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# Fast Evaluation of the Biot-Savart Integral using FFT for Electrical Conductivity Imaging

Hassan Yazdanian<sup>1</sup>, Guilherme B Saturnino<sup>2</sup>, Axel Thielscher<sup>3</sup>, *Member, IEEE*, and Kim Knudsen<sup>3</sup>

**Abstract**—Magnetic resonance electrical impedance tomography (MREIT) and current density imaging (MRCDI) are emerging as new methods to non-invasively assess the electric conductivity of and current density distributions within biological tissue in vivo. The accurate and fast computation of magnetic fields caused by low frequency electrical currents is a central component of the development, evaluation and application of reconstruction methods that underlie the estimations of the conductivity and current density, respectively, from the measured MR data. However, methods for performing these computations have not been well established in the literature. In the current work, we describe a fast and efficient technique to evaluate the Biot-Savart integral based on the fast Fourier transform (FFT), and evaluate its convergence. We show that the method can calculate magnetic fields in realistic human head models in one minute on a standard computer, while keeping error below 2%.

**Index Terms**—Biot-Savart Integral, Conductivity, FFT, Forward problem, Magnetic field

## I. INTRODUCTION

NOVEL magnetic resonance (MR) methods have been leveraging the high sensitivity of MR to inhomogeneities in the main magnetic field of the scanner to image the electrical impedance of biological tissue at low frequency (MR electric impedance tomography - MREIT) [1] or to assess current density distributions inside the tissue (MR current density imaging - MRCDI) [2] [3]. In these applications, an external current source produces a static or quasi-static *imaging current* inside a body part, and the MR scan is optimized to be sensitive to the magnetic fields produced by these currents. Due to the nature of MR imaging, typically only a single component of the magnetic field can be measured. Therefore, inverse methods are needed to reconstruct the electrical impedance (MREIT) or the current flow (MRCDI) inside the volume conductor from the measurements [4].

The inverse methods typically require the calculation of magnetic fields caused by the externally applied *imaging current* (the forward problem) [2], [5]–[8]. This can be done by direct evaluation of the Biot-Savart integral [5], [9], [10], solving a Poisson’s equation for magnetic field [9], [10], or

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using the fast fourier transform (FFT) to compute the Biot-Savart integral [10], [11]. This last method has been shown to be the fastest among the three [10]. Other methods such as Fast multipole method (FMM) can also be applied to efficiently compute the Biot-Savart integral [12], but can be difficult to implement and is generally slower than FFT [13]. However, to the best of our knowledge, the numerical accuracy of the FFT method for the Biot-Savart integral calculations in MREIT and MRCDI forward calculation applications has not been evaluated in detail so far.

In the current work, we describe an FFT-based method for calculating the Biot-Savart integral and assess its speed and accuracy using both phantoms with analytical solutions and a realistic head model. We show that the method performs well in both conditions, being able to produce accurate solutions in reasonable time, and is therefore a good candidate to compute the magnetic field in forward simulation of MREIT and MRCDI. The method will be integrated in the free and open source software SimNIBS [14], which is currently capable of simulating electrical potential, electrical field, and current density field inside biological tissues, turning it into an integrated tool for MREIT and MRCDI forward calculations.

## II. METHODS

### A. Low frequency current-injected bioelectromagnetic phenomena

Assuming the human body to be an ohmic conductor with electrical conductivity  $\sigma(\mathbf{x})$ , the electric potential  $u(\mathbf{x})$  caused by an externally injected low frequency current is governed by the Laplace equation with the Neumann boundary conditions:

$$\begin{aligned} \nabla \cdot (\sigma(\mathbf{x})\nabla u(\mathbf{x})) &= 0 \quad \text{in } \Omega, \\ -\sigma\nabla u \cdot \mathbf{n} &= g \quad \text{on } \partial\Omega, \end{aligned} \quad (1)$$

where the physical body corresponds to a bounded domain  $\Omega \subset \mathbb{R}^3$  with a smooth boundary  $\partial\Omega$ ,  $\mathbf{x}$  is a position vector in  $\mathbb{R}^3$ ,  $g$  is the normal current flux through the boundary, and  $\mathbf{n}$  is the outward unit normal vector on  $\partial\Omega$ . A unique solution  $u$  for (1) can be obtained by choosing a grounding of the potential, for instance as  $u(\mathbf{x}_0) = 0$  for  $\mathbf{x}_0 \in \Omega$ .

The electric field inside  $\Omega$  is given by

$$\mathbf{E} = -\nabla u \quad (2)$$

and the current density field inside  $\Omega$  is given by Ohm’s law

$$\mathbf{J} = \sigma\mathbf{E} = -\sigma\nabla u. \quad (3)$$

Outside  $\Omega$  we have  $\mathbf{J} = 0$ .

The presence of  $\mathbf{J}$  inside  $\Omega$  gives rise to a magnetic field  $\mathbf{B}$  obtained by the Biot-Savart integral

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_{\Omega} \mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}', \quad (4)$$

where  $\mu_0$  is the magnetic permeability of vacuum. Note that the domain for integration is the bounded set  $\Omega$ .

In complex geometries such as the ones found in the human body, we must resort to numerical methods such as the Finite Element Method (FEM) for computing the electric potential caused by the injected currents, and therefore of the other quantities derived from it (Eqs. (1)-(4)). There are many readily available open source tools for calculating electric potentials, electric fields and current density fields in bioelectric applications such as SimNIBS [14], EIDORS [15], and OpenMEEG [16]. However, the MREIT forward problem is given in terms of the mapping from  $\sigma$  to  $\mathbf{B}$  and the MRCDI forward problem is given by the mapping  $\mathbf{J} \mapsto \mathbf{B}$ , and thus both require the evaluation of magnetic fields in large sections of the domain (Eq. (4)), something which is not readily available in an efficient form in the cited packages.

### B. Fast Evaluation of the Biot-Savart Integral

Our approach for fast evaluation of the Biot-Savart integral (Eq. (4)) using FFT is inspired by similar work for a 2D problem [17].

We can rewrite (4) in Cartesian coordinate functions  $\mathbf{B} = (B_1, B_2, B_3)$  as a convolution:

$$\begin{aligned} B_1(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int_{\Omega} \frac{(x_3 - x'_3)J_2(\mathbf{x}') - (x_2 - x'_2)J_3(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}' \\ &= \mu_0(G_3 * J_2 - G_2 * J_3)(\mathbf{x}), \\ B_2(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int_{\Omega} \frac{(x_1 - x'_1)J_3(\mathbf{x}') - (x_3 - x'_3)J_1(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}' \\ &= \mu_0(G_1 * J_3 - G_3 * J_1)(\mathbf{x}), \\ B_3(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int_{\Omega} \frac{(x_2 - x'_2)J_1(\mathbf{x}') - (x_1 - x'_1)J_2(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}' \\ &= \mu_0(G_2 * J_1 - G_1 * J_2)(\mathbf{x}), \end{aligned} \quad (5)$$

where

$$G_n = \frac{1}{4\pi} \frac{x_n}{|\mathbf{x}|^3}, \quad n = 1, 2, 3 \quad (6)$$

are the Biot-Savart integral kernels and  $\mathbf{J} = (J_1, J_2, J_3)$  is the current density field, which has compact support.

The convolutions in (5)

$$C_{n,m}(\mathbf{x}) = \mu_0 G_n(\mathbf{x}) * J_m(\mathbf{x}), \quad n \neq m \quad (7)$$

can be calculated numerically in several ways. Consider we have  $N$  sampling points where we have evaluated the current density field and want to calculate the magnetic field in the same set of points. Naïve computation of the convolution has the complexity  $O(N^2)$ . However, the computation can be done more efficiently in the frequency domain. Using the

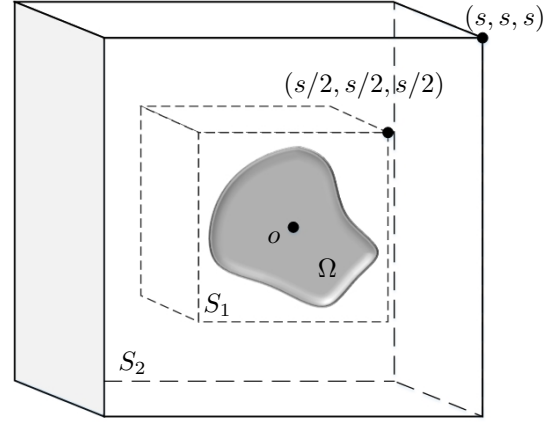


Figure 1. The large cube  $S_2$  determines the periodization of the function. The small cube  $S_1$  shows that the function has compact support.  $\Omega$  is bounded by the small cube. Both of the cubes are centered at the origin.

Fourier transform  $\mathcal{F}$  and the inverse Fourier transform  $\mathcal{F}^{-1}$ , we obtain:

$$C_{n,m}(\mathbf{x}) = \mu_0 \mathcal{F}^{-1} \{ \mathcal{F} \{ G_n \} \cdot \mathcal{F} \{ J_m \} \} (\mathbf{x}). \quad (8)$$

In this domain, the computation than has linear complexity,  $O(N)$ . The Biot-Savart kernel  $G_n$  has an analytical form (details are given in Appendix)

$$\mathcal{F} \{ G_n \} = -j \frac{k_n}{|\mathbf{k}|^2}. \quad (9)$$

where  $\mathbf{k} = (k_1, k_2, k_3)$  is the spatial frequency in radian per meter and  $j$  is the imaginary unity. The current density field  $\mathbf{J}$  does not generally have an analytical form, so we must resort to numerical methods for computing the Fourier transform. The FFT algorithm has a complexity of  $O(N \log(N))$ , which means that by performing the calculations using the Fourier domain we are able to reduce complexity from  $O(N^2)$  to  $O(N \log(N))$ .

There are, however, a few caveats that need to be addressed in order to properly use the FFT to evaluate (7).

- 1)  $\mathbf{J}$  needs to be sampled on a regularly spaced grid. So, if the original simulations were performed on an irregular grid, they need to be interpolated to obtain a discretized current density field  $\bar{\mathbf{J}}$  on an equidistant grid. The resolution of such grid is directly related to the frequency spectra captured by the FFT and therefore the accuracy of the procedure, but also changes the computation time. Therefore, finding an appropriate grid resolution is an important step towards obtaining accurate, yet fast, evaluations of the Biot-Savart integral.
- 2) The FFT calculates a circular convolution, and not the linear convolution required for evaluating (7). However, the linear convolution can be calculated by introducing a *periodization* of the discretized current density field  $\bar{\mathbf{J}}$  [18]. As the current density field is compactly supported in the domain  $\Omega$ , we define a cube  $S_1 = (-s/2, s/2)^3$  in  $\mathbb{R}^3$  centered at the origin  $o$ , such that  $\Omega \subset S_1$ . Then, for  $\mathbf{x} \in S_1$ , the integral (7) only involves values of  $G_n$  on  $S_2$ , where  $S_2 = (-s, s)^3$  is a domain with double

size of  $S_1$  centered at the origin. Thus, to compute the discrete linear convolution by FFT, we should calculate the FFT of  $\bar{\mathbf{J}}$  in  $S_2$ , where the values for  $\bar{\mathbf{J}} \notin \Omega$  are set to zero. The result of this operation for  $\mathbf{x} \in S_1$ , will be identical to the linear convolution. See Fig. 1 for a description of the domains defined above.

After the two steps above are taken into account, we obtain the following equation for calculating the discrete convolution values  $\bar{C}_{n,m}$  inside  $S_2$ :

$$\bar{C}_{n,m} = \mu_0 \text{iFFT}(\overline{\mathcal{F}\{G_n\}} \cdot \text{FFT}(\bar{J}_m)). \quad (10)$$

where  $\overline{\mathcal{F}\{G_n\}}$  is the analytical Fourier transform (9) evaluated at the spatial frequencies  $\mathbf{k}$  present in the FFT transformation. In order to avoid the singularity at  $\mathbf{k} = \mathbf{0}$ , we set

$$\overline{\mathcal{F}\{G_i\}}[\mathbf{k} = \mathbf{0}] = 0. \quad (11)$$

### III. NUMERICAL EXPERIMENTS

In this section we present some numerical examples for the evaluation of the developed technique. In the first examples we have an analytic expression of the fields  $\mathbf{J}$  and  $\mathbf{B}$ , and in the final example we used a current density field calculated on a realistic head model using SimNIBS.

#### A. Benchmark tests

In this section we employ two benchmark tests in order to validate the technique developed to compute the Biot-Savart integral. In both tests the magnetic flux density is known analytically, and therefore the performance of the numerical algorithm can be compared to a ground truth. The first example is a smooth phantom, while the second phantom is piecewise smooth.

We start with an explicit magnetic vector potential  $\mathbf{A}(\mathbf{x}) = (A_1(\mathbf{x}), A_2(\mathbf{x}), A_3(\mathbf{x}))$ , which in the ball  $B(0, h)$  is given analytically, and outside is set to 0. We then calculate the magnetic field by  $\mathbf{B} = \nabla \times \mathbf{A}$ . In this way, we obtain a divergence-free magnetic field  $\mathbf{B}$ , needed for consistency with Gauss's law for magnetism ( $\nabla \cdot \mathbf{B} = 0$ ). We further calculate the current density field  $\mathbf{J}$  by  $\nabla \times \nabla \times \mathbf{A} = \mathbf{J}$  which is obtained by using Ampere's law,  $\nabla \times \mathbf{B} = \mu \mathbf{J}$ , assuming  $\mu = 1$ .

We consider two different functions  $\mathbf{A}$ , both divergence-free. The first  $\mathbf{A}$  has a continuous second derivative everywhere and therefore leads to a smooth  $\mathbf{J}$ , and was defined as

$$\begin{aligned} A_1(\mathbf{x}) &= x_2 x_3 \exp\left(\frac{-1}{h^2 - |\mathbf{x}|^2}\right), \\ A_2(\mathbf{x}) &= x_1 x_3 \exp\left(\frac{-1}{h^2 - |\mathbf{x}|^2}\right), \\ A_3(\mathbf{x}) &= -2x_1 x_2 \exp\left(\frac{-1}{h^2 - |\mathbf{x}|^2}\right). \end{aligned}$$

Generally, however, the solutions for the Laplace equation with Neumann boundary conditions (Eq. (1)) produces values of  $\mathbf{J}$  which can be discontinuous on the boundary  $\partial\Omega$  and where there are discontinuities in the conductivity  $\sigma(\mathbf{x})$ . As the Fourier components of discontinuous function fall off slowly

and we can only compute truncated terms of the Fourier transform, we need to evaluate how the smoothness of  $\mathbf{J}$  affects the convergence of our computations. Therefore, we also used a second  $\mathbf{A}$  which has a discontinuous second derivative and therefore a discontinuous  $\mathbf{J}$  in the outer boundary  $|\mathbf{x}| = h$ . It was defined as

$$A_1(\mathbf{x}) = x_2(h - |\mathbf{x}|)^2$$

$$A_2(\mathbf{x}) = x_3(h - |\mathbf{x}|)^2$$

$$A_3(\mathbf{x}) = x_1(h - |\mathbf{x}|)^2.$$

We used both phantoms to evaluate the FFT method by calculating  $\mathbf{B}$  numerically based on analytical calculations of the current field density  $\mathbf{J}$ . We set  $h = 1$ , the computational domain had a size  $s = 2$  and was discretized using  $M = 32, 64, 128, 256, \text{ and } 512$  equally spaced voxels along each dimensions.

#### B. Real head model and simulation setup

We use SimNIBS to simulate a current density field in a realistic situation. The performance of SimNIBS for calculating  $u$  and  $\mathbf{J}$  has been previously validated in [19].

The realistic simulation was run on a head model of a healthy volunteer from the SimNIBS example dataset ([www.simnibs.org](http://www.simnibs.org)). For more information regarding the generation of the head model, please see [20]. This model is a tetrahedral head mesh with  $\approx 7 \times 10^5$  nodes and  $\approx 5 \times 10^6$  tetrahedral elements segmented into six tissue types: white matter, gray matter, cerebrospinal fluid, skull, scalp and eyes, with conductivities of 0.126 S/m, 0.275 S/m, 1.645 S/m, 0.01 S/m, 0.465 S/m and 0.5 S/m respectively [21].

We used SimNIBS to model two single layer rectangular electrodes of  $50\text{mm} \times 50\text{mm} \times 4\text{mm}$  and used them to deliver a 1 mA current. The anode was placed above the C6 position and the cathode above the C5 position according to the EEG 10-20 system. For each electrode, the complete upper surface was set to a common electric potential, corresponding to a highly conductive top layer to distribute the currents [22]. The conductivity of the electrode bodies was set to 1 S/m [22].

All simulations were implemented in Python 2.7 and the MKL-linked *numpy* package was used for the FFT evaluations. The code was executed on a core i7 3.6 GHz desktop computer with 16 GB RAM.

## IV. RESULTS AND DISCUSSION

#### A. Benchmark tests

Fig. 2 shows slices of the  $B_3$  field calculated analytically and numerically, as well as the errors in the phantoms with smooth  $\mathbf{A}$ , while Fig. 3 shows the same quantities for the phantom with piecewise smooth  $\mathbf{A}$ . We have chosen  $B_3$  because some MREIT and MRCDI are only sensitive to a single component of the magnetic field, usually denoted  $B_z$  or  $B_3$  [3], [23], [24]. Fig. 2a and Fig. 3a depict the analytical  $B_3$  corresponding to smooth and piecewise smooth phantoms, respectively. As can be seen, by increasing the number of grid

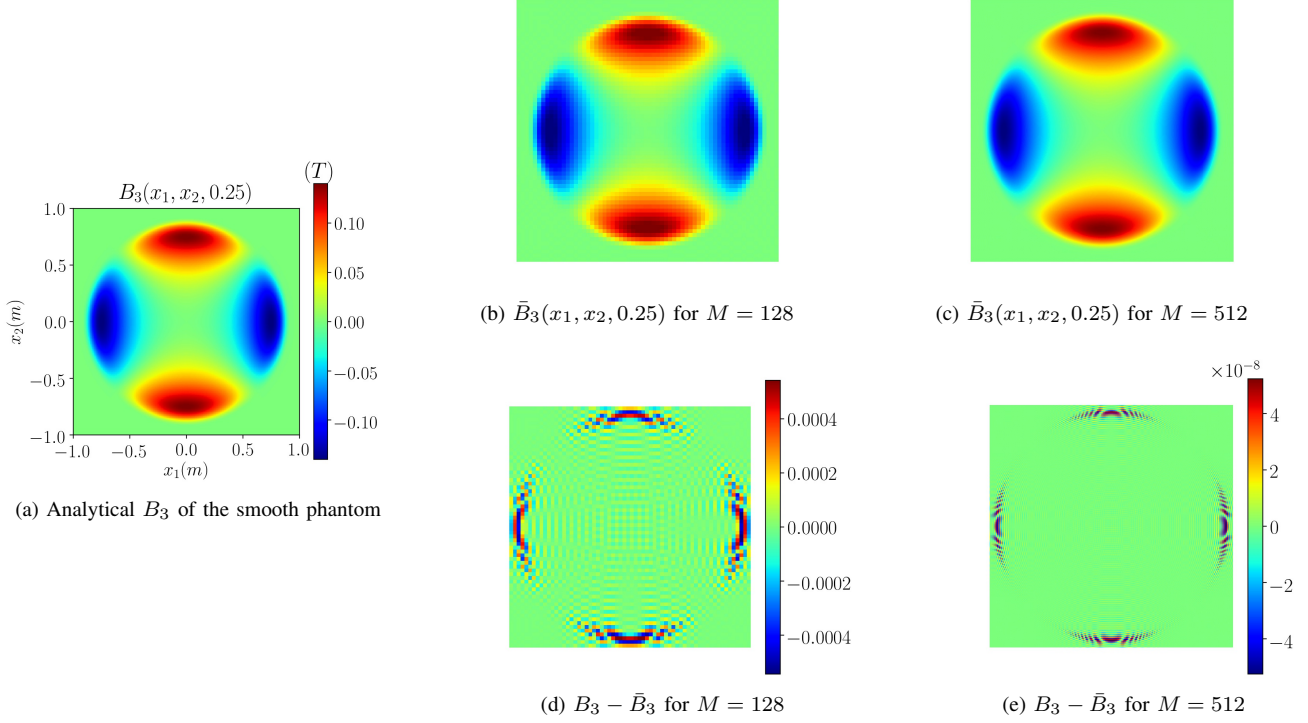


Figure 2. (a) Slice ( $x_3 = 0.25$ ) of the analytical  $B_3$  corresponding to the smooth phantom at  $x_3 = 0.25$ . Numerical results for  $\bar{B}_3$  for (b)  $M = 128$  and (c)  $M = 512$ . (d) and (e) are the difference images between  $B_3$  and  $\bar{B}_3$  for  $M = 128$  and  $M = 512$ , respectively.

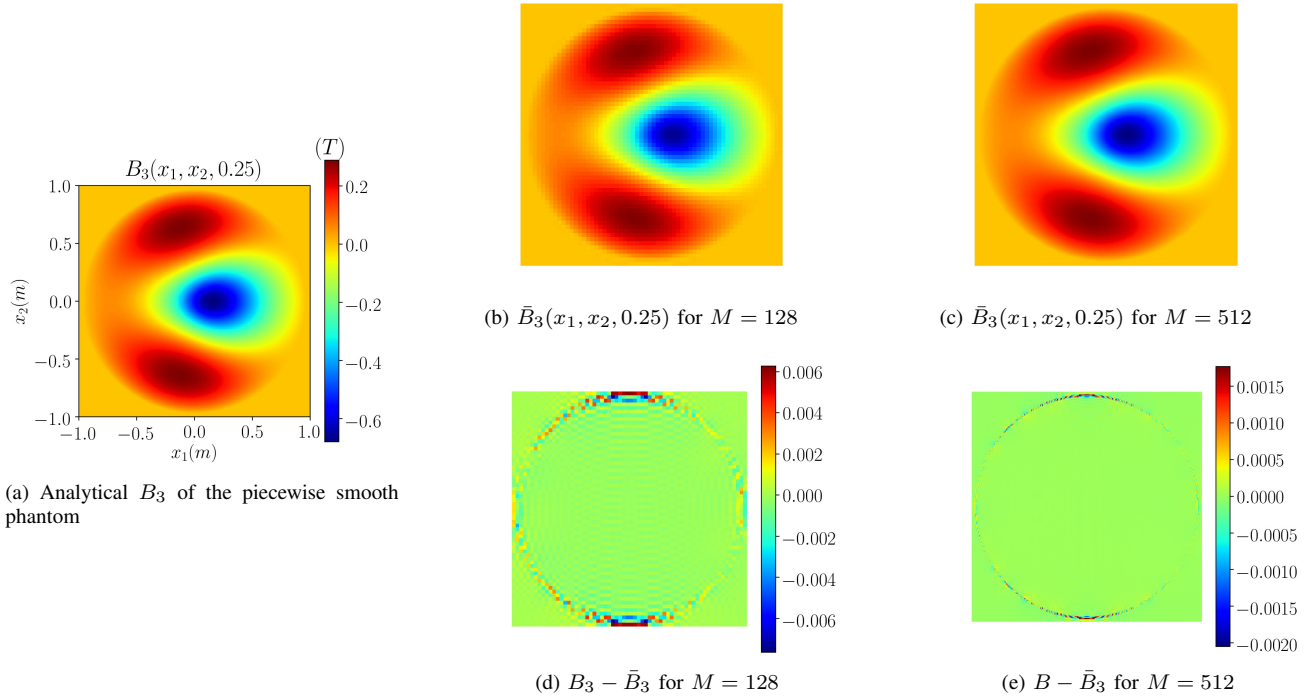


Figure 3. (a) Slice ( $x_3 = 0.25$ ) of the analytical  $B_3$  corresponding to the piecewise smooth phantom at  $x_3 = 0.25$ . Numerical results for  $\bar{B}_3$  for (b)  $M = 128$  and (c)  $M = 512$ . (d) and (e) are the difference images between  $B_3$  and  $\bar{B}_3$  for  $M = 128$  and  $M = 512$ , respectively.

points from  $M = 128$  in Fig. 2b and Fig. 3b to  $M = 512$  in Fig. 2c and Fig. 3c, the quality of the computed field is improved. By increasing  $M$ , the error in  $\bar{B}_3$  dramatically

decreases in both the central region and near the boundary region of the smooth phantom (Fig. 2d and Fig. 2e). As expected, the error for the second (piecewise smooth  $\mathbf{A}$ )



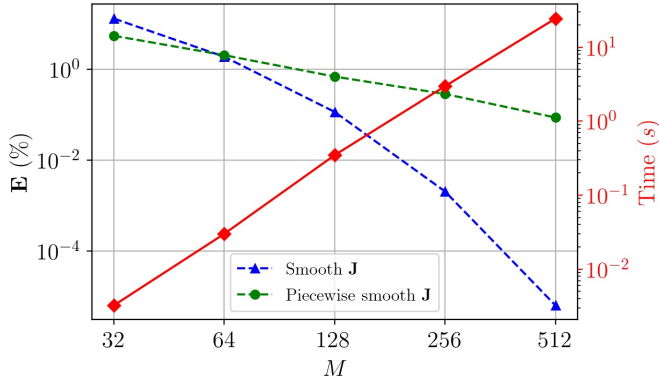


Figure 4. Error analysis of the benchmark tests. Relative error of  $\mathbf{B}$  versus number of grid points along each axis ( $M$ ) using smooth  $\mathbf{J}$  (blue dashed line) and piecewise smooth  $\mathbf{J}$  (green dashed line). The red line is computation time versus  $M$ .

phantom decreases much slower than the error for the smooth phantom, especially close to the discontinuous outer boundary (Fig. 3d and Fig. 3e). This arises from the piecewise nature of  $\mathbf{J}$  which causes the Fourier transform to decay slowly. Thus, truncation of the Fourier transform slows down the convergence rate.

Fig. 4 shows the error analysis for the benchmark tests. The dashed lines depict the relative error of  $\mathbf{B}$  versus the number of grid points along each axis ( $M$ ). The blue dashed line is error for smooth phantom and the green dashed line with dots one is error for piecewise smooth  $\mathbf{J}$ . The error was calculated by:

$$\mathbf{E}(\%) = 100 \times \frac{\|\mathbf{B} - \bar{\mathbf{B}}\|}{\|\mathbf{B}\|} \quad (12)$$

where  $\mathbf{B}$  and  $\bar{\mathbf{B}}$  are the analytical and numerical magnetic field, respectively, and  $\|\cdot\|$  denotes the  $L_2$ -norm calculated in the discrete domain.

As can be seen from Fig. 4, the results are converging for both tests. However, the convergence rate is different. It can be said that convergence rate and smoothness are inversely related. For  $M = 512$ , the error for smooth  $\mathbf{J}$  was below  $5 \times 10^{-8} \%$  and for piecewise smooth  $\mathbf{J}$  was below 0.5%. The computation time for this number of grid points ( $N = M^3 = 134217728$ ) was around 27 seconds.

Table I  
VOXEL SIZES AND CORRESPONDING NUMBER OF GRID POINTS FOR CALCULATION ON THE REALISTIC HEAD MODEL

Voxel size ( $\text{mm}^3$ )	M	N ( $M^3$ )
3.2	160	4,096,000
2.56	200	8,000,000
2	256	16,777,216
1.6	320	32,768,000
1	512	134,217,728
0.5 (Ground-truth)	1,024	1,073,741,824

## B. Realistic simulation

Fig. 5a shows the distribution of the norm of the current density field,  $|\mathbf{J}|$ , in an axial slice of the realistic head model. The first row of Fig. 5b displays the corresponding  $B_3$  in the same slice and for different voxel sizes. The second row of Fig. 5b depicts the error in the computed  $B_3$  for each voxel size compared to the magnetic field computed at a high resolution with a voxel size of  $0.5 \text{ mm}^3$ . As can be seen, the error decreases by increasing the resolution. In addition, as we expected, the maximum error occurred near the place of electrodes due to discontinuity in the current density field.

To compute  $\mathbf{B}$  by FFT-based methods, it is required to convert  $\mathbf{J}$  defined on the elements of the tetrahedral mesh to the points of the defined regular grid. This was done using the `interpolate_to_grid` function in SimNIBS, which grids the data using the superconvergent patch recovery (SPR)-based interpolation described in [19]. The computation time for this function was approximately 40 seconds for all field resolutions, as there is a high overhead (the nodal recovery process) associated with the gridding at all resolutions.

To evaluate the FFT-based method in the realistic simulation situation, we considered two approaches:

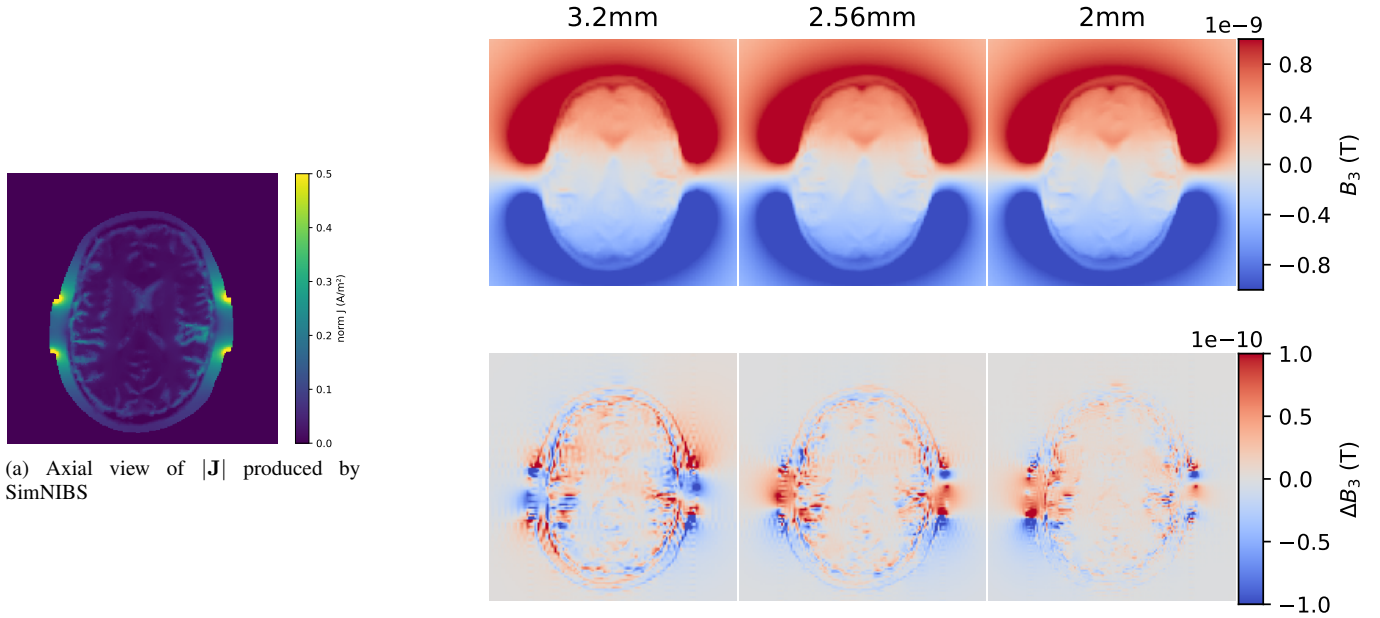
- 1) First, we tried different voxel sizes to see how the method converges by decreasing the voxel size (Table I). In this approach, we considered the voxel size of  $0.5 \text{ mm} \times 0.5 \text{ mm} \times 0.5 \text{ mm}$  as a ground-truth and calculated the error of magnetic field using (12).
- 2) In the second approach, we used a back-and-forth procedure. We computed the magnetic field using the FFT-based method for the current density field produced by SimNIBS,  $\mathbf{J}_{sim}$ . Then, using Ampere's law, we reproduced the current density distribution by numerically applying the curl operator to the computed magnetic field. However, the curl of a vector field is solenoidal (divergence free) whereas the initial  $\mathbf{J}_{sim}$  is non-solenoidal due to the discontinuity at the electrode surfaces. To resolve this problem, we used the Helmholtz decomposition [25] to compute the solenoidal part of  $\mathbf{J}_{sim}$ ,  $\mathbf{J}_{so}$ . Based on the Helmholtz decomposition we can write  $\mathbf{J}_{so} = \mathbf{J}_{sim} - \mathbf{J}_{ir}$ , where  $\mathbf{J}_{ir}$  is the irrotational (curl free) part of  $\mathbf{J}_{sim}$ . We calculated  $\mathbf{J}_{ir}$  in the frequency domain as follows [25]

$$\mathcal{F}\{\mathbf{J}_{ir}\} = \mathbf{k} \frac{\mathbf{k} \cdot \mathcal{F}\{\mathbf{J}_{sim}\}}{|\mathbf{k}|^2}. \quad (13)$$

Then, we subtracted (13) from  $\mathcal{F}\{\mathbf{J}_{sim}\}$  to obtain the Fourier transform of  $\mathbf{J}_{so}$ . By applying  $\mathcal{F}^{-1}\{\cdot\}$  to the result, we obtained an approximation of  $\mathbf{J}_{so}$ . Finally, we considered the resultant field as the input of the back-and-forth procedure.

For both approaches, we considered a computation domain of size  $512 \text{ mm} \times 512 \text{ mm} \times 512 \text{ mm}$ , enough to encompass the entire head and the periodization of the domain.

Fig. 6 shows the results for both approaches. The blue dashed line with triangles is the error of magnetic field in the first approach and the green dashed line with dots is the error of current density field in the second one. As can be seen



(a) Axial view of  $|\mathbf{J}|$  produced by SimNIBS

(b) Third component of magnetic field  $B_3$  and its error  $\Delta B_3$

Figure 5. Results of the realistic simulation. (a) Axial view of  $|\mathbf{J}|$  produced by SimNIBS. (b) (First row) Third component of the magnetic field for voxel sizes of  $3.2 \text{ mm}^3$  (first column),  $2 \text{ mm}^3$  (second column) and  $1 \text{ mm}^3$  (third column). (Second row) Corresponding error with relation to the  $B_3$  field computed at a high resolution ( $0.5 \text{ mm}^3$ )

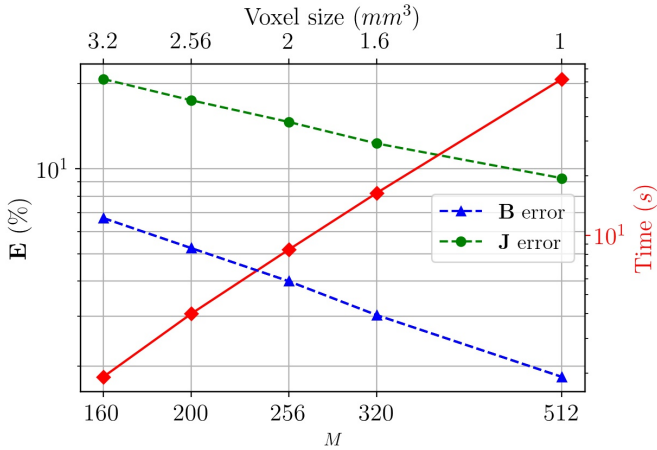


Figure 6. Relative error versus  $M$  using the first approach (blue dashed line) and the second approach (green dashed line). The red line is computation time versus  $M$ .

from Fig. 6, the results are converging for both approaches. Furthermore, the convergence rates are approximately the same. However, the error values for the second approach are higher. We believe this is because of the error arising from the additional numerical differentiation performed in the back-and-forth procedure.

The red line with squares in Fig. 6 show the computation time of magnetic field  $\mathbf{B}$  for different voxel sizes. The elapsed

time for a voxel size of  $0.5 \text{ mm}^3$  was about 1 minute while the maximum memory usage in this resolution was 12.3 GB. For a voxel size of  $2 \text{ mm}^3$  these amounts decrease dramatically to a computation time of 8.5 seconds and a memory usage of 181 MB.

## V. CONCLUSION

In this paper, we developed a computationally efficient technique for evaluation of the Biot-Savart integral. This integral has the form of a convolution, and thus lends itself well to an efficient numerical evaluation by means of the FFT algorithm. We performed three numerical experiments to evaluate the technique, using two phantoms with known analytical solutions and in addition a realistic head model. The results show a good convergence in all three experiments, suggesting that the method is appropriate to, for example, forward modelling of MREIT and MRCDI applications.

By integrating this new feature in SimNIBS, a fast and comprehensive package for modeling of low frequency exogenic bioelectromagnetic phenomena is available. It can compute the electrical potential  $u$ , current density field  $\mathbf{J}$ , and magnetic field  $\mathbf{B}$  in a realistic head domain with typical pixel sizes in less than three minutes. This opens the door for the development of new methods for current and tissue conductivity reconstruction from imaging and tomography data based on low-frequency electromagnetics, such as MREIT and MRCDI.

We can obtain further improvement for this technique by using DSP techniques such as FFTW [26] to decrease the

computation time or overlap add/save methods [27] to save memory.

## APPENDIX

### FOURIER TRANSFORM OF THE BIOT-SAVART KERNEL

In this section the details for extraction of (9) are presented. Applying the Fourier transform to  $G_i$  yields

$$\begin{aligned}\mathcal{F}\{G_i\} &= \mathcal{F}\left\{\frac{1}{4\pi} \frac{x_i}{|\mathbf{x}|^3}\right\} \\ &= \mathcal{F}\left\{-\frac{\partial}{\partial x_i} \frac{1}{4\pi|\mathbf{x}|}\right\} \\ &= -jk_i \mathcal{F}\left\{\frac{1}{4\pi|\mathbf{x}|}\right\}.\end{aligned}\quad (\text{A1})$$

Now  $1/4\pi|\mathbf{x}|$  is the fundamental solution for the negative Laplace operator and its Fourier transform is given by [28]

$$\mathcal{F}\left\{\frac{1}{4\pi|\mathbf{x}|}\right\} = \frac{1}{|\mathbf{k}|^2}.\quad (\text{A2})$$

By substituting (A2) into (A1), we obtain (9):

$$\mathcal{F}\{G_i\} = -\frac{jk_i}{|\mathbf{k}|^2}.\quad (\text{A3})$$

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