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THE COMPLETE LIST OF GENERA OF QUOTIENTS OF THE $\mathbb{F}_{q^2}$ -MAXIMAL HERMITIAN CURVE FOR $q \equiv 1 \pmod{4}$

MARIA MONTANUCCI AND GIOVANNI ZINI

ABSTRACT. Let $\mathbb{F}_{q^2}$ be the finite field with q^2 elements. Most of the known $\mathbb{F}_{q^2}$ -maximal curves arise as quotient curves of the $\mathbb{F}_{q^2}$ -maximal Hermitian curve $\mathcal{H}_q$. After a seminal paper by Garcia, Stichtenoth and Xing, many papers have provided genera of quotients of $\mathcal{H}_q$, but their complete determination is a challenging open problem. In this paper we determine completely the spectrum of genera of quotients of $\mathcal{H}_q$ for any $q \equiv 1 \pmod{4}$.

Keywords: Hermitian curve, unitary groups, quotient curves, maximal curves

2000 MSC: 11G20

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1. INTRODUCTION

Let $q = p^n$ be a prime power, $\mathbb{F}_{q^2}$ the finite field with q^2 elements, and $\mathcal{X}$ be a projective, absolutely irreducible, non-singular algebraic curve of genus g defined over $\mathbb{F}_{q^2}$. The curve $\mathcal{X}$ is called $\mathbb{F}_{q^2}$ -maximal if the number $|\mathcal{X}(\mathbb{F}_{q^2})|$ of its $\mathbb{F}_{q^2}$ -rational points attains the Hasse-Weil upper bound $q^2 + 1 + 2gq$. Maximal curves have been investigated for their applications in Coding Theory. Surveys on maximal curves are found in [12, 13, 14, 16, 37, 38] and [24, Chapter 10].

A well-known and important example of an $\mathbb{F}_{q^2}$ -maximal curve is the Hermitian curve $\mathcal{H}_q$. It is defined as any $\mathbb{F}_{q^2}$ -rational curve which is projectively equivalent to the plane curve $Y^{q+1} - X^{q+1} + Z^{q+1} = 0$. For fixed q , the curve $\mathcal{H}_q$ has the largest genus $g(\mathcal{H}_q) = q(q-1)/2$ that an $\mathbb{F}_{q^2}$ -maximal curve can have. The full automorphism group $\text{Aut}(\mathcal{H}_q)$ is isomorphic to $\text{PGU}(3, q)$, the group of projectivities of $\text{PG}(2, q^2)$ commuting with the unitary polarity associated with $\mathcal{H}_q$. The automorphism group $\text{Aut}(\mathcal{H}_q)$ is extremely large with respect to the value $g(\mathcal{H}_q)$. Indeed it is known that the Hermitian curve is the unique curve of genus $g \geq 2$ up to isomorphisms admitting an automorphism group of order at least equal to $16q^4$.

By a result commonly referred to as the Kleiman-Serre covering result, see [28] and [29, Proposition 6], a curve $\mathcal{X}$ defined over $\mathbb{F}_{q^2}$ which is $\mathbb{F}_{q^2}$ -covered by an $\mathbb{F}_{q^2}$ -maximal curve is $\mathbb{F}_{q^2}$ -maximal as well. In particular, $\mathbb{F}_{q^2}$ -maximal curves can be obtained as Galois $\mathbb{F}_{q^2}$ -subcovers of an $\mathbb{F}_{q^2}$ -maximal curve $\mathcal{X}$, that is, as quotient curves $\mathcal{X}/G$ where G is a finite automorphism group of $\mathcal{X}$. Most of the known maximal curves are Galois covered by the Hermitian curve; see for instance [18, 6, 19, 34, 32, 33] and the references therein.

The first example of a maximal curve not Galois covered by the Hermitian curve is due to Garcia and Stichtenoth [17]. This curve is $\mathbb{F}_{36}$ -maximal and it is not Galois covered by $\mathcal{H}_{27}$. It is a special case of the $\mathbb{F}_{q^6}$ -maximal GS curve, which was later shown not to be Galois covered by $\mathcal{H}_{q^3}$ for any $q > 3$, [22, 30]. Giulietti and Korchmáros [21] provided an $\mathbb{F}_{q^6}$ -maximal curve, nowadays referred to as the GK curve, which is not covered by the Hermitian curve $\mathcal{H}_{q^3}$ for any $q > 2$. Two generalizations of the GK curve were introduced by Garcia, Güneri and Stichtenoth [15] and by Beelen and Montanucci [3]. Both these generalizations are $\mathbb{F}_{q^{2n}}$ -maximal curves, for any q and odd $n \geq 3$. Also, they are not Galois covered by the Hermitian curve $\mathcal{H}_{q^n}$.

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for $q > 2$ and $n \geq 5$, see [11, 3]; the Garcia-Güneri-Stichtenoth's generalization is also not Galois covered by $\mathcal{H}_{2^n}$ for $q = 2$, see [22].

A challenging open problem is the determination of the spectrum $\Gamma(q^2)$ of genera of $\mathbb{F}_{q^2}$ -maximal curves, for given q . Apart from the examples listed above, most of the known values in $\Gamma(q^2)$ have been obtained from quotient curves $\mathcal{H}_q/G$ of the Hermitian curve, which have been investigated in many papers. The most significant cases are the following:

- G fixes an $\mathbb{F}_{q^2}$ -rational point of $\mathcal{H}_q$; see [2, 18, 1].
- G fixes a self-polar triangle in $\text{PG}(2, q^2) \setminus \mathcal{H}_q$; see [8].
- G normalizes a Singer subgroup of $\mathcal{H}_q$ acting on three $\mathbb{F}_{q^6}$ -rational points of $\mathcal{H}_q$; see [18, 7].
- G has prime order; see [6].
- G fixes neither points nor triangles in $\text{PG}(2, q^6)$; see [33].

From the results already obtained in the literature (see [8, 33] and the references therein), in order to obtain the complete list of genera of quotients $\mathcal{H}_q/G$, $G \leq \text{Aut}(\mathcal{H}_q)$, only the following cases still have to be analyzed:

- (1) G fixes an $\mathbb{F}_{q^2}$ -rational point $P \notin \mathcal{H}_q$, $p > 2$.
- (2) G fixes a point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$, $p = 2$ and $|G| = p^\ell d$ where $p^\ell \leq q$ and $d \mid (q - 1)$.

In this paper a complete analysis of Case 1 is given provided that q is congruent to 1 modulo 4. This provides the complete list of genera of quotient of the Hermitian curve under this assumption.

More precisely, this paper is organized as follows. Section 2 provides a collection of necessary preliminary results on the Hermitian curve and its automorphism group. In Section 3 a complete analysis of Case 1 is given for $q \equiv 1 \pmod{4}$. Section 4 contains the complete list of genera of quotients of the Hermitian curve for $q \equiv 1 \pmod{4}$ joining our results with the ones already obtained in the literature.

2. PRELIMINARY RESULTS

Throughout this paper $q = p^n$, where p is a prime number and n is a positive integer. The Deligne-Lusztig curves defined over a finite field $\mathbb{F}_q$ were originally introduced in [9]. Other than the projective line, there are three families of Deligne-Lusztig curves, named Hermitian curves, Suzuki curves and Ree curves. The Hermitian curve $\mathcal{H}_q$ arises from the algebraic group ${}^2A_2(q) = \text{PGU}(3, q)$ of order $(q^3 + 1)q^3(q^2 - 1)$. It has genus $q(q - 1)/2$ and is $\mathbb{F}_{q^2}$ -maximal. This curve is $\mathbb{F}_{q^2}$ -isomorphic to the following curves:

- (1) $X^{q+1} - Y^{q+1} - Z^{q+1} = 0;$
- (2) $X^q Z + X Z^q - Y^{q+1} = 0.$

The automorphism group $\text{Aut}(\mathcal{H}_q)$ is isomorphic to the projective unitary group $\text{PGU}(3, q)$, and it acts on the set $\mathcal{H}_q(\mathbb{F}_{q^2})$ of all $\mathbb{F}_{q^2}$ -rational points of $\mathcal{H}_q$ as $\text{PGU}(3, q)$ in its usual 2-transitive permutation representation. The combinatorial properties of $\mathcal{H}_q(\mathbb{F}_{q^2})$ can be found in [25]. The size of $\mathcal{H}_q(\mathbb{F}_{q^2})$ is equal to $q^3 + 1$, and a line of $\text{PG}(2, q^2)$ has either 1 or $q + 1$ common points with $\mathcal{H}_q(\mathbb{F}_{q^2})$, that is, it is either a tangent line or a chord of $\mathcal{H}_q(\mathbb{F}_{q^2})$. Furthermore, a unitary polarity is associated with $\mathcal{H}_q(\mathbb{F}_{q^2})$ whose isotropic points are those of $\mathcal{H}_q(\mathbb{F}_{q^2})$ and isotropic lines are the tangent lines to $\mathcal{H}_q(\mathbb{F}_{q^2})$, that is, the tangents to $\mathcal{H}_q$ at the points of $\mathcal{H}_q(\mathbb{F}_{q^2})$.

The following classification of subgroups of $\text{PGU}(3, q)$ goes back to Mitchell [31] and Hartley [23]; see also [33].

Theorem 2.1. *Let G be a nontrivial subgroup of $\text{PGU}(3, q)$. Then one of the following cases holds.*

- (i) G is contained in the maximal subgroup of $\text{PGU}(3, q)$ of order $q^3(q^2 - 1)$ which stabilizes an $\mathbb{F}_{q^2}$ -rational point of $\mathcal{H}_q$.
- (ii) G is contained in the maximal subgroup $\mathcal{M}_q$ of $\text{PGU}(3, q)$ of order $q(q - 1)(q + 1)^2$ which stabilizes an $\mathbb{F}_{q^2}$ -rational point of $\text{PG}(2, q^2) \setminus \mathcal{H}_q$; equivalently, $\mathcal{M}_q$ stabilizes a chord of $\mathcal{H}_q(\mathbb{F}_{q^2})$.

- (iii) G is contained in the maximal subgroup of $\text{PGU}(3, q)$ of order $6(q+1)^2$ which stabilizes a self-polar triangle of $\text{PG}(2, q^2) \setminus \mathcal{H}_q$ with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$.
- (iv) G is contained in the maximal subgroup of $\text{PGU}(3, q)$ of order $3(q^2 - q + 1)$ which stabilizes a triangle in $\mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$ invariant under the Frobenius collineation $\Phi_{q^2} : (X, Y, Z) \mapsto (X^{q^2}, Y^{q^2}, Z^{q^2})$ of $\text{PG}(2, \mathbb{F}_q)$.
- (v) G stabilizes no points nor triangles in $\text{PG}(2, \bar{\mathbb{F}}_{q^2})$.

We also recall the classification of subgroups of $\text{SL}(2, q)$ and $\text{PGL}(2, q)$.

Theorem 2.2. (see [36, Theorem 6.17]) *Up to isomorphism, the complete list of subgroups of $\text{SL}(2, q)$ is the following.*

• *Tame subgroups:*

- (1) $\text{SL}(2, 5)$, when $p \geq 7$ and $q^2 \equiv 1 \pmod{5}$.
- (2) $\hat{\Sigma}_4$, the representation group of S_4 in which the transpositions correspond to the elements of order 4, when $p \geq 5$ and $q^2 \equiv 1 \pmod{16}$; $\hat{\Sigma}_4$ is isomorphic to $\text{SmallGroup}(48, 28)$.
- (3) $\text{SL}(2, 3)$, when $p \geq 5$.
- (4) The cyclic group C_d of order d , where $d \mid (q \pm 1)$.
- (5) The dicyclic group $\text{Dic}_m = \langle \delta, \epsilon \mid \delta^{2m} = 1, \epsilon^2 = \delta^m, \epsilon^{-1}\delta\epsilon = \delta^{-1} \rangle \cong C_{2m} \circ C_4$ of order $4m$, where $1 < m \mid \frac{q \pm 1}{2}$.

• *Non-tame subgroups:*

- (1) $E_{p^k} \rtimes C_d$, where $d \mid \gcd(p^k - 1, q - 1)$, $k \leq n$, and E_{p^k} is elementary abelian of order p^k .
- (2) $\text{SL}(2, 5)$, when $p = 3$ and $q^2 \equiv 1 \pmod{5}$;
- (3) $\text{SL}(2, p^k)$, where $k \mid n$;
- (4) $\text{TL}(2, p^k) \cong \langle \text{SL}(2, p^k), d_\pi \rangle$, where $k \mid n$, n/k is even,

$$d_\pi = \begin{pmatrix} w & 0 & 0 \\ 0 & w^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$w = \xi^{\frac{p^k+1}{2}}, \text{ and } \mathbb{F}_{p^{2k}}^* = \langle \xi \rangle.$$

Theorem 2.3. (see [10, Chapt. XII, Par. 260], [26, Kap. II, Hauptsatz 8.27], [24, Thm. A.8]) *The following is the complete list of subgroups of $\text{PGL}(2, q)$ up to conjugacy:*

- (i) the cyclic group of order h with $h \mid (q \pm 1)$;
- (ii) the elementary abelian p -group of order p^f with $f \leq n$;
- (iii) the dihedral group of order $2h$ with $h \mid (q \pm 1)$;
- (iv) the alternating group $\mathbf{A}_4$ for $p > 2$, or $p = 2$ and n even;
- (v) the symmetric group $\mathbf{S}_4$ for $16 \mid (q^2 - 1)$;
- (vi) the alternating group $\mathbf{A}_5$ for $p = 5$ or $5 \mid (q^2 - 1)$;
- (vii) the semidirect product of an elementary abelian p -group of order p^k by a cyclic group of order h , with $k \leq n$ and $h \mid \gcd(p^f - 1, q - 1)$;
- (viii) $\text{PSL}(2, p^f)$ for $f \mid n$;
- (ix) $\text{PGL}(2, p^f)$ for $f \mid n$.

In our investigation it is useful to know how an element of $\text{PGU}(3, q)$ of a given order acts on $\text{PG}(2, \bar{\mathbb{F}}_q)$, and in particular on $\mathcal{H}_q(\mathbb{F}_{q^2})$. This can be obtained as a corollary of Theorem 2.1, and is stated in Lemma 2.2 with the usual terminology of collineations of projective planes; see [25]. In particular, a linear collineation σ of $\text{PG}(2, \bar{\mathbb{F}}_q)$ is a (P, ℓ) -perspectivity, if σ preserves each line through the point P (the *center* of σ), and fixes each point on the line ℓ (the *axis* of σ). A (P, ℓ) -perspectivity is either an *elation* or a *homology* according to $P \in \ell$ or $P \notin \ell$. This classification result was obtained in [34].

Lemma 2.4. *For a nontrivial element $\sigma \in \text{PGU}(3, q)$, one of the following cases holds.*

- (A) $\text{ord}(\sigma) \mid (q+1)$ and σ is a homology, with center $P \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$ and axis ℓ which is a chord of $\mathcal{H}_q(\mathbb{F}_{q^2})$; (P, ℓ) is a pole-polar pair with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$.
- (B) $p \nmid \text{ord}(\sigma)$ and σ fixes the vertices P_1, P_2, P_3 of a non-degenerate triangle $T \subset \text{PG}(2, q^6)$.
 - (B1) $\text{ord}(\sigma) \mid (q+1)$, $P_1, P_2, P_3 \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$, and the triangle T is self-polar with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$.
 - (B2) $\text{ord}(\sigma) \mid (q^2 - 1)$ and $\text{ord}(\sigma) \nmid (q+1)$; $P_1 \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$ and $P_2, P_3 \in \mathcal{H}_q(\mathbb{F}_{q^2})$.
 - (B3) $\text{ord}(\sigma) \mid (q^2 - q + 1)$ and $P_1, P_2, P_3 \in \mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$.
- (C) $\text{ord}(\sigma) = p$ and σ is an elation with center $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and axis ℓ which is tangent to $\mathcal{H}_q$ at P , such that (P, ℓ) is a pole-polar pair with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$.
- (D) $\text{ord}(\sigma) = p$ with $p \neq 2$, or $\text{ord}(\sigma) = 4$ and $p = 2$; σ fixes a point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and a line ℓ which is tangent to $\mathcal{H}_q$ at P , such that (P, ℓ) is a pole-polar pair with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$.
- (E) $\text{ord}(\sigma) = p \cdot d$ where d is a nontrivial divisor of $q+1$; σ fixes two points $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and $Q \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$, the polar line PQ of P , and the polar line of Q which is another line through P .

Throughout the paper, a nontrivial element of $\text{PGU}(3, q)$ is said to be of type (A), (B), (B1), (B2), (B3), (C), (D), or (E), as given in Lemma 2.4.

To compute the genus of a quotient curve we make use of the Riemann-Hurwitz formula; see [35, Theorem 3.4.13]. For any subgroup G of $\text{PGU}(3, q)$, the cover $\mathcal{H}_q \rightarrow \mathcal{H}_q/G$ is a Galois cover defined over $\mathbb{F}_{q^2}$ and the degree Δ of the different divisor is given by the Riemann-Hurwitz formula, namely $\Delta = (2g(\mathcal{H}_q) - 2) - |G|(2g(\mathcal{H}_q/G) - 2)$. On the other hand, $\Delta = \sum_{\sigma \in G \setminus \{id\}} i(\sigma)$, where $i(\sigma) \geq 0$ is given by the Hilbert's different formula [35, Thm. 3.8.7], namely $i(\sigma) = \sum_{P \in \mathcal{H}_q(\mathbb{F}_q)} v_P(\sigma(t) - t)$, where t is a local parameter at P .

By analyzing the geometric properties of the elements $\sigma \in \text{PGU}(3, q)$, it turns out that there are only a few possibilities for $i(\sigma)$. This is obtained as a corollary of Lemma 2.4 and stated in the following theorem, which is proved in [34].

Theorem 2.5. *For a nontrivial element $\sigma \in \text{PGU}(3, q)$ one of the following cases occurs.*

- (1) If $\text{ord}(\sigma) = 2$ and $2 \mid (q+1)$, then σ is of type (A) and $i(\sigma) = q+1$.
- (2) If $\text{ord}(\sigma) = 3$, $3 \mid (q+1)$ and σ is of type (B3), then $i(\sigma) = 3$.
- (3) If $\text{ord}(\sigma) \neq 2$, $\text{ord}(\sigma) \mid (q+1)$ and σ is of type (A), then $i(\sigma) = q+1$.
- (4) If $\text{ord}(\sigma) \neq 2$, $\text{ord}(\sigma) \mid (q+1)$ and σ is of type (B1), then $i(\sigma) = 0$.
- (5) If $\text{ord}(\sigma) \mid (q^2 - 1)$ and $\text{ord}(\sigma) \nmid (q+1)$, then σ is of type (B2) and $i(\sigma) = 2$.
- (6) If $\text{ord}(\sigma) \neq 3$ and $\text{ord}(\sigma) \mid (q^2 - q + 1)$, then σ is of type (B3) and $i(\sigma) = 3$.
- (7) If $p = 2$ and $\text{ord}(\sigma) = 4$, then σ is of type (D) and $i(\sigma) = 2$.
- (8) If $\text{ord}(\sigma) = p$, $p \neq 2$ and σ is of type (D), then $i(\sigma) = 2$.
- (9) If $\text{ord}(\sigma) = p$ and σ is of type (C), then $i(\sigma) = q+2$.
- (10) If $\text{ord}(\sigma) = p \cdot d$ with $1 \neq d \mid (q+1)$, then σ is of type (E) and $i(\sigma) = 1$.

In order to characterize the genera of $\mathcal{H}_q/G$ for any subgroup G of $\text{PGU}(3, q)$ under the assumption $q \equiv 1 \pmod{4}$, it is sufficient to consider the case $G \leq \mathcal{M}_q$, where $\mathcal{M}_q$ is the maximal subgroup (ii) in Theorem 2.1. In fact, if $G \not\leq \mathcal{M}_q$:

- the genera $g(\mathcal{H}_q/G)$ for the subgroups $G \leq \text{PGU}(3, q)$ stabilizing an $\mathbb{F}_{q^2}$ -rational point of $\mathcal{H}_q$ are characterized in [2, Theorem 1.1];
- the genera $g(\mathcal{H}_q/G)$ for the subgroups $G \leq \text{PGU}(3, q)$ stabilizing a self-polar triangle of $\text{PG}(2, q^2)$ are characterized in [8, Section 3];
- the genera $g(\mathcal{H}_q/G)$ for the subgroups $G \leq \text{PGU}(3, q)$ stabilizing a Frobenius-invariant triangle in $\mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$ are characterized in [6, Proposition 4.2];
- the genera $g(\mathcal{H}_q/G)$ for the subgroups $G \leq \text{PGU}(3, q)$ which do not stabilize any point or triangle are characterized in [33].

3. THE MAXIMAL SUBGROUP $\mathcal{M}_q$ FOR $q \equiv 1 \pmod{4}$

Let $\mathcal{M}_q$ be the maximal subgroup of $\text{PGU}(3, q)$ stabilizing a point $P \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$ and its polar line ℓ , which is a chord of $\mathcal{H}_q(\mathbb{F}_{q^2})$. For any odd q , the structure of $\mathcal{M}_q$ was already given in [6, Section 3] and [32, Section 3] as a semidirect product $\mathcal{M}_q = \Gamma \rtimes C$, where Γ is the commutator subgroup of $\mathcal{M}_q$ and is isomorphic to $\text{SL}(2, q)$, while C is a cyclic group of order $q+1$; this description was used to compute the genera $g(\mathcal{H}_q/G)$ for some $G \leq \mathcal{M}_q$, namely when G is a tame subgroup of Γ in [6], and when $G = (G \cap \Gamma) \rtimes (G \cap C)$ in [32].

Henceforth, we assume $q \equiv 1 \pmod{4}$ and use a different description of $\mathcal{M}_q$. Let $\mathcal{H}_q$ be given by the model (1). Up to conjugation in $\text{PGU}(3, q)$, we can assume that $P = (0, 0, 1)$ and ℓ has equation $Z = 0$. As pointed out in the proof of Theorem 2.5 in [3],

$$\mathcal{M}_q = \left\{ \begin{pmatrix} a & \zeta c^q & 0 \\ c & \zeta a^q & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a, c \in \mathbb{F}_{q^2}, a^{q+1} - c^{q+1} = 1, \zeta^{q+1} = 1 \right\}$$

and the commutator subgroup of $\mathcal{M}_q$ is

$$H = \left\{ \begin{pmatrix} a & c^q & 0 \\ c & a^q & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a, c \in \mathbb{F}_{q^2}, a^{q+1} - c^{q+1} = 1 \right\} \cong \text{SL}(2, q).$$

The center Z of $\mathcal{M}_q$ is cyclic of order $q+1$ and is made by the elements of type (A) with center P ; see [32, Section 3]. Hence,

$$Z = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a^{q+1} = 1 \right\}.$$

The intersection of $I = H \cap Z$ has order 2 and is generated by the unique involution

$$\iota = \text{diag}(-1, -1, 1)$$

of H . Since $\frac{q+1}{2}$ is odd, the group generated by H and Z is a direct product $HZ = H \times \Omega$, where $\Omega \cong C_{\frac{q+1}{2}}$ is the subgroup of Z of order $\frac{q+1}{2}$.

The group HZ has index 2 in $\mathcal{M}_q$, and contains exactly one involution ι . Let β be any involution of $\mathcal{M}_q$ different from ι , for instance $\beta = \text{diag}(-1, 1, 1)$; obviously, β normalizes both $H = \mathcal{M}'_q$ and $Z = Z(\mathcal{M}_q)$. Then

$$\mathcal{M}_q = (H \rtimes \langle \beta \rangle) \times \Omega \cong (\text{SL}(2, q) \rtimes C_2) \times C_{\frac{q+1}{2}}.$$

With the notation of [20, Section 9] we also denote $H \rtimes \langle \beta \rangle$ by $\text{SU}^\pm(2, q)$, meaning that $H \rtimes \langle \beta \rangle$ consists of the elements of $\mathcal{M}_q$ with determinant 1 or -1 ; here, the determinant of an element $\alpha \in \mathcal{M}_q$ is the determinant of the representative matrix of α having entry 1 in the third row and column. Note that $H \rtimes \langle \beta \rangle$ is conjugated to $H \rtimes \langle \gamma \rangle$ in $\mathcal{M}_q$ for any involution $\gamma \in \mathcal{M}_q \setminus \{\iota\}$, because γ and β are conjugated in $\mathcal{M}_q$ (see [27, Lemma 2.2]) and H is normal in $\mathcal{M}_q$. Note also that $\mathcal{M}_q$ contains no elements of type (B3) or (D).

For any $G \leq \mathcal{M}_q$, we will use the following notation:

$$G_\pm = G \cap \text{SU}^\pm(2, q), \quad G_H = G \cap H, \quad G_\Omega = G \cap \Omega, \quad \omega = |G_\Omega|.$$

We determine in the following proposition the subgroups of $\text{SU}^\pm(2, q)$.

Proposition 3.1. *The following is the complete list of subgroups of $\text{SU}^\pm(2, q) \leq \mathcal{M}_q$ for $q \equiv 1 \pmod{4}$.*

- The subgroups of H , listed in Theorem 2.2.
- Cyclic groups of order 2.
- Groups $\text{SL}(2, 3) \rtimes C_2 \cong \text{SmallGroup}(48, 29)$ when $p \geq 5$ and $8 \nmid (q-1)$.
- Cyclic groups of order $2d > 2$, where either $d \mid (q-1)$ and $d \nmid \frac{q-1}{2}$, or $d \mid (q+1)$.
- Abelian groups $C_d \times C_2$, where $d \mid (q+1)$ and d is even.

- *Dihedral groups of order $2d$, where $d \mid (q \pm 1)$.*
- *Groups $\text{Dic}_m = \langle \alpha, \epsilon \mid \alpha^{4m} = 1, \epsilon^2 = \alpha^{2m}, \epsilon^{-1}\alpha\epsilon = \alpha^{2m-1} \rangle$ of order $8m$ extending a subgroup Dic_m of H , when $m \mid \frac{q-1}{2}$ and $m \nmid \frac{q+1}{4}$.*
- *Groups $\text{Dic}_m \rtimes C_2$ of order $8m$ extending a subgroup Dic_m of H , when $1 < m \mid \frac{q+1}{2}$.*
- *Groups $E_{p^k} \rtimes C_{2d}$, where $k \leq n$, $d \mid \gcd(p^k - 1, q - 1)$, d is even, E_{p^k} is elementary abelian of order p^k , and C_{2d} is cyclic of order $2d$.*
- *Groups $\text{SU}^\pm(2, p^k) \cong \text{SL}(2, p^k) \rtimes C_2$, where $k \mid n$ and n/k is odd.*

Proof. Let $G \leq H \rtimes \langle \beta \rangle$ and assume $G \not\leq H$, so that $G_H = G \cap H$ has index 2 in G . We may assume that G has order greater than 2, that is G_H is a nontrivial subgroup of H . If ι is the unique involution of H , we denote by $\bar{G}$ and $\bar{G}_H$ the images of G and G_H under the canonical epimorphism $\text{SU}^\pm(2, q) \rightarrow \text{SU}^\pm(2, q)/\langle \iota \rangle$. Since ι is the kernel of the action of $\text{SU}^\pm(2, q)$ on the line ℓ , the action of $\text{SU}^\pm(2, q)/\langle \iota \rangle$ on the $q+1$ points of $\mathcal{H}_q \cap \ell$ is equivalent to the action of a subgroup of $\text{PGL}(2, q)$; as they have the same order, $\text{SU}^\pm(2, q)/\langle \iota \rangle = \text{PGL}(2, q)$. Note that $|\bar{G}| = |G_H|$ or $|\bar{G}| = 2|G_H|$ according to $|G_H|$ being even or odd, respectively. We classify G_H according to Theorem 2.2.

- Suppose $G_H = \text{SL}(2, 5)$. Then $\text{PGL}(2, q)$ has a subgroup $\bar{G}$ of order 120 and $\bar{G}$ contains $\bar{G}_H \cong \mathbf{A}_5$, a contradiction to Theorem 2.3.
- Suppose $G_H = \hat{\Sigma}_4$. As in the previous point, $\text{PGL}(2, q)$ has a subgroup of order 48 containing a subgroup isomorphic to $\mathbf{S}_4$, a contradiction to Theorem 2.3.
- Suppose $G_H = E_{p^k} \rtimes C_d$ with $k \leq n$ and $d \mid \gcd(p^k - 1, q - 1)$. The unique Sylow p -subgroup E_{p^k} of G is normal in G , and hence G fixes the unique fixed point $Q \in \ell$ of E_{p^k} on $\mathcal{H}_q$; see [24, Lemma 11.129]. If d is odd, then $|G| = 2|G_H|$ and $G \setminus G_H$ contains an involution ω , which is of type (A) and has center on ℓ ; this is a contradiction, since ω cannot fix any point on $\ell \cap \mathcal{H}_q$ by Lemma 2.4. Then d is even. By [24, Lemma 11.44], $G = E_{p^k} \rtimes C_{2d}$. Such a group $G = E_{p^k} \rtimes C_{2d}$ actually exists in $\mathcal{M}_q$: for instance, as in [18], use the model (2) for $\mathcal{H}_q$, assume up to conjugacy that $P = (0, 1, 0)$, and define

$$G = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid c^{p^k} + c = 0 \right\} \rtimes \left\{ \begin{pmatrix} a^{p^k+1} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a^{2d} = 1 \right\} \cong E_{p^k} \rtimes C_{2d}.$$

Then $G \leq \mathcal{M}_q$. Since $\mathcal{M}_q = \text{SU}^\pm(2, q) \times C_{\frac{q+1}{2}}$ and $\gcd(|G|, \frac{q+1}{2}) = 1$, this implies $G \leq \text{SU}^\pm(2, q)$.

- Suppose $G_H = \text{SL}(2, p^k)$ with $k \mid n$. Then $|\bar{G}| = |\text{PGL}(2, p^k)|$ and $\bar{G}$ contains $\bar{G}_H \cong \text{PSL}(2, p^k)$, so that $\bar{G} = \text{PGL}(2, p^k)$ by Theorem 2.3. Let G_1 be a subgroup of $\text{SU}^\pm(2, q)$ with $G_H \leq G_1$ and $[G_1 : G_H] = 2$. Clearly, $\bar{G}_1 = \bar{G} = \text{PGL}(2, p^k)$; we show that $G_1 = G$. Choose $\delta \in \bar{G}$ of order $o(\delta) = p^k - 1$ if $4 \mid (p^k - 1)$, or $o(\delta) = p^k + 1$ if $4 \mid (p^k + 1)$. Let $\alpha \in G$ and $\bar{\alpha}_1 \in G_1$ be preimages of δ , that is $\bar{\alpha} = \bar{\alpha}_1 = \delta$; their order is $o(\alpha) = o(\alpha_1) = 2 \cdot o(\delta)$ and divides $2(q - 1)$. This implies in particular $\alpha, \alpha_1 \notin G_H$, so that $G = \langle G_H, \alpha \rangle$ and $G_1 = \langle G_H, \alpha_1 \rangle$. The group $\text{PGU}(3, q)$ contains a unique cyclic subgroup C such that $\delta \in C$ and $[C : \langle \delta \rangle] = 2$, see Lemma 2.4; hence, $\langle \alpha \rangle = \langle \alpha_1 \rangle$. Thus, $G_1 = G$: there is at most one subgroup of $\text{SU}^\pm(2, q)$ containing G_H with index 2.

If n/k is even, then $G = \text{TL}(2, p^k)$ by Theorem 2.2. Assume that n/k is odd. Then the group

$$(3) \quad G = \left\{ \begin{pmatrix} a & c^q & 0 \\ c & a^q & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a, c \in \mathbb{F}_{p^{2k}}, a^{p^k+1} - c^{p^k+1} = 1 \right\} \rtimes \left\{ \begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \zeta = \pm 1 \right\}$$

is a subgroup of $\text{SU}^\pm(2, q)$; this can be proved in analogy to the proof of [33, Lemma 3.10], i.e. showing by induction on n/k that the condition $a^{p^k+1} - c^{p^k+1} = 1$ is equivalent to $a^{q+1} - c^{q+1} = 1$ for any $a, c \in \mathbb{F}_{p^{2k}}$. Clearly, $G \cong \text{SU}^\pm(2, p^k) = \text{SL}(2, p^k) \rtimes C_2$.

- Suppose $G_H = \text{TL}(2, p^k)$. Then $\text{PGL}(2, q)$ has a subgroup $\bar{G}$ such that $|\bar{G}| = p^k(p^{2k} - 1)$ and $\text{PGL}(2, p^k) = \bar{G}_H \leq \bar{G}$, a contradiction to Theorem 2.3.
- Suppose $G_H = C_d = \langle \alpha \rangle$ cyclic of order $d \mid (q \pm 1)$, so that $|G| = 2d$. Firstly, we prove that the conditions in the statement when G is abelian or dihedral are necessary for the existence of G ; secondly, we show that such conditions are also sufficient.

Note that, if $\alpha \in \text{SU}^\pm(2, q) \setminus H$ and $o(\alpha) > 2$, then $o(\alpha) \nmid (q-1)$. In fact, if $2 < o(\alpha) \mid (q-1)$, then α is of type (B2) and fixes exactly two points other than P , say $Q, R \in \ell \cap \mathcal{H}_q$; but the pointwise stabilizer S of $\{Q, R\}$ in $\text{PGU}(3, q)$ is cyclic of order $q^2 - 1$ (see Lemma 2.4), and $|S \cap H| = q - 1$, which implies $\alpha \in H$. Hence, if $d \mid (q-1)$ and G is cyclic, then $d \nmid \frac{q-1}{2}$.

We can assume that $d > 2$. A generator δ of G_H is either of type (B1) or of type (B2) and has three fixed points P, Q, R , where $Q, R \in \ell$; since G_H is normal in G , G acts on $\{Q, R\}$. Let $\gamma \in G \setminus G_H$. If $\gamma(Q) = Q, \gamma(R) = R$, and $d \mid (q-1)$, then G is cyclic because the pointwise stabilizer of $\{P, Q, R\}$ in $\text{PGU}(3, q)$ is cyclic. If $\gamma(Q) = Q, \gamma(R) = R$, and $d \mid (q+1)$, then G is contained in the pointwise stabilizer $C_{q+1} \times C_{q+1}$ of $\{P, Q, R\} \subset \text{PG}(2, q^2) \setminus \mathcal{H}_q$, see [8, Section 3]; hence, $G = C_d \times C_2$ (which is C_{2d} when d is odd). Assume that γ interchanges Q and R . Then γ^2 fixes ℓ pointwise, so that either γ is an involution or $\gamma^2 = \iota$, the unique element of type (A) in H . If $\gamma^2 = \iota$, then γ has order 4 $\mid (q-1)$, and hence $\gamma \in G_H$, a contradiction. Then γ is an involution, and G is a semidirect (eventually direct) product $G_H \rtimes \langle \gamma \rangle$; denote by $\tilde{Q}, \tilde{R} \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$ the two points of ℓ fixed by γ (which are the center of γ and the intersection between ℓ and the axis of γ ; see Lemma 2.4). Since $d > 2$ and δ acts with orbits of length d on $\ell \setminus \{Q, R\}$, δ cannot commute with γ unless $\{Q, R\} = \{\tilde{Q}, \tilde{R}\}$, which implies δ being of type (B1) and hence $d \mid (q+1)$. If δ and γ do not commute, then G induces a dihedral group $\bar{G} \leq \text{PGL}(2, q)$ of order $\frac{d}{\gcd(2, d)} \cdot 2$, and G is dihedral itself.

Conversely, we show that abelian and dihedral groups G as in the statement actually exist in $\text{SU}^\pm(2, q) \setminus H$. To this aim, we make use of other models of $\mathcal{H}_q$ and provide groups G which fix P whose order is coprime to $\frac{q+1}{2}$; this assures that $G \leq \text{SU}^\pm(2, q)$.

A cyclic group G of order $2d$ with $d \mid (q-1)$ and $d \nmid \frac{q-1}{2}$ is provided as follows: $\mathcal{H}_q$ has equation (2), $P = (0, 1, 0)$, and G is generated by $\text{diag}(a^{q+1}, a, 1)$, where $0(a) = 2d$; if $G \leq H$, multiply its generator with $\text{diag}(-1, 1, 1) \in \text{SU}^\pm(2, q) \setminus H$.

A dihedral group G of order $2d$ with $d \mid (q-1)$ is provided as follows: $\mathcal{H}_q$ has equation (2), $P = (0, 1, 0)$, and G is generated by $\text{diag}(a^{q+1}, a, 1)$ and

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where $o(a) = d$.

A cyclic group G of order $2d$ with $2 < d \mid (q+1)$ is provided as follows: $\mathcal{H}_q$ has equation (1), $P = (0, 0, 1)$, and G is generated by $\text{diag}(\lambda, \lambda^i, 1)$, where $o(\lambda) = 2d$ and $\gcd(2d, i) = 1$. After λ is chosen, there exists only one value $\bar{i} \in \{2, \dots, 2d-1\}$ such that $\text{diag}(\lambda, \lambda^{\bar{i}}, 1) \in H$; hence we can choose i such that $G \not\leq H$.

An abelian group $G = C_d \times C_2$ with $d \mid (q+1)$ is provided as follows: $\mathcal{H}_q$ has equation (1), $P = (0, 0, 1)$, G_H is generated by $\text{diag}(\lambda, \lambda^i, 1)$ as in the previous point, and G is generated by G_H together with $\text{diag}(-1, 1, 1) \in \text{SU}^\pm(2, q) \setminus H$.

A dihedral group of order $2d$ with $d \mid (q+1)$ is provided as follows: $\mathcal{H}_q$ has equation (1), $P = (0, 0, 1)$, G_H is generated by $\text{diag}(\lambda, \lambda^i, 1)$ as in the previous point, and G is generated by G_H together

with

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Suppose $G_H = \text{Dic}_m = \langle \delta, \epsilon \rangle$ with $m \mid \frac{q-1}{2}$, $o(\delta) = 2m$, and $o(\epsilon) = 4$. Note that both δ and ϵ are of type (B2); let Q and R be the points on $\ell \cap \mathcal{H}_q$ fixed by δ . Let $\alpha \in G \setminus G_H$. Since $\langle \delta \rangle$ is normal in G , α acts on $\{Q, R\}$. Up to replacing α with $\alpha\epsilon$, we can assume that $\alpha(Q) = Q$ and $\alpha(R) = R$. Since the pointwise stabilizer S of $\{Q, R\}$ in $\text{PGU}(3, q)$ is cyclic, $\langle \delta, \alpha \rangle$ is cyclic of order $4m$; up to replacing α with a generator of $\langle \delta, \alpha \rangle$, we can assume that α is of type (B2) and has order $4m$. Therefore $G = \langle \alpha, \epsilon \rangle$ with $\alpha^{2m} = \epsilon^2$. Since $G \not\leq H$ and $|S \cap H| = q - 1$, we have $o(\alpha) = 4m \nmid (q - 1)$.

Such a subgroup G actually exists in $\text{SU}^\pm(2, q)$ and is determined uniquely up to conjugation, as follows. Let $\mathcal{H}_q$ have equation (2); up to conjugation we have $P = (0, 1, 0)$, $Q = (1, 0, 0)$, $R = (0, 0, 1)$. The only element of order $4m$ in $\text{PGU}(3, q)$ fixing $\{P, Q, R\}$ pointwise is $\alpha = \text{diag}(a^{q+1}, a, 1)$, where a is a primitive $4m$ -th root of unity. Any element of order 4 in $\text{PGU}(3, q)$ fixing P and interchanging Q and R has the form

$$\epsilon = \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 1 & 0 \\ -\gamma^{-1} & 0 & 0 \end{pmatrix},$$

where $\gamma^{q+1} = 1$. By direct checking, $\epsilon^{-1}\alpha\epsilon = \alpha^{2m-1}$ and $G = \langle \alpha, \epsilon \rangle$ has order $8m$. Since $G \leq \mathcal{M}_q$ and $\gcd(|G|, \frac{q+1}{2}) = 1$, we have $G \leq \text{SU}^\pm(2, q)$. Also, the assumptions $m \mid \frac{q-1}{2}$ and $m \nmid \frac{q-1}{4}$ imply $\alpha \notin H$ and $\alpha^2 \in H$, so that $G_H = \langle \alpha^2, \epsilon \rangle \cong \text{Dic}_m$. The elements of G are α^i and $\alpha^i\epsilon$, with $i = 0, \dots, 4m - 1$. By direct checking, α^{2m} and $\alpha^i\epsilon$ with odd i are involutions and hence of type (A), while α^j with $j \neq 0, 2m$ and $\alpha^k\epsilon$ with even k are of type (B2).

- Suppose $G_H = \text{Dic}_m = \langle \delta, \epsilon \rangle$ with $m \mid \frac{q+1}{2}$, $o(\delta) = 2m$, and $o(\epsilon) = 4$. Denote by Q and R the points other than P fixed by δ ; we have $Q, R \in \ell \setminus \mathcal{H}_q$. Let $\alpha \in G \setminus G_H$. If $o(\alpha) = 4m$, then $o(\alpha) \mid (q^2 - 1)$ and $o(\alpha) \nmid (q + 1)$, so that α is of type (B2) and α^2 is of type (A), a contradiction to the fact that ι is the only element of type (A) in H ; hence, $o(\alpha) \neq 4m$. Since $\langle \delta \rangle$ is normal in G , the subgroup $K = \langle \delta, \alpha \rangle$ has order $4m$. As K is not cyclic, we have shown above that either K is a direct product $C_{2m} \times C_2$, or K is a dihedral group $C_{2m} \rtimes C_2$. We can then assume that α is an involution.

We show that we can also assume $K = \langle \delta \rangle \times \langle \alpha \rangle \cong C_{2m} \times C_2$. The number of subgroups of order 4 generated elements of $G_H \setminus \langle \delta \rangle$ is equal to m and hence is odd. This implies that α normalizes $\langle \zeta \rangle$ for some $\zeta \in G_H \setminus \langle \delta \rangle$ with $o(\zeta) = 4$; up to conjugation, $\zeta = \epsilon$. Let $\mathcal{H}_q$ have equation (1) and assume up to conjugation that $P = (0, 0, 1)$, $Q = (1, 0, 0)$, and $R = (0, 1, 0)$. If $\alpha(Q) = Q$ and $\alpha(R) = R$, then α is represented by a diagonal matrix and commutes with δ . Suppose that α and δ do not commute, so that α interchanges Q and R . Then

$$(4) \quad \delta = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & a & 0 \\ a^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & e & 0 \\ -e^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for some $(q + 1)$ -th roots of unity λ, a, e with $o(\lambda) = 2m$. Since α normalizes ϵ , either $a^2 = -e^2$ if $\alpha\epsilon\alpha = \epsilon$, or $a^2 = e^2$ if $\alpha\epsilon\alpha = \epsilon^{-1}$. In the former case, $\alpha\epsilon$ is a diagonal matrix, so that $\alpha\epsilon$ fixes $\{Q, R\}$ pointwise; but $o(\alpha\epsilon) = 4$, so that $\alpha\epsilon$ is of type (B2): a contradiction. Hence, $\alpha\epsilon\alpha = \epsilon^{-1}$; this implies $o(\alpha\epsilon) = 2$, and either $\alpha\epsilon = \text{diag}(1, -1, 1)$ or $\alpha\epsilon = \text{diag}(-1, 1, 1)$. We can then replace α with $\alpha\epsilon$, so that $K = \langle \delta \rangle \times \langle \alpha \rangle$ and G is a product $K\langle \epsilon \rangle \cong (C_{2m} \times C_2)C_4$, with $|K \cap \langle \epsilon \rangle| = 2$.

Such a group $G = (\langle \delta \rangle \times \langle \alpha \rangle)\langle \epsilon \rangle \cong (C_{2m} \times C_2)C_4$ actually exists in $\text{SU}^\pm(2, q)$, as the choice in (4) shows. The elements in $\langle \delta \rangle \times \langle \alpha \rangle$ of order greater than 2 are of type (B1); the three involutions in $\langle \delta \rangle \times \langle \alpha \rangle$ are of type (A); the elements $\delta^i\epsilon$ of $G_H \setminus \langle \delta \rangle$ have order 4 and are of type (B2); the remaining elements have the form $\delta^i\alpha\epsilon$, are involutions, and are of type (A).

- Suppose $G_H = \text{SL}(2, 3)$ with $p \geq 5$. Then $\bar{G}$ is a subgroup of order 24 with a subgroup $\bar{G}_H \cong \mathbf{A}_4$; from Theorem 2.3, $\bar{G} \cong \mathbf{S}_4$. By direct checking with MAGMA [4], the unique groups L admitting a normal subgroup isomorphic to $\text{SL}(2, 3)$ and such that the factor group of L over the unique involution of $\text{SL}(2, 3)$ is isomorphic to $\mathbf{S}_4$ are $\text{SmallGroup}(48, 28) \cong \hat{\Sigma}_4$ and $\text{SmallGroup}(48, 29)$.

We show that, if L_1 and L_2 are subgroups of $\text{SU}^\pm(2, q)$ containing G_H with index $[L_1 : G_H] = [L_2 : G_H] = 2$, then $L_1 = L_2$. By direct inspection on $\text{SmallGroup}(48, 28)$ and $\text{SmallGroup}(48, 29)$, both L_1 and L_2 are generated by G_H together with any element of order 8, whose square lies in G_H . Also, any cyclic subgroup C_4 of order 4 of G_H is contained both in a cyclic subgroup C_8^1 of order 8 of L_1 and in a cyclic subgroup C_8^2 of order 8 of L_2 . The group C_4 is generated by an element of type (B2) with two fixed points $Q, R \in \ell \cap \mathcal{H}_q$; thus, C_8^1 and C_8^2 act on $\{Q, R\}$. If a generator α_i of C_8^i ($i \in \{1, 2\}$) interchanges Q and R , then α_i^2 is of type (A), a contradiction to Lemma 2.4. Thus, both C_8^1 and C_8^2 fix $\{Q, R\}$ pointwise. Since the pointwise stabilizer of $\{Q, R\}$ in $\text{PGU}(3, q)$ is cyclic, this implies $C_8^1 = C_8^2$ and hence $L_1 = L_2$.

If $8 \mid (q-1)$, then by Theorem 2.2 there H already contains a subgroup $\hat{\Sigma}_4 \cong \text{SmallGroup}(48, 28)$ having a subgroup isomorphic to $\text{SL}(2, 3)$. Hence, there exists no subgroup G of $\text{SU}^\pm(2, q)$ with $G_H \cong \text{SL}(2, 3)$.

If $8 \nmid (q-1)$, then $G \leq \text{SU}^\pm(2, q)$ with $G_H \cong \text{SL}(2, 3)$ does exist, and can be constructed as follows. Let $\mathcal{H}_q$ be given by Equation (2); up to conjugation, $P = (0, 1, 0)$ and $\ell : Y = 0$. Choose $\lambda, \mu \in \mathbb{F}_q$ and $c, e \in \mathbb{F}_{q^2}$ such that $\lambda^2 = -1$, $\mu^2 = \lambda$, $c^2 = \frac{\lambda+1}{2}$, and $e = \mu c$. Define

$$\alpha_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & -\lambda c \\ 0 & 1 & 0 \\ -\frac{\lambda}{c} & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & c \\ 0 & 1 & 0 \\ -\frac{1}{c} & 0 & 0 \end{pmatrix},$$

$$\xi = \begin{pmatrix} -\frac{\lambda+1}{2} & 0 & \frac{\lambda-1}{2}c \\ 0 & 1 & 0 \\ c & 0 & \frac{1-\lambda}{2} \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 & e \\ 0 & 1 & 0 \\ e^{-1} & 0 & 0 \end{pmatrix}.$$

By direct checking, the following holds.

- $\alpha_1, \alpha_2, \alpha_3$ have order 4; $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \iota = \text{diag}(1, -1, 1)$; $\alpha_1 \alpha_2 \alpha_3 = \text{id}$. Hence, $Q_8 = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is a quaternion group.
- $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}_q$ since they preserve the equation (2) of $\mathcal{H}_q$ and fix P ; $\alpha_1, \alpha_2, \alpha_3$ are of type (B2). The fixed points other than P are $P_1 = (1, 0, 0)$ and $P_3 = (0, 0, 1)$ for α_1 ; $Q_1 = (-c, 0, 1)$ and $Q_3 = (c, 0, 1)$ for α_2 ; $R_1 = (-\lambda c, 0, 1)$ and $R_3 = (\lambda c, 0, 1)$ for α_3 .
- ξ has order 3 and $\xi \in \mathcal{M}_q$. For $i \in \{1, 3\}$, we have $\xi(P_i) = Q_i$, $\xi(Q_i) = R_i$, and $\xi(R_i) = P_i$; since the pointwise stabilizer of two points in $\ell \cap \mathcal{H}_q$ is cyclic, this implies that ξ normalizes Q_8 . Also, ξ does not commute with α_j . Then $\langle \alpha_1, \alpha_2, \alpha_3, \xi \rangle = Q_8 \rtimes C_3$ is isomorphic to $\text{SL}(2, 3)$.
- ξ has no fixed points in $\text{PG}(2, \mathbb{F}_{q^2}) \setminus \ell$; this implies by Lemma 2.4 that ξ is of type (B1).
- γ is an involution of $\mathcal{M}_q$ satisfying $\gamma(P_1) = P_3$, $\gamma(Q_1) = R_1$, and $\gamma(Q_3) = R_3$; this implies that γ normalizes Q_8 . Also, $\gamma \xi \gamma = \alpha_2 \xi^2 \in Q_8 \rtimes C_3$; this implies that γ normalizes $Q_8 \rtimes C_3 \cong \text{SL}(2, 3)$.

Therefore, $G = \langle \alpha_1, \alpha_2, \alpha_3, \xi, \gamma \rangle$ is a subgroup of $\mathcal{M}_q$ of order 48, with a subgroup of index 2 isomorphic to $\text{SL}(2, 3)$. Since G contains a Klein four-group $\langle \iota, \gamma \rangle$, we have $G \cong \text{SmallGroup}(48, 29)$. We have $\gcd(|G|, \frac{q+1}{2}) \in \{1, 3\}$; recall that $\mathcal{M}_q \times C_{\frac{q+1}{2}}$. If $\gcd(|G|, \frac{q+1}{2}) = 1$, then $G \leq \text{SU}^\pm(2, q)$. If $\gcd(|G|, \frac{q+1}{2}) = 3$, then all elements of order 3 in G are of type (B1) as they are conjugated to ξ ; this implies again $G \leq \text{SU}^\pm(2, q)$, because any element ψ of order 3 in $\mathcal{M}_q \setminus \text{SU}^\pm(2, q)$ is of type (A).

In fact, let $\mathcal{H}_q$ have equation (1); up to conjugation, $P = (0, 0, 1)$ and ψ fixes $Q = (1, 0, 0)$ and $R = (0, 1, 0)$. This implies $\psi = \text{diag}(u, v, 1)$ with $u^3 = v^3 = 1$. If ψ is not of type (A), then $u \neq 1$, $v \neq 1$, $u \neq v$; hence, $v = u^{-1}$ and $\psi \in \text{SU}^\pm(2, q)$.

□

We will make use of the following remark, which can be easily proved in analogy to [8, Remark 4.1].

Remark 3.2. *Let $G \leq \mathcal{M}_q$ be such that G/G_Ω is generated by elements whose order is coprime to $|G_\Omega|$. Then $G = G_\pm \times G_\Omega$. If in addition G/G_Ω is generated by elements of odd order, then $G = G_H \times G_\Omega$.*

Now we compute the genera of quotient curves $\mathcal{H}_q/G$ for all subgroups G of $\mathcal{M}_q = \mathrm{SU}^\pm(2, q) \times \Omega$. The factor group G/G_Ω is isomorphic to a subgroup of $\mathrm{SU}^\pm(2, q)$. Hence, we will consider the different possibilities for G/G_Ω given by Proposition 3.1 and Theorem 2.2.

Lemma 3.3. *Let $G \leq \mathcal{M}_q$ be such that one of the following cases holds:*

- (1) G/G_Ω is cyclic of order a divisor of $q+1$ different from 2.
- (2) G/G_Ω is dicyclic of order $4m$ with $1 < m \mid \frac{q+1}{2}$.
- (3) $G/G_\Omega \cong C_d \times C_2$ where $d \mid (q+1)$ and d is even.
- (4) G/G_Ω is dihedral of order $2d$ with $1 < d \mid (q+1)$.
- (5) $G/G_\Omega \cong \mathrm{Dic}_m \rtimes C_2$ with $1 < m \mid \frac{q+1}{2}$.

Then G is contained in the maximal subgroup of $\mathrm{PGU}(3, q)$ stabilizing a self-polar triangle $T \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$.

If $G \leq \mathcal{M}_q$ is such that $G/G_\Omega \cong E_{p^k} \rtimes C$, where E_{p^k} is elementary abelian of order p^k and C is cyclic, then G is contained in the maximal subgroup of $\mathrm{PGU}(3, q)$ stabilizing a point of $\mathcal{H}_q(\mathbb{F}_{q^2})$.

Proof. If $\langle \alpha_1 G_\Omega, \dots, \alpha_r G_\Omega \rangle$ is a normal subgroup of G/G_Ω , then $\langle \alpha_1, \dots, \alpha_r \rangle$ is a normal subgroup of G , because G_Ω is central in G .

- Let $G/G_\Omega \cong E_{p^k} \rtimes C$. Then E_{p^k} has a unique fixed point on $\mathcal{H}_q(\mathbb{F}_{q^2})$; see [24, Lemma 11.129]. As E_{p^k} is normal in G , G fixes this point.
- Let $G/G_\Omega = \langle \alpha G_\Omega \rangle \times \langle \gamma G_\Omega \rangle$ satisfy assumption (3), with $\alpha^d, \gamma^2 \in G_\Omega$. We show that $o(\alpha), o(\gamma) \mid (q+1)$; this implies that α and γ are of type (A) or (B1), $\langle \alpha, \gamma \rangle$ fixes pointwise a triangle $T = \{P, Q, R\}$ with $Q, R \in \ell(\mathbb{F}_{q^2}) \setminus \mathcal{H}_q$, and hence G fixes T pointwise, the claim.

Suppose by contradiction that $o(\alpha) \nmid (q+1)$. By Lemma 2.4, $o(\alpha) \mid 2(q+1)$, α is of type (B2), and α^2 is of type (A) with center P . Let $Q, R \in \ell \cap \mathcal{H}_q$ be the fixed points of α other than P . Since the pointwise stabilizer of $\{Q, R\}$ is cyclic unlikely G/G_Ω , γ interchanges Q and R . Let $\mathcal{H}_q$ have equation (2); up to conjugation, $P = (0, 1, 0)$, $Q = (1, 0, 0)$, and $R = (0, 0, 1)$; hence,

$$\alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 & c \\ 0 & 1 & 0 \\ d & 0 & 0 \end{pmatrix},$$

where $c^{q+1} = d^{q+1} = a^{2(q+1)} = 1 \neq a^{q+1}$. Then $\alpha\gamma \neq \gamma\alpha$, a contradiction. Suppose that $o(\gamma) \nmid (q+1)$. Then swap the roles of α and γ in the argument above to obtain a contradiction.

- Let $G/G_\Omega = \langle \alpha G_\Omega \rangle$ satisfy assumption (1), with $|G/G_\Omega| = d$. By Lemma 2.4 $o(\alpha) \mid (q^2 - 1)$. Together with $o(\alpha) \mid d(q+1)$, this yields $o(\alpha) \mid 2(q+1)$. If $o(\alpha) \nmid (q+1)$, then α is of type (B2) and $\alpha^2 \in G_\Omega$, a contradiction. Hence, $o(\alpha) \mid (q+1)$, and α is of type (A) or (B1). Since $\langle \alpha \rangle$ is normal in G , G acts on the points fixed by α ; as G fixes P , this implies that G stabilizes a self-polar triangle $\{P, Q, R\} \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$.
- Let $G/G_\Omega = \langle \alpha G_\Omega, \epsilon G_\Omega \rangle \cong \mathrm{Dic}_n$ or $G/G_\Omega = \langle \alpha G_\Omega, \epsilon G_\Omega, \xi G_\Omega \rangle \cong \mathrm{Dic}_n \rtimes C_2$ satisfy assumption (2) or (5), respectively, with $6 \leq o(\alpha G_\Omega) = 2n \mid (q+1)$. Using the normality of $\langle \alpha \rangle$ in G and arguing as in the previous point, we have that α is of type (A) or (B1), and G stabilizes a self-polar triangle $T \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$.
- Let $G/G_\Omega = \langle \alpha \rangle \rtimes \langle \gamma \rangle$ satisfy assumption (4), with $\alpha^d, \gamma^2 \in G_\Omega$. If $d = 2$, then G satisfies also assumption (3) and the claim was already proved. If $d > 2$, then $\langle \alpha \rangle$ is normal in G ; arguing as in the previous point, α is of type (A) or (B1), and G stabilizes a self-polar triangle $T \subset \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$.

□

Lemma 3.3 provides cases for G/G_Ω which do not need to be considered in the following, since G is contained in a maximal subgroup of $\text{PGU}(3, q)$ for which $g(\mathcal{H}_q/G)$ has already been computed in the literature. Namely, if $G/G_\Omega \cong E_{p^k} \rtimes C$, then $g(\mathcal{H}_q/G)$ is computed in [2, Theorem 1.1]; if G/G_Ω satisfies one of the assumptions (1) to (5), then $g(\mathcal{H}_q/G)$ is computed in [8, Proposition 3.4].

Proposition 3.4. *Let $G \leq \mathcal{M}_q$ be such that $G/G_\Omega \cong \text{SL}(2, 5)$, with $q^2 \equiv 1 \pmod{5}$. Then*

$$g(\mathcal{H}_q/G) = \frac{(q+1)(q-1-2\omega) + 180\omega - 20r - 48s}{240\omega},$$

where

$$r = \begin{cases} 4\omega & \text{if } 3 \mid (q-1), \\ q+1+2\omega & \text{if } 3 \mid q, \\ 0 & \text{if } 3 \mid (q+1), 3 \nmid \omega, \\ 2(q+1) & \text{if } 3 \mid \omega; \end{cases} \quad s = \begin{cases} 2\omega & \text{if } 5 \mid (q-1), \\ 0 & \text{if } 5 \mid (q+1), 5 \nmid \omega, \\ q+1 & \text{if } 5 \mid \omega. \end{cases}$$

Proof. By Remark 3.2, $G = G_H \times G_\Omega$ with $G_H \cong \text{SL}(2, 5)$. The nontrivial elements of G are as follows.

- 20 elements of order 3 in G_H ; they are of type (B2), (C), or (B1), according to $3 \mid (q-1)$, $3 \mid q$, or $3 \mid (q+1)$, respectively.
- $2\omega - 1$ elements in $\langle \iota \rangle \times G_\Omega$; they are of type (A).
- 30ω elements obtained as the product of an element of order 4 in G_H by an element of G_Ω ; they are of type (B2).
- 20ω elements obtained as the product of an element of order 6 in G_H by an element of G_Ω ; they are of type (B2), (E), or (B1), according to $3 \mid (q-1)$, $3 \mid q$, or $3 \mid (q+1)$, respectively.
- 48ω elements obtained as the product of an element $\eta \in G_H$ of order 5 or 10 by an element $\theta \in G_\Omega$. If $5 \mid (q-1)$, they are of type (B2). If $5 \mid (q+1)$ and $5 \nmid \omega$, they are of type (B1).

If $5 \mid \omega$, 48 of them are of type (A), the other ones are of type (B1). Namely, if $o(\eta) = 10$ then $\eta\theta$ is of type (B1); if $o(\eta) = 5$ and $\{P, Q, R\}$ are the fixed points of η , then there are exactly 2 choices for $\theta \in G_\Omega$ such that $\eta\theta$ is of type (A), one with center Q , the other with center R .

- $20(\omega - 1)$ elements obtained as the product of an element of order 3 in G_H by a nontrivial element of G_Ω ; they are of type (B2) or (E) if $3 \mid (q-1)$ or $3 \mid q$, respectively. If $3 \mid (q+1)$, either they are all of type (B1) or there are 48 of them of type (A), according to $5 \nmid \omega$ or $5 \mid \omega$, arguing as in the previous case.

The claim follows by direct computation using the Riemann-Hurwitz formula and Theorem 2.5. $\square$

Proposition 3.5. *Let $G \leq \mathcal{M}_q$ be such that $G/G_\Omega \cong \hat{\Sigma}_4 \cong \text{SmallGroup}(48, 28)$, with $p \geq 5$ and $8 \mid (q-1)$. Then*

$$g(\mathcal{H}_q/G) = \frac{(q+1)(q-1-2\omega) + 36\omega - 16r}{96\omega}, \quad \text{where } r = \begin{cases} 2\omega & \text{if } 3 \mid (q-1), \\ 0 & \text{if } 3 \mid (q+1), 3 \nmid \omega, \\ q+1 & \text{if } 3 \mid \omega. \end{cases}$$

Proof. By Remark 3.2 and Proposition 3.1, $G = G_H \times G_\Omega$ with $G_H \cong \text{SmallGroup}(48, 28)$. The nontrivial elements of G are as follows. Since $SU^\pm(2, q)$ has no elements of type (A) with odd order, the type of any element in G_H is uniquely determined by its order. The product of ι by an element of G_Ω has type (A), and the product of an element of type (B2) in G_H by an element of G_Ω has type (B2). The product of an element η of type (B1) in G_H by an element of G_Ω has type (B1), unless $o(\eta) \mid \omega$; if $o(\eta) \mid \omega$, then G_Ω contains exactly 2 elements θ_1, θ_2 such that $\eta\theta_1$ and $\eta\theta_2$ are of type (A); here, this argument applies to the elements of order 3. Now the claim follows by direct computation with the Riemann-Hurwitz formula and Theorem 2.5. $\square$

Proposition 3.6. *Let $G \leq \mathcal{M}_q$ be such that $G/G_\Omega \cong \mathrm{SL}(2, 3)$ with $p \geq 5$. If $3 \mid (q-1)$, then*

$$(5) \quad g(\mathcal{H}_q/G) = \frac{(q+1)(q-1-2\omega) + 4\omega}{48\omega}.$$

If $3 \mid (q+1)$, then one of the following cases holds:

$$(6) \quad g(\mathcal{H}_q/G) = \frac{(q+1)(q-1-2\omega) + 36\omega - 8(q+1)(\gcd(3, \omega) - 1)}{48\omega};$$

$$(7) \quad g(\mathcal{H}_q/G) = \frac{(q+1)(q-9-2\omega) + 36\omega}{48\omega}, \quad \text{with } 3 \nmid \omega;$$

$$(8) \quad g(\mathcal{H}_q/G) = \frac{(q+1)(q-1-2\omega) + 36\omega}{48\omega}, \quad \text{with } 3 \mid \omega, \quad 3 \mid \frac{q+1}{\omega}.$$

All cases (5) to (8) actually occur, for some G as in the assumptions.

Proof. Assume $3 \mid (q-1)$. By Remark 3.2, $G = G_H \times G_\Omega$ with $G_H \cong \mathrm{SL}(2, 3)$. The nontrivial elements of G are as follows: $2\omega - 1$ elements of type (A) in $\langle \iota \rangle \times G_\Omega$; 22ω elements of order divisible by 3 or 4, which are of type (B2). Equation (5) follows by the Riemann-Hurwitz formula and Theorem 2.5.

For the rest of the proof, assume $3 \mid (q+1)$. Let $G/G_\Omega = Q_8 \rtimes \langle \xi G_\Omega \rangle$, where Q_8 is a quaternion group and $\xi \notin G_\Omega$ satisfies $\xi^3 \in G_\Omega$. Since Q_8 and G_Ω have coprime orders, Q_8 is induced by a subgroup $\langle \alpha_1, \alpha_2 \rangle$ of G isomorphic to Q_8 .

Suppose that there exists $\xi \in G_H$ inducing ξG_Ω ; this can be assumed when $3 \nmid \omega$. Then $G = G_H \times G_\Omega$, and the nontrivial elements of G are as follows: $2\omega - 1$ elements of type (A) in $\langle \iota \rangle \times G_\Omega$; 6ω elements of order divisible by 4, which are of type (B2). If $3 \nmid \omega$, any other element is of type (B1). If $3 \mid \omega$, then there are $8 \cdot 2$ elements of order 3 and type (A): namely, for any element $\eta \in G_H$ of order 3 there are exactly 2 elements $\theta_1, \theta_2 \in G_\Omega$ such that $\eta\theta_1, \eta\theta_2$ are of type (A); any other element is of type (B1). Equation (6) follows by the Riemann-Hurwitz formula and Theorem 2.5.

For the rest of the proof, we can assume that $\xi \notin G_H$ for any $\xi \in G$ inducing ξG_Ω . As the subgroups of $\mathrm{SL}(2, 3)$ of order 3 are conjugated by elements of Q_8 , no element of G_H induces an element of order 3 in G/G_Ω .

- Suppose that $o(\xi) = 3$. Since $\xi \notin G_H$ and $\xi \notin G_\Omega$, this implies that ξ is of type (A) with center on ℓ and axis passing through P . Note that $G_H = Q_8$. Note also that $3 \nmid \omega$; otherwise, G_Ω has an element ρ of order 3, and either $\xi\rho$ or $\xi\rho^2$ is an element of type (B1) and order 3 lying in G_H , a contradiction. The nontrivial elements of G are as follows: $2\omega - 1$ elements of type (A) in $\langle \iota \rangle \times G_\Omega$; 8 elements of order 3 and type (A); 6ω elements of order a multiple of 4 and type (B2); any other element is of type (B1). Equation (7) follows by the Riemann-Hurwitz formula and Theorem 2.5.

Such a group G actually exists in $\mathcal{M}_q$. In fact, let $3 \nmid \omega$ and $(Q_8 \rtimes C_3) \times C_\omega$ be the subgroup isomorphic to $\mathrm{SL}(2, 3) \times G_\Omega$ constructed above, with $Q_8 \rtimes C_3 \leq H$. Let η be a generator of $C_3 \leq H$ and ρ be an element of order 3 in Ω . Then $\eta\rho$ is an element of order 3 and type (A) not in Ω , such that $G := (Q_8 \rtimes \langle \eta\rho \rangle) \times C_\omega$ is the desired group.

- Now suppose that $o(\xi) > 3$. Up to composing with an element of G_Ω , we can assume that ξ is a 3-element, of order 3^k , $k \geq 2$. As ξ^3 is a nontrivial element of G_Ω , ξ is not of type (A), and hence is of type (B1). As G_Ω is cyclic, $\langle \xi^3 \rangle$ is the Sylow 3-subgroup of G_Ω . Then $G \cong (Q_8 \rtimes C_{3^k}) \times C_{\omega/3^{k-1}}$, where $C_{3^k} = \langle \xi \rangle$ and $3^k \nmid \omega$. The nontrivial elements of G are as follows.

- Elements of $Q_8 \rtimes C_{3^k}$. There are $2 \cdot 3^{k-1} - 1$ elements of type (A) in $\langle \iota \rangle \times C_{3^{k-1}}$; $6 \cdot 3^{k-1}$ elements of type (B2), as the product of an element of order 4 in Q_8 by an element of $C_{3^{k-1}}$; $2(3^k - 3^{k-1})$ elements of type (B1) in $(\langle \iota \rangle \times C_{3^k}) \setminus (\langle \iota \rangle \times C_{3^{k-1}})$.

The $6(3^k - 3^{k-1})$ elements σ obtained as the product of an element α of order 4 in Q_8 by an element γ of order 3^k in C_{3^k} are of type (B1). In fact, $o(\sigma) \in \{3^k, 2 \cdot 3^k, 4 \cdot 3^k\}$. If $o(\sigma) = 4 \cdot 3^k$, then σ is of type (B2) and $o(\sigma^4) \mid (q+1)$, so that $\sigma^4 \in G_H$ with $o(\sigma^4) = 3^k$, a contradiction. If

- $o(\sigma) = 2 \cdot 3^k \mid (q+1)$, then σ cannot be of type (A) with center on ℓ , because $\sigma^{3^k} = \iota$ has center P ; hence, σ is of type (B1). If $o(\sigma) = 3^k$, then by Schur-Zassenhaus Theorem $\langle \sigma \rangle$ is conjugated to C_{3^k} ; hence, σ is of type (B1).
- Elements of $G \setminus (Q_8 \rtimes C_{3^k})$. There are $2 \cdot 3^{k-1} \cdot \left(\frac{\omega}{3^{k-1}} - 1\right)$ elements of type (A) in $\langle \iota \rangle \times C_{3^{k-1}} \times C_{\omega/3^{k-1}}$; $6 \cdot 3^{k-1} \cdot \left(\frac{\omega}{3^{k-1}} - 1\right)$ elements of type (B2) obtained as the product of an element of type (B2) in $Q_8 \rtimes C_{3^k}$ by a non trivial element of $C_{\omega/3^{k-1}}$; $8(3^k - 3^{k-1}) \left(\frac{\omega}{3^{k-1}} - 1\right)$ elements of type (B1) obtained as the product of an element either trivial or of type (B1) in $Q_8 \rtimes C_{3^k}$ by a nontrivial element of $C_{\omega/3^{k-1}}$.

Equation (8) follows by the Riemann-Hurwitz formula and Theorem 2.5.

Such a group actually exists in $\mathcal{M}_q$. In fact, consider a group $(Q_8 \rtimes C_3) \times C_{\omega/3^{k-1}}$ with $Q_8 \rtimes C_3 \leq H$ and $C_{\omega/3^{k-1}} \leq \Omega$, as described above; let ξ be a generator of C_3 , and σ be an element of Ω of order 3^k . Then replace ξ with $\xi\sigma$, that is, define $G = (Q_8 \rtimes \langle \xi\sigma \rangle) \times C_{\omega/3^{k-1}}$.

□

Proposition 3.7. *Let $G \leq \mathcal{M}_q$ be such that G/G_Ω is cyclic of order d with $d \mid (q-1)$. Then either $d = 2$ and G stabilizes pointwise a self-polar triangle $\{P, Q, R\} \subset \text{PG}(2, q^2) \setminus \mathcal{H}_q$, or*

$$g(\mathcal{H}_q/G) = \frac{(q-1)(q+1-\omega \cdot \gcd(d, 2))}{2d\omega}.$$

Both cases occur.

Proof. Let αG_Ω be a generator of G/G_Ω . If $d = 1$, the claim is trivial. Suppose $d = 2$. Since $|G_\Omega|$ is odd, G is cyclic of order $2d \mid (q+1)$ and fixes pointwise a self-polar triangle $\{P, Q, R\} \subset \text{PG}(2, q^2) \setminus \mathcal{H}_q$. Suppose $d > 2$. Then α is of type (B2) and G is cyclic. The number of elements of type (A) is either $\omega - 1$ or $2\omega - 1$, according to d odd or d even, respectively; any other nontrivial element is of type (B2).

The claim follows by direct computation with Theorem 2.5. □

Proposition 3.8. *Let $G \leq \mathcal{M}_q$ be such that $G/G_\Omega \cong \text{Dic}_n$, with $1 < n \mid \frac{q-1}{2}$. Then*

$$g(\mathcal{H}_q/G) = \frac{(q-1)(q+1-2\omega)}{8n\omega}.$$

Proof. By Remark 3.2 and Proposition 3.1, $G = G_H \times G_\Omega$ with $G_H \cong \text{Dic}_n$. Any nontrivial element $\sigma \in G$ is of type (A) if $\sigma \in \langle \iota \rangle \times G_\Omega$, and of type (B2) otherwise. The claim follows by Theorem 2.5. □

Proposition 3.9. *Let $G \leq \mathcal{M}_q$ be such that $G/G_\Omega \cong \text{SL}(2, p^k)$ with $k \mid n$, $r = n/k$. Then*

$$g(\mathcal{H}_q/G) = 1 + \frac{q^2 - q - 2 - \Delta}{2p^k(p^{2k} - 1)\omega},$$

where

$$\begin{aligned} \Delta = & (p^{2k} - 1)(q + 2) + p^{2k} - 1 + q + 1 + p^k(p^k + 1)(p^k - 3)\omega + p^k(p^k - 1)^2(\gcd(r, 2) - 1) + 2(p^{2k} - 1)(\omega - 1) \\ & + 2(\omega - 1)(q + 1) + p^k(p^k - 1)^2(\omega - 1)(\gcd(r, 2) - 1) + (\gcd(\omega, p^k + 1) - 1)p^k(p^k - 1)(q + 1)(2 - \gcd(r, 2)). \end{aligned}$$

Proof. By Remark 3.2, $G = G_H \times G_\Omega$ with $G_H \cong \text{SL}(2, p^k)$. Clearly $(p^k - 1) \mid (q - 1)$, while $(p^k + 1) \mid (q - 1)$ or $(p^k + 1) \mid (q + 1)$ according to r even or r odd, respectively. The nontrivial elements in G_H are classified in the proof of [32, Proposition 4.3] as follows:

- $p^{2k} - 1$ elements of order p and type (C);
- $p^{2k} - 1$ elements of order p times a nontrivial divisor of $q + 1$, which are of type (E);
- 1 involution ι , of type (A);
- $\frac{p^k(p^k+1)(p^k-3)}{2}$ elements of order a divisor of $p^k - 1$ different from 2, which are of type (B2);

- $\frac{p^k(p^k-1)^2}{2}$ elements of order a divisor of $p^k + 1$ different from 2, which are of type (B1) or (B2) according to r odd or r even, respectively.

The product of an element $\sigma \in G_H$ by a nontrivial element $\tau \in G_\Omega$ is as follows. If σ is of type (C) or (E), then $\sigma\tau$ is of type (E). If $\sigma \in \langle \iota \rangle$, then $\sigma\tau$ is of type (A). If σ is of type (B2), then $\sigma\tau$ is of type (B2).

If σ is of type (B1) and $o(\sigma) \mid |G_\Omega|$, then G_Ω contains exactly 2 elements τ_1, τ_2 such that $\sigma\tau_1, \sigma\tau_2$ are of type (A); otherwise, $\sigma\tau$ is of type (B1). If r is even, there are no elements of type (B1). Assume r odd. The elements of type (B1) together with $\langle \iota \rangle$ form $\frac{p^k(p^k-1)}{2}$ cyclic groups of order $p^k + 1$ which intersect pairwise in $\langle \iota \rangle$. Then, the number of elements $\sigma\tau$ of type (A) with σ of type (B1) is $\frac{p^k(p^k-1)}{2} \cdot (\gcd(\omega, p^k + 1) - 1) \cdot 2$.

Now the claim follows by direct computation with Theorem 2.5. $\square$

Proposition 3.10. *Let $G \leq \mathcal{M}_q$ be such that $G/G_\Omega \cong \text{TL}(2, p^k)$, where $k \mid n$ and n/k is even. Then*

$$g(\mathcal{H}_q/G) = 1 + \frac{q^2 - q - 2 - \Delta}{4p^k(p^{2k} - 1)\omega},$$

where

$$\begin{aligned} \Delta = & (p^{2k} - 1)(q + 2) + p^{2k} - 1 + q + 1 + p^k(p^k + 1)(p^k - 3)\omega + p^k(p^k - 1)^2 + 2(p^{2k} - 1)(\omega - 1) \\ & + 2(\omega - 1)(q + 1) + p^k(p^k - 1)^2(\omega - 1) + 2p^k(p^{2k} - 1)\omega. \end{aligned}$$

Proof. By Remark 3.2 and Proposition 3.1, $G = G_H \times G_\Omega$ with $G_H = \langle L, \delta \rangle \cong \text{TL}(2, p^k)$, $L \cong \text{SL}(2, p^k)$, $o(\delta) = 2(p^k - 1)$. The nontrivial elements in $L \times G_\Omega$ are already classified according to their type in the proof of Proposition 3.9. Every element in $G_H \setminus L$ is of type (B2); see the proof of [32, Proposition 4.4]. Hence, for every $\sigma \in G_H \setminus L$ and $\tau \in G_\Omega$, $\sigma\tau$ is of type (B2). The claim now follows by direct computation with Theorem 2.5. $\square$

The case G/G_Ω isomorphic to a cyclic subgroup of $\text{SU}^\pm(2, q)$ of order 2 not in H has already been considered in Proposition 3.7.

Proposition 3.11. *Let $G \leq \mathcal{M}_q$ be such that $G/G_\Omega \cong \text{SL}(2, 3) \rtimes C_2 \cong \text{SmallGroup}(48, 29)$, with $p \geq 5$ and $8 \nmid (q - 1)$. Then*

$$g(\mathcal{H}_q/G) = \frac{(q + 1)(q - 2\omega - 13) + 60\omega - 16r}{96\omega}, \quad \text{where } r = \begin{cases} 2\omega & \text{if } 3 \mid (q - 1), \\ 0 & \text{if } 3 \mid (q + 1), 3 \nmid \omega, \\ q + 1 & \text{if } 3 \mid \omega. \end{cases}$$

Proof. By Remark 3.2 and Proposition 3.1, $G = G_\pm \times G_\Omega$ with $G_\pm \cong \text{SmallGroup}(48, 29)$ and $G_H \cong \text{SL}(2, 3)$. By Lemma 2.4, $G_\pm$ contains 13 involutions, of type (A); 18 elements of order 8 or 4, of type (B2); 16 elements of order 6 or 3, which are of type (B2) or (B1) according to $3 \mid (q - 1)$ or $3 \mid (q + 1)$. The element $\sigma\tau$, where $\sigma \in G_\pm$ and $\tau \in G_\Omega \setminus \{id\}$, is as follows. If σ is the unique involution ι of G_H , then $\sigma\tau$ is of type (A). If σ is an involution different from ι , then $\sigma\tau$ is of type (B1). If σ is of type (B2), then $\sigma\tau$ is of type (B2). If σ has order 6 and is of type (B1), then $\sigma\tau$ is of type (B1). If σ has order 3 and is of type (B1), then there are $\gcd(|G_\Omega|, 3) - 1$ elements of G_Ω such that $\sigma\tau$ is of type (A), while for any other $\tau \in G_\Omega$ $\sigma\tau$ is of type (B1); in fact, $\sigma\tau$ is of type (A) if and only if $o(\tau) = 3$.

The claim follows by the Riemann-Hurwitz formula and Theorem 2.5. $\square$

Proposition 3.12. *Let $G \leq \mathcal{M}_q$ be such that G/G_Ω is cyclic of order $2d > 2$, where either $d \mid (q - 1)$ and $d \nmid \frac{q-1}{2}$, or $d \mid (q + 1)$. Assume also that G does not stabilize any self-polar triangle $T \subset \text{PG}(2, q^2) \setminus \mathcal{H}_q$. Then either $d \mid (q - 1)$ and $d \nmid \frac{q-1}{2}$, or $d = 2$; in both cases,*

$$(9) \quad g(\mathcal{H}_q/G) = \frac{(q - 1)(q + 1 - 2\omega)}{4d\omega}.$$

Whenever d satisfies the numerical assumptions, a subgroup $G \leq \mathcal{M}_q$ with $g(\mathcal{H}_q/G)$ as in Equation (9) exists.

Proof. Let αG_Ω be a generator of G/G_Ω .

Suppose that $d \mid (q-1)$ and $d \nmid \frac{q-1}{2}$. Then $o(\alpha) \mid (q^2-1)$ and α is of type (B2). The group G fixes pointwise the 2 fixed points $Q, R \in \ell \cap \mathcal{H}_q$ other than P . Hence, G is cyclic; we can assume that $G = \langle \alpha \rangle$. Since $\gcd(o(\alpha), q+1) = 2\omega$, G contains $2\omega-1$ elements of type (A) and $2d\omega-2\omega$ elements of type (B2); Equation (9) follows. Such a group G does exist in $\mathcal{M}_q$, being generated by any element of type (B2) and order $2d\omega$.

Suppose that $d \mid (q+1)$. If $2d \mid (q+1)$, then Lemma 3.3 implies that G stabilizes a self-polar triangle $T \subset \text{PG}(2, q^2) \setminus \mathcal{H}_q$; hence, we can assume that $2d \nmid (q+1)$, that is, d is even and $d/2$ is odd. Then $4 \mid o(\alpha)$ and α is of type (B2). Also, $o(\alpha^2) \mid (q+1)$ implies that $\alpha^4 \in G_\Omega$, so that $d = 2$. As above, G is cyclic, and we can assume that α is a generator of G . The group G has $2\omega-1$ elements of type (A) and $4\omega-2\omega$ elements of type (B2); Equation (9) follows. Such a group G does exist in $\mathcal{M}_q$, being generated by any element of type (B2) and order 4ω . $\square$

Proposition 3.13. *Let $G \leq \mathcal{M}_q$ be such that G/G_Ω is dihedral of order $2d$, with $2 < d \mid (q-1)$. Then*

$$g(\mathcal{H}_q/G) = \frac{(q+1)(q-1 - \gcd(d, 2) \cdot \omega - d) + 2\omega(d + \gcd(d, 2))}{4d\omega}.$$

Proof. By Remark 3.2, $G = G_\pm \times G_\Omega$, where $G_\pm = \langle \alpha G_\Omega, \gamma G_\Omega \rangle$ is dihedral; we can assume that $o(\alpha) = d$ and $o(\gamma) = 2$. Such a group G actually exists in $\mathcal{M}_q$, as shown in the proof of Proposition 3.1 using the model (2) of $\mathcal{H}_q$. The nontrivial elements of G are as follows: $\gcd(d, 2) \cdot \omega - 1$ elements of type (A) and center P in G_Ω or $\langle \iota \rangle \times G_\Omega$, according to d odd or d even; d elements of type (A) with center on ℓ , in $G_\pm$; $(d - \gcd(d, 2))\omega$ elements of type (B2) in $\langle \alpha \rangle \times G_\Omega$; $d(\omega-1)$ elements of type (B1) as the product of an involution in $G_\pm \setminus \langle \iota \rangle$ by an element of G_Ω . The claim follows by direct computation with Theorem 2.5. $\square$

Proposition 3.14. *Let $G \leq \mathcal{M}_q$ be such that $G/G_\Omega \cong \hat{D}ic_m$, where $m \mid \frac{q-1}{2}$ and $m \nmid \frac{q-1}{4}$. Then*

$$g(\mathcal{H}_q/G) = 1 + \frac{q^2 - q - 2 - [(2m + 2\omega - 1)(q + 1) + 4\omega(3m - 1)]}{16m\omega}.$$

Proof. By Remark 3.2, $G = G_\pm \times G_\Omega$ with $G_\pm \cong \hat{D}ic_m$. From the proof of Proposition 3.1, the nontrivial elements $G_\pm$ are exactly: 1 involution ι ; $2m$ other involutions of type (A) with center on ℓ ; $6m-2$ elements of type (B2). The nontrivial elements of G_Ω are of type (A). The product $\sigma\tau$ with $\sigma \in G_\pm$ and $\tau \in G_\Omega \setminus \{id\}$ is as follows: of type (A), if $\sigma = \iota$; of type (B1), if σ is an involution different from σ ; of type (B2), if σ is of type (B2). The claim follows by direct computation with Theorem 2.5. $\square$

Proposition 3.15. *Let $G \leq \mathcal{M}_q$ be such that $G/G_\Omega \cong \text{SU}^\pm(2, p^k) \cong \text{SL}(2, p^k) \rtimes C_2$, where $k \mid n$ and n/k is odd. Then*

$$g(\mathcal{H}_q/G) = 1 + \frac{q^2 - q - 2 - \Delta}{4p^k(p^{2k} - 1)\omega},$$

where

$$\begin{aligned} \Delta = & (q+1) + p^k(p^k+1)(p^k-3) + (p^{2k}-1)(q+3) + p^k(p^k-1)(q+1) + p^k(p^{2k}-1) + (2\omega-2)(q+1) \\ & + 2(p^{2k}-1)(\omega-1) + 2p^k(p^k+1)(p^k-2)(\omega-1) + 2p^k(p^k-1)(q+1)(\gcd(p^k+1, \omega)-1). \end{aligned}$$

Proof. By Remark 3.2 and Proposition 3.1, $G = G_\pm \times G_\Omega$ with $G_\pm \cong \text{SU}^\pm(2, p^k)$ and $G_H \cong \text{SL}(2, p^k)$.

The nontrivial elements of G_H are classified according to their type in the proof of [32, Proposition 4.3]. Namely, G_H contains exactly: 1 element ι of type (A); $\frac{p^k(p^k-1)^2}{2}$ elements of type (B1), forming $\frac{p^k(p^k-1)}{2}$ cyclic groups of order p^k+1 with pairwise intersection $\langle \iota \rangle$; $\frac{p^k(p^k+1)(p^k-3)}{2}$ elements of type (B2), forming $\binom{p^k+1}{2}$ cyclic groups of order p^k-1 with pairwise intersection $\langle \iota \rangle$; $p^{2k}-1$ elements of type (C), forming

$p^k + 1$ elementary abelian groups of order p^k with trivial pairwise intersection; $p^{2k} - 1$ elements of type (E), contained in $p^k + 1$ cyclic groups of order $2p^k$ with pairwise intersection $\langle \iota \rangle$.

The elements of $G_{\pm} \setminus G_H$ are classified as follows.

- $G_{\pm} \setminus G_H$ contains exactly $p^k(p^k - 1)$ elements of type (A) and $\frac{p^k(p^k - 1)^2}{2}$ elements of type (B1).
In fact, let $\alpha \in \mathcal{M}_q \setminus (\langle \iota \rangle \times \Omega)$ be of type (A) or (B1), and $\{Q, R\} \in \text{PG}(2, p^{2k}) \setminus \mathcal{H}_q$ be the fixed points of α on ℓ . Let $\mathcal{H}_q$ have equation (1); up to conjugation, $G_{\pm}$ is made by $\mathbb{F}_{p^{2k}}$ -rational elements, as pointed out in Equation (3). Also, up to conjugation, $P = (0, 0, 1)$, $Q = (1, 0, 0)$, and $R = (0, 1, 0)$, so that $\alpha = \text{diag}(\lambda, \mu, 1)$ with $\lambda^{o(\alpha)} = \mu^{o(\alpha)} = 1$. For any $(p^k + 1)$ -th root of unity λ , $\alpha \in G_{\pm} \setminus G_H$ if and only if $\det(\alpha) = -1$, i.e. $\mu = -\lambda^{-1}$. Note that $-\lambda^{-1} \neq \lambda$ since $4 \nmid (p^k + 1)$. Hence, after the choice of $\{Q, R\}$, there are exactly 2 elements of type (A) (namely, $\alpha = \text{diag}(1, -1, 1)$ and $\alpha = \text{diag}(-1, 1, 1)$) and $p^k - 1$ elements of type (B1) (namely, $\alpha = \text{diag}(\lambda, -\lambda^{-1}, 1)$ with $\lambda \neq \pm 1$).
- There are exactly $\frac{p^{2k+1} - (p^k + 1)}{2} = \frac{p^k(p^k - 1)}{2}$ choices for $\{Q, R\}$. In fact, Q can be chosen as anyone of the $\mathbb{F}_{p^{2k}}$ -rational points of ℓ which are not on $\mathcal{H}_q$; then R is uniquely determined as the pole of the line PQ with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$.
- As $p \nmid [G : G_H]$, there are no p -elements in $G \setminus G_H$. Also, any element of type (E) in $\mathcal{M}_q$ is the product $\sigma\tau$ of a p -element of type (C) by an element τ of type (A) in $\langle \iota \rangle \times \Omega$ (i.e., τ has center P). Since $G_{\pm} \cap (\langle \iota \rangle \times \Omega) = \langle \iota \rangle$, there are no elements of type (E) in $G_{\pm} \setminus G_H$.
- Any other element of $G_{\pm} \setminus G_H$ is of type (B2); their number is

$$p^k(p^{2k} - 1) - p^k(p^k - 1) - \frac{p^k(p^k - 1)^2}{2} = \frac{p^k(p^k - 1)(2p^k + 2 - 2 - p^k + 1)}{2} = \frac{p^k(p^{2k} - 1)}{2}.$$

The nontrivial elements of G_{Ω} are of type (A). The elements $\sigma\tau$ with $\sigma \in G_{\pm} \setminus \{id\}$ and $\tau \in G_{\Omega} \setminus \{id\}$ are classified as follows.

- If σ is of type (C) or (E), then $\sigma\tau$ is of type (E).
- If $\sigma = \iota$, then $\sigma\tau$ is of type (A).
- If σ is an involution different from ι , then $\sigma\tau$ is of type (B1).
- If σ is of type (B2), then $\sigma\tau$ is of type (B2).
- Let σ be of type (B1) and $\{Q, R\}$ be the points fixed by σ on ℓ ; we have $\frac{p^k(p^k - 1)}{2}$ choices for $\{Q, R\}$. Arguing as above, we can use the model (1) for $\mathcal{H}_q$, assume that $\{P, Q, R\}$ is the fundamental triangle, and that $\sigma = \text{diag}(\lambda, \lambda^{-1}, 1)$ or $\sigma = \text{diag}(\lambda, -\lambda^{-1}, 1)$, with $\lambda^{p^k+1} = 1$, $\lambda \neq \pm 1$. If $o(\lambda) \mid \omega$, then there exists exactly one $\tau \in G_{\Omega} \setminus \{id\}$ such that $\sigma\tau$ is of type (A); otherwise, $\sigma\tau$ is of type (B1). Altogether, when σ ranges over the elements of type (B1), there are exactly $\frac{p^k(p^k - 1)}{2} \cdot (\gcd(\omega, p^k + 1) - 1) \cdot 4$ elements $\sigma\tau$ of type (A); the other ones are of type (B1).

Now the claim follows by direct computation with Theorem 2.5. $\square$

Remark 3.16. *The results of Section 3 provide new genera for quotients of the Hermitian curve, and more generally new genera for maximal curves. For instance, the integers 60, 146, 2148, 2160, 5460, 10920 are genera of $\mathbb{F}_{729^2}$ -maximal curves which can be obtained respectively from Propositions 3.10, 3.15, 3.4, 3.9, 3.10, 3.9; up to our knowledge, they are new in the literature as elements of the spectrum of genera of $\mathbb{F}_{729^2}$ -maximal curves.*

4. THE COMPLETE LIST OF GENERA OF QUOTIENTS OF $\mathcal{H}_q$ FOR $q \equiv 1 \pmod{4}$

This section provides the explicit complete list of genera of quotients $\mathcal{H}_q/G$, $G \leq \text{PGU}(3, q)$ for $q = p^n \equiv 1 \pmod{4}$, and hence it gives a collection of the results obtained in this paper together with the results already obtained in the literature. It will be divided into subsections corresponding to the maximal subgroup of $\text{PGU}(3, q)$ containing G . The case in which $G \leq \text{PGU}(3, q)_Q$, $Q \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$ will not be repeated here as it was already described in the previous sections.

4.1. $G \leq \text{PGU}(3, q)_P$, $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$.

Let $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ and suppose that $G \leq \text{PGU}(3, q)_P$, the stabilizer of P in $\text{Aut}(\mathcal{H}_q) = \text{PGU}(3, q)$. Since $\text{PGU}(3, q)$ acts transitively on $\mathcal{H}_q(\mathbb{F}_{q^2})$ we can assume that $P = P_\infty$, that is, the unique point at infinity of the model $y^{q+1} = x^q + x$ of $\mathcal{H}_q$. The complete list for odd values of q of genera of quotients $\mathcal{H}_q/G$ where $G \leq \text{PGU}(3, q)_{P_\infty}$ was determined in [2] and partially in [18].

In these papers, an element $\sigma \in \text{PGU}(3, q)_{P_\infty}$ is uniquely described as a triple of elements in $\mathbb{F}_{q^2}$, $\sigma = [a, b, c]$. The authors associate to G three sets, namely

$$G_1 = \{a \mid [a, b, c] \in G\}, \quad G_2 = \{b \mid [1, b, c] \in G\}, \quad G_3 = \{c \mid [1, 0, c] \in G\}$$

with $|G_1| = g_1$, $|G_2| = p^{g_2}$, and $|G_3| = p^{g_3}$.

In [18] it is proved that the genus of the quotient $\mathcal{H}_q/G$ depends just on the values g_i . Namely, the following theorem is proved.

Theorem 4.1. (see [18] and [2, Lemma 4.1]) *Let $G \leq \text{PGU}(3, q)_{P_\infty}$, G_i , and g_i be defined as above, and let $d = \gcd(g_1, q + 1)$. Then*

$$(10) \quad g(\mathcal{H}_q/G) = \frac{q - p^{g_3}}{2|G|} (q - (d - 1)p^{g_2}).$$

Defining r and u to be the smallest integers such that $p^r \equiv 1 \pmod{g_1}$ and $p^u \equiv 1 \pmod{g_1/d}$ respectively, then $|G| = h_1 p^{g_2 r + g_3 u}$.

At this point the authors provide necessary and sufficient conditions to the triple (g_1, g_2, g_3) to be associated with an existing subgroup G of $\text{PGU}(3, q)_{P_\infty}$, so that Theorem 4.1 gives the complete list of genera when G is in $\text{PGU}(3, q)_P$ for some $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$. The authors observed that when $g_1 \mid (q^2 - 1)$ is fixed it is sufficient to consider the cases $0 \leq g_2 \leq 2n/r$ and $0 \leq g_3 < n/u$ as otherwise $\mathcal{H}_q/G$ would be rational.

Theorem 4.2. ([2, Theorems 4.3 and 5.8]) *Define r and u as above. Let $M_G(p, n) = \{(g_1, g_2, g_3) : g_1 \mid (q^2 - 1), 0 \leq g_2 \leq 2n/r, \text{ and } 0 \leq g_3 < n/u \text{ for some } G \leq \text{PGU}(3, q)_{P_\infty}\}$. Then*

- If $g_1 \mid (q - 1)$ then for every $0 \leq g_2 \leq n/r$ and $0 \leq i < n/u$, $(g_1, g_2, i) \in M_G(p, n)$.
- If $g_1 \mid (q^2 - 1)$ but $g_1 \nmid (q - 1)$ then for every $0 \leq i < n/u$, $(g_1, 0, i) \in M_G(p, n)$.
- If $g_1 \mid (q^2 - 1)$ but $g_1 \nmid (q - 1)$ then for every $0 < g_2 \leq 2n/r$ and $\mu_{g_2} \leq i < n/u$, $(g_1, g_2, i) \in M_G(p, n)$, where

$$\mu_{g_2} = \frac{r}{2u} \min_{\ell \mid (2n/r)} \left(2 \left\lceil \frac{g_2}{\ell} \right\rceil - 1 \right) \ell.$$

Moreover, this is the complete list of elements in $M_G(p, n)$.

4.2. $G \leq \text{PGU}(3, q)_T$, T a self-polar triangle in $\text{PG}(2, q^2) \setminus \mathcal{H}_q$.

Let $T = \{P_1, P_2, P_3\}$ be a self-polar triangle in $\text{PG}(2, q^2) \setminus \mathcal{H}_q$ and $\text{PGU}(3, q)_T$ be the maximal subgroup of $\text{PGU}(3, q)$ stabilizing T . We have $\text{PGU}(3, q)_T = (C_{q+1} \times C_{q+1}) \rtimes \mathbf{S}_3$, where $C_{q+1} \times C_{q+1}$ stabilizes T pointwise and $\mathbf{S}_3$ acts faithfully on T .

The genera of quotients $\mathcal{H}_q/G$ where $G \leq \text{PGU}(3, q)_T$ are completely classified in [8] as follows according to the action of G in T . We define $G_T = G \cap (C_{q+1} \times C_{q+1})$.

Theorem 4.3. ([8, Theorem 3.1]) *Let $q + 1 = \prod_{i=1}^\ell p_i^{r_i}$ be the prime factorization of $q + 1$.*

- (i) *For any divisors $a = \prod_{i=1}^\ell p_i^{s_i}$ and $b = \prod_{i=1}^\ell p_i^{t_i}$ of $q + 1$ ($0 \leq s_i, t_i \leq r_i$), let $c = \prod_{i=1}^\ell p_i^{u_i}$ be such that, for all $i = 1, \dots, \ell$, we have $u_i = \min\{s_i, t_i\}$ if $s_i \neq t_i$, and $s_i \leq u_i \leq r_i$ if $s_i = t_i$. Define $d = a + b + c - 3$. Let $e = \frac{abc}{\gcd(a, b)} \cdot \prod_{i=1}^\ell p_i^{v_i}$, where for all i 's v_i satisfies $0 \leq v_i \leq r_i - \max\{s_i, t_i, u_i\}$.*

We also require that, if $p_i = 2$ and either $2 \nmid abc$ or $2 \mid \gcd(a, b, c)$, then $v_i = 0$. Then there exists a subgroup G of $C_{q+1} \times C_{q+1}$ such that

$$(11) \quad g(\mathcal{H}_q/G) = \frac{(q+1)(q-2-d) + 2e}{2e}.$$

- (ii) Conversely, if $G \leq C_{q+1} \times C_{q+1}$, then the genus of $\mathcal{H}_q/G$ is given by Equation (11), where $e = |G|$ and d is the number of elements of type (A) in G ; d, e satisfy the numerical assumptions in point (i), for some a, b, c .

Proposition 4.4. ([8, Proposition 3.4]) *Let q be odd.*

- (i) *Let ℓ, a, c , and e be positive integers satisfying $e \mid (q+1)^2$, $c \mid (q+1)$, $\ell \mid c$, $a \mid c$, $ac \mid e$, $\frac{e}{a} \mid (q+1)$, and $\gcd(\frac{e}{ac}, \frac{c}{a}) = 1$. If $2 \mid a$ or $2 \nmid c$, we also require that $2 \nmid \frac{e}{ac}$. Then there exists a subgroup $G \leq (C_{q+1} \times C_{q+1}) \rtimes \mathbf{S}_3$ of order $2e$ such that $|G \cap (C_{q+1} \times C_{q+1})| = e$ and*

$$(12) \quad g(\mathcal{H}_q/G) = \frac{(q+1)(q-2a-c+1-h) - 2k + 4e}{4e},$$

where

$$(h, k) = \begin{cases} (\frac{e}{c}, \frac{e}{2}) & \text{if } 2a \nmid (q+1); \\ (\frac{e}{c}, 0) & \text{if } 2a \mid (q+1), 2a \nmid c; \\ (0, e) & \text{if } 2a \mid c, 2\ell \nmid (q+1); \\ (0, 0) & \text{if } 2a \mid c, 2\ell \mid (q+1), 2\ell \nmid c; \\ (\frac{2e}{c}, 0) & \text{if } 2a \mid c, 2\ell \mid c. \end{cases}$$

- (ii) *Conversely, if $G \leq (C_{q+1} \times C_{q+1}) \rtimes \mathbf{S}_3$ and $G \cap (C_{q+1} \times C_{q+1})$ has index 2 in G , then the genus of $\mathcal{H}_q/G$ is given by Equation (12), where: $e = |G|/2$; without loss of generality, $a-1$ is the number of homologies in G with center P_1 which is equal to the number of homologies in G with center P_2 , and $c-1$ is the number of homologies in G with center P_3 ; $\ell = \frac{\alpha(\beta)}{2}$ for some $\beta \in G \setminus G_T$; ℓ, a, c, e satisfy the numerical assumptions in point (i).*

Proposition 4.5. ([8, Proposition 3.5]) *Let q be such that $3 \nmid (q+1)$.*

- (i) *Let a and e be positive integers satisfying $e \mid (q+1)^2$, $a^2 \mid e$, $\frac{e}{a} \mid (q+1)$, and $2 \nmid \frac{e}{a^2}$. We also require that there exist two positive integers r and m such that $m \leq r$, $\frac{e}{a^2} \mid \frac{r}{\gcd(r, m)}$, $r \mid \frac{e}{a}$, and $r \mid (m^2 - m + 1)$. Then there exists a subgroup $G \leq (C_{q+1} \times C_{q+1}) \rtimes \mathbf{S}_3$ of order $3e$ such that $|G \cap (C_{q+1} \times C_{q+1})| = e$ and*

$$(13) \quad g(\mathcal{H}_q/G) = \frac{(q+1)(q-3a+1) + 2e}{6e}.$$

- (ii) *Conversely, if $G \leq (C_{q+1} \times C_{q+1}) \rtimes \mathbf{S}_3$ and $G \cap (C_{q+1} \times C_{q+1})$ has index 3 in G , then the genus of $\mathcal{H}_q/G$ is given by Equation (13), where: $e = |G|/3$; the number of homologies in G with center P_i is $a-1$ for $i = 1, 2, 3$; there exists m such that a, e, m satisfy the numerical assumptions in point (i).*

Proposition 4.6. ([8, Proposition 3.6]) *Let q be such that $3 \mid (q+1)$.*

- (i) *Let a, e , and ℓ be positive integers satisfying $e \mid (q+1)^2$, $a^2 \mid e$, $\frac{e}{a} \mid (q+1)$, $2 \nmid \frac{e}{a^2}$, and $\ell \mid (q+1)$. We also require that there exist two positive integers r and m such that $m \leq r$, $\frac{e}{a^2} \mid \frac{r}{\gcd(r, m)}$, $r \mid \frac{e}{a}$, and $r \mid (m^2 - m + 1)$. Then there exists a subgroup $G \leq (C_{q+1} \times C_{q+1}) \rtimes \mathbf{S}_3$ of order $3e$ such that $|G \cap (C_{q+1} \times C_{q+1})| = e$ and*

$$(14) \quad g(\mathcal{H}_q/G) = \frac{(q+1)(q-3a+1) + h \cdot e}{6e}, \quad \text{with} \quad h = \begin{cases} 2 & \text{if } a \nmid \frac{q+1}{3}, \\ 0 & \text{if } a \mid \frac{q+1}{3}, \ell \nmid \frac{q+1}{3}, \\ 6 & \text{if } a \mid \frac{q+1}{3}, \ell \mid \frac{q+1}{3}. \end{cases}$$

- (ii) Conversely, if $G \leq (C_{q+1} \times C_{q+1}) \rtimes \mathbf{S}_3$ and $G \cap (C_{q+1} \times C_{q+1})$ has index 3 in G , then the genus of $\mathcal{H}_q/G$ is given by Equation (14), where: $e = |G|/3$; the number of homologies in G with center P_i is $a - 1$ for $i = 1, 2, 3$; there exist ℓ and m such that a, e, ℓ, m satisfy the numerical assumptions in point (i).

Proposition 4.7. ([8, Proposition 3.7])

- (i) Let a be a divisor of $q+1$. We choose $e = a^2$ if $3 \nmid (q+1)$ or $3 \mid a$; $e \in \{a^2, 3a^2\}$ if $3 \mid (q+1)$ and $3 \nmid a$. Then there exists a subgroup $G \leq (C_{q+1} \times C_{q+1}) \rtimes \mathbf{S}_3$ of order $6e$ such that $|G \cap (C_{q+1} \times C_{q+1})| = e$ and

$$(15) \quad g(\mathcal{H}_q/G) = \frac{(q+1)(q-3a+1-\frac{3e}{a})-2r-3s-t+12e}{12e},$$

where

$$r = \begin{cases} \frac{7e}{2} & \text{if } q \equiv 0 \text{ or } 1 \pmod{3}, q \text{ is odd, } a \nmid \frac{q+1}{2}, \\ \frac{3e}{2} & \text{if } q \equiv 2 \pmod{3} \text{ } q \text{ is odd, } a \nmid \frac{q+1}{2}, \\ 2e & \text{if } q \equiv 0 \text{ or } 1 \pmod{3} \text{ either } q \text{ is even or } a \mid \frac{q+1}{2}, \\ 0 & \text{if } q \equiv 2 \pmod{3} \text{ either } q \text{ is even or } a \mid \frac{q+1}{2}; \end{cases}$$

$$s = \begin{cases} \frac{4e}{3} & \text{if } q \equiv 2 \pmod{3} \text{ and } a \nmid \frac{q+1}{3}, \\ 0 & \text{otherwise;} \end{cases} \quad t = \begin{cases} 0 & \text{if } q \text{ is odd,} \\ 3e & \text{if } q \text{ is even.} \end{cases}$$

- (ii) Conversely, if $G \leq (C_{q+1} \times C_{q+1}) \rtimes \mathbf{S}_3$ and $G \cap (C_{q+1} \times C_{q+1})$ has index 6 in G , then the genus of $\mathcal{H}_q/G$ is given by Equation (15), where: $e = |G|/6$; the number of homologies in G with center P_i is $a - 1$ for $i = 1, 2, 3$; a and e satisfy the numerical assumptions in point (i).

4.3. $G \leq \text{PGU}(3, q)_T$, T a triangle in $\mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$.

Let $T = \{P_1, P_2, P_3\}$ be a triangle in $\mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$ which is invariant under the Frobenius collineation Φ_{q^2} of $\text{PG}(2, q^6)$, and $\text{PGU}(3, q)_T$ be the maximal subgroup of $\text{PGU}(3, q)$ stabilizing T . We have $\text{PGU}(3, q)_T = S \rtimes C_3$, where $S \cong C_{q^2-q+1}$ is a Singer subgroup stabilizing T pointwise and acting semiregularly on $\text{PG}(2, q^2)$; C_3 acts faithfully on T .

The genera of quotients $\mathcal{H}_q/G$ where $G \leq \text{PGU}(3, q)_T$ are completely classified in [6] whenever $p \nmid |G|$. The case $p \mid |G|$ happens only if $p = 3$. This case was not considered in [6], and can be dealt with using Theorem 2.5 and the fact that each element in $G \setminus G_T$ has order 3.

Theorem 4.8. ([6, Proposition 4.2])

- (i) Let $\nu \mid (q^2 - q + 1)$. Then there exists $G \leq \text{PGU}(3, q)_T$ such that $|G_T| = \nu$ and one of the following holds:

$$(16) \quad g(\mathcal{H}_q/G) = \begin{cases} \frac{1}{2} \left(\frac{q^2-q+1}{\nu} - 1 \right); \\ \frac{q^2-q+1-\nu}{6\nu}, & q \equiv 2 \pmod{3} \text{ and } 3 \mid \nu, \text{ or } q \equiv 0, 1 \pmod{3}; \\ \frac{q^2-q+1-3\nu}{6\nu}, & q \equiv 2 \pmod{3} \text{ and } 3 \nmid \nu; \\ \frac{q^2-q+1+3\nu}{6\nu}, & q \equiv 2 \pmod{3} \text{ and } 3 \nmid \nu. \end{cases}$$

When $q \equiv 2 \pmod{3}$ and $3 \nmid \nu$, then both the third and the fourth line in Equation (16) are obtained for some $G \leq \text{PGU}(3, q)$.

- (ii) Conversely, let $G \leq \text{PGU}(3, q)_T$. If $G = G_T$, then $g(\mathcal{H}_q/G)$ is given by the first line in Equation (16) with $\nu = |G|$. If $G \neq G_T$, then $g(\mathcal{H}_q/G)$ is given by the second or third or fourth line in Equation (16) with $\nu = |G|/3$.

4.4. $G \leq \text{PGU}(3, q)$ has no fixed points or triangles.

The genus of quotient $\mathcal{H}_q/G$ with $G \leq \text{PGU}(3, q)$ was computed in [33] whenever G has no fixed points or triangles. The equations of this section describe the genera of such quotients $\mathcal{H}_q/G$.

Theorem 4.9. ([33]) *For any integer $\bar{g}$ provided by one of the Equations (17) to (27), there exists $G \leq \text{PGU}(3, q)$ such that $g(\mathcal{H}_q/G) = \bar{g}$ and G has no fixed points or triangle.*

Conversely, if $G \leq \text{PGU}(3, q)$ has no fixed points or triangles, then $g(\mathcal{H}_q/G)$ is given by one of the Equations (17) to (27).

$$(17) \quad \frac{q^2 - 34q - 48\gamma - 54\delta}{432}, \quad \frac{q^2 - 10q + 133 - 54\delta}{144}, \quad \frac{q^2 - 10q + 61 - 18\delta}{72},$$

$$\text{with } \gamma = \begin{cases} 0, & \text{if } 9 \mid (q+1), \\ 3, & \text{if } 9 \nmid (q+1), \end{cases} \quad \delta = \begin{cases} 0, & \text{if } 4 \mid (q+1), \\ 2, & \text{if } 4 \mid (q-1), \end{cases}$$

when q is odd and $3 \mid (q+1)$, where $G \cong \text{PGU}(3, 2)$, $G \cong \text{PSU}(3, 2)$, $G \cong \text{SmallGroup}(36, 9)$, respectively.

$$(18) \quad \frac{q^2 - 16q + 103 - 24\gamma - 20\delta}{120}, \quad \text{when } q \text{ is odd; } p = 5 \text{ or } 5 \mid (q^2 - 1); \quad G \cong \mathbf{A}_5;$$

$$\delta = \begin{cases} 2, & \text{if either } p = 3 \text{ or } 3 \mid (q-1), \\ 0, & \text{if } 3 \mid (q+1); \end{cases} \quad \text{and } \gamma = \begin{cases} 0, & \text{if } 5 \mid (q+1), \\ 2, & \text{if } p = 5 \text{ or } 5 \mid (q-1). \end{cases}$$

$$(19) \quad \frac{q^2 - q - 2 - \Delta}{\bar{q}(\bar{q}+1)(\bar{q}-1)} + 1,$$

where q is odd, $q = \bar{q}^h$, $\bar{q} \neq 3$, $G \cong \text{PSL}(2, \bar{q})$, and

- $\Delta = +2(\bar{q} - 2)(\bar{q} + 1) + 2\frac{\bar{q}(\bar{q}+1)}{2}\left(\frac{\bar{q}-1}{2} - 2\right) + \frac{\bar{q}(\bar{q}+1)}{2}(q+1) + \delta\frac{\bar{q}(\bar{q}-1)}{2}\left(\frac{\bar{q}+1}{2} - 1\right)$, if $\bar{q} \equiv 1 \pmod{4}$,
- $\Delta = +2(\bar{q} - 1)(\bar{q} + 1) + 2\frac{\bar{q}(\bar{q}+1)}{2}\left(\frac{\bar{q}-1}{2} - 1\right) + \frac{\bar{q}(\bar{q}+1)}{2}(q+1) + \delta\frac{\bar{q}(\bar{q}-1)}{2}\left(\frac{\bar{q}+1}{2} - 2\right)$, if $\bar{q} \equiv 3 \pmod{4}$,

$$\text{with } \delta = \begin{cases} 2, & \text{if } h \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

$$(20) \quad \frac{q^2 - q - 2 - \Delta}{2\bar{q}(\bar{q}+1)(\bar{q}-1)} + 1,$$

where q is odd, $q = \bar{q}^h$, $\bar{q} \neq 3$, $G \cong \text{PGL}(2, \bar{q})$, and

$$\Delta = 2(\bar{q} - 1)(\bar{q} + 1) + \frac{\bar{q}(\bar{q}+1)}{2}(q+1) + \frac{\bar{q}(\bar{q}-1)}{2}(q+1) + 2\frac{\bar{q}(\bar{q}+1)}{2}(\bar{q}-1-2) + \delta\frac{\bar{q}(\bar{q}-1)}{2}(\bar{q}+1-2),$$

$$\text{with } \delta = \begin{cases} 2, & \text{if } h \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

$$(21) \quad \frac{q^2 - 22q + 313 - 56\alpha - 48\beta - 42\gamma}{336}, \quad \text{when } p = 7 \text{ or } \sqrt{-7} \notin \mathbb{F}_q,$$

$$\text{where } \alpha = \begin{cases} 0, & \text{if } 3 \mid (q+1), \\ 2, & \text{otherwise;} \end{cases} \quad \beta = \begin{cases} 0, & \text{if } 7 \mid (q+1), \\ 3, & \text{if } 7 \mid (q^2 - q + 1), \\ 2, & \text{otherwise;} \end{cases} \quad \gamma = \begin{cases} 0, & \text{if } 4 \mid (q+1), \\ 2, & \text{otherwise.} \end{cases}$$

$$(22) \quad \frac{q^2 - 10q + 25}{72}, \quad \frac{q^2 - 16q + 55}{120}, \quad \frac{q^2 - 10q + 25}{144},$$

$$\frac{q^2 - 46q + 205}{720}, \quad \frac{q^2 - 46q + 205}{1440}, \quad \text{when } q = 5^n, n \text{ is odd,}$$

where $G \cong \text{SmallGroup}(36, 9)$, $G \cong A_5$, $G \cong \text{PSU}(3, 2)$, $G \cong A_6$, $G \cong \text{SmallGroup}(720, 765)$, respectively.

$$(23) \quad \frac{q^2 - 46q + 673 - 80\alpha - 90\beta - 144\gamma}{720}, \quad \frac{q^2 - 16q + 103 - 20\alpha - 24\gamma}{120}, \quad \frac{q^2 - 10q + 61 - 18\beta}{72},$$

when either $p = 3$ and n is even, or $\sqrt{5} \in \mathbb{F}_q$ and $\mathbb{F}_q$ contains no primitive cube roots of unity, with

$$\alpha = \begin{cases} 2, & \text{if } p = 3, \\ 0, & \text{otherwise,} \end{cases} \quad \beta = \begin{cases} 2, & \text{if } p = 3 \text{ or } q \equiv 1 \pmod{4}, \\ 0, & \text{otherwise,} \end{cases} \quad \gamma = \begin{cases} 0, & \text{if } 5 \mid (q + 1), \\ 2, & \text{otherwise,} \end{cases}$$

and $G \cong A_6$, $G \cong A_5$, $G \cong \text{SmallGroup}(36, 9)$, respectively.

$$(24) \quad \frac{q^2 - 106q + 2665 - 720\beta}{5040}, \quad \frac{q^2 - 46q + 205}{720}, \quad \frac{q^2 - 22q + 229 - 48\beta}{336},$$

$$(25) \quad \frac{q^2 - 26q + 105}{240}, \quad \frac{q^2 - 16q + 55}{120}, \quad \frac{q^2 - 10q + 25}{72},$$

where

$$q = 5^n, n \text{ is odd,} \quad \beta = \begin{cases} 0, & \text{if } 7 \mid (q + 1), \\ 3, & \text{otherwise,} \end{cases}$$

and $G \cong A_7$, $G \cong A_6$, $G \cong \text{PSL}(2, 7)$, $G \cong A_5 \rtimes C_2$, $G \cong A_5$, $G \cong \text{SmallGroup}(36, 9)$, respectively.

$$(26) \quad 1 + \frac{q^2 - q - 2 - \Delta}{2\bar{q}^3(\bar{q}^3 + 1)(\bar{q}^2 - 1)}, \quad \text{when } \bar{q} = p^k, k \mid n, n/k \text{ is odd, } G \cong \text{PGU}(3, p^k),$$

where

$$\Delta = (\bar{q} - 1)(\bar{q}^3 + 1) \cdot (q + 2) + (\bar{q}^3 - \bar{q})(\bar{q}^3 + 1) \cdot 2 + \bar{q}(\bar{q}^4 - \bar{q}^3 + \bar{q}^2) \cdot (q + 1)$$

$$+ (\bar{q}^2 - \bar{q} - 2) \frac{(\bar{q}^3 + 1)\bar{q}^3}{2} \cdot 2 + (\bar{q} - 1)\bar{q}(\bar{q}^3 + 1)\bar{q}^2 \cdot 1 + (\bar{q}^2 - \bar{q}) \frac{\bar{q}^6 + \bar{q}^5 - \bar{q}^4 - \bar{q}^3}{3} \cdot \gamma,$$

with

$$\gamma = \begin{cases} 3, & \text{if } (\bar{q}^2 - \bar{q} + 1) \mid (q^2 - q + 1) \text{ and } \bar{q} \neq 2, \\ 0, & \text{if } (\bar{q}^2 - \bar{q} + 1) \mid (q + 1) \text{ and } \bar{q} \neq 2, \\ 3, & \text{if } \bar{q} = 2 \text{ and } 3 \mid (q + 1) \text{ and } 3 \nmid n, \\ 0, & \text{if } \bar{q} = 2 \text{ and } 3 \mid (q + 1) \text{ and } 3 \mid n. \end{cases}$$

$$(27) \quad \frac{3(q^2 - q - 2 - \Delta)}{2\bar{q}^3(\bar{q}^2 - 1)(\bar{q}^3 + 1)} + 1, \quad \text{when } \bar{q} = p^k, k \mid n, n/k \text{ is odd, } 3 \mid (q + 1), G \cong \text{PSU}(3, p^k),$$

where

$$\Delta = (\bar{q} - 1)(\bar{q}^3 + 1) \cdot (q + 2) + (\bar{q}^3 - \bar{q})(\bar{q}^3 + 1) \cdot 2 + ((\bar{q} + 1)/3 - 1)(\bar{q}^4 - \bar{q}^3 + \bar{q}^2) \cdot (q + 1)$$

$$+ ((\bar{q}^2 - 1)/3 - (\bar{q} + 1)/3) \frac{(\bar{q}^3 + 1)\bar{q}^3}{2} \cdot 2 + (\bar{q} - 1)((\bar{q} + 1)/3 - 1)(\bar{q}^3 + 1)\bar{q}^2 \cdot 1 + ((\bar{q}^2 - \bar{q} + 1)/3 - 1) \frac{\bar{q}^6 + \bar{q}^5 - \bar{q}^4 - \bar{q}^3}{3} \cdot \delta,$$

with

$$\delta = \begin{cases} 3, & \text{if } (\bar{q}^2 - \bar{q} + 1)/3 \mid (q^2 - q + 1), \\ 0, & \text{if } (\bar{q}^2 - \bar{q} + 1)/3 \mid (q + 1). \end{cases}$$

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