorem 3.3.2. *For each $r \geq 4$, there exists an r-regular connected graph which has no decomposition into a spanning tree, an even graph, and a matching.*

Proof. Let $r \geq 4$ be given and let G be the graph obtained from K_{r+1} by subdividing each edge once. Let G' be a graph obtained from K_{r+1} by subdividing $r-2$ edges once and adding an edge between each pair of vertices of degree 2. For each vertex v of degree 2 in G, let G_v denote a copy of G'. Now let G'' be obtained from the disjoint union of G and all the graphs G_v by adding edges between v and the vertices of degree $r-1$ in G_v, for each vertex v of degree 2 in G. Note that G'' is r-regular and any decomposition of G'' into a spanning tree, an even graph, and a matching also induces such a decomposition of G. Clearly, the even graph cannot contain any edges of G, therefore this corresponds to a decomposition of G into a spanning tree and a matching. The graph G has $r(r+1)$ edges, and every spanning tree of G has $r + \frac{r(r+1)}{2}$ edges, thus the matching has to contain at least $\frac{r(r-1)}{2} \geq r+2$ edges. However, the size of a maximal matching in G is $r+1$, so G cannot be decomposed into a spanning tree and a matching. □

The above shows that we also need to adjust the requirement for the "error term" (the matching) in our 3-decomposition if we want to obtain a true statement for general connected graphs. So we now relax the condition on the error term and only require it to form a star forest. With this modification we do indeed obtain a true 3-decomposition statement for general connected graphs:

Theorem 3.3.3. *Every connected graph can be decomposed into a spanning tree, an even subgraph, and a star forest.*

Note that the construction in the proof of Theorem 3.3.2 shows that for r-regular graphs the size of the stars in the forest in Theorem 3.3.3 grows at least linearly in r.

In order to use induction it is more convenient to prove a slightly more technical version of Theorem 3.3.3. In order to formulate this we will need a few definitions. A cycle C in a connected graph G is called edge-separating if $G - E(C)$ is disconnected.

Definition 3.3.4 (fragile)**.** *A graph G is called **fragile**, if G is connected and every cycle of G is edge-separating.*

It suffices to prove that any fragile graph can be decomposed into a spanning tree and a star forest, since we can decompose any connected graph into an even graph and a fragile graph: given a connected graph G we can construct a sequence of connected subgraphs of $G = G_0 \supset G_1 \supset \ldots \supset G_n$ such that each G_i is obtained from G_{i-1} by removing the edges of a cycle and such that G_n is fragile (the removed edges will then induce an even graph and $E(G_i)$ induces a fragile graph). So in the following we will assume that the given graph is fragile.

Definition 3.3.5 (starlit)**.** *A spanning tree T of a graph G is called **starlit** if $G - E(T)$ is a star forest.*

Definition 3.3.6 (v-full)**.** *A spanning tree T of a graph G is called v**-full** for some vertex v in G, if all edges incident with v in G are also in T.*

We are now ready to formulate and prove a statement which implies Theorem 2.2.5. The extension which makes the induction work smoothly is that we can prescribe all edges incident to some given vertex to be in the spanning tree.

Theorem 3.3.7. *A fragile graph G has a starlit v-full spanning tree for any $v \in V(G)$.*

Proof. Let G be a counterexample of minimal size and let $v \in V(G)$ be such that G has no v-full spanning tree.

Claim 1: G is 2-connected.
Proof of the claim. Suppose the claim is false and u is a cut-vertex in G. Let K be a component of $G - u$, let G_1 be the subgraph of G induced by $K \cup \{u\}$, and let G_2 denote the graph induced by the edges in $G - E(G_1)$. We can assume that $v \in V(G_1)$. Clearly G_1 and G_2 are fragile and contain fewer edges than G, so G_1 contains a starlit v-full spanning tree T_1, and G_2 contains a starlit u-full spanning tree T_2. Now the union of T_1 and T_2 is a starlit v-full spanning tree in G. $\diamond$

Note that Claim 1 implies that the minimum degree of G is at least 2.

Claim 2: There are no adjacent vertices of degree 2 in G.
Proof of the claim. Suppose x and y are two adjacent vertices of degree 2 and let z denote the neighbour of y different from x. We may assume without loss of generality that $v \neq y$. The graph $G' = G - xy$ is fragile, so by minimality of G there exists a starlit v-full spanning tree T' of G'. If $v \neq x$, then $T = T'$ is also a starlit v-full spanning tree of G. If $v = x$, then we choose instead a starlit z-full spanning tree T'' of G'. Now $T'' + xy - yz$ is a starlit v-full spanning tree of G. $\diamond$

Let H be the subgraph of G induced by the vertices of degree at least 3.

Claim 3: H contains no isolated vertices and no cycles of length 3.
Proof of the claim. Suppose u is an isolated vertex in H. That is, u is a vertex of degree at least 3 in G all of whose neighbours have degree 2.
First, suppose $u = v$. Let x be a neighbour of u, and y the neighbour of x different from u. By Claim 2, y has degree at least 3 and is therefore not adjacent to u. Let G' be the graph obtained from G by removing x and adding the edge uy. Since u has only one neighbour of degree greater than 2 in G', every cycle through u is still edge-separating. Thus, G' is fragile and contains a starlit u-full spanning tree T'. Now $T = T' - uy + ux + xy$ is a starlit u-full spanning tree of G.
By the above we can assume $u \neq v$. The graph $G' = G - u$ is connected by Claim 1. Clearly G' is fragile and therefore contains a v-full starlit spanning tree T'. If v is a neighbour of u, then $T = T' + uv$ is a starlit v-full spanning tree of G. If v is not a neighbour of u, then adding an arbitrary edge incident with u to T' results in a starlit v-full spanning tree of G. This contradiction shows that the minimum degree of H is at least 1.
Finally, suppose H contains a cycle C of length 3. Since every vertex of C has degree at least 3, and since G is 2-connected, it is easy to see that C is not edge-separating, which contradicts G being fragile. $\diamond$

Claim 4: If u is a vertex in H different from v, then $d_H(u) \geq 2$.
Proof of the claim. Suppose u is a vertex of degree 1 in H, $u \neq v$, and x is the neighbour of u in H. First, suppose that v is not a degree 2 vertex adjacent to u in G. By Claim 1, the graph $G' = G - u$ is fragile, so it has a starlit v-full spanning tree T' by minimality of G. Now $T = T' + ux$ is a v-full starlit spanning tree of G. Thus, we can assume that v has degree 2 and is a neighbour of u in G. Let T'' be a starlit x-full spanning tree of G'. Clearly $T = T'' + uv$ is a v-full spanning tree of G. Since T'' is x-full, the spanning tree T is also starlit, contradicting our choice of G. $\diamond$

Claim 4 implies that there exists a cycle in H. The following claim shows that there are at most two vertices in H which have degree less than 3 in H.

Claim 5: If $d_H(u) = 2$, then either $u = v$ or $d_G(v) = 2$ and $uv \in E(G)$.
Proof of the claim. Suppose u is a vertex of degree 2 in H, $u \neq v$, and $d_G(v) \geq 3$ or $uv \notin E(G)$. Let x and y denote the neighbours of u in H. Note that all other neighbours of u in G have degree 2. Let G' be the graph obtained from $G - u$ by adding the edge xy. Claim 1 implies that G is connected and Claim 3 implies that G' has no multiple edges. For a cycle C' in G' containing xy, the corresponding cycle C in G, which is obtained from C' by replacing xy with the path xuy, is edge-separating if and only if C' is edge-separating. Thus, G' is fragile and contains a starlit v-full spanning tree T'. If $xy \in E(T')$, then $T = T' - xy + ux + uy$ is a starlit v-full spanning tree in G. Thus, we can assume $xy \notin E(T')$. Since $G' - E(T')$ is a star forest, at least one of x and y has degree 1 in $G' - E(T')$, say x. Now $T = T' + uy$ is a starlit v-full spanning tree in G. $\diamond$

Let C be a cycle in H for which the component K of $G - E(C)$ containing v has maximal size.

Claim 6: H is contained in the subgraph of G induced by $V(K) \cup V(C)$.
Proof of the claim. Suppose u is a vertex of degree at least 3 which is not in K or C. Let L denote the component of $G - E(C)$ containing u. There exists no cycle in $L \cap H$ since that cycle would contradict the choice of C. Claim 4 now implies that L contains a path P joining two vertices a and b on C such that all intermediate vertices are in $V(H) \setminus V(C)$. Let P_1 and P_2 be the two edge-disjoint subpaths of C joining a and b. We may assume P_2 contains some vertices of K. Now the cycle formed by the union of P and P_1 contradicts the choice of C. $\diamond$

Since G is fragile, the graph $G - E(C)$ is disconnected so there is a vertex u on C which is not in K. Clearly C is induced, so u has exactly two neighbours on C. Claim 6 implies that all neighbours of u not on C have degree 2. Now $d_H(u) = 2$ and $u \neq v$. By Claim 5, we have $d_G(v) = 2$ and $uv \in E(G)$, which implies that u is in K, contradicting the choice of u. $\square$

3.4 Paths whose Lengths Differ by 1 or 2

The material presented in this section essentially consists of a part of one research article [Lyna].

In this section we will study the existence of paths whose lengths differ by 1 or 2 and which join two prescribed vertices in a 2-connected subcubic graph. As also mentioned in Section 1.1 a result by Fan [Fan02] implies that if x and y are vertices in a 2-connected graph G where $d_G(z) = 3$ for all $z \in V(G) \setminus \{x, y\}$, then there exist two $x - y$ paths in G whose lengths differ by 1 or 2. In this section we will show that under some mild additional conditions we can allow up to two additional vertices besides x and y to also have degree 2 and still be guaranteed the existence of two $x - y$ paths whose lengths differ by 1 or 2.

To simplify parts of the proofs in this section we introduce the following definition.

Definition 3.4.1 ($f_C(x, y)$)**.** *Let C be a cycle in a graph and let x, y be two distinct vertices of C. We define $f_C(x, y)$ as the absolute difference of the lengths of the two $x - y$ paths on C.*

The first theorem we will prove is the following.

Theorem 3.4.2. *Let x, y, z be three distinct vertices in a subcubic 2-connected graph G. If $d(v) = 3$ for all $v \in V(G) \setminus \{x, y, z\}$, then there are two $x - y$ paths P_1, P_2 with $1 \le |E(P_1)| - |E(P_2)| \le 2$.*

Theorem 3.4.2 is an extension of the result implied by Fan's result mentioned above, since here we allow a third vertex to possibly have degree 2. Before we prove Theorem 3.4.2 it is convenient to prove the following slightly technical lemma.

Lemma 3.4.3. *Let G be a 2-connected graph which is not a 3-cycle, let $xy \in E(G)$ and let $z \in V(G) \setminus \{x, y\}$. If $d_G(x) \le 4$, $d_G(y) \le 4$, $d_G(z) \le 3$ and $d_G(v) = 3$ for all $v \in V(G) \setminus \{x, y, z\}$, then there are two $x - y$ paths P_1, P_2 in $G - xy$ with $1 \le |E(P_1)| - |E(P_2)| \le 2$.*

Proof. Suppose the theorem is false and let (G, x, y, z) be a counterexample where $|V(G)| + |E(G)|$ is minimum. Clearly we have that $|V(G)| \ge 4$. Let $G' = G - xy$ and note that $d_{G'}(v) \ge 2$ for every $v \in V(G') \setminus \{x, y\}$. By the choice of G, x, and y there are no two $x - y$ paths in G' whose lengths differ by 1 or 2.

Claim 1: G' is 2-connected.
Proof of the claim. First suppose G' is not 2-connected. Since G is not a cycle, there exists a 2-connected block B in G'. Let Q_1 be an $x - B$ path and let Q_2 be a $B - y$ path in G'. Since G is 2-connected, the block graph of G' is a path, so we must have $Q_1 \cap Q_2 = \emptyset$. Let q_1 and q_2 be the endvertices of Q_1 and Q_2 in B, respectively. Note that the only vertices of degree 2 in B are q_1, q_2, and possibly z. Let $B' = B + q_1q_2$

if $q_1q_2 \notin E(B)$ and $B' = B$ otherwise. If B' is a triangle, then $B' = B$ and there are two $q_1 - q_2$ paths R_1, R_2 in B of lengths 1 and 2. If B' is not a triangle, then by minimality of G there are two $q_1 - q_2$ paths R_1, R_2 in B whose lengths differ by 1 or 2. Now $P_1 = Q_1 \cup R_1 \cup Q_2$ and $P_2 = Q_1 \cup R_2 \cup Q_2$ are two $x - y$ paths in G' whose lengths differ by 1 or 2. $\diamond$

A 2-edge-cut $\{e, f\}$ in G' is called *non-trivial* if both components of $G - e - f$ contains at least two vertices.

Claim 2: There are no non-trivial 2-edge-cuts in G'.
Proof of the claim. Suppose the claim is false and let $e, f \in E(G')$ be a non-trivial 2-edge-cut for which the component K of $G' - e - f$ containing at most one of x, y, z has minimum order. By the choice of e and f the component K is 2-connected. Let e_K and f_K denote the ends of e and f in K, respectively. We may assume $x \notin V(K)$. First suppose that also $y \notin V(K)$. By Claim 1, the graph G' is 2-connected so there are two disjoint $\{x, y\} - \{e_K, f_K\}$ paths Q_1 and Q_2 in G'. If K is a triangle, then let P_1, P_2 denote the $e_K - f_K$ paths in K of lengths 1 and 2, respectively. If K is not a triangle, then by minimality of G, there are two $e_K - f_K$ paths P_1, P_2 in K whose lengths differ by 1 or 2. Now $Q_1 \cup P_1 \cup Q_2$ and $Q_1 \cup P_2 \cup Q_2$ are two $x - y$ paths in G' whose lengths differ by 1 or 2.
By the above we may assume $y \in V(K)$ and hence $z \notin V(K)$. By possibly renaming e and f we can also assume that $y \neq e_K$. If K is a triangle, then let P_1, P_2 denote the $e_K - y$ paths in K of lengths 1 and 2, respectively. If K is not a triangle, then by minimality of G there are two $e_K - y$ paths P_1, P_2 in K whose lengths differ by 1 or 2. Let Q be an $x - e_K$ path in G' having no edges in K. Now $Q \cup P_1$ and $Q \cup P_2$ are two $x - y$ paths in G' whose lengths differ by 1 or 2. $\diamond$

Note that Claim 1 implies that if $u, v \in \{x, y, z\}$ and $d_G(u) = d_G(v) = 2$, then u and v are non-adjacent. Let G'' denote the graph obtained from G' by suppressing the vertices of degree 2 and note that G'' is cubic. If G'' is not simple, then Claim 2 implies that G'' consists of three parallel edges. In this case G' must be a 3-rail with 4 or 5 vertices and it is easy to see that there are two $x - y$ paths in G' whose lengths differ by 1 or 2. Thus, we may assume that G'' is simple.
By Claim 2, the graph G'' is 3-connected, so by Theorem 3.1.3 there exists an induced non-separating cycle C in G' containing x and not containing y. Let $w \in V(C) \setminus \{z\}$ be such that $f_C(x, w) \in \{1, 2\}$. Let P_1 and P_2 denote the two $x - w$ paths in C, and let Q be a $w - y$ path intersecting C only in w. Now $P_1 \cup Q$ and $P_2 \cup Q$ are two $x - y$ paths in G' whose lengths differ by 1 or 2. $\square$

With Lemma 3.4.3 available we can easily prove Theorem 3.4.2:

Proof of Theorem 3.4.2. The statement is trivial if G is a 3-cycle. If $xy \in E(G)$ and G is not a 3-cycle, then by Lemma 3.4.3 there exist two $x - y$ paths in $G - xy$ whose lengths differ by 1 or 2. If $xy \notin E(G)$, then $G' = G + xy$ satisfies the conditions of Lemma 3.4.3. Thus there are two $x - y$ paths in G whose lengths differ by 1 or 2. $\square$

The next we will look at is the case where there are exactly four vertices x_1, x_2, y, z of degree 2 in a 2-connected subcubic graph G. We still want to conclude that there are two $x_1 - x_2$ paths in G whose lengths differ by 1 or 2. However, for this to be true it turns out that we need further assumptions. Therefore we add the assumptions that x_1 and x_2 are non-adjacent and that x_1 and x_2 are not contained in a 4-cycle in G. These additional assumptions are necessary as can be seen by the graph G in Figure 3.10. Clearly this graph G shows that x_1 and x_2 must not be opposite vertices in a 4-cycle.

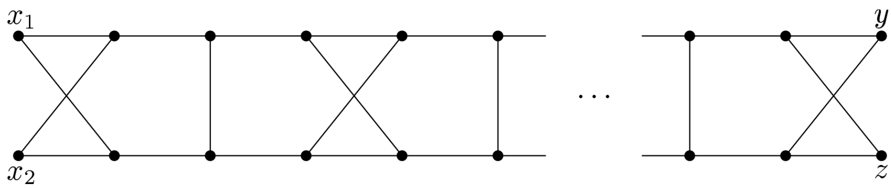


Figure 3.10: A graph where the lengths of no two $x_1 - x_2$ paths differ by 1 or 2.

To see that it is also necessary to assume that x_1 and x_2 are non-adjacent, consider the graph G' obtained from the graph G in Figure 3.10 by removing the 4-cycle C containing x_1 and x_2, and let x_1' and x_2' denote the two vertices of degree 2 in G' which had degree 3 in G. Note that x_1' and x_2' are adjacent and not contained in a 4-cycle in G'. It is easy to check that there are no two $x_1' - x_2'$ paths in G' whose lengths differ by 1 or 2. This shows that it is indeed necessary to assume that x_1 and x_2 are non-adjacent in the following theorem.

Theorem 3.4.4. *Let x_1, x_2, y, z be four distinct vertices of degree 2 in a 2-connected subcubic graph G. If $d(v) = 3$ for all $v \in V(G) \setminus \{x_1, x_2, y, z\}$, the vertices x_1 and x_2 are not adjacent and x_1 and x_2 are not opposite vertices in a 4-cycle in G, then there are two $x_1 - x_2$ paths P_1, P_2 with $|E(P_1)| - |E(P_2)| \in \{1, 2\}$.*

Proof. Suppose the theorem is false and let (G, x_1, x_2, y, z) be a counterexample where G has minimum size. Clearly we can assume that $V(G) \geq 5$.

Claim 1: $x_1y, x_1z, x_2y, x_2z \notin E(G)$.
Proof of the claim. Suppose the claim is false and assume $x_1y \in E(G)$. Let x_1' be the neighbour of x_1 distinct from y and let y' be the neighbour of y distinct from x_1'. Since G is 2-connected and $x_1x_2 \notin E(G)$ we have $x_1' \neq y'$ and $x_1' \neq x_2$.
First suppose $x_1'y' \in E(G)$. In this case, since G is 2-connected and not a 4-cycle, both x_1' and y' have degree 3 in G. We can assume that there are no two $x_1' - x_2$ paths in $G' = G - x_1 - y$ whose lengths differ by 1 or 2, since otherwise there are also two $x_1 - x_2$ paths in G whose lengths differ by 1 or 2. Thus, by minimality of G, we have $x_1'x_2 \in E(G')$ or x_1' and x_2 are contained in a 4-cycle. Similarly, we can assume that there are no two $y' - x_2$ paths in G' whose lengths differ by 1 or 2 and again, the minimality of G implies that $y'x_2 \in E(G')$ or y' and x_2 are contained in

a 4-cycle. Note that x_2 cannot be adjacent to both x_1' and y' since in this case G is a 5-cycle with a chord and contains only three vertices of degree 2. Thus x_2 is contained in a 4-cycle which also contains x_1' and it follows by 2-connectivity of G that G is a 6-cycle with the chord $x_1'y'$. It is easy to see that in this case there are two $x_1 - x_2$ paths whose lengths differ by 1 or 2.
By the above we can assume $x_1'y' \notin E(G)$. Let G'' be the graph obtained from $G - x_1 - y$ by adding the edge $e = x_1'y'$. Note that G'' is 2-connected and x_2, z are the only vertices of degree 2 in G''. By Theorem 3.4.2, there are two $x_1' - x_2$ paths Q_1, Q_2 in G'' whose lengths differ by 1 or 2. If Q_1 does not contain e let P_1 be the $x_1 - x_2$ path consisting of x_1x_1' and Q_1. If Q_1 contains e, let P_1 be the $x_1 - x_2$ path we obtain from Q_1 by replacing e with the path x_1yy'. We analogously define an $x_1 - x_2$ path P_2 using Q_2. Note that $|E(P_1)| = |E(Q_1)| + 1$ and $|E(P_2)| = |E(Q_2)| + 1$, so P_1 and P_2 are as desired. $\diamond$

Recall that a 2-edge-cut $\{e, f\}$ in G is called *non-trivial* if both components of $G - e - f$ contains at least two vertices.

Claim 2: There are no non-trivial 2-edge-cuts in G.
Proof of the claim. First suppose there exists a non-trivial 2-edge-cut $\{e, f\}$ such that $G - e - f$ has a component K containing at most one of x_1, x_2, y, z. We choose such a 2-edge-cut $\{e, f\}$ for which the component K containing at most one vertex in $\{x_1, x_2, y, z\}$ has minimal size. Note that by this choice of $\{e, f\}$, the component K is 2-connected. Let e_K, f_K denote the ends of e and f in K. If K does not contain any of x_1, x_2, then, since G is 2-connected, there exist two disjoint $\{x_1, x_2\} - \{e_K, f_K\}$ paths P_1, P_2 in $G - E(K)$. Since K is 2-connected, by Theorem 3.4.2 there are two $e_K - f_K$ paths Q_1, Q_2 in K whose lengths differ by 1 or 2. Now $P_1 \cup Q_1 \cup P_2$ and $P_1 \cup Q_2 \cup P_2$ are two $x_1 - x_2$ paths whose lengths differ by 1 or 2. Thus, we may assume $x_1 \in V(K)$, and note that the minimality of K implies that $x_1 \neq e_K$. Again, by Theorem 3.4.2, there are two $x_1 - e_K$ paths Q_1, Q_2 in K whose lengths differ by 1 or 2. Let P be an $e_K - x_2$ path in $G - E(K)$. Now $Q_1 \cup P$ and $Q_2 \cup P$ are two $x_1 - x_2$ paths whose lengths differ by 1 or 2. Thus, we have established the following:

(*) For every non-trivial 2-edge-cut $\{e, f\}$ in G, each component of $G - e - f$ contains two vertices in $\{x_1, x_2, y, z\}$.

Let $\{e, f\}$ be a non-trivial 2-edge-cut for which the component K of $G - e - f$ containing x_1 has minimal size. Now K is 2-connected by (*), Claim 1, and since x_1 is not adjacent to x_2. Let e_K and f_K denote the ends of e and f in K. Let L be the component of $G - e - f$ different from K. First suppose $x_2 \in V(K)$. By (*), we have $y \in V(L)$ and $z \in V(L)$, so the minimality of G implies that there are two $x_1 - x_2$ paths in K whose lengths differ by 1 or 2. Thus we may assume $x_2 \in V(L)$. By (*), we may assume $y \in V(K)$ and $z \in V(L)$. If x_1 is not adjacent to e_K and x_1, e_K are not opposite vertices in a 4-cycle, then by minimality of G there are two $x_1 - e_K$ paths in K whose lengths differ by 1 or 2. Since these paths can be extended to two desired $x_1 - x_2$ paths in G, we may assume that x_1 is adjacent to e_K or x_1 and e_K are opposite vertices in a 4-cycle. Similarly we can assume that either x_1 is adjacent to f_K or x_1 and f_K are opposite vertices in a 4-cycle. If x_1 is adjacent to both e_K and f_K, then K is a 4-cycle where x_1 and y are opposite vertices. In this case there exist $x_1 - e_K$ paths in K of lengths 1 and 3. Thus, we may assume that x_1 and f_K are not adjacent and therefore they must be opposite vertices in a 4-cycle. If also x_1 and e_K are opposite vertices in a 4-cycle, then there are $x_1 - e_K$ paths of length 2 and 4 in K. If x_1 is adjacent to e_K then there are $x_1 - e_K$ paths of length 1 and 3 in K. So in any case there are two $x_1 - e_K$ paths whose lengths differ by 1 or 2 in K and these paths can be extended to two desired $x_1 - x_2$ paths in G. ◇

Let G' denote the graph obtained from G by suppressing the vertices of degree 2. By Claim 2 and the fact that G' is cubic, the graph G' is simple and 3-connected. Now Theorem 3.1.3 implies that there is a non-separating induced cycle C in G containing x_1 and not containing z.
First suppose $x_2 \notin V(C)$. There are two different vertices $v_1, v_2 \in V(C)$ such that $f_C(x_1, v_1) = f_C(x_1, v_2) \in \{1, 2\}$. In particular, there exists a vertex $v \in V(C)$ different from y for which $f_C(x_1, v) \in \{1, 2\}$. Let P be a $v - x_2$ path in $G - E(C)$, and let Q_1, Q_2 be the two $x_1 - v$ paths on C. Now $Q_1 \cup P$ and $Q_2 \cup P$ are two $x_1 - x_2$ paths whose lengths differ by 1 or 2.
By the above we may assume $x_2 \in V(C)$. Let C_1 and C_2 be the two $x_1 - x_2$ paths on C. We may assume $|E(C_1)| \leq |E(C_2)|$. If $|E(C_1)| = |E(C_2)|$, we may assume that C_2 does not contain y. Let v be the neighbour of x_2 on C_1. Note that $v \in V(C) \setminus \{x_1, x_2, y\}$. For a vertex $w \in V(C_2)$, let $Q_1(w) = x_1 C_2 w$ and $Q_2(w) = w C_2 x_2$. Moreover, for $w \in V(C_2)$, let

$$f(w) = |E(C_1)| - 2 + |E(Q_2(w))| - |E(Q_1(w))|\,.$$

Note that as we move from x_1 to x_2 along C_2, the function f decreases by 2 at every vertex. We have $f(x_1) = |E(C)| - 2$ so since $|E(C)| \geq 5$, we get $f(x_1) \geq 3$.
By definition, $f(x_2) = |E(C_1)| - |E(C_2)| - 2 \leq -2$. If $f(x_2) < -2$, then there are two vertices $w_1, w_2 \in V(C_2) \setminus \{x_1, x_2\}$ such that $|f(w_1)| = |f(w_2)| \in \{1, 2\}$. In particular, there exists a vertex $w \in V(C_2) \setminus \{x_1, x_2, y\}$ with $|f(w)| \in \{1, 2\}$. If $f(x_2) = -2$, then $|E(C_1)| = |E(C_2)|$ and y is not contained in C_2. Also in this case there exists a vertex $w \in V(C_2) \setminus \{x_1, x_2, y\}$ with $|f(w)| \in \{1, 2\}$.
In each case, we can choose $w \in V(C_2)$ such that $d(w) = 3$ and $|f(w)| \in \{1, 2\}$. Let P be a $v - w$ path in G', see Figure 3.11. Now $(C_1 - vx_2) \cup P \cup Q_2(w)$ and $Q_1(w) \cup P \cup \{vx_2\}$ are two $x_1 - x_2$ paths and the difference of their lengths is

$$|E(C_1)| - 2 + |E(Q_2(w))| - |E(Q_1(w))| = |f(w)|,$$

which is 1 or 2 by our choice of w. □

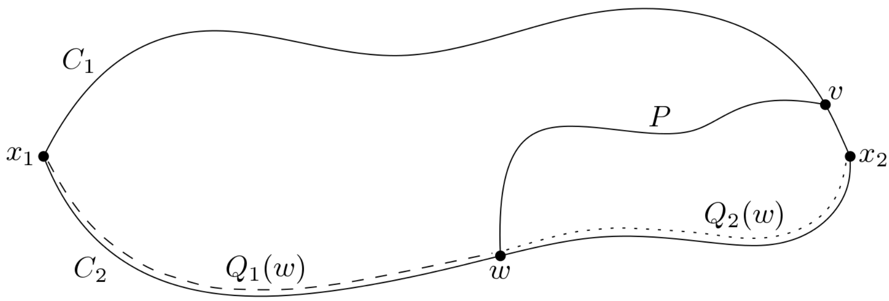


Figure 3.11: Proof of Theorem 3.4.4.

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