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Marino, Giuseppe; Montanucci, Maria; Zullo, Ferdinando

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MRD-codes arising from the trinomial

\[ x^q + x^{q^3} + cx^{q^5} \in \mathbb{F}_{q^6}[x] \]

Giuseppe Marino, Maria Montanucci and Ferdinando Zullo

December 16, 2019

Abstract

In [10], the existence of \( \mathbb{F}_{q^6} \)-linear MRD-codes of \( \mathbb{F}_{q^6}^{6 \times 6} \), with dimension 12, minimum distance 5 and left idealiser isomorphic to \( \mathbb{F}_{q^6} \), defined by a trinomial of \( \mathbb{F}_{q^6}[x] \), when \( q \) is odd and \( q \equiv 0, \pm 1 \pmod{5} \), has been proved. In this paper we show that this family produces \( \mathbb{F}_{q^6} \)-linear MRD-codes of \( \mathbb{F}_{q^6}^{6 \times 6} \), with the same properties, also in the remaining \( q \) odd cases, but not in the \( q \) even case. These MRD-codes are not equivalent to the previously known MRD-codes. We also prove that the corresponding maximum scattered \( \mathbb{F}_{q^6} \)-linear sets of PG(1, \( q^6 \)) are not PΓL(2, \( q^6 \))-equivalent to any previously known linear set.

AMS subject classification: 51E20, 05B25, 51E22

Keywords: Scattered subspace, MRD-code, Linear set

1 Introduction and preliminary results

Let \( \text{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n}) := \text{Hom}_q(\mathbb{F}_{q^n}, \mathbb{F}_{q^n}) \) be the set of all \( \mathbb{F}_{q^n} \)-linear maps of \( \mathbb{F}_{q^n} \) in itself. It is well-known that each element \( f \) of \( \text{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \) can be represented in a unique way as a \( q \)-polynomial over \( \mathbb{F}_{q^n} \) of degree less than or equal to \( q^{n-1} \), that is \( f(x) = \sum_{i=0}^{n-1} a_i x^q^i \), with coefficients in \( \mathbb{F}_{q^n} \). Such polynomials are also called linearized. The set of \( q \)-polynomials over \( \mathbb{F}_{q^n} \), say \( \mathcal{L}_{n,q} \), considered modulo \( (x^{q^n} - x) \), and endowed with the addition and composition of polynomials in \( \mathbb{F}_{q^n} \) and scalar multiplication by elements in \( \mathbb{F}_q \), forms an
\(F_q\)-subalgebra of the algebra of \(F_q\)-linear transformations of \(F_q^n\). Hence, we can define the kernel of \(f\) as the kernel of the corresponding \(F_q\)-linear transformation of \(F_q^n\), which is the same as the set of roots of \(f\) in \(F_q^n\); and the rank of \(f\) as the rank of the corresponding \(F_q\)-linear transformation of \(F_q^n\).

For \(f \in \mathcal{L}_{n,q}\) with \(\deg f = q^k\), we call \(k\) the \(q\)-degree of \(f\) and we denote it by \(\deg_q f\). It is clear that in this case the kernel of \(f\) has dimension at most \(k\) and the rank of \(f\) is at least \(n - k\).

In [8], the \(q\)-polynomials \(f\) such that \(\dim_{F_q} \ker f = \deg_q f\) are called \(q\)-polynomials with maximum kernel. Also in [8, Theorem 1.2], sufficient and necessary conditions for the coefficients of a \(q\)-polynomial \(f\) over \(F_q^n\) ensuring \(f\) has maximum kernel are given (see also [28]).

The set \(F_{m \times n}^q\) of all \(m \times n\) matrices over \(F_q\) is a rank metric \(F_q\)-space with the rank metric or the rank distance defined by

\[
d(A, B) = \text{rank}(A - B),
\]

for any \(A, B \in F_{m \times n}^q\). A subset \(C \subseteq F_{m \times n}^q\) with respect to the rank metric is usually called a rank-metric code or a rank-distance code (or RD-code for short). When \(C\) contains at least two elements, the minimum distance of \(C\) is given by

\[
d(C) = \min_{A, B \in C, A \neq B} \{d(A, B)\}.
\]

When \(C\) is an \(F_q\)-linear subspace of \(F_{m \times n}^q\), we say that \(C\) is an \(F_q\)-linear code and its dimension \(\dim_{F_q}(C)\) is defined to be the dimension of \(C\) as a subspace over \(F_q\). For any \(C \subseteq F_{m \times n}^q\) with \(d(C) = d\), it is well-known that

\[
\#C \leq q^{\max\{m, n\}(\min\{m, n\} - d + 1)},
\]

which is a Singleton like bound for the rank metric ([14]). When equality holds, we call \(C\) a maximum rank-distance (MRD for short) code.

In this paper we only consider \(F_q\)-linear RD and MRD-codes with \(m = n\).

Two \(F_q\)-linear rank-distance codes \(C_1\) and \(C_2\) in \(F_{n \times n}^q\) are equivalent if there exist \(A, B \in \text{GL}(n, q)\) and \(\rho \in \text{Aut}(F_q)\) such that \(C_2 = \{AM^\rho B : M \in C_1\}\).

In general, it is a difficult task to tell whether two given rank-distance codes are equivalent or not. The idealizers of an RD-code are useful invariants which may help us to distinguish them (see [23, 27, 17]). Given an \(F_q\)-linear rank-distance code \(C \subseteq F_{n \times n}^q\), following [23] its left and right idealisers are defined as

\[
L(C) = \{M \in F_{n \times n}^q : MC \in C \text{ for all } C \in C\},
\]

\[
R(C) = \{M \in F_{n \times n}^q : CM \in C \text{ for all } C \in C\}.
\]
and
\[ R(\mathcal{C}) = \{ M \in \mathbb{F}_q^{n \times n} : CM \in \mathcal{C} \text{ for all } C \in \mathcal{C} \}, \]
respectively.

The adjoint of an \( \mathbb{F}_q \)-linear RD-code \( \mathcal{C} \subseteq \mathbb{F}_q^{n \times n} \) is the \( \mathbb{F}_q \)-linear code
\[ C^\top := \{ C^T \in \mathbb{F}_q^{n \times n} : C \in \mathcal{C} \}, \]
where \((.)^T\) denotes the transpose operation. Note that the adjoint operation also preserves rank distance, implying that an \( \mathbb{F}_q \)-linear RD-code and its adjoint have the same minimum distance. Also \( L(\mathcal{C}) = R(\mathcal{C}^T) \) and \( R(\mathcal{C}) = L(\mathcal{C}) \) ([27, Prop. 4.2]).

The Delsarte dual code of an \( \mathbb{F}_q \)-linear code \( \mathcal{C} \subseteq \mathbb{F}_q^{n \times n} \) is
\[ \mathcal{C}^\perp := \{ M \in \mathbb{F}_q^{n \times n} : \text{Tr}(MN^T) = 0 \text{ for all } N \in \mathcal{C} \}, \]
where \((.)^T\) denotes the transpose operation. If \( \mathcal{C} \) is a linear MRD-code then \( \mathcal{C}^\perp \) is also a linear MRD-code as it was proved by Delsarte [14]. Also from [14], if \( \mathcal{C} \) is an \( \mathbb{F}_q \)-linear code \( \mathcal{C} \subseteq \mathbb{F}_q^{n \times n} \) with dimension \( k \) and minimum distance \( d \), then \( \mathcal{C}^\perp \) has dimension \( n(n-k) \) and minimum distance \( k+1 \).

It is well-known that two linear rank-distance codes are equivalent if and only if their adjoint codes (or their Delsarte duals) are equivalent.

Two MRD-codes in \( \mathbb{F}_q^{n \times n} \) with minimum distance \( n \) are equivalent if and only if the corresponding semifields are isotopic [22, Theorem 7]. In contrast, it appears to be difficult to obtain inequivalent MRD-codes in \( \mathbb{F}_q^{n \times n} \) with minimum distance strictly less than \( n \). So far, the known inequivalent MRD-codes in \( \mathbb{F}_q^{n \times n} \) of minimum distance strictly less than \( n \), can be divided into two types.

1. The first type of constructions consists of MRD-codes of minimum distance \( d \) for arbitrary \( 2 \leq d \leq n \).

   • The first construction of MRD-codes was given by Delsarte [14] and rediscovered independently by Gabidulin [16]. This construction was generalized by Kshevetskiy and Gabidulin [20] with the nowadays commonly called (generalized) Gabidulin codes. In 2016, Sheekey [32] found the so-called (generalized) twisted Gabidulin codes. They can be generalized into additive MRD-codes [29]. Very recently, by using skew polynomial rings Sheekey [33] proved that they can be further generalized into a quite large family and all the MRD-codes mentioned above can be obtained in this way.
• The non-additive family constructed by Otal and Özbudak in [30].
• The family appeared in [35].

2. The second type of constructions provides us MRD-codes of minimum distance \( d = n - 1 \).

• Non-linear MRD-codes by Cossidente, the second author and Pavese [13] which were later generalized by Durante and Siciliano [15].
• Linear MRD-codes associated with maximum scattered linear sets of \( PG(1, q^6) \) and \( PG(1, q^8) \) presented in [6] and [10].

Very recently, new MRD-codes of minimum distance \( d = n - 2 \) and \( n \in \{7, 8\} \) have been constructed in [5].

For the relationship between MRD-codes and other geometric objects such as linear sets and Segre varieties, we refer to [24] and also to [34].

Since the metric space \( \mathbb{F}_q^{n \times n} \) is isomorphic to the metric space \( \text{End}_{\mathbb{F}_q}(\mathbb{F}_q^n) \) with rank distance defined as \( d(f, g) := \text{rk}(f - g) \), taking into account the previous algebra isomorphism between \( \text{End}_{\mathbb{F}_q}(\mathbb{F}_q^n) \) and \( \mathcal{L}_{n,q} \), it is clear that each \( \mathbb{F}_q \)-linear RD-code \( C \) can be regarded as an \( \mathbb{F}_q \)-vector subspace of \( \mathcal{L}_{n,q} \). Hence, in terms of linearized polynomial, an RD-code of \( \mathbb{F}_q^{n \times n} \), with minimum distance \( d = \min \{ d(f, g) : f, g \in C, f \neq g \} \), also two given \( \mathbb{F}_q \)-linear MRD-codes \( C_1 \) and \( C_2 \) are equivalent if and only if there exist \( \varphi_1, \varphi_2 \in \mathcal{L}_{n,q} \) permuting \( \mathbb{F}_q^n \) and \( \rho \in \text{Aut}(\mathbb{F}_q) \) such that

\[
\varphi_1 \circ f^\rho \circ \varphi_2 \in C_2 \text{ for all } f \in C_1,
\]

where \( \circ \) stands for the composition of maps and \( f^\rho(x) = \sum a_i^\rho x^q^i \) for \( f(x) = \sum a_i x^q^i \). For a rank distance code \( C \) given by a set of linearized polynomials, its left and right idealisers can be written as

\[
L(C) = \{ \varphi \in \mathcal{L}_{n,q} : \varphi \circ f \in C \text{ for all } f \in C \},
\]

and

\[
R(C) = \{ \varphi \in \mathcal{L}_{n,q} : f \circ \varphi \in C \text{ for all } f \in C \},
\]

respectively.

Consider the non-degenerate symmetric bilinear form of \( \mathbb{F}_q^n \) over \( \mathbb{F}_q \) defined by \( \langle x, y \rangle := \text{Tr}_{q^n/q}(xy) \), for each \( x, y \in \mathbb{F}_q^n \). Then the adjoint \( \hat{f} \) of the linearized polynomial \( f(x) = \sum_{i=0}^{n-1} a_i x^q^i \in \mathcal{L}_{n,q} \) with respect to the bilinear
form $<,>$ is $\hat{f}(x) = \sum_{i=0}^{n-1} a_i x^{q^{n-i}}$. We will refer to $\hat{f}$ simply as the adjoint of $f$, omitting the bilinear form involved. Hence, we may define the adjoint of a rank distance code $C$ given by a set of linearized polynomials as follows $C^\top := \{\hat{f} : f \in C\}$.

In [10], the authors proved that the set $C = \langle x, x^q + x^q + cx^q^3 \rangle_{F_{q^6}}$, $q$ odd, $c^2 + c = 1$, $q \equiv 0, \pm 1 \pmod 5$ is an $F_q$-linear MRD-code of $L_{6,q}$ of dimension 12, minimum distance 5 and left idealiser isomorphic to $F_{q^6}$. The right idealiser of $C$ is isomorphic to $F_{q^2}$ ([36, Appendix B]). In this paper we further investigate the set $C$ with the same assumption $c^2 + c = 1$ and for each value of $q$ (odd and even), obtaining the following result.

**Theorem 1.1.** The set of $q$-polynomials of $L_{6,q}$

$$C = \langle x, x^q + x^q + cx^q^3 \rangle_{F_{q^6}},$$

with $c^2 + c = 1$, is an $F_q$-linear MRD-code of $L_{6,q}$ with dimension 12, minimum distance 5, left idealiser isomorphic to $F_{q^6}$ and right idealiser isomorphic to $F_{q^2}$, if and only if $q$ is odd. Moreover, when $q$ is odd and $q \equiv \pm 2 \pmod 5$, $C$ is not equivalent to the previously known MRD-codes.

Since both the adjoint and the Delsarte dual operations preserve the equivalence of MRD-codes, we have also that the MRD-codes presented in Theorem 1.1 are not equivalent neither to the adjoint nor to the Delsarte dual of any previously known MRD-code.

## 2 $F_q$-linear MRD-codes and maximum scattered $F_q$-subspaces

An $F_q$-subspace $U$ of rank $n$ of a 2-dimensional $F_{q^n}$-space $V$ is maximum scattered if it defines a scattered $F_q$-linear set of the projective line $\text{PG}(V, F_{q^n})$, i.e. $\dim_{F_q}(U \cap \langle \mathbf{v} \rangle_{F_{q^n}}) \leq 1$ for each $\mathbf{v} \in V \setminus \{0\}$. Let $V = \mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$, up to the action of the group $\text{GL}(2, q^n)$, an $F_q$-subspace $U$ of $V$ of rank $n$ can be written as $U = U_f = \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$, for some $f \in L_{n,q}$.

Sheekey in [32] made a breakthrough in the construction of new linear MRD-codes using linearized polynomials (see also [26]).

In [32], the author proved the following result (which have been generalized in [24, Section 2.7] and [34], see also [9, Result 4.7]).
Result 2.1. \( C \) is an \( \mathbb{F}_q \)-linear MRD-code of \( \mathcal{L}_{n,q} \) with minimum distance \( n - 1 \) and with left-idealiser isomorphic to \( \mathbb{F}_{q^n} \) if and only if up to equivalence

\[
C = \langle x, f(x) \rangle_{\mathbb{F}_{q^n}}
\]

for some \( f \in \mathcal{L}_{n,q} \) and the \( \mathbb{F}_q \)-subspace

\[
U_C = \{ (x, f(x)) : x \in \mathbb{F}_{q^n} \}
\]

is a maximum scattered \( \mathbb{F}_q \)-subspace of \( \mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \).

Also, two \( \mathbb{F}_q \)-linear MRD-codes \( C \) and \( C' \) of \( \mathcal{L}_{n,q} \), with minimum distance \( n - 1 \) and with left idealisers isomorphic to \( \mathbb{F}_{q^n} \), are equivalent if and only if \( U_C \) and \( U_{C'} \) are \( \Gamma \text{L}(2, q^n) \)-equivalent.

So far, the known non-equivalent (under \( \Gamma \text{L}(2, q^n) \)) maximum scattered \( \mathbb{F}_q \)-subspaces, yielding to the known non-equivalent \( \mathbb{F}_q \)-linear MRD-codes with left idealiser isomorphic to \( \mathbb{F}_{q^n} \), are

1. \( U_{s,n}^1 := \{ (x, x^{q^s}) : x \in \mathbb{F}_{q^n} \}, 1 \leq s \leq n - 1, \gcd(s, n) = 1, \) see [11] [12];
2. \( U_{s,\delta}^{2,n} := \{ (x, \delta x^{q^s} + x^{q^{n-s}}) : x \in \mathbb{F}_{q^n} \}, n \geq 4, N_{q^n/q}(\delta) \notin \{0, 1\} \) \[1\]
   \( \gcd(s, n) = 1, \) see [25] for \( s = 1, [32, 26] \) for \( s \neq 1; \)
3. \( U_{s,\delta}^{3,n} := \{ (x, \delta x^{q^s} + x^{q^{s+n/2}}) : x \in \mathbb{F}_{q^n} \}, n \in \{6, 8\}, \gcd(s, n/2) = 1, \)
   \( N_{q^n/q^{n/2}}(\delta) \notin \{0, 1\}, \) for the precise conditions on \( \delta \) and \( q \) see [6] Theorems 7.1 and 7.2 \[3\]
4. \( U_c^4 := \{ (x, x^q + x^{q^2} + cx^{q^5}) : x \in \mathbb{F}_{q^n} \}, q \text{ odd, } c^2 + c = 1, q \equiv 0, 1 \pmod{5}, \) see [10].

In this paper, we further investigate the \( \mathbb{F}_q \)-subspaces \( U_f \) arising from the trinomial

\[
f(x) = x^q + x^{q^2} + cx^{q^5} \in \mathbb{F}_{q^n}[x],
\]

with the same assumption \( c^2 + c = 1 \) and for each value of \( q \) (odd and even), showing that the \( \mathbb{F}_q \)-subspace \( U_f \) of \( \mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \) is maximum scattered also for \( q \) odd and \( q \equiv \pm 2 \pmod{5} \), whereas it is not scattered for \( q \) even.

To do this, as we will see in Section 4, studying the Delsarte dual of the code arising from \( U_f \) was successful.

---

1This condition implies \( q \neq 2. \)
2Also here \( q > 2, \) otherwise \( L_{s,\delta}^{3,n} \) is not scattered.
3 The Delsarte dual of an RD-code

In terms of linearized polynomials, the Delsarte dual of a rank distance code $C$ of $L_{n,q}$ can be interpreted as follows

$$C^\perp = \{ f \in L_{n,q} : b(f, g) = 0 \ \forall g \in C \},$$

where $b(f, g) = \text{Tr}_{q^n/q} \left( \sum_{i=0}^{n-1} a_i b_i \right)$ for $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$, $g(x) = \sum_{i=0}^{n-1} b_i x^{q^i} \in \mathbb{F}_{q^n}[x]$ and $\text{Tr}_{q^n/q}$ denotes the trace function from $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.

The following result has been proved in [14].

**Result 3.1.** Let $C$ be an $\mathbb{F}_q$-linear RD-code of $L_{n,q}$. Then $C$ is an $\mathbb{F}_q$-linear MRD-code if and only if $C^\perp$ is an $\mathbb{F}_q$-linear MRD-code.

Let us consider the set of $L_{6,q}$

$$C = \langle x, x^{q^2} + x^{q^3}, cx^{q^2} \rangle_{\mathbb{F}_{q^6}},$$

with $c^2 + c = 1$. By [10] Theorem 5.1 and Proposition 6.1, $C$ is an $\mathbb{F}_q$-linear MRD-code of $L_{6,q}$ with dimension 12, minimum distance 5, left idealiser isomorphic to $\mathbb{F}_{q^6}$ and right idealiser isomorphic to $\mathbb{F}_{q^2}$ when $q$ is odd and $q \equiv 0, \pm 1 \pmod{5}$.

In order to investigate the remaining cases, by Result [3.1] we can consider the Delsarte dual RD-code of $C$, which is equivalent to

$$D = \langle x^{q^2}, x^{q^3}, -x + x^{q^2}, cx^{q^2} - x^{q^4} \rangle_{\mathbb{F}_{q^6}}.$$

Our aim is now to establish under which conditions the RD-code

$$D = \langle x^{q^2}, x^{q^3}, -x + x^{q^2}, cx - x^{q^4} \rangle_{\mathbb{F}_{q^6}}$$

is an MRD-code$^3$

The $\mathbb{F}_q$-linear RD-code $D$ is an MRD if and only if for each nonzero element $f \in D$ we get $\dim_{\mathbb{F}_q} \ker f \leq 3$. Since the maximum $q$-degree of the polynomials in $D$ is 4 it suffices that do not exist $\alpha, \beta$ and $\gamma$ in $\mathbb{F}_{q^6}$ such that the kernel of

$$f(x) = \alpha x^{q^2} + \beta x^{q^3} + \gamma(-x + x^{q^2}) + cx - x^{q^4} =$$

$^3$We write $c$ instead of $c^q$, since $c^q$ satisfies $x^2 + x = 1$. 

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\[ = (\gamma + c)x + \alpha x^q + \gamma x^{q^2} + \beta x^{q^3} - x^{q^4} \]

has dimension 4. Taking into account the characterization of maximum kernel \( q \)-polynomials when \( k = 4 \) and \( n = 6 \) (cf. [8, Section 3.4]) we have the following result.

**Proposition 3.2.** The set of \( q \)-polynomials

\[ \mathcal{C} = \langle x, x^q + x^{q^3} + cx^{q^5} \rangle_{\mathbb{F}_{q^6}}, \]

with \( c^2 + c = 1 \) is an \( \mathbb{F}_q \)-linear MRD-code of \( \mathcal{L}_{6,q} \) with dimension 12, minimum distance 5 and left idealiser isomorphic to \( \mathbb{F}_{q^6} \), if and only if the system

\[
\begin{cases}
\alpha \neq 0 \\
(\gamma + c)^{q^5-1} = 1 \\
(\gamma + c)(-\gamma + c)^{q^4+q^2} = \beta^{q^5+q^4}(-\gamma + c)^{q^4+q^2} + \beta^{q^2+q^4} = 1 \\
\alpha = -(\gamma + c)^{q^4+1} \beta^{q^2} \\
\gamma = -(-\gamma + c)^{q^4+1} + \beta^{q^3+q^2}(-\gamma + c)^{q^2+q^4+1} \\
\beta = (\gamma + c)^{q^3+q^2+1} \beta^{q^4} + \beta^{q^2}(-\gamma + c)^{q^3+q^4+1} - \beta^{q^4+q^3+q^2}(-\gamma + c)^{q^4+q^2+q^4+1} 
\end{cases}
\]

has no solutions \( \alpha, \beta \) and \( \gamma \) in \( \mathbb{F}_{q^6} \).

In the next section we will study System (1) when \( q \) is odd, \( q \equiv \pm 2 \) (mod 5) and when \( q \) is even separately.

## 4 Proof of Theorem 1.1

In this section we prove the main theorem of the paper, showing that System (1) has solutions in \( \mathbb{F}_{q^6} \) if and only if \( q \) is even. By Result 3.1, taking [10, Theorem 5.1 and Proposition 6.1] into account, System (1) has no solutions \( \alpha, \beta \) and \( \gamma \) in \( \mathbb{F}_{q^6} \) when \( q \equiv 0, \pm 1 \) (mod 5). Hence, we have to investigate the remaining cases. The resultants presented in this section have been obtained by using the software package MAGMA [2].

### 4.1 The \( q \) odd case, \( q \equiv \pm 2 \) (mod 5)

From [19, Section 1.5 (xiv)] it follows that \( c \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \), and so \( c \) and \( c^q \) are the two distinct roots of \( x^2 + x - 1 \). Also \( c^{q+1} = c + c^q = -1 \)
Our aim now is to show that the system

\[
\begin{align*}
\left\{ \begin{array}{l}
\gamma = \gamma - q + c \\
\gamma = -(\gamma + c)q^{2+q^2} + \beta q^{q^2} - \gamma + c)
\end{array} \right. \\
\text{has no solutions in the variables } \gamma \text{ and } \beta \text{ over } \mathbb{F}_{q^6}, \text{ and as a consequence System (1) does not have solutions. It is straightforward to see that the previous system admits solutions if and only if the following system}
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
\gamma = -\gamma q^2 (q^2 + q^3) \\
\left( \frac{1}{\gamma + c} - \gamma q^2 \right)^{q^6 - 1} = 1
\end{array} \right.
\end{align*}
\]

admits \( \mathbb{F}_{q^6} \)-rational solutions in the variable \( \gamma \). Clearly, System (2) may be written as

\[
\begin{align*}
\left\{ \begin{array}{l}
(-\gamma q^2 (q^2 + q^3))^{q^2 + q^3} = 1 \\
\left( \frac{1}{\gamma + c} - \gamma q^2 \right)^{q^6 - 1} = 1
\end{array} \right.
\end{align*}
\]

The following preliminary result holds.

**Proposition 4.1.** If \( x \) is a solution of System (4), then

\[
x = \frac{2}{\lambda q^2 + q(c + 2) - \lambda q^2 c + c},
\]

for some \( \lambda \in \mathbb{F}_{q^3}^\ast \) such that \( \lambda q^2 + q^3 = 1 \).

**Proof.** By the second equation of (4) it follows that \( x \neq 0 \) and

\[
\frac{1}{-x + c} - x q^2
\]

is a \((q + 1)\)-th power in \( \mathbb{F}_{q^6}^\ast \). Therefore, there exists \( y \in \mathbb{F}_{q^6}^\ast \) such that

\[
y^{q+1} = \frac{1}{-x + c} - x q^2,
\]

and hence

\[
-x + c = \frac{1}{y^{q+1} + x q^2}.
\]
The first equation of (4) reads
\[
\left( -\frac{x^{q-1}}{y^{q+1} + x^{q^2}} \right)^{q^2+q+1} = 1.
\]

Hence, this is equivalent to the existence of \( \lambda \in \mathbb{F}_{q^3}^* \) with \( \lambda^{q^2+q+1} = 1 \) and
\[
-\lambda x^{q-1} = y^{q+1} + x^{q^2}.
\]

By Equations (5) and (6) we have
\[
-x^{q^2} - \lambda x^{q-1} = \frac{1}{-x + c} - x^{q^2},
\]
and hence
\[
\frac{1}{\lambda} \left( \frac{1}{x} \right)^q + c \frac{1}{x} - 1 = 0.
\]

Denoting \( T := \frac{1}{x} \), the above equation becomes
\[
T^q + c\lambda T - \lambda = 0. \tag{7}
\]

By [19, Theorem 1.22], since
\[
(-\lambda c)^{q^2+q^4+q^3+q^2+q+1} = \lambda^{(q^2+q+1)(q^3+1)c^3(q+1)} = -1 \neq 1,
\]
Equation (7) has one solution and it is
\[
T := \lambda(-\lambda c)^{q^2+q^4+q^3+q^2+q} + \lambda^{q^4+q^3+q^2} + \\
+ \lambda^{q^2}(-\lambda c)^{q^4+q^3+q} + \lambda^{q^4}(-\lambda c)^{q^3+q^2} + \lambda^{q^2}.
\]

Since \( \lambda \in \mathbb{F}_{q^3}^* \) and \( c^2 + c - 1 = 0 \) we have
\[
\bar{T} = \frac{c + \lambda^{q^2+q}(c + 2) - c\lambda^{q^2}}{2}
\]
and the assertion follows.

By way of contradiction, suppose that System (3) admits at least one solution in \( \gamma \), say \( x \). Hence \( x \) is as in Proposition 4.1 and the variables of System (4) are \( \lambda \) and \( c \). In particular the first equation becomes \( \lambda^{q^2+q+1} = 1 \).
Our aim is to show that a solution $\lambda$ of the new system obtained in this way does not exist.

Looking at each $q$-power of $\lambda$ as a distinct variable, we define

$$L := \lambda, M := \lambda^q, N := \lambda^{q^2}, C := c$$

and consider the Frobenius images $\lambda^{q^i}$, with $i = 0, 1, 2$ as variables in System (4). Hence, we have a weaker system, say $\Sigma$, in the variables $L, M, N, C$. We want to show that System $\Sigma$ has no solutions over $\mathbb{F}_{q^6}$. Denote by $EQ1$ and by $NEQ2$ the first equation and the numerator of the system $\Sigma$, respectively. Hence

$$EQ1 := LMN - 1 = 0.$$ 

We have that

$$\text{Resultant} (\text{Resultant} (NEQ2, C^2 + C - 1, C), EQ1, L) = 2^{14} N^8 \cdot M^{12} \cdot \text{COND1}^2 \cdot \text{COND2}^2 \cdot \text{COND3}^2,$$

where

$$\text{COND1} := M^2 N^2 - 2MN^2 - 4MN + N^2 + 4N - 1,$$

$$\text{COND2} := M^2 N^2 + 4MN^2 - 4MN - N^2 + 2N - 1,$$

and

$$\text{COND3} := M^2 N^2 + 4MN^2 - 2MN - N^2 - 4N + 1.$$ 

Hence, three cases occur.

- **COND1 = 0.**
  
  Let $Z := NM - N$, we get
  
  $$Z^2 - 4Z - 1 = 0,$$

  which implies $Z \in \mathbb{F}_q$ since $\lambda \in \mathbb{F}_{q^3}$. Therefore $Z - Z^q = 0$ and hence the following two resultants should be zero
  
  $$R1 := \text{Resultant} (Z - Z^q, EQ1, N) = 0$$

  and
  
  $$R2 := \text{Resultant} (\text{COND1}, EQ1, N) = 0.$$

  Also,
  
  $$\text{Resultant} (R1, R2, M) = 4L^2 (L^2 - L - 1) = 0,$$
i.e. $\lambda \in \mathbb{F}_{q^2} \cap \mathbb{F}_{q^3} = \mathbb{F}_q$, which implies $\lambda^3 = 1$. This means that either $\lambda = 1$ or $q \equiv 1 \pmod{3}$ and $\lambda^2 + \lambda + 1 = 0$. But both cases contradict condition $\lambda^2 - \lambda - 1 = 0$.

- COND2 = 0.
  In this case the following resultants should be zero

$$Q_1 := \text{Resultant}(\text{COND2}, EQ1, N) = 0$$

and

$$Q_2 := \text{Resultant}(\text{COND2}^q, EQ1, N) = 0.$$  

This implies that

$$\text{Resultant}(Q_1, Q_2, Lq) = 2^4 L^6 (L^2 + 5L - 5) = 0.$$  

Again $\lambda \in \mathbb{F}_q$ and we get a contradiction when $q \equiv \pm 2 \pmod{5}$.

- COND3 = 0.
  Then

$$S_1 := \text{Resultant}(\text{COND3}, EQ1, N) = 0$$

and

$$S_2 := \text{Resultant}(\text{COND3}, EQ1, N) = 0,$$

and then

$$\text{Resultant}(S_1, S_2, Lq) = 2^4 L^4 (5L^2 - 5L + 1) = 0,$$

again a contradiction.

The proof is now complete.

4.2 The $q$ even case

Differently from what happens in the case $q$ odd, we want to show that, when $q$ is even, System (1) admits at least a solution of type $(\alpha, \beta, 0) \in \mathbb{F}_q^3$, with $\alpha$ and $\beta$ not zero. Indeed, substituting the value $\gamma = 0$ in System (1) we get
\[
\begin{align*}
\alpha &\neq 0 \\
\frac{c^q}{q+1} &= 1 \\
c[q^4+q^2 + \beta q^2q^3 + q^2 + q^2] &= 1 \\
\alpha &= c^{q+1}\beta q^2 \\
0 &= c^{q+1} + \beta q^2q^3+q+1 \\
\beta &= c^{q+1}q+1 + \beta q^2q^3+q+1 + \beta q^3q^4q+q^2+q+1 \\
\end{align*}
\]

(8)

Note that the conditions on \(\alpha\) will be automatically satisfied once we define \(\alpha := c^{q+1}\beta q^2\) forcing \(\beta \neq 0\). Also, the second equation is trivially satisfied since \(c \in \mathbb{F}_4^*\) and \(c \in \mathbb{F}_{q2}\). Hence System (8) has solutions if and only if the following system admits solutions

\[
\begin{align*}
c[q^4+q^2 + \beta q^2q^3 + q^2 + q^2] &= 1 \\
1 &= \beta q^2q^3 \\
\beta &= c^{q+2}\beta q^4 + \beta q^2q^3+q+1 + \beta q^3q^4c^2q^3+q^2+q+1 \\
\end{align*}
\]

(9)

Since \(c^2 + c + 1 = 0\), then \(c \in \mathbb{F}_{q2}\) and \(c^3 = 1\). Then there exists \(\beta \in \mathbb{F}_{q^6}\) such that \(\beta^{q+1} = 1/c\). Hence \(\beta^{q^3+q^2} = (1/c^q)^q = 1/c^q\) and the second equation of (9) is satisfied. Also, the first equation of System (9) reads

\[
1 = c \left[ c^2 + \frac{1}{c}c^{q+2} + \frac{1}{c} \right],
\]

and hence it is fulfilled. At this point the third equation of (9) becomes

\[
1 = c^{q+2}\beta q^4+1 + c^{q+1}\beta q^3+1 + \beta q^4+q^2+q^3+q^2+1c^2q^3+q^2+q+1,
\]

and using that \(\beta^{q+1} = 1/c^q\) and \(c^3 = 1\), we get that it is satisfied.

4.3 The right idealiser of RD-codes of Theorem 1.1

Following the computations in [36, Appendix B]), we show that the right idealiser of the RD-codes presented in Theorem 1.1 is isomorphic to \(\mathbb{F}_{q2}\). Indeed, let \(\varphi(x)\) be an element of \(R(C)\). Since \(C\) contains the identity map \(\varphi(x) \in C\) and hence there exist \(\alpha, \beta \in \mathbb{F}_{q^6}\) such that \(\varphi(x) = \alpha x + \beta x^q + \beta x^{q^2} + \beta cx^q\). Also,

\[
(x^q + x^{q^2} + cx^q) \circ \varphi(x) = \varphi(x)^q + \varphi(x)^{q^2} + c\varphi(x)^{q^3} \in C
\]
implying the existence of $a, b \in \mathbb{F}_{q^6}$ such that
\[
\alpha q^3 x^q + \beta q^3 (x^q + x^{q^2} + c^q x^{q^3}) + \alpha q^4 x^{q^2} + \beta q^4 (x^{q^2} + x^{q^4} + c^{q^2} x^{q^5}) + c(\alpha q^5 x^{q^5} + \beta q^5 (x + x^{q^3} + c^{q^5} x^{q^4})) = ax + b(x^q + x^{q^3} + cx^{q^5}),
\]
which is a polynomial identity in $x$. By comparing the coefficients of terms of degree $q$ and $q^3$ we get $\alpha \in \mathbb{F}_{q^2}$, and by comparing the coefficients of the terms of degree $q^2$ and $q^4$, taking into account that $c \in \mathbb{F}_{q^2}$, we get
\[
\beta q^4 + c\beta q^3 + c\beta q^3 = 0 \text{ and } \beta q^4 + \beta q^3 + c^q + 1 \beta q^5 = 0.
\]
Subtracting the second equation to the first one, we get $(c^q - 1)(\beta q^3 - c\beta q^5) = 0$. Since $c \neq 1$, then $\beta q^4 = c^q \beta$ and this equation admits a nonzero solution $\beta \in \mathbb{F}_{q^6}$ if and only if $c^3 = 1$, contradicting the condition $c^2 + c - 1 = 0$.

### 4.4 The equivalence issue

We want to finish this part of the paper showing that the $\mathbb{F}_{q^6}$-linear MRD-codes of $L_{6,q}$ defined in Theorem 1.1 are not equivalent to the previously known MRD-codes.

From [6, Section 6] and [10, Theorem 6.1], the previously known $\mathbb{F}_{q^6}$-linear MRD-codes of $L_{6,q}$ with dimension 12, minimum distance 5 and left idealiser isomorphic to $\mathbb{F}_{q^6}$, up to equivalence, arise from one of the following maximum scattered subspaces of $\mathbb{F}_{q^6} \times \mathbb{F}_{q^6}$: $U^{1,6}_{s}, U^{2,6}_{s,\delta}, U^{3,6}_{s,\delta}$ and $U^{4}_{c}$. Also, from Result 2.1, two $\mathbb{F}_{q^6}$-linear MRD-codes $C$ and $C'$ of $L_{6,q}$, with minimum distance 5 and with left-idealisers isomorphic to $\mathbb{F}_{q^6}$, are equivalent if and only if $U_C$ and $U_{C'}$ are $\Gamma L(2, q^6)$-equivalent.

The stabilisers of the $\mathbb{F}_{q^6}$-subspaces above in the group $GL(2, q^6)$ were determined in [6, Sections 5 and 6] and in [10, Proposition 5.2]. They have the following orders:

1. for $U^{1,6}_{s}$ we have a group of order $q^6 - 1$,
2. for $U^{2,6}_{s,\delta}$ and $U^{4}_{c}$ we have a group of order $q^2 - 1$,
3. for $U^{3,6}_{s,\delta}$ we have a group of order $q^3 - 1$.

Also, since the $\Gamma L(2, q^6)$-equivalence preserves the order of such stabilisers and since the results of [10] Propositions 5.2 and 5.3 do not depend on the congruence of $q$ odd, using the same arguments we prove the last part of Theorem 1.1.
5 New maximum scattered $\mathbb{F}_q$-linear sets of $\text{PG}(1, q^6)$

A point set $L$ of a line $\Lambda = \text{PG}(W, \mathbb{F}_{q^n}) = \text{PG}(1, q^n)$ is said to be an $\mathbb{F}_q$-linear set of $\Lambda$ of rank $n$ if it is defined by the non-zero vectors of an $n$-dimensional $\mathbb{F}_q$-vector subspace $U$ of the two-dimensional $\mathbb{F}_{q^n}$-vector space $W$, i.e.

$$L = L_U := \{(u)_{\mathbb{F}_{q^n}} : u \in U \setminus \{0\}\}.$$ 

One of the most natural questions about linear sets is their equivalence. Two linear sets $L_U$ and $L_V$ of $\text{PG}(1, q^n)$ are said to be PΓL-equivalent (or simply equivalent) if there is an element in PΓL$(2, q^n)$ mapping $L_U$ to $L_V$. In the applications it is crucial to have methods to decide whether two linear sets are equivalent or not. This can be a difficult problem and some results in this direction can be found in [11, 3].

Linear sets of rank $n$ of $\text{PG}(1, q^n)$ have size at most $(q^n - 1)/(q - 1)$. A linear set $L_U$ of rank $n$ whose size achieves this bound is called maximum scattered. For applications of these objects we refer to [31] and [21].

To make notation easier, by $L_{1,n}$, $L_{s,n}$ and $L_{c,n}$ we will denote the $\mathbb{F}_q$-linear set defined by $U_{1,n}^i$, $U_{s,n}^i$ and $U_{c,n}^i$, respectively. The $\mathbb{F}_q$-linear sets PΓL$(2, q^n)$-equivalent to $L_{1,n}$ are called of pseudoregulus type. It is easy to see that $L_{1,n}^i = L_{s,n}^i$ for any $s$ with gcd$(s, n) = 1$ and that $U_{s,n}^i$ is GL$(2, q^n)$-equivalent to $U_{2,n}^i$.

In [25, Theorem 3] Lunardon and Polverino proved that $L_{1,5}^2$ and $L_{1}^1$ are not PΓL$(2, q^n)$-equivalent when $q > 3$, $n \geq 4$. For $n = 5$, in [4] it is proved that $L_{2,5}^2$ is PΓL$(2, q^5)$-equivalent neither to $L_{1,5}^2$ nor to $L_{1,5}^1$.

In [10, Theorem 4.4], the authors proved that for $n = 6, 8$ the linear sets $L_{1,n}^1$, $L_{s,n}^2$ and $L_{s,n}^3$ are pairwise non-equivalent for any choice of $s, s', \delta, \delta'$. Also in [10, Theorem 5.4] it has been proved that the linear set $L_{s,n}^1$ for $q$ odd and $q \equiv 0, \pm 1 \pmod{5}$ is not equivalent to the aforementioned maximum scattered linear sets of $\text{PG}(1, q^6)$. This result has been obtained by [10, Proposition 5.3], where the congruences of $q$ odd plays no role. Hence, using the same arguments, we have the following result.

**Theorem 5.1.** The $\mathbb{F}_q$-linear set $L_c$ of rank 6 of $\text{PG}(1, q^6)$ defined by the $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^6} \times \mathbb{F}_{q^6}$

$$U_c = \{(x, x^q + x^{q^3} + cx^{q^5}) : x \in \mathbb{F}_{q^6}\},$$
with \( c^2 + c = 1 \), is scattered if and only if \( q \) is odd. Also, when \( q \equiv \pm 2 \pmod{5} \), \( L_c \) is not PTL(2, \( q^6 \))-equivalent to the previously known maximum scattered \( \mathbb{F}_q \)-linear sets of \( \text{PG}(1, q^6) \).

**Final remark**

In this paper we have proved that the RD-code \( \mathcal{C} = \langle x, f(x) \rangle_{\mathbb{F}_{q^n}} \) of \( L_{c_2} \), with \( f(x) = x^q + x^{q^3} + c x^{q^5} \in \mathbb{F}_{q^6} \) (\( n = 3 \)) and \( c^2 + c + 1 = 0 \), is an MRD-code with dimension 12, minimum distance 5 and left idealiser isomorphic to \( \mathbb{F}_{q^6} \) if and only if \( q \) is odd. Computational results show that, for suitable choices of \( c \in \mathbb{F}_{q^6} \setminus (\mathbb{F}_{q^2} \cup \mathbb{F}_{q^3}) \) the previous trinomial produces MRD-codes also when \( q \leq 64 \) is even.

We strongly believe that the previous MRD-codes belong to a larger class of MRD-codes, arising from polynomials of type \( f(x) = x^q + \sum_{i=1}^{n-1} a_{2i+1} x^{q^{2i+1}} \in \mathbb{F}_{q^n}[x] \), under suitable assumptions on the coefficients \( a_j \)'s. In Table 1 we provide some explicit examples. They are the results of our successful searches using the software package MAGMA [2] for small values of \( n \) and \( q \). When a parameter \( a_i \) appears in a row of the table it means that there exist explicit values of \( a_i \in \mathbb{F}_{q^n} \) for which the polynomial \( f(x) \) gives rise to an MRD-code. Certainly, a careful study of the corresponding \( \mathbb{F}_q \)-subspaces of \( \mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \) should be undertaken in order to establish whether the MRD-codes are equivalent to the previously known ones. The authors are currently beginning work on these two projects.

<table>
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<th>( n )</th>
<th>( q )</th>
<th>( f(x) )</th>
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<tr>
<td>3</td>
<td>( q \leq 64 ), even</td>
<td>( x^q + x^{q^3} + a_5 x^{q^5} )</td>
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<tr>
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<td>( x^q - x^{q^3} + a_5 x^{q^5} )</td>
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<tr>
<td>4</td>
<td>4</td>
<td>( x^q + a_5^2 x^{q^3} + a_3 x^{q^5} + x^{q^7} )</td>
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<td>5</td>
<td>3</td>
<td>( x^q + a_3 x^{q^3} + a_5 x^{q^5} + a_7 x^{q^7} + a_9 x^{q^9} )</td>
</tr>
</tbody>
</table>

Table 1: Computational results
References


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Giuseppe Marino  
Dipartimento di Matematica e Applicazioni “Renato Caccioppoli”  
Università degli Studi di Napoli “Federico II”  
Via Cintia, Monte S.Angelo I-80126 Napoli  
Italy  
giuseppe.marino@unina.it

Maria Montanucci  
Technical University of Denmark  
Asmussens Allé  
Building 303B, room 150  
2800 Kgs. Lyngby  
Denmark  
marimo@dtu.dk

Ferdinando Zullo  
Dipartimento di Matematica e Fisica  
Università degli Studi della Campania Luigi Vanvitelli  
Viale lincoln, 5  
36100 Caserta  
Italy  
ferdinando.zullo@unicampania.it