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Multi-domain Stability and Modal Analysis Applied to Gas Foil Bearings: Three Approaches

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Abstract

The dynamic characteristics of rotors supported by Gas Foil Bearings (GFBs) are commonly assessed using linear force coefficients, which are usually derived from an analytical perturbation in combination with an implicit compliance model. This remains common practice even though discrepancies have been reported between the Onset Speed of Instability (OSI) predicted using such linear coefficients and observations from non-linear time integration. For the first time, the present paper pinpoints the root cause of this discrepancy by demonstrating an extended perturbation, akin to that used for tilting pad journal bearings, to predict exactly the same OSIs as obtained from an eigenvalue analysis based on the Jacobian of the full non-linear system of equations. To demonstrate this, the OSIs predicted using (i) the classical perturbation, (ii) the extended perturbation, and (iii) the Jacobian eigenvalues are compared over a range of compliance levels covering essentially rigid bearings to modern heavily loaded industrial GFBs. Using carefully aligned implementations, the deficiency of the classical method is demonstrated to increase with the compliance level (up to 13 % for the benchmark case), while the extended perturbation provides OSIs in agreement with the Jacobian eigenvalues for all compliance levels. Furthermore, the mode shapes attainable from the three approaches, encompassing only the rotor, the rotor and foil, and the rotor, foil and pressure respectively, are compared.

Keywords:

Gas Foil Bearings, Perturbation, Stability

Nomenclature

DOF	Degree of Freedom	$e_{x,y}, \varepsilon_{x,y}$	Journal eccentricity components, $\varepsilon = eC^{-1}$
FE	Finite Element	E_b	Young's modulus of bump foil material
GFB	Gas Foil Bearing	f_α	Functions $f_{x,y}$ and $f_{w_j} = W_j(\theta)$
NR	Newton–Raphson	$f_{x,y}$	Functions $f_x = \cos \theta$ and $f_y = \sin \theta$
OSI	Onset Speed of Instability	$f_{h_c}(\theta, p)$	Axially uniform compliance function
RE	Reynolds Equation	g	Gravitational acceleration
RPM	Revolutions per Minute	$h, \tilde{h}$	Film height, $\tilde{h} = hC^{-1}$
SEFM	Simple Elastic Foundation Model	$h_0, \tilde{h}_0$	Static film height, $\tilde{h}_0 = h_0C^{-1}$
($\cdot$)	Time derivative, $d/d\tau$	h_b	Bump foil height
($\cdot$) ²	Time derivative, $d^2/d\tau^2$	$h_c, \tilde{h}_c$	Film height (compliant), $\tilde{h}_c = h_cC^{-1}$
($\cdot$)	Non-dimensional quantity	$h_r, \tilde{h}_r$	Film height (rigid), $\tilde{h}_r = h_rC^{-1}$
($\cdot$) ^e	Quantity at element level	k_0	Nominal bump stiffness per unit area
A	In-film element area	k	Bump stiffness per unit area
C	Radial clearance		

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k_c	Dynamic foil stiffness
$L, \tilde{L}$	Bearing length, $\tilde{L} = LR^{-1}$
l_0	Bump half length
m'_f	Average foil mass per circumferential length
m_r	Rotor mass
n_θ	Number of circumferential nodes
n_z	Number of axial nodes
$P_c(\lambda)$	Characteristic polynomial
$p, \tilde{p}$	Film pressure, $\tilde{p} = p_a^{-1}p$
$p_0, \tilde{p}_0$	Static pressure, $\tilde{p}_0 = p_a^{-1}p_0$
$p_\gamma, \tilde{p}_\gamma$	Dynamic pressure, rotor DOF γ , $\tilde{p}_\gamma = p_a^{-1}p_\gamma$
$p_{wj}, \tilde{p}_{wj}$	Dynamic pressure, foil DOF j , $\tilde{p}_{wj} = p_a^{-1}p_{wj}$
p_a	Ambient pressure, fluid pressure scaling
$\tilde{p}_m$	Mean axial pressure
R	Journal radius
R_b	Bump radius of curvature
S_τ	Non-dimensional group, $S_\tau = 6R^2\mu\omega_\tau C^{-2}p_a^{-1}$
S_b	Bump foil pitch
t	Physical time
t_b	Thickness of bump foil
$w_j, \tilde{w}_j$	j -th discrete foil deformation, $\tilde{w}_j = w_j C^{-1}$
W_j	j -th foil DOF film height weight function
W_x	Static vertical load
X	Oscillation amplitude
$z, \tilde{z}$	Axial gas film coordinate, $\tilde{z} = zR^{-1}$
z_j	j -th axial nodal coordinate
$\Delta\theta_j, \Delta\tilde{z}_j$	Span associated with j -th grid location
δ	Logarithmic decrement
$\varepsilon_x, \varepsilon_y$	Eccentricity ratios
η	Hysteretic foil damping constant
Λ_γ	Rotor perturbation, $\Delta\varepsilon_\gamma e^{i\tilde{\omega}_\tau\tau}$, $\gamma = x, y$
Λ_j	Foil perturbation, $\Delta w_j e^{i\tilde{\omega}_\tau\tau}$, $j = 1, \dots, n_\theta$
λ	Eigenvalue
$\hat{\lambda}$	Pseudo eigenvalue
μ	Dynamic gas viscosity
ν_b	Poisson's ratio of bump foil material
$\Omega, \tilde{\Omega}$	Angular velocity $\tilde{\Omega} = \omega_\tau^{-1}\Omega$
Ω_{OSI}	Angular velocity at the OSI
$\omega_f, \tilde{\omega}_f$	Foil oscillation frequency, $\tilde{\omega}_f = \omega_\tau^{-1}\omega_f$
$\omega_s, \tilde{\omega}_s$	Perturbation/whirl frequency, $\tilde{\omega}_s = \omega_\tau^{-1}\omega_s$
ω_τ	Characteristic frequency
ρ_b	Density of bump foil material
τ	Dimensionless time, $\tau = \omega_\tau t$
θ	Circumferential angle
θ_b	Bump half angle
θ_i	i -th circumferential nodal coordinate
θ_l	Pad leading edge angle
θ_t	Pad trailing edge angle
ζ	Damping ratio
$\mathbf{f}_b, \tilde{\mathbf{f}}_b$	Bearing force vector, $\tilde{\mathbf{f}}_b = p_a^{-1}R^{-2}\mathbf{f}_b$

$\mathbf{f}_G$	Coupled system residual function
$\mathbf{f}_{G,t}$	Unsteady part of $\mathbf{f}_G$
$\mathbf{f}_{G,s}$	Steady non-constant part of $\mathbf{f}_G$
$\mathbf{f}_{G,0}$	Constant part of $\mathbf{f}_G$
$\mathbf{f}_p, \tilde{\mathbf{f}}_p$	Foil structure pressure loads, $\tilde{\mathbf{f}}_p = p_a^{-1}R^{-2}\mathbf{f}_p$
$\mathbf{f}_{rw}, \tilde{\mathbf{f}}_{rw}$	Rotor static force vector, $\tilde{\mathbf{f}}_{rw} = p_a^{-1}R^{-2}\mathbf{f}_{rw}$
$\mathbf{h}_r, \tilde{\mathbf{h}}_r$	Rigid film heights at FE nodes
$\mathbf{h}_c, \tilde{\mathbf{h}}_c$	Compliant film heights at FE nodes
$\mathbf{p}, \tilde{\mathbf{p}}$	FE nodal pressures, $\tilde{\mathbf{p}} = p_a^{-1}\mathbf{p}$
$\tilde{\mathbf{r}}$	Coupled system residual
$\tilde{\mathbf{r}}_f$	Foil structure residual
$\tilde{\mathbf{r}}_{p_\gamma}$	FE source term, 1 st order RE
$\tilde{\mathbf{r}}_{\text{RE}}$	FE residual
$\tilde{\mathbf{r}}_r$	Rotor residual vector
$\mathbf{v}_i$	i -th eigenvector
$\mathbf{w}, \tilde{\mathbf{w}}$	Foil structure DOF vector, $\tilde{\mathbf{w}} = C^{-1}\mathbf{w}$
$\mathbf{w}_0, \tilde{\mathbf{w}}_0$	Foil structure static equilibrium vector
$\mathbf{x}$	Generic DOF vector
$\mathbf{x}_f, \tilde{\mathbf{x}}_f$	Foil structure DOF vector, $\tilde{\mathbf{x}}_f = C^{-1}\mathbf{x}_f$
$\mathbf{x}_r, \tilde{\mathbf{x}}_r$	Rotor DOF vector, $\tilde{\mathbf{x}}_r = C^{-1}\mathbf{x}_r$
$\mathbf{x}_{r0}, \tilde{\mathbf{x}}_{r0}$	Rotor static equilibrium
$\mathbf{z}$	Coupled system state vector
$\mathbf{z}_0$	Static system equilibrium
$\mathbf{z}_f, \mathbf{z}'_f$	Foil structure state vector
$\mathbf{z}_r$	Rotor state vector
$\mathbf{A}_f$	Foil structure matrix
$\mathbf{A}'_f$	Massless foil structure matrix
$\mathbf{A}_G$	Global linearised system matrix
$\mathbf{A}_{p_\gamma}$	FE coefficient matrix, 1 st order RE
$\mathbf{A}_\lambda$	Lambda matrix
$\mathbf{A}_{\text{RE}}$	RE FE matrix, Couette & Poiseuille terms
$\mathbf{A}_r$	Rotor state-space inertia
$\mathbf{B}$	Shape function derivatives matrix
$\mathbf{B}_f$	Foil structure stiffness/damping
$\mathbf{B}'_f$	Massless foil structure stiffness/damping
$\mathbf{B}_{\text{RE}}$	RE FE matrix, squeeze term
$\mathbf{B}_r$	Rotor state-space stiffness/damping
$\mathbf{D}(\omega_s, \Omega)$	Generic frequency dependent damping
$\mathbf{D}_f, \tilde{\mathbf{D}}_f$	Foil structure damping, $\tilde{\mathbf{D}}_f = \omega_\tau C p_a^{-1} R^{-2} \mathbf{D}_f$
$\tilde{\mathbf{D}}_{g,2\text{DOF}}$	Gas damping, two-DOF perturbation
$\tilde{\mathbf{D}}_{g,x\text{DOF}}$	Gas damping, extended perturbation
$\mathbf{D}_r, \tilde{\mathbf{D}}_r$	Rotor damping, $\tilde{\mathbf{D}}_r = \omega_\tau C p_a^{-1} R^{-2} \mathbf{D}_r$
$\mathbf{C}_{\text{RE}}$	RE FE matrix, local expansion term
$\mathbf{H}_r$	Mapping from $\tilde{\mathbf{x}}_r$ to $\tilde{\mathbf{h}}_r$
$\mathbf{H}_{h_c p}$	Pressure to compliant film height mapping
$\mathbf{H}_{f_b p}$	Bearing pressure to force mapping
$\mathbf{I}$	Identity matrix
$\mathbf{J}_{z,s}$	Jacobian of $\mathbf{f}_{G,s}$ with respect to $\mathbf{z}$
$\mathbf{J}_{z,t}$	Jacobian of $\mathbf{f}_{G,t}$ with respect to $\mathbf{z}$
$\mathbf{J}_{\dot{z}}$	Jacobian of $\mathbf{f}_{G,t}$ with respect to $\dot{\mathbf{z}}$

$\mathbf{K}(\omega_s, \Omega)$	Generic frequency dependent stiffness	$\mathbf{M}$	Generic mass matrix
$\tilde{\mathbf{K}}_{g,2\text{DOF}}$	Gas stiffness, two-DOF perturbation	$\mathbf{M}_f, \tilde{\mathbf{M}}_f$	Foil structure mass, $\tilde{\mathbf{M}}_f = \omega_r^2 C p_a^{-1} R^{-2} \mathbf{M}_f$
$\tilde{\mathbf{K}}_{g,x\text{DOF}}$	Gas stiffness, extended perturbation	$\mathbf{M}_r, \tilde{\mathbf{M}}_r$	Rotor mass, $\tilde{\mathbf{M}}_r = \omega_r^2 C p_a^{-1} R^{-2} \mathbf{M}_r$
$\mathbf{K}_f, \tilde{\mathbf{K}}_f$	Foil structure stiffness, $\tilde{\mathbf{K}}_f = C p_a^{-1} R^{-2} \mathbf{K}_f$	$\mathbf{N}$	Shape function matrix
		$\mathbf{Z}$	Extended perturbation complex stiffness

1. Introduction

The Gas Foil Bearing (GFB) offers a mechanically simple, self-acting and oil-free means for support of lightweight high-speed rotating machinery, making it a perfect and environmentally friendly fit for a wide range of green technologies. As GFBs provide only limited damping, reliable tools for prediction of stability limits and other rotor-bearing system characteristics remain essential to their application. Fundamentally, the stability limit can be assessed using non-linear or linear analysis. In the former case, the full non-linear system of equations is formulated and solved either in a decoupled or simultaneous fashion. For a decoupled model where the physical domains are treated separately, the stability limit can be assessed using brute-force by integrating in time with steps sufficiently small to compensate for the decoupling. For a simultaneous formulation, stability assessment using time integration is likewise possible [1, 2], but it is more efficient in this case to base the analysis on the Jacobian eigenvalues [3]. Having obtained these eigenvalues, it has recently been established that Campbell diagrams can furthermore be extracted [4, 5]. In a linear analysis, the gas film forces are represented by equivalent force coefficients dependent on the operational conditions and excitation frequency. Such force coefficients are found by perturbing the rotor from its static equilibrium either numerically or analytically as proposed for rigid gas bearings by Lund [6] in 1968. The analytical approach was extended to compliant bearings by Peng and Carpino [7], who incorporated the compliance implicitly through the film height function to effectively eliminate the foil equations of motion—an approach that has since been applied extensively [7–16].

Previous studies [1, 2, 8, 17, 18] have investigated the accuracy of the Onset Speed of Stability (OSI) predicted from the perturbation method for compliant type bearings and found varying levels of discrepancy. The main contribution of the present work is, for the first time, to pinpoint the root cause of this discrepancy by demonstrating an extended version of the perturbation method to predict OSIs in agreement with results from an eigenvalue analysis based on the full non-linear system Jacobian.

Similar comparative investigations have previously been published by the authors [1, 2] indicating that the discrepancy of the perturbation OSI increases with the compliance level. In previous works, the coupled system OSI was assessed by time integration, whereas in the present work it is evaluated through system Jacobian eigenvalues. Having expressions for the Jacobian matrices available, this is far more efficient, but it does not change the predicted OSIs. More importantly, three issues related to the consistency between the perturbation and time domain models have been identified and addressed. The first and most critical change is for the assumption of rotor-synchronous viscous foil damping in the time domain model to be replaced by an iterative scheme ensuring that the foil oscillation frequency matches the frequency of the unstable system mode. This means that the equivalent time domain foil damping at the OSI now matches the foil damping applied in the first order Reynolds Equation (RE). This has previously been found to be negligible for a very stiff bearing [3], but has recently been further investigated for higher compliance levels by Pronobis and Liebich [17] proving this inconsistency to account for a significant portion of the reported OSI discrepancies. In [17] the rotor-synchronous foil damping is maintained, while the frequency domain damping expression is consistently adjusted leading to an unprecedented level of agreement between the predicted OSIs (though still deteriorating with compliance level). The strategy for obtaining consistency in [17] is hence contrary to the one adopted in the present work where the damping in both models is based on the instantaneous oscillation frequency. The second model change is related to the enforcement of the Gümbel condition in the first order perturbation equations. This is now performed during the dynamic pressure integration rather than through constraints on the dynamic pressure solution (which would be adequate for a Reynolds-type boundary condition). Thirdly, the compliant film height due to the dynamic pressure in the first order perturbation equations is now calculated from the axially averaged dynamic pressure, exactly as for the static film height in the zeroth order perturbation equation and for the film height in the coupled model.

With the above model changes in place, the present paper presents OSIs calculated for a benchmark rotor–GFB system for various compliance levels in three different ways:

1. Using force coefficients obtained through the classical perturbation with an implicit film height description to provide force coefficients related to the rotor Degrees of Freedom (DOFs) only.
2. Using force coefficients obtained through an extended perturbation incorporating the foil compliance explicitly through a number of dedicated foil DOFs akin to the pad DOFs (pitch, radial displacement and tangential displacement) used in analysis of tilting pad journal bearings.
3. Using the eigenvalues of the Jacobian matrix of the full non-linear system of equations.

Lastly, a secondary contribution of the presented work is the calculation and visualization of the multi-domain mode shapes obtainable from the coupled system formulation and, partly, from the extended perturbation. The article thus presents the multi-domain mode shapes encompassing intertwined rotor, foil and pressure states and compares these to the mode shapes attainable from the force coefficient methods.

2. Bearing Configuration and Basic Modelling

To facilitate an unambiguous study of the pressure–compliance dependency, the present work is based on the simplest feasible rotor–GFB system. This comprises a widely studied bearing configuration, the Simple Elastic Foundation Model (SEFM) for the compliant structure and a point mass rotor. The mathematical models applied in each domain are treated in the following sections and the properties of the studied system are given in table 1. All quantities in all three sub domains are non-dimensionalized using common scalings: the frequency ω_τ , ambient pressure p_a , the clearance C for radial lengths and the radius R for axial and circumferential lengths.

2.1. Fluid Film

Assuming ideal gas behaviour and defining the circumferential and axial coordinates θ and $\tilde{z}$, the isothermal, compressible and transient Reynolds equation (RE) can be written in terms of pressure p and film height h as

$$\frac{\partial}{\partial \theta} \left(\tilde{p} \tilde{h}^3 \frac{\partial \tilde{p}}{\partial \theta} \right) + \frac{\partial}{\partial \tilde{z}} \left(\tilde{p} \tilde{h}^3 \frac{\partial \tilde{p}}{\partial \tilde{z}} \right) = S_\tau \tilde{\Omega} \frac{\partial}{\partial \theta} (\tilde{p} \tilde{h}) + 2S_\tau \frac{\partial}{\partial \tau} (\tilde{p} \tilde{h}) \quad \text{where} \quad S_\tau = \frac{6R^2 \mu \omega_\tau}{C^2 p_a} \quad (1)$$

and μ is the gas viscosity. Following the Bubnov–Galerkin finite element (FE) approach, eq. (1) is discretized over half the bearing surface (to exploit symmetry conditions) using a regular grid of $n_\theta \times n_z$ nodes connected by bilinear quadrilateral elements as previously described by the authors [16, 20]. The FE approach is straightforward to combine with both implicit and explicit compliance models and performs well despite the fact that it does not guarantee local conservation, as could have been achieved using finite volume. Introducing the non-dimensional nodal pressure vector $\tilde{\mathbf{p}}$, the resulting element level Galerkin residual equations are

$$\int_{A^e} -\mathbf{B}^T \tilde{p} \tilde{h}^3 \mathbf{B} + S_\tau \tilde{\Omega} \mathbf{B}^T \tilde{h} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \mathbf{N} - 2S_\tau \mathbf{N}^T \tilde{h} \mathbf{N} dA \tilde{\mathbf{p}}^e - 2S_\tau \int_{A^e} \mathbf{N}^T \tilde{h} \mathbf{N} dA \dot{\tilde{\mathbf{p}}}^e = \tilde{\mathbf{r}}_{\text{RE}}^e, \quad (2)$$

where $\mathbf{N}$ is the element shape function matrix, $\tilde{\mathbf{r}}_{\text{RE}}^e$ is the element level residual and $\mathbf{B} = \left[\partial \mathbf{N}^T / \partial \theta \quad \partial \mathbf{N}^T / \partial \tilde{z} \right]^T$. The system level residual equations are obtained by the usual element summation. The film height $\tilde{h}$ is composed of a rigid and a compliant contribution with the former given from the rotor DOF vector $\tilde{\mathbf{x}}_r$, while the latter depends either on the pressure $\tilde{\mathbf{p}}$ for an implicit formulation or on the foil DOF vector $\tilde{\mathbf{x}}_f$ for an explicit formulation. It is convenient to partition eq. (2) as

$$\mathbf{A}_{\text{RE}} (\tilde{\mathbf{p}}, \tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) \tilde{\mathbf{p}} + \mathbf{B}_{\text{RE}} (\tilde{\mathbf{p}}, \tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) \tilde{\mathbf{p}} + \mathbf{C}_{\text{RE}} (\tilde{\mathbf{p}}, \tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) \dot{\tilde{\mathbf{p}}} = \tilde{\mathbf{r}}_{\text{RE}}, \quad (3)$$

such that $\mathbf{A}_{\text{RE}}$ represents the Poiseuille and Couette terms, $\mathbf{B}_{\text{RE}}$ holds the squeeze term contribution and $\mathbf{C}_{\text{RE}}$ the local expansion term. The Jacobians needed in order to solve eq. (3) are given in Appendix A, while the boundary conditions are given as

$$\tilde{p}(\theta, 0) = \tilde{p}(0, \tilde{z}) = \tilde{p}(2\pi, \tilde{z}) = 1 \quad \text{and} \quad \partial \tilde{p} / \partial \tilde{z}|_{\tilde{z}=\tilde{L}/2} = 0. \quad (4)$$

Table 1: Properties of the GFB configuration, mainly from [19].

Journal			
Load, $W_x = m_r g$	30 N	Mass, m_r	3.059 kg
Bearing configuration			
Bearing radius, R	19.05 mm	Pad leading edge, θ_l	0°
Bearing length, L	38.10 mm	Pad trailing edge, θ_t	360°
Radial clearance, C	$31.80 \mu\text{m}$		
Fluid properties			
Ambient pressure, p_a	101.3 kPa	Viscosity, μ	$1.950 \times 10^{-5} \text{ Pa s}$
Foil structure properties			
Foil thickness, t_b	0.1016 mm	Bump pitch, S_b	4.572 mm
Bump half length, l_0	1.778 mm	Bump height, h_b	0.5080 mm
Young's modulus E_b	207.0 GPa	Poisson's ratio, ν_b	0.3
Density, ρ_b	8280 kg m^{-3}	Loss factor, η	0.2

Figure 1: (a) Weight function associated with the j -th foil compliance DOF; (b) Schematics of the GFB and illustration of the perturbed DOFs.

2.2. Rotor

The rotor is modelled as a point mass, i.e. with two DOFs and no gyroscopic effect. The rotor model can be cast in state space residual form using the rotor DOF vector $\tilde{\mathbf{x}}_r = \{\varepsilon_x \ \varepsilon_y\}^T$ as

$$\underbrace{\begin{bmatrix} \mathbf{0} & \tilde{\mathbf{M}}_r \\ \mathbf{I} & \mathbf{0} \end{bmatrix}}_{\mathbf{A}_r} \underbrace{\begin{Bmatrix} \dot{\tilde{\mathbf{x}}}_r \\ \ddot{\tilde{\mathbf{x}}}_r \end{Bmatrix}}_{\mathbf{z}_r} + \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}}_{\mathbf{B}_r} \underbrace{\begin{Bmatrix} \tilde{\mathbf{x}}_r \\ \dot{\tilde{\mathbf{x}}}_r \end{Bmatrix}}_{\mathbf{z}_r} - \begin{Bmatrix} \tilde{\mathbf{f}}_{rw} + \tilde{\mathbf{f}}_b \\ 0 \end{Bmatrix} = \tilde{\mathbf{r}}_r \quad \in \mathbb{R}^4, \quad (5)$$

where $\tilde{\mathbf{M}}_r = \omega_r^2 C p_a^{-1} R^{-2} \text{diag}(\{m_r \ m_r\})$ is the rotor mass matrix, $\tilde{\mathbf{f}}_{rw} = p_a^{-1} R^{-2} \{W_x \ 0\}$ holds the static load and $\tilde{\mathbf{f}}_b$ contains the bearing forces from the integrated gas film pressure. The rotor is assumed to be perfectly aligned to the bearing at all times (it includes no rotational DOFs) giving a one-dimensional rigid film height contribution. Referring to fig. 1b, this is given in the usual form as

$$\tilde{h}_r(\theta) = 1 + \varepsilon_x \cos \theta + \varepsilon_y \sin \theta, \quad (6)$$

where the trigonometric functions can be evaluated offline at the circumferential positions needed by the gas film model allowing the rigid film heights to be evaluated using the constant mapping

$$\tilde{\mathbf{h}}_r = 1 + \mathbf{H}_r \tilde{\mathbf{x}}_r. \quad (7)$$

The bearing forces are found from the usual integration of the projected gas film pressure approximated using a weighted sum over the nodal pressures as

$$\tilde{\mathbf{f}}_b = \int_0^L \int_0^{2\pi} (\tilde{p} - 1) \begin{Bmatrix} \cos \theta \\ \sin \theta \end{Bmatrix} dA \approx \sum_i^{n_\theta} \Delta \theta_i \begin{Bmatrix} \cos \theta_i \\ \sin \theta_i \end{Bmatrix} \sum_j^{n_z} \Delta \tilde{z}_j (\tilde{p}(\theta_i, \tilde{z}_j) - 1), \quad (8)$$

where $\Delta \theta_i$ and $\Delta \tilde{z}_j$ are the weights assigned to a given nodal position by the trapezoidal rule (chosen for consistency to the linear shape functions). The integration of forces can hence be written as a mapping

$$\begin{Bmatrix} \tilde{\mathbf{f}}_b \\ 0 \end{Bmatrix}^T = \mathbf{H}_{fbp} (\tilde{\mathbf{p}} - 1), \quad (9)$$

which is convenient when formulating the coupled system Jacobian. As is common for GFBs, the Gmbel condition is applied meaning that sub ambient pressures are discarded during the rotor force integration. This requires the columns of $\mathbf{H}_{fbp}$ corresponding to sub ambient nodal pressures to be zeroed, effectively making the mapping pressure dependent. Notice that the Gmbel condition is one possible (and simple) way of treating the sub ambient regions. The current research is motivated by previous work by the authors [1, 15, 16, 20] on a three-pad bearing for which the Gmbel condition is well-suited, hence the authors have a special interest in this particular case. In general, one should carefully consider the foil configuration in question (fixed leading/trailing edge, position of pad fixation points compared to load direction etc.) to decide on the most appropriate treatment of sub ambient regions as discussed in e.g. [4, 21].

2.3. Compliant Structure

The compliant structure is modelled using the commonly applied SEFM [22]. This is essentially a Winkler foundation model assuming neighbouring points to be independent and the top foil to be inseparable from the bump foil. Furthermore, the foil is assumed to deform uniformly in the axial direction as a function of the axially averaged pressure, which has previously been shown to be a reasonable assumption [23]. For more realistic foil modelling, a variety of elaborate foil structure models are available in the literature [24–27], but the added complexity of these would complicate a strictly consistent implementation across the different modelling approaches and blur the present comparison.

The SEFM is implemented using a uniform nominal stiffness given from the widely applied expression [28]

$$k_0 = \frac{E_b}{2S_b} \left(\frac{t_b}{l_0} \right)^3 (1 - \nu_b^2)^{-1}, \quad (10)$$

where E_b , S_b , t_b , l_0 and ν_b are the bump foil's Young's modulus, pitch, thickness, half bump length and Poisson's ratio, respectively. The stiffness predicted by eq. (10) does not account for bump–bump interaction or friction and has been found to underestimate the bump foil stiffness [29, 30]. For the present work, however, the accuracy of the predicted nominal value is of limited importance as the foil stiffness is varied between $1/10k_0$ and $1000k_0$.

The structural damping of the foil is assumed viscous as quantified by the hysteretic damping constant $\eta = k^{-1}d\omega_f$ with ω_f being the foil oscillation frequency and d the equivalent viscous damping coefficient. When used with harmonic perturbation methods, it is common [8, 13, 16] to combine the foil stiffness and damping into a complex dynamic stiffness as

$$k_c = k(i\omega_s\omega_f^{-1}\eta + 1) = k(i\eta + 1) \in \mathbb{C}, \quad (11)$$

where the foil oscillation frequency ω_f is dictated by the perturbation frequency ω_s , yielding the second equals sign. For time domain models, however, the simulated response will usually contain a variety of frequency components, but one still needs to fix ω_f at a certain value. For want of anything better, the most common choice is to assume 1X foil oscillation [1, 3, 8, 18, 31, 32], i.e. $\omega_f = \Omega$. This might be exact for a non-resonant unbalance response, but only a rough approximation in the presence of significant non-synchronous vibrations. Bonello and Pham [3] investigated the influence of this issue for self-excited limit cycles by comparing non-linear time integrations to results from a Harmonic Balance analysis and found it to be negligible. This conclusion was, however, drawn for a practically rigid bearing (236.5 GN m^{-3}) with a small clearance ($15.9 \mu\text{m}$) and low foil damping ($\eta = 0.1$) in which case the compliant structure has a vanishing influence on the system dynamics compared to the gas film. More recently, Pronobis and Liebich [17] investigated the same issue for higher compliance levels (up to around $C/2$) where they proved it to have a significant impact. In [17], consistency is achieved by setting $\omega_f = \Omega$ in eq. (11), effectively giving $k_c = k_c(\omega_s)$, for use in the perturbation method while defining the time domain viscous damping coefficient as $\eta k \Omega^{-1}$ (contrary to $\eta k \omega_f^{-1}$ as used in the present work, see eq. (13) below). The approach taken in [17] allows the damping to be defined a priori and consistently for both models providing a significant improvement of the agreement in predicted OSIs. Though being consistent, the damping model in [17] causes the steady state energy dissipation per oscillation cycle for a given amplitude X to become frequency dependent as $\pi k \eta \frac{\omega_s}{\Omega} X^2$ which, depending on the interpretation of η , may or may not be desirable. In the present work, k_c is kept constant as defined from eq. (11) providing the constant energy dissipation per cycle $\pi k \eta X^2$, but requiring the time domain OSI to be calculated iteratively since the set value for ω_f must match the resulting self-excited oscillation frequency. As most GFB vibrations of interest appear around $\Omega/2$, the difference in provided damping between the two approaches is small compared to the remaining uncertainties related to the loss factor, but results should not be compared directly between the two approaches. Notice that the damping model applied in [17] allows consistent, but frequency dependent, damping to be prescribed simultaneously across all modes, while the present approach provides consistency at the particular frequency ω_f only. This is not a major concern when investigating the OSI as this is governed solely by the leading mode, but the iterative approach required for each frequency would become increasingly impractical if multiple modes were to be investigated.

In the present work, the SEFM is implemented both in an implicit and an explicit version. In the former case (used for the perturbation methods), the compliant film height is a direct function of the axially averaged pressure $\tilde{p}_m$ (approximated using trapezoidal integration) as

$$\tilde{h}_c(\theta) = \frac{1}{\tilde{k}_c} (\tilde{p}_m(\theta) - 1) = \underbrace{\frac{1}{\tilde{L}\tilde{k}_c} \int_0^{\tilde{L}} (\tilde{p}(\theta, \tilde{z}) - 1) d\tilde{z}}_{\equiv f_{h_c}(\theta, \tilde{p}-1)} \approx \frac{1}{\tilde{L}\tilde{k}_c} \sum_{j=1}^{n_z} \Delta \tilde{z}_j (\tilde{p}(\theta, \tilde{z}_j) - 1). \quad (12)$$

In the explicit case, the compliance is given from $\mathbf{x}_f$ containing n_θ foil DOFs representing pointwise radial deformations at the pressure nodes' circumferential positions. In this case, the stiffness and damping matrices can be written as

$$\begin{aligned} \mathbf{K}_f &= L R k \text{diag}(\Delta\theta_1 \dots \Delta\theta_{n_\theta}) \in \mathbb{R}^{n_\theta \times n_\theta}, \\ \mathbf{D}_f &= \eta \omega_f^{-1} \mathbf{K}_f \in \mathbb{R}^{n_\theta \times n_\theta}, \end{aligned} \quad (13)$$

where L is the axial length of the bearing and $\Delta\theta_j$ is the angular segment ascribed to the j -th foil DOF as illustrated in fig. 1b. The foil equation of motion providing exactly the same dynamics as the implicit model can hence be written

in residual form as

$$\underbrace{\tilde{\mathbf{D}}_f}_{\mathbf{A}'_f} \underbrace{\dot{\mathbf{x}}_f}_{\dot{\mathbf{z}}_f} + \underbrace{\tilde{\mathbf{K}}_f}_{\mathbf{B}'_f} \underbrace{\mathbf{x}_f}_{\mathbf{z}_f} - \tilde{\mathbf{f}}_p = \tilde{\mathbf{r}}'_f \in \mathbb{R}^{n_\theta}, \quad (14)$$

representing a first order system with time constant $\eta\omega_f^{-1}$. The vector $\tilde{\mathbf{f}}_p$ holds the loading from the axially averaged gas film pressure minus the ambient pressure acting on the back of the foil. This is obtained using trapezoidal integration as

$$\tilde{\mathbf{f}}_p = \begin{Bmatrix} \int_0^{\tilde{L}} \int_{\theta_1 - \frac{1}{2}\Delta\theta_1}^{\theta_1 + \frac{1}{2}\Delta\theta_1} \tilde{p}(\theta, \tilde{z}) - 1 d\theta d\tilde{z} \\ \vdots \\ \int_0^{\tilde{L}} \int_{\theta_{n_\theta} - \frac{1}{2}\Delta\theta_{n_\theta}}^{\theta_{n_\theta} + \frac{1}{2}\Delta\theta_{n_\theta}} \tilde{p}(\theta, \tilde{z}) - 1 d\theta d\tilde{z} \end{Bmatrix} \approx \tilde{L} \begin{Bmatrix} \Delta\theta_1 \sum_{j=1}^{n_z} (\tilde{p}(\theta_1, \tilde{z}_j) - 1) \Delta\tilde{z}_j \\ \vdots \\ \Delta\theta_{n_\theta} \sum_{j=1}^{n_z} (\tilde{p}(\theta_{n_\theta}, \tilde{z}_j) - 1) \Delta\tilde{z}_j \end{Bmatrix} = \mathbf{H}_{f,p}(\tilde{\mathbf{p}} - 1). \quad (15)$$

Even though this is expected to be insignificant to the overall system dynamics, the proposed extended perturbation method furthermore requires the inclusion of foil mass. A simple foil mass matrix can be obtained similarly to $\mathbf{K}_f$ and $\mathbf{D}_f$ in eq. (13) as

$$\mathbf{M}_f = m'_f R \text{diag}(\Delta\theta_1 \dots \Delta\theta_{n_\theta}) \in \mathbb{R}^{n_\theta \times n_\theta}, \quad (16)$$

where m'_f is the average mass of the foil structure per unit circumferential length (see Appendix B). The foil equation of motion including mass can hence be written in state space residual form as

$$\underbrace{\begin{bmatrix} \mathbf{0} & \tilde{\mathbf{M}}_f \\ \mathbf{I} & \mathbf{0} \end{bmatrix}}_{\mathbf{A}_f} \underbrace{\begin{Bmatrix} \dot{\mathbf{x}}_f \\ \mathbf{x}_f \end{Bmatrix}}_{\dot{\mathbf{z}}_f} + \underbrace{\begin{bmatrix} \tilde{\mathbf{K}}_f & \tilde{\mathbf{D}}_f \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}}_{\mathbf{B}_f} \underbrace{\begin{Bmatrix} \mathbf{x}_f \\ \dot{\mathbf{x}}_f \end{Bmatrix}}_{\mathbf{z}_f} - \begin{Bmatrix} \tilde{\mathbf{f}}_p \\ 0 \end{Bmatrix} = \tilde{\mathbf{r}}_f \in \mathbb{R}^{2n_\theta}. \quad (17)$$

3. Perturbation Methods

3.1. The Classical two-DOF Perturbation

This section gives a brief review of the classical two-DOF perturbation employed with various variations by e.g. [7–16, 18]. As originally introduced in the appendix of [6], the fundamental assumption is for the pressure field $\tilde{p}$ to be given as a function of rotor position $\tilde{\mathbf{x}}_r$ and velocity $\dot{\tilde{\mathbf{x}}}_r$ only. Assuming the rotor exhibits small harmonic oscillations with frequency $\tilde{\omega}_s$ around a stationary point $\tilde{\mathbf{x}}_{r0}$ as

$$\tilde{\mathbf{x}}_r = \tilde{\mathbf{x}}_{r0} + \Delta\tilde{\mathbf{x}}_r e^{i\tilde{\omega}_s \tau}, \quad (18)$$

a Taylor series expansion leads to a perturbed pressure field given as

$$\tilde{p} = \tilde{p}_0 + \tilde{p}_x \Lambda_x + \tilde{p}_y \Lambda_y, \quad (19)$$

where

$$\tilde{p}_0 = \tilde{p}(\tilde{\mathbf{x}}_{r0}) \in \mathbb{R}, \quad \tilde{p}_\gamma = \left(\frac{\partial \tilde{p}}{\partial \varepsilon_\gamma} \Big|_{\tilde{\mathbf{x}}_{r0}} + i\tilde{\omega}_s \frac{\partial \tilde{p}}{\partial \dot{\varepsilon}_\gamma} \Big|_{\tilde{\mathbf{x}}_{r0}} \right) \in \mathbb{C} \text{ for } \gamma = x, y \quad \text{and} \quad \Lambda_\gamma = \Delta\varepsilon_\gamma e^{i\tilde{\omega}_s \tau} \in \mathbb{C} \text{ for } \gamma = x, y. \quad (20)$$

This perturbation is then substituted into RE from eq. (1), while the foil compliance is introduced implicitly through the film height function as

$$\tilde{h}(\theta) = \tilde{h}_r(\theta) + f_{h_c}(\theta, \tilde{p} - 1), \quad (21)$$

where f_{h_c} provides the axial average of $\tilde{p} - 1$ at the circumferential position θ as defined from eq. (12). Inserting the film height description and pressure perturbation from eqs. (19) and (21) into eq. (1), the zeroth order equation for $\tilde{p}_0$ can be extracted by collecting terms of order Λ_γ^0 as

$$\frac{\partial}{\partial \theta} \left(\tilde{p}_0 \tilde{h}_0^3 \frac{\partial \tilde{p}_0}{\partial \theta} \right) + \frac{\partial}{\partial \tilde{z}} \left(\tilde{p}_0 \tilde{h}_0^3 \frac{\partial \tilde{p}_0}{\partial \tilde{z}} \right) = S_\tau \tilde{Q} \frac{\partial}{\partial \theta} (\tilde{p}_0 \tilde{h}_0), \quad (22)$$

while the first order equations for $\tilde{p}_\gamma$ correspond to the terms of order Λ_γ^1 as

$$\begin{aligned} \frac{\partial}{\partial \theta} \left(\tilde{p}_0 \tilde{h}_0^3 \frac{\partial \tilde{p}_\gamma}{\partial \theta} \right) + \frac{\partial}{\partial \tilde{z}} \left(\tilde{p}_0 \tilde{h}_0^3 \frac{\partial \tilde{p}_\gamma}{\partial \tilde{z}} \right) + \frac{\partial}{\partial \theta} \left(\tilde{h}_0^2 \left(\tilde{p}_\gamma \tilde{h}_0 + 3 \tilde{p}_0 f_{h_c}(\theta, \tilde{p}_\gamma) \right) \frac{\partial \tilde{p}_0}{\partial \theta} \right) + \frac{\partial}{\partial \tilde{z}} \left(\tilde{h}_0^2 \left(\tilde{p}_\gamma \tilde{h}_0 + 3 \tilde{p}_0 f_{h_c}(\theta, \tilde{p}_\gamma) \right) \frac{\partial \tilde{p}_0}{\partial \tilde{z}} \right) \\ - S_\tau \tilde{\Omega} \frac{\partial}{\partial \theta} \left(\tilde{p}_\gamma \tilde{h}_0 + \tilde{p}_0 f_{h_c}(\theta, \tilde{p}_\gamma) \right) - 2i S_\tau \tilde{\omega}_s \left(\tilde{p}_\gamma \tilde{h}_0 + \tilde{p}_0 f_{h_c}(\theta, \tilde{p}_\gamma) \right) \quad (23) \\ = 2i S_\tau \tilde{\omega}_s \tilde{p}_0 f_\gamma + S_\tau \tilde{\Omega} \frac{\partial}{\partial \theta} \left(\tilde{p}_0 f_\gamma \right) - \frac{\partial}{\partial \theta} \left(3 \tilde{p}_0 \tilde{h}_0^2 f_\gamma \frac{\partial \tilde{p}_0}{\partial \theta} \right) - \frac{\partial}{\partial \tilde{z}} \left(3 \tilde{p}_0 \tilde{h}_0^2 f_\gamma \frac{\partial \tilde{p}_0}{\partial \tilde{z}} \right), \quad \gamma = x, y, \end{aligned}$$

where $f_x = \cos(\theta)$, $f_y = \sin(\theta)$ and

$$\tilde{h}_0(\theta) = 1 + \varepsilon_{x0} \cos \theta + \varepsilon_{y0} \sin \theta + f_{h_c}(\theta, \tilde{p}_0 - 1). \quad (24)$$

The zeroth order equation is a real-valued (η is set to zero) non-linear equation corresponding to the steady part of eq. (2) with $\tilde{\mathbf{p}} = \tilde{\mathbf{p}}_0$. It is solved in a segregated fashion using two nested Newton–Raphson (NR) solvers. The outer solver searches for the rotor equilibrium $\tilde{\mathbf{x}}_{r0}$ satisfying eq. (5) using finite difference approximations to the bearing force Jacobian $\partial \tilde{\mathbf{f}}_b / \partial \tilde{\mathbf{x}}_r \in \mathbb{R}^{2 \times 2}$ (incorporating the Gmbel condition as given from eq. (9)). The inner solver solves eq. (22) for a given eccentricity using the analytical expressions for the Jacobian $\partial \tilde{\mathbf{r}}_{RE} / \partial \tilde{\mathbf{p}}$ provided in Appendix A.

The first order equation eq. (23) is linear and complex valued and can be discretized using FE following the same procedure as for the zeroth order equation to yield

$$\mathbf{A}_{p_\gamma} \tilde{\mathbf{p}}_\gamma = \tilde{\mathbf{r}}_{p_\gamma}, \quad (25)$$

where $\mathbf{A}_{p_\gamma} \in \mathbb{C}^{n_\theta n_z \times n_\theta n_z}$ and $\tilde{\mathbf{p}}_\gamma, \tilde{\mathbf{r}}_{p_\gamma} \in \mathbb{C}^{n_\theta n_z}$. On the element level, these are given as

$$\begin{aligned} \mathbf{A}_{p_\gamma}^e \tilde{\mathbf{p}}_\gamma = \int_{A^e} \tilde{p}_0 \tilde{h}_0^3 \mathbf{B}^T \mathbf{B} + \tilde{h}_0^3 \mathbf{B}^T \mathbf{B} \tilde{\mathbf{p}}_0^e \mathbf{N} - S_\tau \tilde{\Omega} \tilde{h}_0 \mathbf{B}^T \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \mathbf{N} + 2i \tilde{\omega}_s S_\tau \tilde{h}_0 \mathbf{N}^T \mathbf{N} dA \tilde{\mathbf{p}}_\gamma^e \\ + \int_{A^e} 3 \tilde{p}_0 \tilde{h}_0^2 \mathbf{B}^T \mathbf{B} \tilde{\mathbf{p}}_0^e \mathbf{N} - S_\tau \tilde{\Omega} \tilde{p}_0 \mathbf{B}^T \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \mathbf{N} + 2i \tilde{\omega}_s S_\tau \tilde{p}_0 \mathbf{N}^T \mathbf{N} dA \mathbf{H}_{h_c p}^e \tilde{\mathbf{p}}_\gamma \end{aligned} \quad (26)$$

$$\tilde{\mathbf{r}}_{p_\gamma}^e = \int_{A^e} S_\tau \tilde{\Omega} \tilde{p}_0 f_\gamma \mathbf{B}^T \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - 3 \tilde{p}_0 \tilde{h}_0^2 f_\gamma \mathbf{B}^T \mathbf{B} \tilde{\mathbf{p}}_0^e - 2i \tilde{\omega}_s S_\tau \tilde{p}_0 f_\gamma \mathbf{N}^T dA. \quad (27)$$

which is solved using the LAPACK routines zgbtrf and zgbtrs [33]. The coupling to $\tilde{\mathbf{p}}_\gamma$, i.e. to pressure DOFs not included in $\tilde{\mathbf{p}}_\gamma^e$, in eq. (26) is due to the axial pressure averaging given by $\mathbf{H}_{h_c p}^e$ which is detailed in Appendix A, while the boundary conditions are given as

$$\tilde{p}_\gamma(\theta, 0) = \tilde{p}_\gamma(0, \tilde{z}) = \tilde{p}_\gamma(2\pi, \tilde{z}) = 0 \quad \text{and} \quad \partial \tilde{p}_\gamma / \partial \tilde{z}|_{\tilde{z}=L/2} = 0. \quad (28)$$

Having obtained the dynamic pressure fields, the gas film reaction forces can be integrated to obtain the bearing coefficients as

$$\underbrace{\begin{bmatrix} \tilde{k}_{xx} & \tilde{k}_{xy} \\ \tilde{k}_{yx} & \tilde{k}_{yy} \end{bmatrix}}_{\tilde{\mathbf{K}}_{g,2DOF}} + i \tilde{\omega}_s \underbrace{\begin{bmatrix} \tilde{d}_{xx} & \tilde{d}_{xy} \\ \tilde{d}_{yx} & \tilde{d}_{yy} \end{bmatrix}}_{\tilde{\mathbf{D}}_{g,2DOF}} = \int_0^L \int_0^{2\pi} \begin{bmatrix} \tilde{p}_x \cos \theta & \tilde{p}_y \cos \theta \\ \tilde{p}_x \sin \theta & \tilde{p}_y \sin \theta \end{bmatrix} d\theta d\tilde{z} \in \mathbb{C}^{2 \times 2}, \quad (29)$$

using which, the linearised equations of motion for the two rotor DOFs around the static equilibrium can be written as

$$\tilde{\mathbf{M}}_r \ddot{\tilde{\mathbf{x}}}_r + \tilde{\mathbf{D}}_{g,2DOF}(\tilde{\omega}_s, \tilde{\Omega}) \dot{\tilde{\mathbf{x}}}_r + \tilde{\mathbf{K}}_{g,2DOF}(\tilde{\omega}_s, \tilde{\Omega}) \tilde{\mathbf{x}}_r = \mathbf{0}. \quad (30)$$

Compared to previous work by the authors [1, 2, 16], two important changes should be mentioned. First, the point wise expression for the dynamic compliant film height $\tilde{p}_\gamma \tilde{k}_c^{-1}$ has been replaced by $f_{h_c}(\theta, \tilde{p}_\gamma)$ in eq. (23) to provide axial averaging in agreement with the calculation of the static film height $\tilde{h}_0$ as also described in [13]. Second, the Gmbel condition implies that sub ambient regions should not contribute to the dynamic forces, i.e. $\tilde{p}_\gamma = 0$ where $\tilde{p}_0 < 1$. Previously, this was introduced as constraints in eq. (25) before its solution, while the dynamic pressure in sub

ambient regions is now discarded only during the integration in eq. (29) analogously to the static pressure integration in eqs. (8) and (9). It should be emphasised that the truncation of the dynamic pressure field $\tilde{p}_y$ based on the static pressure $\tilde{p}_0$ is only correct if the coefficients are calculated at the static equilibrium, e.g. to assess its stability from the eigenvalues. If the coefficients are calculated along a trajectory in a transient simulation, the truncation should instead be based on the instantaneous pressure field at each point of evaluation.

3.2. An Extended Perturbation

A natural extension of the classical two-DOF perturbation is to introduce an explicit dependency of the pressure field on a number of foil states, akin to the pad DOFs used when analysing tilting pad journal bearings [34, 35]. Effectively, this means that the pressure is assumed to vary as a function of foil deformation even in the absence of rotor movement. To allow a direct comparison, this is implemented using n_θ discrete radial deformations $\tilde{w}_j$ located at the same circumferential positions as the pressure nodes. The deformed shape between these points is evaluated using linear interpolation consistent with the linear shape functions of the FE discretization of RE making the explicit formulation equivalent to the implicit one in terms of the static deflection. A weight function describing the film height contribution from each foil DOF is hence defined and illustrated in fig. 1a as

$$W_j(\theta) = \begin{cases} \frac{\theta - \theta_{j-1}}{\theta_j - \theta_{j-1}} & \text{if } \theta_{j-1} < \theta < \theta_j \\ \frac{\theta_{j+1} - \theta}{\theta_{j+1} - \theta_j} & \text{if } \theta_j \leq \theta < \theta_{j+1} \\ 0 & \text{otherwise,} \end{cases} \quad (31)$$

such that the continuous film height can be written as

$$\tilde{h}(\theta) = 1 + \varepsilon_x \cos \theta + \varepsilon_y \sin \theta + \sum_{j=1}^{n_\theta} \tilde{w}_j W_j(\theta), \quad (32)$$

where the discrete deformation at each foil DOF is calculated using the axially averaged pressure defined in eq. (12) as

$$\tilde{w}_j = f_{h_c}(\theta_j, \tilde{p} - 1). \quad (33)$$

The assumption of small harmonic oscillations, which was applied to the rotor DOFs in eq. (18), is now applied also directly to the foil DOFs as

$$\tilde{\mathbf{w}} = \tilde{\mathbf{w}}_0 + \Delta \tilde{\mathbf{w}} e^{i\tilde{\omega}_s \tau} \quad \text{where} \quad \tilde{\mathbf{w}} = \{w_1 \quad \dots \quad w_{n_\theta}\}^T, \quad (34)$$

such that a first order Taylor series expansion of the pressure field $\tilde{p} = \tilde{p}(\tilde{\mathbf{x}}_r, \tilde{\mathbf{w}})$ now provides $3 + n_\theta$ terms as

$$\tilde{p} = \tilde{p}_0 + \tilde{p}_x \Lambda_x + \tilde{p}_y \Lambda_y + \sum_{j=1}^{n_\theta} \tilde{p}_{w_j} \Lambda_j, \quad (35)$$

where $\tilde{p}_0$, $\tilde{p}_x$, $\tilde{p}_y$, Λ_x and Λ_y are defined from eq. (20) as previously, except that the former three should now be evaluated at $\tilde{\mathbf{x}}_{r0}$, $\tilde{\mathbf{w}}_0$. The dynamic pressure fields related to the foil DOFs are defined as

$$\tilde{p}_{w_j} = \left(\frac{\partial \tilde{p}}{\partial \tilde{w}_j} \Big|_{\tilde{\mathbf{x}}_{r0}, \tilde{\mathbf{w}}_0} + i\tilde{\omega}_s \frac{\partial \tilde{p}}{\partial \dot{\tilde{w}}_j} \Big|_{\tilde{\mathbf{x}}_{r0}, \tilde{\mathbf{w}}_0} \right) \in \mathbb{C} \quad j = 1, \dots, n_\theta \quad \text{and} \quad \Lambda_j = \Delta \tilde{w}_j e^{i\tilde{\omega}_s \tau} \in \mathbb{C} \quad j = 1, \dots, n_\theta. \quad (36)$$

Notice that $\tilde{p}_{w_j}$ provides the pressure derivative at any point with respect to the j -th foil deformation DOF at the frequency $\tilde{\omega}_s$. Fundamentally, this is no different from the $\tilde{p}_x$ and $\tilde{p}_y$ fields supplying the pressure derivatives with respect to the two rotor DOFs.

Inserting the film height description and extended perturbation from eqs. (32) and (35) into RE from eq. (1), the zeroth and first order equations can be separated as was also the case for the two-DOF perturbation. The zeroth order equation is unchanged, except for an alternative, but equivalent, static film height expression

$$\tilde{h}_0(\theta) = 1 + \varepsilon_{x0} \cos \theta + \varepsilon_{y0} \sin \theta + \sum_{j=1}^{n_\theta} \tilde{w}_{j0} W_j(\theta), \quad (37)$$

where $\tilde{w}_{j0}$ is evaluated with $\eta = 0$ as for the implicit formulation. Compared to the two-DOF perturbation, the first order equations for the rotor DOF dynamic pressure fields $\tilde{p}_\gamma$ are slightly simplified corresponding to setting $f_{h_c}(\theta, \tilde{p}_\gamma) = 0$ in eq. (23), reflecting that the dynamic compliance is no longer dictated by the rotor movement. Additionally, n_θ first order equations for the dynamic pressure fields related to the foil DOFs are obtained, giving in total $2 + n_\theta$ first order equations as

$$\begin{aligned} & \frac{\partial}{\partial \theta} \left(\tilde{p}_0 \tilde{h}_0^3 \frac{\partial \tilde{p}_\alpha}{\partial \theta} \right) + \frac{\partial}{\partial \tilde{z}} \left(\tilde{p}_0 \tilde{h}_0^3 \frac{\partial \tilde{p}_\alpha}{\partial \tilde{z}} \right) + \frac{\partial}{\partial \theta} \left(\tilde{h}_0^3 \tilde{p}_\alpha \frac{\partial \tilde{p}_0}{\partial \theta} \right) + \frac{\partial}{\partial \tilde{z}} \left(\tilde{h}_0^3 \tilde{p}_\alpha \frac{\partial \tilde{p}_0}{\partial \tilde{z}} \right) - S_\tau \tilde{\Omega} \frac{\partial}{\partial \theta} (\tilde{p}_\alpha \tilde{h}_0) - 2iS_\tau \tilde{\omega}_s (\tilde{p}_\alpha \tilde{h}_0) \\ & = 2iS_\tau \tilde{\omega}_s \tilde{p}_0 f_\alpha + S_\tau \tilde{\Omega} \frac{\partial}{\partial \theta} (\tilde{p}_0 f_\alpha) - \frac{\partial}{\partial \theta} \left(3\tilde{p}_0 \tilde{h}_0^2 f_\alpha \frac{\partial \tilde{p}_0}{\partial \theta} \right) - \frac{\partial}{\partial \tilde{z}} \left(3\tilde{p}_0 \tilde{h}_0^2 f_\alpha \frac{\partial \tilde{p}_0}{\partial \tilde{z}} \right), \quad \alpha = x, y, w_1, \dots, w_{n_\theta}, \end{aligned} \quad (38)$$

where $f_x = \cos(\theta)$, $f_y = \sin(\theta)$ and $f_{w_j} = W_j(\theta)$. These can be FE discretized and solved analogously to the two-DOF first order equations from eq. (23). Having obtained $\tilde{p}_0$, $\tilde{p}_\gamma$ and $\tilde{p}_{w_j}$, these can be integrated to obtain the gas film reaction forces for each DOF as

$$\begin{Bmatrix} \tilde{\mathbf{f}}_b \\ \tilde{\mathbf{f}}_p \end{Bmatrix} = \int_0^{\tilde{L}} \begin{Bmatrix} \int_0^{2\pi} (\tilde{p} - 1) \begin{Bmatrix} \cos \theta \\ \sin \theta \end{Bmatrix} d\theta \\ \int_{\theta_1 - \frac{1}{2}\Delta\theta_1}^{\theta_1 + \frac{1}{2}\Delta\theta_1} (\tilde{p} - 1) d\theta \\ \vdots \\ \int_{\theta_{n_\theta} - \frac{1}{2}\Delta\theta_{n_\theta}}^{\theta_{n_\theta} + \frac{1}{2}\Delta\theta_{n_\theta}} (\tilde{p} - 1) d\theta \end{Bmatrix} d\tilde{z} = \int_0^{\tilde{L}} \begin{Bmatrix} \int_0^{2\pi} (\tilde{p}_0 - 1) \begin{Bmatrix} \cos \theta \\ \sin \theta \end{Bmatrix} d\theta \\ \int_{\theta_1 - \frac{1}{2}\Delta\theta_1}^{\theta_1 + \frac{1}{2}\Delta\theta_1} (\tilde{p}_0 - 1) d\theta \\ \vdots \\ \int_{\theta_{n_\theta} - \frac{1}{2}\Delta\theta_{n_\theta}}^{\theta_{n_\theta} + \frac{1}{2}\Delta\theta_{n_\theta}} (\tilde{p}_0 - 1) d\theta \end{Bmatrix} d\tilde{z} + \underbrace{\begin{bmatrix} \mathbf{Z}_{x_r, x_r} & \mathbf{Z}_{x_r, w} \\ \mathbf{Z}_{w, x_r} & \mathbf{Z}_{w, w} \end{bmatrix}}_{\equiv \mathbf{Z}} \begin{Bmatrix} \tilde{\mathbf{x}}_r \\ \tilde{\mathbf{w}} \end{Bmatrix} e^{i\tilde{\omega}_s \tau}, \quad (39)$$

where the complex stiffness matrix $\mathbf{Z} \in \mathbb{C}^{(2+n_\theta) \times (2+n_\theta)}$ is given as

$$\begin{aligned} \mathbf{Z} &= \int_0^{\tilde{L}} \begin{bmatrix} \int_0^{2\pi} \begin{bmatrix} \tilde{p}_x \cos \theta & \tilde{p}_y \cos \theta \\ \tilde{p}_x \sin \theta & \tilde{p}_y \sin \theta \end{bmatrix} d\theta & \int_0^{2\pi} \begin{bmatrix} \tilde{p}_{w_1} \cos \theta & \dots & \tilde{p}_{w_{n_\theta}} \cos \theta \\ \tilde{p}_{w_1} \sin \theta & \dots & \tilde{p}_{w_{n_\theta}} \sin \theta \end{bmatrix} d\theta \\ \int_{\theta_1 - \frac{1}{2}\Delta\theta_1}^{\theta_1 + \frac{1}{2}\Delta\theta_1} \begin{bmatrix} \tilde{p}_x & \tilde{p}_y \end{bmatrix} d\theta & \int_{\theta_1 - \frac{1}{2}\Delta\theta_1}^{\theta_1 + \frac{1}{2}\Delta\theta_1} \begin{bmatrix} \tilde{p}_{w_1} & \dots & \tilde{p}_{w_{n_\theta}} \end{bmatrix} d\theta \\ \vdots & \vdots \\ \int_{\theta_{n_\theta} - \frac{1}{2}\Delta\theta_{n_\theta}}^{\theta_{n_\theta} + \frac{1}{2}\Delta\theta_{n_\theta}} \begin{bmatrix} \tilde{p}_x & \tilde{p}_y \end{bmatrix} d\theta & \int_{\theta_{n_\theta} - \frac{1}{2}\Delta\theta_{n_\theta}}^{\theta_{n_\theta} + \frac{1}{2}\Delta\theta_{n_\theta}} \begin{bmatrix} \tilde{p}_{w_1} & \dots & \tilde{p}_{w_{n_\theta}} \end{bmatrix} d\theta \end{bmatrix} d\tilde{z} \\ &= \underbrace{\begin{bmatrix} \tilde{k}_{xx} & \tilde{k}_{xy} & \tilde{k}_{xw_1} & \dots & \tilde{k}_{xw_{n_\theta}} \\ \tilde{k}_{yx} & \tilde{k}_{yy} & \tilde{k}_{yw_1} & \dots & \tilde{k}_{yw_{n_\theta}} \\ \tilde{k}_{w_1x} & \tilde{k}_{w_1y} & \tilde{k}_{w_1w_1} & \dots & \tilde{k}_{w_1w_{n_\theta}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{k}_{w_{n_\theta}x} & \tilde{k}_{w_{n_\theta}y} & \tilde{k}_{w_{n_\theta}w_1} & \dots & \tilde{k}_{w_{n_\theta}w_{n_\theta}} \end{bmatrix}}_{\tilde{\mathbf{K}}_{g,xDOF}} + i\tilde{\omega}_s \underbrace{\begin{bmatrix} \tilde{d}_{xx} & \tilde{d}_{xy} & \tilde{d}_{xw_1} & \dots & \tilde{d}_{xw_{n_\theta}} \\ \tilde{d}_{yx} & \tilde{d}_{yy} & \tilde{d}_{yw_1} & \dots & \tilde{d}_{yw_{n_\theta}} \\ \tilde{d}_{w_1x} & \tilde{d}_{w_1y} & \tilde{d}_{w_1w_1} & \dots & \tilde{d}_{w_1w_{n_\theta}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{d}_{w_{n_\theta}x} & \tilde{d}_{w_{n_\theta}y} & \tilde{d}_{w_{n_\theta}w_1} & \dots & \tilde{d}_{w_{n_\theta}w_{n_\theta}} \end{bmatrix}}_{\tilde{\mathbf{D}}_{g,xDOF}}. \end{aligned} \quad (40)$$

Finally, the system equations of motion, which now comprise the rotor DOFs as well as the n_θ newly introduced foil DOFs, can be assembled as

$$\begin{bmatrix} \tilde{\mathbf{M}}_r & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{M}}_f \end{bmatrix} \begin{Bmatrix} \ddot{\tilde{\mathbf{x}}}_r \\ \ddot{\tilde{\mathbf{w}}} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{D}}_f(\tilde{\omega}_s) \end{bmatrix} \begin{Bmatrix} \dot{\tilde{\mathbf{x}}}_r \\ \dot{\tilde{\mathbf{w}}} \end{Bmatrix} + \tilde{\mathbf{D}}_{g,xDOF}(\tilde{\omega}_s, \tilde{\Omega}) \begin{Bmatrix} \dot{\tilde{\mathbf{x}}}_r \\ \dot{\tilde{\mathbf{w}}} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{K}}_f \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{x}}_r \\ \tilde{\mathbf{w}} \end{Bmatrix} + \tilde{\mathbf{K}}_{g,xDOF}(\tilde{\omega}_s, \tilde{\Omega}) \begin{Bmatrix} \tilde{\mathbf{x}}_r \\ \tilde{\mathbf{w}} \end{Bmatrix} = \mathbf{0}, \quad (41)$$

where the foil damping is given from eq. (13) with $\omega_f = \omega_s$. At present, the foil mass is included in eq. (41) to provide a straightforward format of the extension not treating the foil DOFs any differently from the rotor DOFs. If recasting the rotor equations into state space form, it would be possible to rearrange eq. (41) leaving out the foil mass as shown in Appendix C. This would lead to a slightly altered eigenvalue problem requiring adjustments to the solution procedure described in the following section, but would probably yield similar results. The foil mass is, however, retained since this intuitively matches the assumption of harmonic oscillations applied in eq. (34) and ultimately is shown not to affect the OSI.

3.3. Calculation of Eigenvalues and Eigenvectors

Calculation of eigenvalues from the equations of motion resulting from the perturbation methods is complicated by the frequency dependency of the stiffness and damping matrices. For a generic frequency dependent system

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}(\omega_s, \Omega)\dot{\mathbf{x}} + \mathbf{K}(\omega_s, \Omega)\mathbf{x} = \mathbf{0}, \quad (42)$$

the eigenvalues λ_i at a given speed Ω are the roots of the characteristic polynomial

$$P_c(\lambda_i) = 0 \quad \text{where} \quad P_c(\lambda) = \det \underbrace{\left(\lambda^2 \mathbf{M} + \lambda \mathbf{D}(\text{Im}(\lambda), \Omega) + \mathbf{K}(\text{Im}(\lambda), \Omega) \right)}_{\mathbf{A}_\lambda(\lambda, \Omega)}, \quad (43)$$

while the corresponding eigenvector $\mathbf{v}_i$ (for eigenvalues with multiplicity 1) is found as the null space (or kernel) as

$$\mathbf{v}_i = \text{null}(\mathbf{A}_\lambda(\lambda_i, \Omega)). \quad (44)$$

To find the eigenvalues at a given speed Ω from eq. (43), the matrices $\mathbf{K}(\omega_s, \Omega)$ and $\mathbf{D}(\omega_s, \Omega)$ must be evaluated over a range of ω_s values from which the determinant of $\mathbf{A}_\lambda$ can be calculated over a suitable grid of guessed real and imaginary λ values. The local minima of the resulting response surface (currently found using a simple filter) can then serve as initial guesses from which the eigenvalues λ_i can be found using a standard local minimization routine (here the downhill simplex algorithm as implemented in SciPy's "fmin"), relying either on interpolation between the previously evaluated $\mathbf{K}(\omega_s, \Omega)$ and $\mathbf{D}(\omega_s, \Omega)$ matrices or re-evaluation of these. Notice that a very similar approach to the calculation of eigenvalues has recently been presented by Pronobis and Liebich [17]. Having located an eigenvalue λ_i , finding the corresponding eigenvector then requires a singular value decomposition of the resulting $\mathbf{A}_\lambda(\lambda_i, \Omega)$ matrix to identify its null space. Repeating this procedure to obtain the eigenvalues as functions of Ω , one can construct a Campbell diagram and assess the stability limit.

If only the stability limit is of interest, a graphical approach to the solution is possible as previously applied by the authors [1, 2]. Temporarily relaxing the requirement for ω_s to coincide with $\text{Im}(\lambda_i)$, the eigenvalue problem

$$\left(\hat{\lambda}_i^2 \mathbf{M} + \hat{\lambda}_i \mathbf{D}(\omega_s, \Omega) + \mathbf{K}(\omega_s, \Omega) \right) \hat{\mathbf{v}}_i = \mathbf{0} \quad (45)$$

can be solved for the pseudo eigenvalues $\hat{\lambda}_i$ over a suitable $\omega_s \times \Omega$ grid using a standard eigenvalue solver. Sorting the obtained pseudo eigenvalues consistently, the frequency and speed dependent pseudo eigenvalues $\hat{\lambda}_i(\omega_s, \Omega)$ can be obtained and the stability limit Ω_{OSI} can be found by evaluating the condition

$$\text{Im}(\hat{\lambda}_i(\omega_s, \Omega = \Omega_{\text{OSI}})) = \omega_s \quad \wedge \quad \text{Re}(\hat{\lambda}_i(\omega_s, \Omega = \Omega_{\text{OSI}})) = 0, \quad (46)$$

for each $\hat{\lambda}_i$. The left half of the condition requires the pseudo eigenvalue to satisfy eq. (43) and hence to represent an actual eigenvalue, while the right half requires the same eigenvalue to be marginally stable. The fulfilment of eq. (46) can be found graphically by superimposing the contour of $\text{Im}(\hat{\lambda}_i)/\omega_s = 1$ on the contours of $\text{Re}(\hat{\lambda}_i)$ as shown in fig. 2. It should be noted that the OSI defined by eq. (46) is not restricted to modes with any particular structure. For systems resulting from the extended perturbation, the condition could hypothetically be fulfilled by a mode with vanishing rotor participation. In this case, the static equilibrium would have become unstable, but further analysis would be needed to assess if the OSI (in a rotordynamics sense) had been found. This situation has, however, not been encountered by the authors and all perturbation OSIs presented in the present paper are defined by rotor-dominated modes, i.e. the modes that would appear in a Campbell diagram.

For the 2×2 systems resulting from the two-DOF perturbation, both approaches are relatively easy to apply. Only two eigenvalues are to be found and these are usually well separated by frequency. The sorting of pseudo eigenvalues between the (ω_s, Ω) grid points required to generate the contour plots is hence trivial and the evaluation of the characteristic polynomial in eq. (43) is very cheap. However, for the $(2 + n_\theta) \times (2 + n_\theta)$ systems resulting from the extended perturbation, both methods become significantly more difficult. The number of eigenvalues increases drastically and they are usually much more densely spaced. For the contour plot approach, each individual eigenvalue problem becomes much more expensive and a more elaborate scheme for sorting the eigenvalues (presently based on the eigenvector structures) is required. For the root finding approach, the initial $\text{Re}(\lambda) \times \text{Im}(\lambda)$ grid must be denser

to reliably find and separate the requested eigenvalues, the evaluation of eq. (43) becomes rather expensive and the values of $P_c(\lambda)$ (away from the roots) explode due to the matrix size causing numerical difficulties. The latter issue is currently tackled by calculating the logarithm of the determinant, i.e. $\log(P_c(\lambda_i)) = \log \det(\mathbf{A}_\lambda)$, instead of the determinant directly. Note that if only the OSI is of interest, only a few $\text{Re}(\lambda) \times \text{Im}(\lambda)$ grid points in the $\text{Re}(\lambda)$ direction around zero are sufficient. In the present work, the OSIs are based on $\mathbf{K}$ and $\mathbf{D}$ matrices evaluated with an ω_s resolution of 1 Hz for every 10 RPM followed by linear interpolation to locate the roots to an absolute accuracy of 10^{-4} Hz.

4. Direct Coupling of the Domains

As an alternative to the perturbation approaches, one can formulate the governing equations for the rotor–GFB system as a set of coupled non-linear equations allowing the full system dynamics to be captured. For the present rotor–bearing system, the global state-space and residual vectors can be defined as

$$\mathbf{z} = \left\{ \tilde{\mathbf{p}}^T \quad \mathbf{z}_f^T \quad \mathbf{z}_r^T \right\}^T \quad \text{and} \quad \tilde{\mathbf{r}} = \left\{ \tilde{\mathbf{r}}_{\text{RE}}^T \quad \tilde{\mathbf{r}}_f^T \quad \tilde{\mathbf{r}}_r^T \right\}^T, \quad (47)$$

while the governing equations for the three domains can be combined into the global residual function $\mathbf{f}_G(\mathbf{z}, \dot{\mathbf{z}})$ with transient, steady and constant components $\mathbf{f}_{G,t}(\mathbf{z}, \dot{\mathbf{z}})$, $\mathbf{f}_{G,s}(\mathbf{z})$ and $\mathbf{f}_{G,0}$. This allows the coupled non-linear system of equations to be written in implicit form as

$$\mathbf{f}_G(\mathbf{z}, \dot{\mathbf{z}}) = \underbrace{\begin{Bmatrix} \mathbf{B}_{\text{RE}}(\tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) \tilde{\mathbf{p}} + \mathbf{C}_{\text{RE}}(\tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) \dot{\tilde{\mathbf{p}}} \\ \mathbf{A}_f \dot{\mathbf{z}}_f \\ \mathbf{A}_r \dot{\mathbf{z}}_r \end{Bmatrix}}_{\mathbf{f}_{G,t}(\mathbf{z}, \dot{\mathbf{z}})} + \underbrace{\begin{Bmatrix} \mathbf{A}_{\text{RE}}(\tilde{\mathbf{p}}, \tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) \tilde{\mathbf{p}} \\ \mathbf{B}_f \mathbf{z}_f + [\mathbf{H}_{f,p}^T \quad \mathbf{0}]^T (\tilde{\mathbf{p}} - 1) \\ \mathbf{H}_{f,b,p} (\tilde{\mathbf{p}} - 1) \end{Bmatrix}}_{\mathbf{f}_{G,s}(\mathbf{z})} + \underbrace{\begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \{\tilde{\mathbf{f}}_{rw} \mathbf{0}\}^T \end{Bmatrix}}_{\mathbf{f}_{G,0}} = \tilde{\mathbf{r}}, \quad (48)$$

where the topmost row holds the FE gas film equations from eq. (3), but with the $\tilde{\mathbf{p}}$ and $\dot{\tilde{\mathbf{p}}}$ dependencies of $\mathbf{B}_{\text{RE}}$ and $\mathbf{C}_{\text{RE}}$ omitted since no implicit compliance model is included. Instead, the compliance is governed by the foil equations of motion from eq. (17) (or eq. (14)) placed in the midmost row. Lastly, the lowermost row holds the rotor equations of motion from eq. (5). Notice that if unbalance forces were to be included, $\mathbf{f}_{G,t}$ (and hence $\mathbf{f}_G$) would furthermore contain an explicit time dependency.

If attainable, system formulations on explicit first order form, i.e. where $\mathbf{f}_G(\mathbf{z}) = \dot{\mathbf{z}}$, are usually preferred over the implicit form used in eq. (48). Transforming the rotor and foil equations to explicit form would merely require a one-time inversion of $\mathbf{A}_f$ and $\mathbf{A}_r$, while an explicit formulation of the gas film equations could have been obtained by substitution of the alternative state variable $\psi = \tilde{p}/h$ into the unsteady terms of eq. (1) [36]. The present implementation [37] does, however, encompass a range of foil structure models where the foil mass, the foil damping or both of these are disregarded. The latter results in differential/algebraic (rather than ordinary differential) equation systems where an explicit formulation cannot be obtained hence the implicit form is used.

For eq. (48) to be efficiently solved and to form a firm basis for the stability analysis, it is necessary to formulate analytical expressions for the Jacobian matrices. For the steady-state case where $\mathbf{f}_{G,t}(\mathbf{z}, \dot{\mathbf{z}}) = \mathbf{0}$, the Jacobian with respect to the state vector $\mathbf{z}$ is given as

$$\mathbf{J}_{z,s} = \frac{\partial \mathbf{f}_{G,s}}{\partial \mathbf{z}} = \begin{bmatrix} \frac{\partial \mathbf{A}_{\text{RE}}(\tilde{\mathbf{p}}, \tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) \tilde{\mathbf{p}}}{\partial \tilde{\mathbf{p}}} & \frac{\partial \mathbf{A}_{\text{RE}}(\tilde{\mathbf{p}}, \tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) \tilde{\mathbf{p}}}{\partial \mathbf{z}_f} & \frac{\partial \mathbf{A}_{\text{RE}}(\tilde{\mathbf{p}}, \tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) \tilde{\mathbf{p}}}{\partial \mathbf{z}_r} \\ [\mathbf{H}_{f,p}^T \quad \mathbf{0}]^T & \mathbf{B}_f & \mathbf{0} \\ \mathbf{H}_{f,b,p}(\tilde{\mathbf{p}}) & \mathbf{0} & \mathbf{B}_r \end{bmatrix}, \quad (49)$$

where the upper three elements are state-dependent and given in Appendix A. Notice that the foil residuals are not directly affected by the rotor states and vice versa, which is physically meaningful since the rotor is only coupled to the foil through the gas film. For the transient case, the Jacobian with respect to both the state vector and the state vector's temporal derivative is necessary. The Jacobian with respect to $\mathbf{z}$ receives the additional contribution

$$\mathbf{J}_{z,t} = \frac{\partial \mathbf{f}_{G,t}}{\partial \mathbf{z}} = \begin{bmatrix} \mathbf{B}_{\text{RE}}(\tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) & \frac{\partial \mathbf{C}_{\text{RE}}(\tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) \dot{\tilde{\mathbf{p}}}}{\partial \mathbf{z}_f} & \frac{\partial \mathbf{C}_{\text{RE}}(\tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) \dot{\tilde{\mathbf{p}}}}{\partial \mathbf{z}_r} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (50)$$

representing the pressure and film height dependencies originating from the local expansion and squeeze terms. Finally, the Jacobian with respect to the temporal derivative of the state vector $\dot{\mathbf{z}}$ is given as

$$\mathbf{J}_{\dot{\mathbf{z}}} = \frac{\partial \mathbf{f}_{G,t}}{\partial \dot{\mathbf{z}}} = \begin{bmatrix} \mathbf{C}_{\text{RE}}(\tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) & \frac{\partial \mathbf{B}_{\text{RE}}(\tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) \tilde{\mathbf{p}}}{\partial \dot{\mathbf{z}}_f} & \frac{\partial \mathbf{B}_{\text{RE}}(\tilde{\mathbf{x}}_r, \tilde{\mathbf{x}}_f) \tilde{\mathbf{p}}}{\partial \dot{\mathbf{z}}_r} \\ \mathbf{0} & \mathbf{A}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_r \end{bmatrix}. \quad (51)$$

The static equilibrium of the rotor system can be found by zeroing out the transient component of eq. (48), i.e. setting $\mathbf{f}_{G,t}(\mathbf{z}, \dot{\mathbf{z}}) = \mathbf{0}$, and solving the resulting non-linear algebraic equation system. This is achieved using a C implementation utilising the general purpose algebraic solver KINSOL from the SUNDIALS suite [38]. Having aligned the implementations carefully, this leads to exactly the same equilibrium positions as found by the segregated solver applied for the zeroth order equations (to machine precision). As the segregated approach seems to be common practice in bearing codes, it is worth mentioning that the coupled solver converges in about the same number of iterations (on the order of 10) as used by the segregated solver's outer NR loop alone, thus saving potentially hundreds of iterations taken by the inner solver. Notice, however, that while the linear equation systems arising from the segregated solver's inner NR steps are banded (bandwidth $4n_z$, see eq. (A.2)), those arising from the coupled solver steps are not (see eq. (49)). To benefit computationally from the lower iteration count of the coupled approach, it is hence necessary to employ a sparse linear solver such as KLU [39], which is currently being used.

Having obtained the static equilibrium $\mathbf{z}_0$ (where $\dot{\mathbf{z}}_0 = \mathbf{0}$) at a given speed Ω , the behaviour of the dynamical system in the vicinity of this equilibrium can be assessed from a linearisation, i.e. a first order Taylor series expansion, of eq. (48) as

$$\underbrace{\left. \frac{\partial \mathbf{f}_{G,t}(\mathbf{z}, \dot{\mathbf{z}})}{\partial \dot{\mathbf{z}}} \right|_{\mathbf{z}_0, \dot{\mathbf{z}}_0=\mathbf{0}}}_{\mathbf{J}_{\dot{\mathbf{z}}}(\mathbf{z}_0, \dot{\mathbf{z}}_0=\mathbf{0})} \dot{\mathbf{z}} + \left(\underbrace{\left. \frac{\partial \mathbf{f}_{G,t}(\mathbf{z}, \dot{\mathbf{z}})}{\partial \mathbf{z}} \right|_{\mathbf{z}_0, \dot{\mathbf{z}}_0=\mathbf{0}}}_{\mathbf{J}_{z,t}(\mathbf{z}_0, \dot{\mathbf{z}}_0=\mathbf{0})} + \underbrace{\left. \frac{\partial \mathbf{f}_{G,s}(\mathbf{z})}{\partial \mathbf{z}} \right|_{\mathbf{z}_0}}_{\mathbf{J}_{z,s}(\mathbf{z}_0)} \right) \underbrace{(\mathbf{z} - \mathbf{z}_0)}_{\Delta \mathbf{z}} = \mathbf{0}, \quad (52)$$

which can alternatively be written in the more familiar format

$$\dot{\mathbf{z}} = \mathbf{A}_G \Delta \mathbf{z}, \quad \text{where} \quad \mathbf{A}_G = -\mathbf{J}_{\dot{\mathbf{z}}}(\mathbf{z}_0, \dot{\mathbf{z}}_0 = \mathbf{0})^{-1} (\mathbf{J}_{z,t}(\mathbf{z}_0, \dot{\mathbf{z}}_0 = \mathbf{0}) + \mathbf{J}_{z,s}(\mathbf{z}_0)) \quad (53)$$

allowing the (linear) stability of the full system to be judged from the eigenvalues of $\mathbf{A}_G$ as introduced for foil bearing systems in explicit form by Bonello and Pham [3] and used for systems on linearly implicit form in [40]. It should be emphasised, as it was also discussed in relation to eq. (46), that for an unstable leading eigenvalue of $\mathbf{A}_G$ to define the OSI consistently to the perturbation methods, it must be related to a mode with significant rotor participation. An unstable leading mode dominated by the foil would predict a stability limit undetectable using the two-DOF perturbation, while a pressure dominated unstable leading mode would be invisible to both perturbation techniques. A case in which the stability limit is dictated by a mode with vanishing rotor movement has not been encountered by the authors and it is debatable whether it would ever occur in practice for common GFB configurations. For tilting pad journal bearings, on the contrary, pad flutter would represent exactly such a case, hence potentially devastating instabilities arising from the compliant structure (or other included domains) should not be precluded. In the present form, it is evident that $\mathbf{A}_G$ depends on $\mathbf{J}_{\dot{\mathbf{z}}}$ to be invertible. This is the case when eq. (48) represents a system of ordinary differential equations, as a singular $\mathbf{J}_{\dot{\mathbf{z}}}$ is exactly the defining characteristic of a differential/algebraic equation system. The presented stability analysis can hence not be applied directly to a system containing algebraic equations as originating e.g. from a purely static foil model.

Two notes should be made regarding the foil equations when included into the coupled model. As is, eq. (48) includes the mass-augmented SEFM from eq. (17) in order to match the foil description used in the extended perturbation method. It is, however, straightforward to replace this by the massless SEFM from eq. (14) to produce a foil model matching that used in the classical perturbation method. Secondly, the foil damping entering eq. (48) through $\mathbf{B}_f$ depends on the foil oscillation frequency $\tilde{\omega}_s$ as discussed in section 2.3. For the coupled model to be completely consistent with the perturbation methods at the OSI, $\tilde{\omega}_s$ must match the frequency of the mode becoming unstable. As this is unknown a priori, one needs to locate the coupled system OSI in an iterative fashion while updating the foil oscillation frequency. In the present work, this has been done manually, leading to $\tilde{\omega}_s$ matching the frequency of the mode becoming unstable (± 0.001 Hz) within a few iterations.

Figure 2: Stability maps for a nominal foil stiffness ($k = 4.6417 \text{ GN m}^{-3}$) resulting from a two-DOF perturbation based on a 46×45 (ω_s, Ω) grid. The contours of the logarithmic decrement $\delta = -2\pi \text{Re}(\lambda) / \text{Im}(\lambda)$ are plotted along with a single contour showing the concurrence of the excitation frequency ω_s and the natural frequency $\omega_i = \text{Im}(\lambda)$. Equation (46) is fulfilled at the crossing of $\delta = 0$ thus marking the OSI (for mode 1 at 102.7 Hz, 23.62 kRPM, while mode 2 is stable).

5. Results

The stability limit of the benchmark rotor–bearing system with parameters as given in table 1 is calculated in three different ways using 1) the classical perturbation method, i.e. the eigenvalues of eq. (30); 2) the extended perturbation method, i.e. the eigenvalues of eq. (41); and 3) the coupled system representation, i.e. the eigenvalues of eq. (53). For all three approaches, the bearing varies from very soft to practically rigid using SEFM stiffness values between 1/10 and 1000 times the nominal foil stiffness of 4.6417 GN m^{-3} as calculated from eq. (10).

5.1. Eigenvalues and Mode Shapes from the Classical Two-DOF Perturbation

For the case of nominal foil stiffness, the stability maps for the two eigenvalues resulting from the classical perturbation method are shown in fig. 2. The black line marked $\omega_s = \omega_i$ is the contour where eq. (43) is fulfilled by the pseudo-eigenvalues and hence gives the actual system eigenvalues as a function of rotational speed Ω . The stability is indicated by the contours of the logarithmic decrement showing the OSI condition from eq. (46) to be fulfilled for the lowest mode at 23.62 kRPM with a frequency of 102.7 Hz and the second mode to be well-damped throughout the plotted range. The same limit has been found by solving for the characteristic equation roots based on interpolation between stiffness and damping matrices evaluated over a grid with 10 RPM and 1 Hz resolutions. Omitting the decay factor $e^{\text{Re}(\lambda_i)t}$, the rotor mode shape resulting from a given eigenvector $\mathbf{v}_i$ with associated natural frequency $\omega_i = \text{Im}(\lambda_i)$ can be evaluated as

$$\tilde{\mathbf{x}}_r = \tilde{\mathbf{x}}_{r0} + \mathbf{v}_i e^{i\omega_i t} + \bar{\mathbf{v}}_i e^{-i\omega_i t} \quad (54)$$

providing an oscillation amplitude $|\mathbf{v}_i| + |\bar{\mathbf{v}}_i| = 2|\mathbf{v}_i|$ which has been normalized to an infinity norm of $C/5$. Having solved for the corresponding eigenvectors from eq. (44), the mode shapes associated with the two modes in fig. 2 can thus be plotted over an oscillation period, i.e. with $\omega_i t$ going from 0 to 2π , as shown in fig. 3 along with the static foil deflection. The mode becoming unstable is clearly identified as a forward whirling mode, while the stable mode at 130.7 Hz represents backward whirling. Notice that only the rotor oscillation is plotted, as the classical perturbation provides eigenvectors $\mathbf{v}_i$ corresponding to the rotor DOFs only. Using the relations given by eqs. (19) and (21), it would be possible to reconstruct and plot also the foil and pressure oscillation based on the rotor DOFs, but these would be phase and frequency locked to the rotor along the entire circumference and hence would not contribute additional information.

Figure 3: Static deflection and mode shapes at 23.62 kRPM resulting from a two-DOF perturbation corresponding to the stability maps in fig. 2: (a) Forward mode with $\omega_i = 102.7$ Hz and $\zeta = -2.4 \times 10^{-5}$; (b) backward mode with $\omega_i = 130.7$ Hz and $\zeta = 0.19$. Notice that the decay factor $e^{\text{Re}(\lambda_i)t}$ has been omitted from the mode visualizations.

5.2. Eigenvalues and Mode Shapes from the Extended Perturbation

Performing the same analysis based on the extended perturbation (still with nominal foil stiffness), very similar stability maps can be obtained, but the OSI (still governed by the lowest mode) is not reached until 24.34 kRPM at 103.1 Hz. The corresponding mode shapes are shown in fig. 4, now encompassing the static foil deflection, the rotor movement and the foil oscillation. This is possible as the obtained eigenvectors (now with length $2 + n_\theta$) encompass both rotor and foil DOFs directly, such that a mode shape is now plotted over an oscillation period as

$$\begin{Bmatrix} \tilde{\mathbf{x}}_r \\ \tilde{\mathbf{w}} \end{Bmatrix} = \begin{Bmatrix} \tilde{\mathbf{x}}_{r0} \\ \tilde{\mathbf{w}}_0 \end{Bmatrix} + \mathbf{v}_i e^{i\omega_i t} + \bar{\mathbf{v}}_i e^{-i\omega_i t} \quad (55)$$

where the decay factor $e^{\text{Re}(\lambda_i)t}$ has been omitted and the infinity norm of $2|\mathbf{v}_i|$ has been normalized to $C/5$ as for the rotor-only modes in eq. (54). Again, the mode shapes are visualized over a full oscillation period with $\omega_i t$ running from 0 to 2π allowing the phase between different DOFs to be read from the colours. Notice e.g. from the mode shown in fig. 4a (which is the one dictating the OSI) that the foil oscillation at 180° is approximately in counter phase to the foil at 270° . Considering the rotor alone, the orbits resulting from the extended perturbation are very similar to those from the classical perturbation shown in fig. 3.

5.3. Eigenvalues and Mode Shapes from the Coupled Model

Using the coupled model to solve for equilibrium positions in the range 8–30 kRPM provides the locus shown in fig. 5. At each of these equilibria, the system Jacobian $\mathbf{A}_G$ as given in eq. (53) is assembled and its eigenvalues are calculated. For the present case, the system comprises $n_z \times n_\theta = 18 \times 118 = 2124$ pressure nodes providing 1972 pressure states (having eliminated prescribed nodes), a state space foil model with $2n_\theta = 236$ states and a state space rotor model with 4 states. The natural frequencies and damping ratios of the resulting 2212 eigenvalues are plotted as functions of rotational velocity in fig. 6. Figure 6a reveals a high number of mostly highly damped modes to be scattered sporadically over the range 0–9.3 kHz with a single mode becoming unstable (marked by red stars) after 24.5 kRPM. From the corresponding damping ratios plotted in fig. 6b, two modes with damping ratios between -0.05 and 0.22 are seen to be clearly separated from the rest. Isolating these two modes, the Campbell diagram shown in fig. 7 can be extracted. This filtering technique is simple and works well for relatively uncomplicated systems such as the one presently studied. For more complex systems or systems containing other lowly damped modes, a filtering technique based on eigenvector structures might be necessary as discussed by Bonello [4, 5] when he introduced the Jacobian based Campbell diagrams. As expected, the two modes remaining in the Campbell diagram represent

Figure 4: Static deflection and mode shapes at 24.35 kRPM resulting from the extended perturbation: (a) Forward mode with $\omega_i = 103.2$ Hz and $\zeta = -3.3 \times 10^{-4}$; (b) backward mode with $\omega_i = 130.8$ Hz and $\zeta = 0.18$. Notice that the decay factor $e^{\text{Re}(\lambda_i)t}$ has been omitted from the mode visualizations.

Figure 5: Locus of rotor equilibrium points for a nominal foil stiffness ($k = 4.6417$ GN m $^{-3}$) obtained from the coupled model.

backward and forward whirl and the OSI is seen to be dictated by the forward mode becoming unstable at 24.34 kRPM with a frequency of 103.1 Hz (found by interpolation between the eigenvalues evaluated at steps of 10 RPM). This matches the result from the extended perturbation exactly. Notice from the Campbell diagram, that the backward mode is apparently changing into a forward mode at 8 kRPM as also found in [5]. This is slightly misleading, as the backward orbit is actually collapsing into a line (oriented at approximately $\theta = 15^\circ$) with forward and backward amplitudes of equal magnitude. Similar degenerated orbits was observed, though for a different GFB configuration, as an intermediate stage for modes transitioning from forward to backward whirl in [4]. For such a degenerated orbit, the notion of forward and backward loses its meaning and the difference is only due to numerical differences between the two components' amplitudes as reported in Appendix D. Notice that the nominal case resembles closely the case treated in [5] and that the results agree well despite the slightly higher foil stiffness (4.739 GN m $^{-3}$), exclusion of foil mass, finite difference based RE discretization and rotor-synchronous foil damping used in [5]. Good agreement is observed from the locus and Campbell diagrams and the OSI predicted at 22.5 kRPM in [5] is comparable to the present result. The unfiltered eigenvalue map shown in fig. 6, on the contrary, looks different from the result in [5] due to the high number of heavily damped high frequency modes stemming from the foil inertia.

For the coupled model, each eigenvector $\mathbf{v}_i$ contains information about the oscillation of all three domains, mean-

Figure 6: (a) Natural frequencies and (b) damping ratios of the system Jacobian's 2212 eigenvalues evaluated at the stationary points found at 8–30 kRPM for a nominal foil stiffness ($k = 4.6417 \text{ GN m}^{-3}$).

Figure 7: Campbell diagram extracted from the full set of eigenvalues shown in fig. 6a based on damping ratios. The stability limit is $\Omega_{\text{OSI}} = 24.34 \text{ kRPM}$.

Figure 8: Mode with $\omega_i = 103.2$ Hz and $\zeta = -2.9 \times 10^{-4}$ at 24.35 kRPM extracted from the full system Jacobian using a nominal foil stiffness ($k = 4.6417$ GN m $^{-3}$) where $\Omega_{OSI} = 24.34$ kRPM. The foil oscillation and rotor movement (forward whirl) is shown to the left while the pressure field oscillation at the bearing centre plane is shown to the right. The mode is scaled to provide a maximum displacement amplitude (ϵ_x) of $C/5$ and plotted over a full period with the phase indicated by the colour (dark towards light) Notice that the decay factor $e^{\text{Re}(l_i)t}$ has been omitted from the mode visualizations.

ing that the mode shapes can be evaluated as

$$\mathbf{z} = \begin{Bmatrix} \tilde{\mathbf{p}}^T & \mathbf{z}_f^T & \mathbf{z}_r^T \end{Bmatrix}^T = \mathbf{z}_0 + \mathbf{v}_i e^{i\omega_i t} + \bar{\mathbf{v}}_i e^{-i\omega_i t}. \quad (56)$$

To plot these mode shapes, the full eigenvector is initially normalized to provide a maximum displacement amplitude of $C/5$ (based on the rows representing foil and rotor displacements). The resulting normalized pressure mode shape represents a two-dimensional oscillating field which is awkward to visualize, hence the rows representing the pressure along the bearing mid-plane are extracted. The mode shape associated with the forward mode becoming unstable is thus plotted just after the OSI in fig. 8. This shows (to the left) a rotor orbit and foil oscillation very similar to those from the extended perturbation in fig. 4a and, to the right, the corresponding oscillation of the mid-plane pressure field. The mode shape associated with the backward mode also present in the Campbell diagram is plotted, likewise at 24.35 kRPM, in fig. 9. In this mode shape, the foil is closer to being in-phase along the entire circumference, but a phase difference is still present. This is very similar to the corresponding mode extracted from the extended perturbation in fig. 4b, though it should be emphasized that the coupled model modes are calculated with the foil damping tuned to the leading mode OSI frequency of 103.1 Hz. For the 131.3 Hz mode in fig. 9, the foil damping applied in the coupled model will thus be approximately 30 % higher than for the corresponding mode from the extended perturbation in fig. 4b, possibly explaining the slightly increased damping ratio of $\zeta = 0.20$ predicted by the coupled model compared to $\zeta = 0.18$ from the extended perturbation. To allow direct comparison of other modes than the leading one, the iterative tuning process would either have to be repeated for each mode or a damping model like the one presented in [17] should be used instead. In all cases, it should be highlighted that the rotor orbits and foil deflection shapes related to both the forward and backward modes plotted to the left in figs. 8 and 9 are very similar to those presented in [5].

It can be seen from fig. 6b that the lowest damped mode not included in the Campbell diagram (for the nominal foil stiffness) has a damping ratio of around 0.7, meaning that this mode would contribute marginally to the system response. Furthermore, as seen from the mode shape plotted in fig. 10, the mode is dominated by foil–pressure interaction with virtually no rotor movement. Measuring such a mode experimentally would hence be very difficult and it is reasonable not to include such modes in the Campbell diagram. It should be noted, however, that the damping ratios of the foil dominated modes depend significantly on the foil parameters. If, e.g., the foil stiffness is lowered to 1/10 of the nominal value, the damping ratio of the shown foil mode drops to about 0.4, so it is plausible that a configuration could be found where the foil modes become significant.

5.4. Comparison of OSIs obtained from the three approaches

Figure 9: Mode with $\omega_i = 131.3$ Hz and $\zeta = 0.20$ at 24.35 kRPM extracted from the full system Jacobian using a nominal foil stiffness ($k = 4.6417$ GN m $^{-3}$) where $\Omega_{OSI} = 24.34$ kRPM. The foil oscillation and rotor movement (backward whirl) is shown to the left while the pressure field oscillation at the bearing centre plane is shown to the right. The mode is scaled to provide a maximum displacement amplitude (ε_x) of $C/5$ and plotted over a full period with the phase indicated by the colour (dark towards light). Notice that the decay factor $e^{\text{Re}(\lambda_i)t}$ has been omitted from the mode visualizations.

Figure 10: Mode with $\omega_i = 410.9$ Hz and $\zeta = 0.73$ at 24.35 kRPM extracted from the full system Jacobian using a nominal foil stiffness ($k = 4.6417$ GN m $^{-3}$) where $\Omega_{OSI} = 24.34$ kRPM. The foil oscillation and rotor movement (a forward whirl of vanishing amplitude) is shown to the left while the pressure field oscillation at the bearing centre plane is shown to the right. The mode is scaled to provide a maximum displacement amplitude (the foil deflection at 261.5°) of $C/5$ and plotted over a full period with the phase indicated by the colour (dark towards light)). Notice that the decay factor $e^{\text{Re}(\lambda_i)t}$ has been omitted from the mode visualizations.

Figure 11: Mode with $\omega_i = 109.9\text{ Hz}$ and $\zeta = -2.3 \times 10^{-4}$ at 18.82 kRPM extracted from the full system Jacobian with the nominal foil stiffness multiplied by 1000 ($k = 4641.7\text{ GN m}^{-3}$) giving $\Omega_{\text{OSI}} = 18.81\text{ kRPM}$. The foil oscillation and rotor movement (forward whirl) is shown to the left while the pressure field oscillation at the bearing centre plane is shown to the right. The mode is scaled to provide a maximum displacement amplitude (ε_x) of $C/5$ and plotted over a full period with the phase indicated by the colour (dark towards light). Notice that the decay factor $e^{\text{Re}(\lambda_i)t}$ has been omitted from the mode visualizations.

Table 2: Eccentricity and maximum foil deformation at the OSI from the fully coupled model.

Factor	Foil stiffness $k \text{ (GN m}^{-3}\text{)}$	Ω_{OSI} (kRPM)	ε_x (-)	ε_y (-)	ε (-)	$\max(h_c)$ (-)	$\theta_{\max(h_c)}$ (deg)
1000	4641.7	18.81	0.4747	0.3844	0.6108	0.0002321	200.0
100	464.17	18.88	0.4758	0.3839	0.6113	0.002315	200.0
1	4.6417	24.34	0.6088	0.3730	0.7140	0.2001	196.9
1/2	2.3209	27.36	0.7606	0.3914	0.8554	0.3835	196.9
1/4	1.1604	34.18	1.0496	0.4394	1.1379	0.7394	193.8
1/8	0.58022	57.32	1.5573	0.5430	1.6492	1.4332	190.8
1/10	0.46417	74.59	1.7875	0.5949	1.8839	1.7768	190.8

A main point of interest for the present work is to investigate the discrepancies in predicted OSIs between the three methods at different compliance levels. Varying the SEFM stiffness between 1/10 and 1000 times the nominal stiffness without changing the bearing load, the OSI predicted by the coupled model along with the corresponding static eccentricity and maximum foil compliance is given in table 2. At the nominal foil stiffness, the static eccentricity is 0.7140 and the maximum static foil deformation is about $0.2C$, indicating a bearing with limited influence from the compliance. Increasing the stiffness to $1000k_0$, the bearing behaves essentially rigid, as it can also be seen from the mode shape of the unstable mode shown in fig. 11. Decreasing the foil stiffness, the significance of the compliance increases until the foil deformation contributes more than 90 % of the clearance at $1/10k_0$. The investigated range of foil stiffness values hence covers the full spectrum from rigid gas bearings to modern heavily loaded industrial GFBs.

The OSIs predicted using the coupled system Jacobian eigenvalues and the whirling frequencies of the modes becoming unstable are compared to the perturbation results in table 3. As evident from the calculated deviations shown in table 4 and the comparative plots in fig. 12, the extended perturbation OSIs match the coupled system OSIs for all compliance levels, while the classical two-DOF perturbation matches only for the rigid bearing and underestimates the OSI by $\approx 13\%$ for the softest case.

5.5. Influence of the Foil Oscillation Frequency on the OSI

The coupled domain OSIs are calculated iteratively in order for the foil oscillation frequency ω_f to match the resulting whirling frequency ($\pm 0.001\text{ Hz}$). This differs from the commonly applied assumption of rotor-synchronous

Table 3: Comparison of OSIs obtained from the coupled system Jacobian, the classical perturbation and the extended perturbation.

Factor	Foil stiffness k (GN m ⁻³)	Jacobian with $\omega_f \approx \omega_s$		Two-DOF perturbation		Extended perturbation	
		Ω_{OSI} (kRPM)	ω_s (Hz)	Ω_{OSI} (kRPM)	ω_s (Hz)	Ω_{OSI} (kRPM)	ω_s (Hz)
1000	4641.7	18.81	109.8	18.81	109.8	18.81	109.8
100	464.17	18.88	109.9	18.87	109.9	18.88	109.9
1	4.6417	24.34	103.1	23.62	102.7	24.34	103.1
1/2	2.3209	27.36	94.1	26.08	93.6	27.35	94.2
1/4	1.1604	34.18	80.9	31.59	80.6	34.18	80.9
1/8	0.58022	57.32	66.0	50.57	65.6	57.33	66.1
1/10	0.46417	74.59	61.2	65.06	60.9	74.60	61.2

Table 4: Deviation of perturbation OSIs from the coupled system Jacobian OSIs.

Factor	Foil stiffness k (GN m ⁻³)	Deviation from Jacobian Ω_{OSI} (%)	
		Two-DOF perturbation	Extended perturbation
1000	4641.7	-0.0011	0
100	464.17	-0.033	-0.0011
1	4.6417	-3.0	-0.00082
1/2	2.3209	-4.7	-0.033
1/4	1.1604	-7.6	0.012
1/8	0.58022	-12	0.018
1/10	0.46417	-13	0.021

Figure 12: Comparison of obtained stability limits as a function of foil stiffness: (a) the OSI predicted from the three methods; (b) the deviation between the coupled system OSIs and the perturbation method OSIs.

Table 5: Change in predicted OSIs from the full system Jacobian when assuming rotor-synchronous viscous foil damping, i.e. by setting $\omega_f = \Omega$. The results with $\omega_f = \omega_s \pm 0.001$ Hz (found iteratively) correspond to those in table 3.

Factor	Foil stiffness k (GN m ⁻³)	$\omega_f = \omega_s \pm 0.001$ Hz		$\omega_f = \Omega$		Deviation	
		Ω_{OSI} (kRPM)	ω_s (Hz)	Ω_{OSI} (kRPM)	ω_s (Hz)	Ω_{OSI} (%)	ω_s (%)
1000	4641.7	18.81	109.8	18.81	109.8	-0.0043	-0.0091
100	464.17	18.88	109.9	18.84	109.9	-0.19	-0.045
1	4.6417	24.34	103.1	21.81	99.7	-10	-3.3
1/2	2.3209	27.36	94.1	22.75	90.2	-17	-4.2
1/4	1.1604	34.18	80.9	24.80	77.0	-27	-4.8
1/8	0.580 22	57.32	66.0	31.63	61.9	-45	-6.2
1/10	0.464 17	74.59	61.2	36.46	57.2	-51	-6.5

Table 6: Change in predicted OSIs from the full system Jacobian when removing the foil mass i.e. by degrading eq. (17) into eq. (14). The results with foil mass correspond to those in table 3.

Factor	Foil stiffness k (GN m ⁻³)	SEFM with foil mass		SEFM without foil mass		Deviation	
		Ω_{OSI} (kRPM)	ω_s (Hz)	Ω_{OSI} (kRPM)	ω_s (Hz)	Ω_{OSI} (%)	ω_s (%)
1000	4641.7	18.81	109.8	18.81	109.8	0	0
100	464.17	18.88	109.9	18.88	109.9	0	0
1	4.6417	24.34	103.1	24.34	103.1	-0.0012	-0.000 97
1/2	2.3209	27.36	94.1	27.35	94.1	-0.049	-0.0014
1/4	1.1604	34.18	80.9	34.17	80.9	-0.022	0.0098
1/8	0.580 22	57.32	66.0	57.29	66.1	-0.048	0.016
1/10	0.464 17	74.59	61.2	74.55	61.3	-0.058	0.017

foil damping, but is essential for obtaining OSIs in agreement with the perturbation methods. This is evident from the results in table 5, where the coupled system OSIs previously shown in table 3 are compared to corresponding OSIs calculated without iteration using rotor-synchronous foil damping, i.e. by setting the foil oscillation frequency equal to the rotational speed. For the rigid bearing, where the foil has a vanishing influence on the system dynamics, the difference is negligible, but the importance increases with the foil compliance level leading to a 51 % underestimation of the OSI for $1/10k_0$.

5.6. Influence of the Foil Inertia on the OSI

Lastly, it should be noted that the coupled system OSIs are based on the system formulation from eq. (48) including the foil mass. The foil mass is likewise taken into account by the extended perturbation, but not by the classical two-DOF perturbation, meaning that any influence on the OSI by the foil mass would assign the extended perturbation method an unintended competitive advantage. To make sure this is not the case, the coupled system OSIs have additionally been calculated using the massless, but otherwise equivalent, foil model from eq. (14). The results are compared to the previously shown OSIs from table 3 (which are calculated with foil mass) in table 6, revealing the influence of the foil mass on the OSI to be negligible.

6. Conclusion

Compared to previous work by the authors [1, 2], the discrepancies between the OSIs calculated from the classical perturbation method and those from a full non-linear model are smaller and with opposite sign. This is caused by three model changes:

- The coupled model OSI is now found with a foil frequency matching the frequency of the mode becoming unstable (see table 5). This requires an iterative calculation of the OSI, but provides consistency with the foil oscillation frequency employed in the first order equations. The same issue was treated in a recent publication

by Pronobis and Liebich [17], but here consistency was achieved while retaining the rotor-synchronous viscous damping by setting $\omega_f = \Omega$ in eqs. (11) and (13), effectively providing a frequency dependent hysteretic damping.

- The Gmbel condition is enforced on the dynamic pressure field at the integration stage, as for the static and time domain pressures, rather than through constraints on the solution (as described in [16]).
- The dynamic compliance, i.e. the foil deflection due to the dynamic pressure field, is now based on the axially averaged dynamic pressure. This is consistent with the calculation of the static foil compliance from the axially averaged static pressure and, importantly, with the foil formulation used in the coupled time domain model where no distinction is made between dynamic and static pressures.

For a very stiff bearing, the classical perturbation method has thus been shown to provide exactly the same OSI as the coupled model. This was to be expected since the method was introduced for use with rigid bearings. For the worst case of a very soft bearing, the perturbation leads to an underestimation of the stability limit by 13 % for the present benchmark rotor–bearing system. These findings are comparable to other recently published results [17], where the classical perturbation is found to predict OSIs in good, but declining, agreement to a time domain model up to a compliance level of around $C/3$.

In the present work, it has been demonstrated that this discrepancy can be eliminated entirely for all compliance levels by extending the perturbation to include also the foil states and their governing equations of motion, similarly to the treatment of pads in stability analysis for tilting pad journal bearings. In its present format, the extended perturbation requires the inclusion of foil mass, but this has been verified as not affecting the OSI by comparing with OSIs from a coupled model incorporating a massless SEFM (see table 6). The deficiency of the classical perturbation is hence not rooted in the omitted foil mass, but fundamentally in the elimination of the independent foil states through the implicit compliance description introduced by Peng and Carpino [7] in 1993. Though evidently problematic to the OSI, the motivation for this simplification is easy to understand considering the available computational resources at the time and it should be highlighted that the efforts by Peng and Carpino [7] have enabled GFB analysis for almost three decades.

From the presented benchmark results as well as the results from [17], it could seem that the two-DOF perturbation approach could still be a useful simplification for low compliance levels, but it is not possible to quantify the influence of other bearing design parameters or the bearing geometry on the OSI discrepancy, and thus whether smaller or larger OSI discrepancies are to be expected for more realistic rotor–bearing systems. Furthermore, it has not been investigated how the issue might influence other predicted characteristics, such as the unbalance response. It can, however, be concluded that the elimination of foil states is fundamentally erroneous for stability analysis as the pressure–compliance interaction is neglected. A possible remedy is to apply the presented extended perturbation method, but this is rather elaborate, computationally expensive and difficult to reconcile with more realistic foil models. Furthermore, a main virtue of the perturbation approach is lost, as the computational effort increases drastically due to the higher number of first order equations and subsequent much larger eigenvalue problems. Instead, stability analysis for GFB supported rotor–bearing systems should be based on coupled equation systems, allowing stability assessment either through brute-force time-integration or, ideally, based on Jacobian eigenvalues.

Appendix A. Jacobian matrices

Appendix A.1. Reynolds Equation Finite Element Residual Derivatives

As the element boundaries are fixed, Leibniz’s rule allows the partial derivative of the RE FE residual from eq. (3) with respect to the pressure to be written (on the element level) as

$$\frac{\partial \tilde{r}_{RE}^e}{\partial \tilde{p}} = \frac{\partial \mathbf{A}_{RE}^e(\tilde{p}^e, \tilde{x}_r, \tilde{x}_f) \tilde{p}^e}{\partial \tilde{p}} + \frac{\partial \mathbf{B}_{RE}^e(\dot{\tilde{p}}, \dot{\tilde{x}}_r, \dot{\tilde{x}}_f) \tilde{p}^e}{\partial \dot{\tilde{p}}} + \frac{\mathbf{C}_{RE}^e(\tilde{p}, \tilde{x}_r, \tilde{x}_f) \dot{\tilde{p}}^e}{\partial \dot{\tilde{p}}}, \quad (\text{A.1})$$

where

$$\frac{\partial \mathbf{A}_{\text{RE}}^e \tilde{\mathbf{p}}^e}{\partial \tilde{\mathbf{p}}} = \int_{A^e} \mathbf{B}^T \left(\underbrace{-\tilde{p} \tilde{h}^3 \mathbf{B}}_{\text{I}} - \underbrace{\tilde{h}^3 \mathbf{B} \tilde{\mathbf{p}}^e \mathbf{N}}_{\text{II}} + \underbrace{\left(S_\tau \tilde{\Omega} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \mathbf{N} - 3 \tilde{p} \tilde{h}^2 \mathbf{B} \right)}_{\text{III}} \right) \tilde{\mathbf{p}}^e \frac{\partial \tilde{h}}{\partial \tilde{\mathbf{p}}} + \underbrace{S_\tau \tilde{\Omega} \tilde{h} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \mathbf{N}}_{\text{V}} dA, \quad (\text{A.2})$$

$$\frac{\partial \mathbf{B}_{\text{RE}}^e \tilde{\mathbf{p}}^e}{\partial \tilde{\mathbf{p}}} = -2S_\tau \int_{A^e} \underbrace{\dot{\tilde{h}} \mathbf{N}^T \mathbf{N}}_{\text{VI}} dA, \quad (\text{A.3})$$

$$\frac{\partial \mathbf{C}_{\text{RE}}^e \dot{\tilde{\mathbf{p}}}^e}{\partial \tilde{\mathbf{p}}} = -2S_\tau \int_{A^e} \underbrace{\mathbf{N}^T \mathbf{N} \dot{\tilde{\mathbf{p}}}^e}_{\text{VII}} \frac{\partial \tilde{h}}{\partial \tilde{\mathbf{p}}} dA, \quad (\text{A.4})$$

in which terms I, II, V and VI are non-zero at the element DOFs only, i.e. they are 4×4 matrices corresponding to $\partial/\partial \tilde{\mathbf{p}}^e$, while terms III, IV and VII (if present) provide element Jacobian contributions to pressure DOFs not included in $\tilde{\mathbf{p}}^e$ due to the axial averaging of pressures.

For an implicit film height description where no $\tilde{\mathbf{x}}_f$ is present, compliance is a function of pressure directly, thus giving rise to terms III, IV and VII. For the present implementation where compliance is based on the axially averaged pressure (referring to eq. (12)), the compliant film heights at the element nodal positions resulting from a pressure field $\tilde{p}$ can be written as

$$\tilde{\mathbf{h}}_c^e(\tilde{p}) = \begin{Bmatrix} f_{h_c}(\theta_1^e, \tilde{p}) \\ \vdots \\ f_{h_c}(\theta_4^e, \tilde{p}) \end{Bmatrix} \approx \tilde{L}^{-1} \tilde{k}_c^{-1} \begin{Bmatrix} \sum_{j=0}^{n_z} \tilde{p}(\theta_1^e, \tilde{z}_j) \Delta \tilde{z}_j \\ \vdots \\ \sum_{j=0}^{n_z} \tilde{p}(\theta_4^e, \tilde{z}_j) \Delta \tilde{z}_j \end{Bmatrix} = \mathbf{H}_{h_c, p}^e \tilde{\mathbf{p}}, \quad (\text{A.5})$$

where $\mathbf{H}_{h_c, p}^e$ has dimension $4 \times n_\theta n_z$, but contains non-zero contributions only from the $2n_z$ pressure DOFs affecting the film height in a given element. The film height derivative with respect to pressure required in eqs. (A.2) and (A.4) can be found using the shape functions as

$$\frac{\partial \tilde{h}}{\partial \tilde{\mathbf{p}}} = \frac{\partial \tilde{h}_c}{\partial \tilde{\mathbf{p}}} = \mathbf{N} \frac{\partial \tilde{\mathbf{h}}_c^e}{\partial \tilde{\mathbf{p}}} = \mathbf{N} \mathbf{H}_{h_c, p}^e \in \mathbb{R}^{1 \times n_\theta n_z}, \quad (\text{A.6})$$

meaning that terms III, IV and VII result in $4 \times n_\theta n_z$ matrices with non-zero contributions to the four element residuals from the $2n_z$ pressure DOFs affecting the compliant height within the element.

For the coupled system approach, the compliant film height is given explicitly from the foil states $\tilde{\mathbf{x}}_f$ implying $\partial \tilde{h}/\partial \tilde{\mathbf{p}} = \mathbf{0}$ and thus eliminating terms III, IV and VII. In return, the derivatives with respect to the rotor and foil DOF vectors $\tilde{\mathbf{x}}_r$ and $\tilde{\mathbf{x}}_f$ are required. These can be written as

$$\frac{\partial \tilde{\mathbf{r}}_{\text{RE}}^e}{\partial \tilde{\mathbf{x}}_r} = \frac{\partial \mathbf{A}_{\text{RE}}^e \tilde{\mathbf{p}}^e}{\partial \tilde{\mathbf{x}}_r} + \frac{\mathbf{C}_{\text{RE}}^e \dot{\tilde{\mathbf{p}}}^e}{\partial \tilde{\mathbf{x}}_r} = \int_{A^e} \mathbf{B}^T \left(-3 \tilde{p} \tilde{h}^2 \mathbf{B} + S_\tau \tilde{\Omega} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \mathbf{N} \right) \tilde{\mathbf{p}}^e \frac{\partial \tilde{h}}{\partial \tilde{\mathbf{x}}_r} dA - \int_{A^e} 2S_\tau \mathbf{N}^T \mathbf{N} \dot{\tilde{\mathbf{p}}}^e \frac{\partial \tilde{h}}{\partial \tilde{\mathbf{x}}_r} dA \quad (\text{A.7})$$

$$\frac{\partial \tilde{\mathbf{r}}_{\text{RE}}^e}{\partial \tilde{\mathbf{x}}_f} = \frac{\partial \mathbf{A}_{\text{RE}}^e \tilde{\mathbf{p}}^e}{\partial \tilde{\mathbf{x}}_f} + \frac{\mathbf{C}_{\text{RE}}^e \dot{\tilde{\mathbf{p}}}^e}{\partial \tilde{\mathbf{x}}_f} = \int_{A^e} \mathbf{B}^T \left(-3 \tilde{p} \tilde{h}^2 \mathbf{B} + S_\tau \tilde{\Omega} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \mathbf{N} \right) \tilde{\mathbf{p}}^e \frac{\partial \tilde{h}}{\partial \tilde{\mathbf{x}}_f} dA - \int_{A^e} 2S_\tau \mathbf{N}^T \mathbf{N} \dot{\tilde{\mathbf{p}}}^e \frac{\partial \tilde{h}}{\partial \tilde{\mathbf{x}}_f} dA \quad (\text{A.8})$$

where $\partial \tilde{h}/\partial \tilde{\mathbf{x}}_r$ is simply $[\cos \theta \ \sin \theta]$, while $\partial \tilde{h}/\partial \tilde{\mathbf{x}}_f$ is a $1 \times n_\theta$ row vector with two non-zero elements corresponding to the two foil states coinciding with the circumferential positions of the element's pressure nodes. Finally, the residual formulation requires also the partial derivatives with respect to the state vector temporal derivative which are given as

$$\frac{\partial \tilde{\mathbf{r}}_{\text{RE}}^e}{\partial \dot{\tilde{\mathbf{p}}}} = \frac{\partial \mathbf{C}_{\text{RE}}^e \dot{\tilde{\mathbf{p}}}^e}{\partial \dot{\tilde{\mathbf{p}}}} + \frac{\partial \mathbf{B}_{\text{RE}}^e \tilde{\mathbf{p}}^e}{\partial \dot{\tilde{\mathbf{p}}}} = -2S_\tau \int_{A^e} \mathbf{N}^T \tilde{h} \mathbf{N} + \mathbf{N}^T \mathbf{N} \tilde{\mathbf{p}}^e \frac{\partial \tilde{h}}{\partial \dot{\tilde{\mathbf{p}}}} dA, \quad (\text{A.9})$$

$$\frac{\partial \tilde{\mathbf{r}}_{\text{RE}}^e}{\partial \dot{\tilde{\mathbf{x}}}_r} = \frac{\partial \mathbf{B}_{\text{RE}}^e \tilde{\mathbf{p}}^e}{\partial \dot{\tilde{\mathbf{x}}}_r} = -2S_\tau \int_{A^e} \mathbf{N}^T \mathbf{N} \tilde{\mathbf{p}}^e \frac{\partial \tilde{h}}{\partial \dot{\tilde{\mathbf{x}}}_r} dA, \quad (\text{A.10})$$

$$\frac{\partial \tilde{\mathbf{r}}_{\text{RE}}^e}{\partial \dot{\tilde{\mathbf{x}}}_f} = \frac{\partial \mathbf{B}_{\text{RE}}^e \tilde{\mathbf{p}}^e}{\partial \dot{\tilde{\mathbf{x}}}_f} = -2S_\tau \int_{A^e} \mathbf{N}^T \mathbf{N} \tilde{\mathbf{p}}^e \frac{\partial \tilde{h}}{\partial \dot{\tilde{\mathbf{x}}}_f} dA, \quad (\text{A.11})$$

where $\partial\tilde{h}/\partial\tilde{\mathbf{x}}_r$, $\partial\tilde{h}/\partial\tilde{\mathbf{x}}_f$ and $\partial\tilde{h}/\partial\tilde{\mathbf{p}}$ are found analogously to $\partial\tilde{h}/\partial\tilde{\mathbf{x}}_r$, $\partial\tilde{h}/\partial\tilde{\mathbf{x}}_f$ and $\partial\tilde{h}/\partial\tilde{\mathbf{p}}$. The leftmost term of eq. (A.9) is non-zero only at the columns corresponding to the element DOFs.

Notice that in previous work by the authors [16], only terms I and V of eq. (A.2) were included in the tangent matrix (yielding an inexact NR method) and that the squeeze and local expansion derivative terms VI and VII are present only for transient conditions. Furthermore, it should be mentioned that term VII from eq. (A.4) and the rightmost term in eq. (A.9) are only present if an implicit compliance formulation is used in a transient simulation. This combination is not applied in the present work, but the terms are included for completeness.

Appendix B. Foil Structure Mass

Assuming the top foil shares the bump foil density and thickness, a rough estimate of the collective foil structure's average mass per unit circumferential length can be obtained as

$$m'_f = \underbrace{(2R_b\theta_b + S_b - 2l_0) L t_b \rho_b S_b^{-1}}_{\text{Bump foil}} + \underbrace{t_b L \rho_b}_{\text{Top foil}}, \quad (\text{B.1})$$

where the bump radius of curvature is given as $R_b = \frac{h_b^2 + l_0^2}{2l_0}$ and the bump half angle as $\theta_b = \arccos \frac{R_b - h_0}{R_b}$. For the properties given in table 1 this gives $m'_f = 65.44 \times 10^{-3} \text{ kg m}^{-1}$.

Appendix C. Extended Perturbation Without Foil Mass

Even though this is believed by the authors to present a less intuitive approach, it would be possible to formulate the extended perturbation equations of motion from eq. (41) without including the foil mass. Recasting the rotor equations into state space form, these could be coupled to the first order (massless) rotor equations as

$$\left(\begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \tilde{\mathbf{M}}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{\mathbf{D}}_f(\tilde{\omega}_s) \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{D}}_{\text{g,xDOF}}(\tilde{\omega}_s, \tilde{\Omega}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{Bmatrix} \ddot{\tilde{\mathbf{x}}}_r \\ \ddot{\tilde{\mathbf{x}}}_f \\ \ddot{\tilde{\mathbf{w}}} \end{Bmatrix} + \left(\begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{\mathbf{K}}_f \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{K}}_{\text{g,xDOF}}(\tilde{\omega}_s, \tilde{\Omega}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{Bmatrix} \dot{\tilde{\mathbf{x}}}_r \\ \dot{\tilde{\mathbf{x}}}_f \\ \dot{\tilde{\mathbf{w}}} \end{Bmatrix} = \mathbf{0}, \quad (\text{C.1})$$

leading to an eigenvalue problem requiring a slightly different treatment than currently described in section 3.3. In brief, the characteristic polynomial function from eq. (43) would change into

$$P_c(\lambda) = \det \left(\begin{bmatrix} -\mathbf{I} & \lambda \mathbf{I} & \mathbf{0} \\ \lambda \tilde{\mathbf{M}}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \lambda \tilde{\mathbf{D}}_f(\text{Im}(\lambda)) + \tilde{\mathbf{K}}_f \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \lambda \tilde{\mathbf{D}}_{\text{g,xDOF}}(\text{Im}(\lambda), \tilde{\Omega}) + \tilde{\mathbf{K}}_{\text{g,xDOF}}(\text{Im}(\lambda), \tilde{\Omega}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right), \quad (\text{C.2})$$

and the structure of the obtained eigenvectors would change. This approach has not been further pursued by the authors, but it would possibly have allowed similar results to be obtained from the extended perturbation without including the foil mass.

Appendix D. Mode Shape with Degenerated Orbit

Reading from the Campbell diagram in fig. 7, the second mode is starting out as a forward whirl at 8 kRPM, but reversing into a backward whirl for ≥ 8.5 kRPM. This is slightly misleading, as the orbit is simply collapsing into a straight line at an angle of approximately $\theta = 15^\circ$ as shown in fig. D.13. For such a mode, the backward and forward components of the rotor movement are of equal magnitude, meaning that the categorization as a forward mode is merely due to numerical differences between the calculated forward and backward magnitudes. Deciding on a numerical tolerance, one could hence have introduced a third category for modes showing this kind of degenerated orbit in fig. 7.

Figure D.13: Mode with $\omega_i = 158.6\text{Hz}$ and $\zeta = 0.2041$ at 8kRPM extracted from the full system Jacobian with the nominal foil stiffness $k = 4.6417\text{GN m}^{-3}$. The foil oscillation and rotor movement (an orbit degenerated into a straight line) is shown to the left while the pressure field oscillation at the bearing centre plane is shown to the right. The mode is scaled to provide a maximum displacement amplitude (ε_x) of $C/5$ and plotted over a full period with the phase indicated by the colour (dark towards light). Notice that the decay factor $e^{\text{Re}(\lambda_i)t}$ has been omitted from the mode visualizations.

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