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The Classical Linearization Technique’s Validity for Compliant Bearings

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Abstract. The Gas Foil Bearing (GFB) is a promising and environmentally friendly technology allowing support of high-speed rotating machinery with low power loss and without oil or electronics. Unfortunately, GFBs provide limited damping, making an accurate prediction of the Onset Speed of Instability (OSI) critical. This has traditionally been assessed using linearised coefficients derived from the perturbed Reynolds Equation with compliance included implicitly. Recent work has, however, revealed significant discrepancies between OSIs predicted using these techniques and those observed from nonlinear analysis. In the present work, the perturbation method’s underlying assumption on the pressure field is investigated by including the hitherto neglected pressure–compliance dependency directly. This leads to an extended perturbation akin to that commonly applied to tilting pad bearings and is shown to predict OSIs with much better agreement to time integration results. The extended perturbation method is cumbersome, but serves to highlight the error introduced when applying the classical perturbation method — as developed for rigid bearings by J. W. Lund — to GFBs.

Keywords: Gas Foil Bearings, Perturbation, Stability

1 Introduction

Due to the limited damping provided by GFBs, an accurate prediction of the Onset Speed of Instability (OSI) remains critical to their application. In order to predict the lateral vibration response of GFB-supported rotors, and hence their stability, it is possible to apply linear [6, 8–11, 13, 16, 19, 24] as well as nonlinear [1–3, 7, 12, 20] approaches. In the linear approach, the gas film forces are fundamentally represented by equivalent springs and dampers with coefficients derived from a linearisation of the Reynolds Equation (RE) around one or several states of equilibrium. The calculation of such gas film coefficients can be achieved by a numerical perturbation or analytically as proposed by Lund [17]. Peng and Carpino [19] were among the first to apply Lund’s perturbation technique to GFBs and such analyses have since been performed by numerous authors [6, 10, 24]. Some of these have furthermore compared their results to nonlinear analyses showing varying levels of agreement. In recent work by the

authors [14], this has been investigated by comparing the OSI of an industrial GFB-supported rotor predicted from the classical frequency domain technique to the stability limits observed from nonlinear time integration. Here, using equivalent numerical implementations for the two approaches, a significant discrepancy was demonstrated and shown to be increasing with the level of compliance.

A possible root cause of the observed disagreement can be found in a primary assumption of the classical perturbation approach, namely that the pressure field depends exclusively on the rotor position and velocity. Assuming subsequently the rotor to perform small harmonic oscillations, a Taylor series expansion can be inserted into RE to solve for the eight bearing stiffness and damping coefficients.

In the present work, the importance of the — thus far neglected — pressure–compliance dependency is investigated. The pressure field is thus assumed to depend not only on the rotor position and velocity, but also on the degrees of freedom (DOFs) representing the foil deflection. This additionally requires the solution of a dynamic pressure field for each of the foil DOFs and hence provides a coefficient matrix with contributions from each of these, analogous to the coefficient matrices often used for tilting pad journal bearings. Using a simple and widely studied GFB configuration [21] supporting a point mass as starting point, OSI predictions from the extended perturbation method are compared to results from both a classical perturbation method and a simultaneous nonlinear time integration. These two reference models are identical to those previously presented by the authors [11, 12, 14] and have been experimentally validated. It should be emphasized that the vibrations occurring at the investigated OSI stems from a self-excited instability and thus are related exclusively to the homogeneous part of the equation system. This should not be confused with the onset of forced subsynchronous vibrations caused by the unbalance excitation in conjunction with the nonlinear GFB characteristics. The appearance of the latter is influenced by the level of unbalance, while the former is not.

The additional terms included in the extended perturbation multiplies the effort needed to attain the bearing coefficients and complicates their subsequent interpretation, but the extension is demonstrated to provide results in much better agreement to those obtainable from nonlinear time integration. The challenges of the extension are discussion and the significance of the foil–compliance terms is treated to illuminate the error introduced when neglecting them. In this sense, the novel contribution of the present work is to identify the limits of validity for the classical perturbation method with respect to compliance.

2 The extended perturbation method

In addition to the eccentricities, the pressure field p is assumed to depend on N DOFs. These dependencies can be collected in the vector $\mathbf{q}$ as

$$\mathbf{q} = \{e_x \ e_y \ w_1 \ \cdots \ w_N\}^T \in \mathbb{R}^{2+N}, \quad (1)$$

where e_x and e_y are the rotor eccentricity components and w_j denotes the j -th foil compliance DOF. Using eq. (1), a first order Taylor series expansion of the

pressure field around a state $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ can be written as

$$p = p(\mathbf{q}, \dot{\mathbf{q}}) \approx p(\mathbf{q}_0, \dot{\mathbf{q}}_0) + \left. \frac{\partial p}{\partial \mathbf{q}} \right|_{\mathbf{q}_0, \dot{\mathbf{q}}_0} (\mathbf{q} - \mathbf{q}_0) + \left. \frac{\partial p}{\partial \dot{\mathbf{q}}} \right|_{\mathbf{q}_0, \dot{\mathbf{q}}_0} (\dot{\mathbf{q}} - \dot{\mathbf{q}}_0). \quad (2)$$

Defining $\mathbf{q}_0$ as the stationary point where $\dot{\mathbf{q}} = 0$, and assuming all DOFs to exhibit small harmonic oscillations around this point with frequency ω_s as

$$e_\gamma = e_{\gamma 0} + \Delta e_\gamma e^{i\omega_s t}, \quad \gamma = x, y \quad \text{and} \quad w_j = w_{j0} + \Delta w_j e^{i\omega_s t}, \quad j = 1, \dots, N, \quad (3)$$

the pressure field expansion from eq. (2) becomes

$$p = p_0 + p_x \Lambda_x + p_y \Lambda_y + \sum_{j=1}^N p_{w_j} \Lambda_j, \quad (4)$$

where

$$p_0 = p(\mathbf{q}_0), \quad (5)$$

$$p_\gamma = \left(\left. \frac{\partial p}{\partial e_\gamma} \right|_{\mathbf{q}_0} + i\omega_s \left. \frac{\partial p}{\partial \dot{e}_\gamma} \right|_{\mathbf{q}_0} \right), \quad \gamma = x, y, \quad (6)$$

$$p_{w_j} = \left(\left. \frac{\partial p}{\partial w_j} \right|_{\mathbf{q}_0} + i\omega_s \left. \frac{\partial p}{\partial \dot{w}_j} \right|_{\mathbf{q}_0} \right), \quad j = 1, \dots, N, \quad (7)$$

$$\Lambda_\gamma = \Delta e_\gamma e^{i\omega_s t}, \quad \gamma = x, y, \quad (8)$$

$$\Lambda_j = \Delta w_j e^{i\omega_s t}, \quad j = 1, \dots, N, \quad (9)$$

so that $p_0 \in \mathbb{R}$ and $p_\gamma, p_{w_j}, \Lambda_\gamma, \Lambda_j \in \mathbb{C}$.

2.1 The film height function

For the present purpose of investigating the pressure–compliance dependency, the rotor is assumed to be perfectly aligned and the foil is assumed to deform uniformly in the axial direction. Axial film height variations are hence neglected and the film height function becomes one-dimensional. This has previously been shown to be reasonable [23], but the proposed perturbation method is not restricted to such assumptions.

Referring to fig. 1b, the film height is composed of a rigid contribution,

$$h_r = C + e_x \cos \theta + e_y \sin \theta, \quad (10)$$

and a compliant contribution h_c stemming from the deformation of the foil structure. It is hence necessary to establish a relation between the foil DOFs w_j and the deformed shape of the top foil. The foil DOFs could be representing Fourier series amplitudes or polynomial coefficients, but for now they will represent discrete point deformations directly such that $w_j = h_c(\theta_j)$. To simplify the numerical implementation, the points θ_j are furthermore chosen to be coincident with the circumferential discretization of the fluid film.

Fig. 1: (a) Weight function associated with the j -th foil compliance DOF; (b) Schematics of the GFB and illustration of the perturbed DOFs.

In order to evaluate the deformed shape between the discrete points, a simple linear interpolation is applied that is consistent with the linear shape functions applied for the finite element (FE) discretization of RE. A weight function is hence defined for each foil DOF as illustrated in fig. 1a and given as

$$W_j(\theta) = \begin{cases} \frac{\theta - \theta_{j-1}}{\theta_j - \theta_{j-1}} & \text{if } \theta_{j-1} < \theta < \theta_j \\ \frac{\theta_{j+1} - \theta}{\theta_{j+1} - \theta_j} & \text{if } \theta_j \leq \theta < \theta_{j+1} \\ 0 & \text{otherwise,} \end{cases} \quad (11)$$

such that the continuous compliant film height contribution can be written as

$$h_c = \sum_{j=1}^N w_j W_j(\theta). \quad (12)$$

This finally allows, when introducing the perturbations from eq. (3), the perturbed film height to be written as

$$h = h_r + h_c = h_0 + A_x \cos \theta + A_y \sin \theta + \sum_{j=1}^N A_j W_j(\theta) \quad (13)$$

where the static film height h_0 is given from

$$h_0 = C + e_{x0} \cos \theta + e_{y0} \sin \theta + \sum_{j=1}^N w_{j0} W_j(\theta). \quad (14)$$

2.2 Perturbation of Reynolds Equation

Defining the circumferential and axial coordinates $\Theta = \theta R$ and z in a bearing with radius R , the isothermal, compressible and transient RE can be written as

$$\frac{\partial}{\partial \Theta} \left(\frac{ph^3}{12\mu} \frac{\partial p}{\partial \Theta} \right) + \frac{\partial}{\partial z} \left(\frac{ph^3}{12\mu} \frac{\partial p}{\partial z} \right) = \frac{\partial}{\partial \Theta} (phU) + \frac{\partial}{\partial t} (ph), \quad (15)$$

where μ is the gas viscosity and $U = R\Omega/2$ is the circumferential gas film velocity. Substituting eqs. (4) and (13) into eq. (15) while neglecting higher order terms, the zeroth and first order equations can be separated. The zeroth order equation obtains the familiar form

$$\frac{\partial}{\partial \Theta} \left(\frac{p_0 h_0^3}{12\mu} \frac{\partial p_0}{\partial \Theta} \right) + \frac{\partial}{\partial z} \left(\frac{p_0 h_0^3}{12\mu} \frac{\partial p_0}{\partial z} \right) = \frac{\partial}{\partial \Theta} (p_0 h_0 U), \quad (16)$$

while the two first order equations for p_x and p_y can be written as

$$\begin{aligned} & \frac{\partial}{\partial \Theta} \left(\frac{p_0 h_0^3}{12\mu} \frac{\partial p_\gamma}{\partial \Theta} \right) + \frac{\partial}{\partial z} \left(\frac{p_0 h_0^3}{12\mu} \frac{\partial p_\gamma}{\partial z} \right) + \frac{\partial}{\partial \Theta} \left(\frac{p_\gamma h_0^3}{12\mu} \frac{\partial p_0}{\partial \Theta} \right) + \\ & \frac{\partial}{\partial z} \left(\frac{p_\gamma h_0^3}{12\mu} \frac{\partial p_0}{\partial z} \right) - \frac{\partial}{\partial \Theta} (p_\gamma h_0 U) - i\omega_s p_\gamma h_0 = -\frac{\partial}{\partial \Theta} \left(\frac{p_0 h_0^2 f_\gamma}{4\mu} \frac{\partial p_0}{\partial \Theta} \right) \\ & - \frac{\partial}{\partial z} \left(\frac{p_0 h_0^2 f_\gamma}{4\mu} \frac{\partial p_0}{\partial z} \right) + \frac{\partial}{\partial \Theta} (p_0 U f_\gamma) + i\omega_s p_0 f_\gamma, \quad \gamma = x, y, \end{aligned} \quad (17)$$

where $f_x = \cos \theta$ and $f_y = \sin \theta$, and, finally, the N first order equations for p_{w_j} related to the foil DOFs become

$$\begin{aligned} & \frac{\partial}{\partial \Theta} \left(\frac{p_0 h_0^3}{12\mu} \frac{\partial p_{w_j}}{\partial \Theta} \right) + \frac{\partial}{\partial z} \left(\frac{p_0 h_0^3}{12\mu} \frac{\partial p_{w_j}}{\partial z} \right) + \frac{\partial}{\partial \Theta} \left(\frac{p_{w_j} h_0^3}{12\mu} \frac{\partial p_0}{\partial \Theta} \right) + \\ & \frac{\partial}{\partial z} \left(\frac{p_{w_j} h_0^3}{12\mu} \frac{\partial p_0}{\partial z} \right) - \frac{\partial}{\partial \Theta} (p_{w_j} h_0 U) - i\omega_s p_{w_j} h_0 = -\frac{\partial}{\partial \Theta} \left(\frac{p_0 h_0^2 W_j}{4\mu} \frac{\partial p_0}{\partial \Theta} \right) \\ & - \frac{\partial}{\partial z} \left(\frac{p_0 h_0^2 W_j}{4\mu} \frac{\partial p_0}{\partial z} \right) + \frac{\partial}{\partial \Theta} (p_0 U W_j) + i\omega_s p_0 W_j, \quad j = 1, \dots, N. \end{aligned} \quad (18)$$

2.3 The coefficient matrix

Having obtained p_0 , p_γ and p_{w_j} by a suitable numerical solution scheme, these can be integrated to obtain the fluid film forces $\mathbf{f}$. Using eq. (4), these can be written as

$$\mathbf{f} = \begin{Bmatrix} \int_0^L \int_0^{2\pi} (p - p_a) \begin{Bmatrix} \cos \theta \\ \sin \theta \end{Bmatrix} R d\theta dz \\ \int_0^L \int_{\theta_1 - \frac{1}{2}\Delta\theta_1}^{\theta_1 + \frac{1}{2}\Delta\theta_1} (p - p_a) R d\theta dz \\ \vdots \\ \int_0^L \int_{\theta_N - \frac{1}{2}\Delta\theta_N}^{\theta_N + \frac{1}{2}\Delta\theta_N} (p - p_a) R d\theta dz \end{Bmatrix} = \mathbf{f}_0 + \begin{bmatrix} \mathbf{Z}_{ee} & \mathbf{Z}_{ew} \\ \mathbf{Z}_{we} & \mathbf{Z}_{ww} \end{bmatrix} \tilde{\mathbf{q}} e^{i\omega_s t}, \quad (19)$$

where the static force is given as the vector

$$\mathbf{f}_0 = \begin{Bmatrix} \int_0^L \int_0^{2\pi} (p_0 - p_a) \begin{Bmatrix} \cos \theta \\ \sin \theta \end{Bmatrix} R d\theta dz \\ \int_0^L \int_{\theta_1 - \frac{1}{2}\Delta\theta_1}^{\theta_1 + \frac{1}{2}\Delta\theta_1} (p_0 - p_a) R d\theta dz \\ \vdots \\ \int_0^L \int_{\theta_N - \frac{1}{2}\Delta\theta_N}^{\theta_N + \frac{1}{2}\Delta\theta_N} (p_0 - p_a) R d\theta dz \end{Bmatrix} \in \mathbb{R}^{2+N}, \quad (20)$$

and the dynamic forces are expressed in terms of the perturbation vector

$$\tilde{\mathbf{q}} = \{\Delta e_x \Delta e_y \Delta w_1 \cdots \Delta w_N\}^T \in \mathbb{R}^{2+N}, \quad (21)$$

along with the matrix $\mathbf{Z}$ with blocks given as

$$\mathbf{Z}_{ee} = \int_0^L \int_0^{2\pi} \begin{bmatrix} p_x \cos \theta & p_y \cos \theta \\ p_x \sin \theta & p_y \sin \theta \end{bmatrix} R d\theta dz \in \mathbb{C}^{2 \times 2}, \quad (22)$$

$$\mathbf{Z}_{ew} = \int_0^L \int_0^{2\pi} \begin{bmatrix} p_{w_1} \cos \theta & \cdots & p_{w_N} \cos \theta \\ p_{w_1} \sin \theta & \cdots & p_{w_N} \sin \theta \end{bmatrix} R d\theta dz \in \mathbb{C}^{2 \times N}, \quad (23)$$

$$\mathbf{Z}_{we} = \begin{bmatrix} \int_0^L \int_{\theta_1 - \frac{1}{2}\Delta\theta_1}^{\theta_1 + \frac{1}{2}\Delta\theta_1} [p_x \ p_y] R d\theta dz \\ \vdots \\ \int_0^L \int_{\theta_N - \frac{1}{2}\Delta\theta_N}^{\theta_N + \frac{1}{2}\Delta\theta_N} [p_x \ p_y] R d\theta dz \end{bmatrix} \in \mathbb{C}^{N \times 2}, \quad (24)$$

$$\mathbf{Z}_{ww} = \begin{bmatrix} \int_0^L \int_{\theta_1 - \frac{1}{2}\Delta\theta_1}^{\theta_1 + \frac{1}{2}\Delta\theta_1} [p_{w_1} \cdots p_{w_N}] R d\theta dz \\ \vdots \\ \int_0^L \int_{\theta_N - \frac{1}{2}\Delta\theta_N}^{\theta_N + \frac{1}{2}\Delta\theta_N} [p_{w_1} \cdots p_{w_N}] R d\theta dz \end{bmatrix} \in \mathbb{C}^{N \times N}. \quad (25)$$

Finally, $\mathbf{Z}$ is related to the dynamic stiffness and damping coefficients as

$$\mathbf{K}_{\text{fluid}} = \text{Re}(\mathbf{Z}) = \begin{bmatrix} k_{xx} & k_{xy} & k_{xw_1} & \cdots & k_{xw_N} \\ k_{yx} & k_{yy} & k_{yw_1} & \cdots & k_{yw_N} \\ k_{w_1x} & k_{w_1y} & k_{w_1w_1} & \cdots & k_{w_1w_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{w_Nx} & k_{w_Ny} & k_{w_Nw_1} & \cdots & k_{w_Nw_N} \end{bmatrix} \in \mathbb{R}^{(2+N) \times (2+N)}, \quad (26)$$

$$\mathbf{D}_{\text{fluid}} = \frac{1}{\omega_s} \text{Im}(\mathbf{Z}) = \begin{bmatrix} d_{xx} & d_{xy} & d_{xw_1} & \cdots & d_{xw_N} \\ d_{yx} & d_{yy} & d_{yw_1} & \cdots & d_{yw_N} \\ d_{w_1x} & d_{w_1y} & d_{w_1w_1} & \cdots & d_{w_1w_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{w_Nx} & d_{w_Ny} & d_{w_Nw_1} & \cdots & d_{w_Nw_N} \end{bmatrix} \in \mathbb{R}^{(2+N) \times (2+N)}. \quad (27)$$

2.4 Including the foil stiffness and damping

The bearing coefficients in eqs. (26) and (27) are derived from the dynamic pressure fields and thus contain merely the contributions from the fluid film.

When evaluating the displacement of the j -th foil DOF from its equilibrium w_{j0} , the stiffness and damping of the foil itself should furthermore be considered.

Various elaborate foil structure models are available in the literature [4, 15, 18, 22], but the added complexity of these would not benefit the present comparison. Instead, the Simple Elastic Foundation Model (SEFM), as introduced by Heshmat et al [5], is employed with a uniform baseline stiffness derived from the widely applied expression by Walowit and Anno [25]

$$k = \frac{E_b}{2S_b} \left(\frac{t_b}{l_0} \right)^3 (1 - \nu_b^2)^{-1}, \quad (28)$$

where E_b , S_b , t_b , l_0 and ν_b are the bump foil's Young's modulus, pitch, thickness, half bump length and Poisson's ratio, respectively. The damping is assumed viscous and expressed simply as $d = k\eta/\omega_s$ where η is a loss factor. A fundamental assumption of the SEFM is that all points in the foil behave independently, meaning that the foil structure is assumed not to contribute any cross couplings. Furthermore, the foil is not directly affecting the e_x, e_y DOFs, meaning that the stiffness and damping matrices stemming from the bump foil can be written as

$$\begin{aligned} \mathbf{K}_{\text{foil}} &= LRk \text{diag} [0 \ 0 \ \Delta\theta_1 \ \cdots \ \Delta\theta_N] \in \mathbb{R}^{(2+N) \times (2+N)}, \\ \mathbf{D}_{\text{foil}} &= \eta\omega_s^{-1} \mathbf{K}_{\text{foil}} \in \mathbb{R}^{(2+N) \times (2+N)}, \end{aligned} \quad (29)$$

where L is the axial length of the bearing and $\Delta\theta_j$ is the angular segment ascribed to the j -th foil DOF as illustrated in fig. 1b.

2.5 The mass matrix

The rotor is modelled as a point mass, implying that neither polar nor transverse inertia is considered. Combining the rotor mass with a rough estimate of the foil mass ascribed to each foil DOF, the system mass matrix is constructed as

$$\mathbf{M} = \text{diag} [m_x \ m_y \ m'_{\text{foil}} R \Delta\theta_1 \ \cdots \ m'_{\text{foil}} R \Delta\theta_N] \in \mathbb{R}^{(2+N) \times (2+N)}, \quad (30)$$

where $m_x = m_y$ is the rotor mass and m'_{foil} is the average mass of the foil structure per unit circumferential length.

2.6 System assembly

Combining eqs. (26), (27), (29) and (30), the linearised $2+N$ equations of motion for the rotor-bearing system can be written as

$$\mathbf{M} \ddot{\mathbf{q}} + \underbrace{(\mathbf{D}_{\text{foil}} + \mathbf{D}_{\text{fluid}})}_{\equiv \mathbf{D}(\omega_s, \Omega)} \dot{\mathbf{q}} + \underbrace{(\mathbf{K}_{\text{foil}} + \mathbf{K}_{\text{fluid}})}_{\equiv \mathbf{K}(\omega_s, \Omega)} \mathbf{q} = \mathbf{0}, \quad (31)$$

which can be recast into first order form using the state vector $\mathbf{z} = \{\mathbf{q} \ \dot{\mathbf{q}}\}^T$ as

$$\dot{\mathbf{z}} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{M}^{-1} \mathbf{K}(\omega_s, \Omega) & \mathbf{M}^{-1} \mathbf{D}(\omega_s, \Omega) \end{bmatrix}}_{\equiv \mathbf{A}(\omega_s, \Omega)} \mathbf{z}, \quad (32)$$

where $\mathbf{0}$ and $\mathbf{I}$ denote $(2+N) \times (2+N)$ zero and identity matrices, respectively.

3 Review of the classical two-DOF perturbation

To emphasize the novel aspects of the proposed extended perturbation, a brief review of the traditional two-DOF approach employed with variations by e.g. [6, 8–11, 13, 16, 19, 24] is included. As originally introduced in the appendix of [17], a fundamental assumption is for the pressure field to be given as a function of rotor position and velocity only. This means that a Taylor expansion leads to a perturbed pressure field given as

$$p = p_0 + p_x \Lambda_x + p_y \Lambda_y, \quad (33)$$

which should be compared to eq. (4) of the extended perturbation. The foil compliance is then introduced implicitly through the film height function as

$$h = C + e_x \cos \theta + e_y \sin \theta + f_{h_c}(p), \quad (34)$$

where $f_{h_c}(p)$ is some function supplying the compliant film height as a function of pressure. For the SEFM incorporating a loss factor, this would be

$$f_{h_c}(p) = k_c^{-1}(p - p_a) \quad \text{where } k_c = k(1 + i\eta). \quad (35)$$

Inserting the film height eq. (34) and perturbed pressure eq. (33) into RE from eq. (15), equations for p_0 and p_γ can be separated. The zeroth order equation is identical to the one obtained for the extended perturbation in eq. (16), while the first order equations for p_γ become

$$\begin{aligned} \frac{\partial}{\partial \Theta} \left(\frac{p_0 h_0^3}{12\mu} \frac{\partial p_\gamma}{\partial \Theta} \right) + \frac{\partial}{\partial z} \left(\frac{p_0 h_0^3}{12\mu} \frac{\partial p_\gamma}{\partial z} \right) + \frac{\partial}{\partial \Theta} \left(\frac{p_\gamma h_0^2 (h_0 + [3p_0 k_c^{-1}])}{12\mu} \frac{\partial p_0}{\partial \Theta} \right) \\ + \frac{\partial}{\partial z} \left(\frac{p_\gamma h_0^2 (h_0 + [3p_0 k_c^{-1}])}{12\mu} \frac{\partial p_0}{\partial z} \right) - \frac{\partial}{\partial \Theta} (p_\gamma (h_0 + [p_0 k_c^{-1}])) U \\ - i\omega_s p_\gamma (h_0 + [p_0 k_c^{-1}]) = - \frac{\partial}{\partial \Theta} \left(\frac{p_0 h_0^2 f_\gamma}{4\mu} \frac{\partial p_0}{\partial \Theta} \right) \\ - \frac{\partial}{\partial z} \left(\frac{p_0 h_0^2 f_\gamma}{4\mu} \frac{\partial p_0}{\partial z} \right) + \frac{\partial}{\partial \Theta} (p_0 U f_\gamma) + i\omega_s p_0 f_\gamma, \quad \gamma = x, y \end{aligned} \quad (36)$$

where

$$h_0 = C + e_{x0} \cos \theta + e_{y0} \sin \theta + k_c^{-1}(p_0 - p_a). \quad (37)$$

which should be compared to eq. (17) of the extended perturbation (the differences are framed in square brackets). The classical technique hence requires the solution of a nonlinear equation for p_0 and two linear complex equations for p_γ from which the bearing coefficients can be extracted as

$$\begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} + i\omega_s \begin{bmatrix} d_{xx} & d_{xy} \\ d_{yx} & d_{yy} \end{bmatrix} = \int_0^L \int_0^{2\pi} \begin{bmatrix} p_x \cos \theta & p_y \cos \theta \\ p_x \sin \theta & p_y \sin \theta \end{bmatrix} R d\theta dz \in \mathbb{C}^{2 \times 2}, \quad (38)$$

involving only the two rotor DOFs (per bearing).

4 Solution strategy

The zeroth order eq. (16) is a nonlinear real equation in p_0 where h_0 is given from eq. (14). The assumption of an axially uniform deformation combined with the SEFM means that when solving for p_0 , the discrete foil compliance w_j is calculated in each iteration from the local mean axial static pressure as

$$w_j = \frac{1}{k} \left(\frac{1}{L} \int_0^L p_0(\theta_j) dz - p_a \right). \quad (39)$$

The zeroth order equation is identical to that presented by the authors in [11] and the same solution strategy based on an FE discretization and Newton-Raphson iteration is employed. Notice that sub-ambient pressures are discarded when integrating the pressure fields (the Gumbel condition). As described in [11], this is meant to represent the effect of the top foil separation from the bump foil [5].

The $2 + N$ first order eqs. (17) and (18) for p_γ and p_{w_j} are linear complex equations which are solved using the FE discretization documented in [11]. Notice that while [11] treats the solution to the p_γ equations produced by an implicit treatment of the foil compliance as in eq. (36), this collapses into eq. (17) for $k_c^{-1} = 0$ and the same solution method can be applied.

A converged mesh of 17 axial (over half the bearing length, as symmetry is exploited) and 114 circumferential elements is used giving 2070 nodes. The number of circumferential locations, and hence the number of p_{w_j} equation systems to be solved, is thus $N = 115$.

5 Results & discussion

The proposed method is evaluated by predicting the OSI for the simplest possible case of a point mass supported by a single-pad GFB in the widely studied configuration [21] listed in table 1. The OSI is calculated using: (a) a simultaneous nonlinear time integration; (b) a classical two-DOF perturbation; and (c) the proposed extended perturbation. For the present bearing, eq. (28) predicts a foil stiffness of 4.6417 GN/m^3 (case 2) which is evaluated along with a practically rigid (case 1: 4641.7 GN/m^3) and a more flexible (case 3: 2.3209 GN/m^3) variant. To put these stiffness levels into perspective, the maximum static foil deflection in the three cases at 20 kRPM are 0.00, 0.21 and 0.40 times C respectively.

5.1 Stability analysis: Frequency domain

To assess the linear stability limit of the rotor system described by eq. (32), a solution on the standard form $\tilde{\mathbf{z}} = \tilde{\mathbf{z}}_0 e^{\lambda t}$ is substituted to produce the standard eigenvalue problem $\mathbf{A}(\omega_s, \Omega) \tilde{\mathbf{z}} = \lambda \tilde{\mathbf{z}}$. Having solved this, an eigenvalue λ_i with positive real part then indicates the corresponding eigenmode $\tilde{\mathbf{z}}_i$ to be unstable.

Applying the classical two-DOF perturbation to a single bearing with coefficients given from eq. (38), four eigenvalues in complex conjugated pairs are

Table 1: Properties of the GFB configuration, mainly from [21].

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Load, $W_x = m_x g$	30 N	Mass, $m = m_x = m_y$	3.059 kg
Bearing configuration			
Bearing radius, R	19.05 mm	Pad leading edge, θ_l	0°
Bearing length, L	38.10 mm	Pad trailing edge, θ_t	360°
Radial clearance, C	31.80 μm		
Fluid properties			
Ambient pressure, p_a	101.3 kPa	Viscosity, μ	$1.950 \times 10^5 \text{ Pa s}$
Foil structure properties			
Foil thickness, t_b	0.1016 mm	Bump pitch, S_b	4.572 mm
Bump half length, l_0	1.778 mm	Bump height, h_b	0.5080 mm
Young's modulus E_b	207.0 GPa	Poisson's ratio, ν_b	0.3
Density, ρ_b	8280 kg/m ³	Loss factor, η	0.2

obtained. By contrast, the extended perturbation in its current implementation and with the applied mesh results in $2(2 + 115) = 234$ eigenvalues. As the eigenvalues of both methods depend on the oscillation frequency and the angular velocity, the stability limit, i.e. the OSI, is characterised by the condition

$$\text{Re}(\lambda_i(\omega_s, \Omega = \Omega_{\text{OSI}})) = 0 \quad \wedge \quad \text{Im}(\lambda_i(\omega_s, \Omega = \Omega_{\text{OSI}})) = \omega_s, \quad (40)$$

requiring the eigenvalue problem to be solved over a range of excitation frequencies and angular velocities. Here it should be highlighted that one should keep track of the eigenvalue order in between the (ω_s, Ω) evaluation points. This is a trivial task for the two complex conjugated pairs arising from the two-DOF approach, but becomes a significant challenge for the 117 pairs obtained from the extended perturbation. At present, this has been handled by comparing the structure of the corresponding eigenvectors.

The result is shown in fig. 2, where the contours of the logarithmic decrement (LD) $\delta = -2\pi \text{Re}(\lambda) / \text{Im}(\lambda)$ is plotted along with a single contour showing the concurrence of the excitation frequency ω_s and the damped natural frequency $\omega_d = \text{Im}(\lambda)$. The crossing of the $\delta = 0$ contour is exactly where eq. (40) is fulfilled and thus marks the OSI. The shown plot is for one of the two eigenmodes dominated by the rotor and is the one first becoming unstable. The mass of the foil from eq. (30) used in the eigenvalue solution is a rough estimate, but it has been verified that this has a vanishing influence on the OSI.

A similar plot is shown in fig. 3, but obtained using the traditional two-DOF perturbation method. The maps are similar, but the OSI is shifted more than 4 kRPM and it is worth emphasizing the difference in computational cost. While both stability maps are based on solutions to 2805 eigenvalue problems, the extended perturbation requires solving 117 2070×2070 complex linear equation systems to obtain the coefficients forming each of the 234×234 eigenvalue problems to be solved. In the traditional perturbation, each eigenvalue problem is

Fig. 2: Case 3 stability map based on the extended perturbation and eigenvalue solutions over a 51×55 (ω_s, Ω) grid. The OSI is marked at (92.8 Hz, 27 424 RPM).

merely 4×4 and the coefficients can be extracted from just two complex linear equation system solutions. The effort related to solving the zeroth order equation for each value of Ω is, however, the same for both methods. For the current implementation, the two-DOF perturbation calculations for fig. 3 take around 45 s while the extended perturbation calculations for fig. 2 take 10 times longer.

5.2 Stability analysis: Time domain

In the nonlinear sense, the stability limit is characterised by a breakdown of structural stability due to a small change in the angular velocity. At this point, the system Jacobian evaluated at the singular point representing the static equilibrium obtains an eigenvalue with zero real part. After this point, the behaviour will be governed by the system's nonlinear characteristics and should not be compared to the linear solutions.

To assess the stability limit, eq. (15) is FE discretized and simultaneously integrated in time with zero unbalance as described in [14]. The implementation has been carefully aligned to that used when solving the perturbed equations. A time integration can hence be started from the static equilibrium found from eq. (16) and the development of the oscillation amplitude can be observed. Such analysis is, however, subject to several challenges. First, even for an asymptotically stable state, the vibration amplitude of a numerical integration will never reach zero. Instead, it converges towards a noise floor specific to the numerical implementation. Initiating a time integration from a static equilibrium will hence always imply some oscillation. Second, even before the true structural stability limit is reached at $\Omega = \Omega_{OSI}$, the static equilibrium state's basin of attraction will gradually narrow, increasing the possibility that an incidental numerical perturbation will push the system away from the stable equilibrium.

Fig. 3: Case 3 stability map based on the two-DOF perturbation and eigenvalue solutions over a 51×55 (ω_s, Ω) grid. The OSI is marked at (95.7 Hz, 31 769 RPM).

To demonstrate the uncertainty related to the OSI using time integration, 1 s simulations have been initialized from their static equilibria for every 10 RPM in the range 23–25 kRPM. For each response it is easy to judge whether the oscillation remains within the numerical noise level or grows exponentially. In fig. 4a, it is shown that some responses grow exponentially from around 23 kRPM, but that this is not persistent until 25.5 kRPM. This is, however, highly dependent on the tolerances applied in both the steady state solver and the subsequent time integration. More consistent results were found to be achievable when perturbing the static eccentricity just enough to leave the numerical noise floor. An initial perturbation of 10^{-10} on the eccentricity ratio ($= 31.8 \times 10^{-16}$ m) thus results in tiny, but very clean oscillations from which the decay or growth can be identified. Plotting the LD obtained from curve fitting to each time series shows a monotonically decreasing curve with a distinct zero-crossing as shown in fig. 4b. The OSI identified using this method is denoted $\tilde{\Omega}_{\text{OSI}}$ to distinguish it from the more firmly defined OSI resulting from the frequency domain analysis.

5.3 Comparison of OSIs

The OSIs obtained using the perturbation methods and time integration are listed in table 2 for all three cases of foil stiffness. The excitation frequencies at the OSI are likewise listed for comparison. For the time integration, the latter has been extracted from frequency spectra based on 5 s simulations at angular frequencies slightly above $\tilde{\Omega}_{\text{OSI}}$. The relative OSI discrepancies are plotted in fig. 5 as a function of foil flexibility, and the extended perturbation is seen to provide results better in agreement with the nonlinear time integration results.

Fig. 4: (a) Stability of static equilibria judged from 1 s simulations for every 10 RPM (case 2); (b) LDs obtained by curve fitting to 0.25 s simulation after a rotor perturbation of $C \times 10^{-10} = 31.8 \times 10^{-16}$ m for every 10 RPM (case 2).

Table 2: Comparison of obtained OSIs

Case	Foil stiffness k (GN/m ³)	Time integration $\tilde{\Omega}_{\text{OSI}}$ (RPM)	$\tilde{\omega}_s$ (Hz)	Two-DOF perturb. Ω_{OSI} (RPM)	ω_s (Hz)	Extended perturb. Ω_{OSI} (RPM)	ω_s (Hz)
1	4641.7	20 868	112.8	21 731	107.5	21 731	107.5
2	4.6417	24 015	101.8	27 868	103.4	24 987	101.0
3	2.3209	25 118	91.7	31 769	95.6	27 424	92.8

Fig. 5: Foil structure flexibility versus the discrepancy in the predicted OSI relative to the time integration result for both perturbation methods.

6 Conclusion

Discrepancies in OSI between the traditional perturbation approach and time integration results were pointed out in [14], and this was postulated to be caused by a deficiency in the pressure expansion. This has been investigated in the present work by carrying through the perturbation method while including the additional pressure–compliance terms in the Taylor expansion. In the presented form, these terms are treated straightforwardly using a high number of discrete foil deflections leading to a rather comprehensive model. This approach is debatable, as are the concrete choices of GFB properties and numerical schemes. The extended perturbation method has, however, been demonstrated to predict OSIs in much better agreement to time integration results, indicating the pressure–compliance terms to be significant.

Irrespective of the physical interpretation of the foil DOFs and their subsequent implementation, the fluid film coefficients affecting each foil DOF are eventually added to the contribution from the foil structure itself. The significance of the pressure–compliance terms will hence decrease for an increasingly stiff foil structure, matching the expectation for the two–DOF perturbation to be adequate for rigid bearings. This has also been demonstrated to be the case using the present implementation, as the methods converge for a very stiff foil.

When lowering the foil stiffness, the significance increases as the discrepancy of the two–DOF method grows and the two perturbation methods diverge. The baseline discrepancy of around 4 % is ascribed to the determination of the time integration OSI, while the increased inconsistency of 9.2 % for case 3 should be further investigated. The agreement is, however, still significantly better than the 26 % discrepancy of the classical two–DOF method.

In conclusion, caution should be exercised if the classical perturbation method is applied to compliant bearings and the OSI of these should rather be assessed using either nonlinear methods or an extended perturbation as presented here.

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