Quotients of the Hermitian curve from subgroups of $\text{PGU}(3,q)$ without fixed points or triangles

Montanucci, Maria; Zini, Giovanni

Published in:
Journal of Algebraic Combinatorics

Link to article, DOI:
[10.1007/s10801-019-00905-7](https://doi.org/10.1007/s10801-019-00905-7)

Publication date:
2019

Document Version
Peer reviewed version

[Link back to DTU Orbit](#)

Citation (APA):
Montanucci, M., & Zini, G. (2019). Quotients of the Hermitian curve from subgroups of $\text{PGU}(3,q)$ without fixed points or triangles. *Journal of Algebraic Combinatorics*. <https://doi.org/10.1007/s10801-019-00905-7>

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

QUOTIENTS OF THE HERMITIAN CURVE FROM SUBGROUPS OF $\mathrm{PGU}(3, q)$ WITHOUT FIXED POINTS OR TRIANGLES

MARIA MONTANUCCI AND GIOVANNI ZINI

ABSTRACT. In this paper we deal with the problem of classifying the genera of quotient curves $\mathcal{H}_q/G$, where $\mathcal{H}_q$ is the $\mathbb{F}_{q^2}$ -maximal Hermitian curve and G is an automorphism group of $\mathcal{H}_q$. The groups G considered in the literature fix either a point or a triangle in the plane $\mathrm{PG}(2, q^6)$. In this paper, we give a complete list of genera of quotients $\mathcal{H}_q/G$, when $G \leq \mathrm{Aut}(\mathcal{H}_q) \cong \mathrm{PGU}(3, q)$ does not leave invariant any point or triangle in the plane. Also, the classification of subgroups G of $\mathrm{PGU}(3, q)$ satisfying this property is given up to isomorphism.

Keywords: Hermitian curve, unitary groups, quotient curves, maximal curves

2000 MSC: 11G20

1

1. INTRODUCTION

Let $\mathbb{F}_q$ be a finite field of order q and $\mathcal{X}$ be a projective, irreducible, non-singular algebraic curve of genus g defined over $\mathbb{F}_q$. The problems of determining the maximum number of points over $\mathbb{F}_q$ that $\mathcal{X}$ can have and finding examples of algebraic curves $\mathcal{X}$ with many rational points have been important, not only from the theoretic perspective, but also for applications in Coding Theory; see for instance [11, 32, 33]. The Hasse-Weil Theorem provides an upper bound for the number of rational points $|\mathcal{X}(\mathbb{F}_q)|$ that a curve $\mathcal{X}$ defined over $\mathbb{F}_q$ can have, namely $|\mathcal{X}(\mathbb{F}_q)| \leq q + 1 + 2g\sqrt{q}$. If $|\mathcal{X}(\mathbb{F}_q)| = q + 1 + 2g\sqrt{q}$, then the curve $\mathcal{X}$ is said to be $\mathbb{F}_q$ -maximal. Clearly, $\mathcal{X}$ can be $\mathbb{F}_q$ -maximal only if either g is zero or q is a square. A natural question in this context is: over a finite field $\mathbb{F}_{q^2}$ of square cardinality, which non-negative integers g can be realized as the genera of maximal curves over $\mathbb{F}_{q^2}$?

A leading example of a maximal curve is the Hermitian curve $\mathcal{H}_q$ over $\mathbb{F}_{q^2}$, where q is a power of a prime p . It is defined as the non-singular plane curve admitting one of the following birational equivalent plane models: $X^{q+1} + Y^{q+1} + Z^{q+1} = 0$ and $X^q Z + X Z^q = Y^{q+1}$.

For fixed q , the curve $\mathcal{H}_q$ has the largest genus $g(\mathcal{H}_q) = q(q-1)/2$ that an $\mathbb{F}_{q^2}$ -maximal curve can have; see [12] and the references therein. The full automorphism group $\mathrm{Aut}(\mathcal{H}_q)$ is isomorphic to $\mathrm{PGU}(3, q)$, the group of projectivities of $\mathrm{PG}(2, q^2)$ commuting with the unitary polarity associated with $\mathcal{H}_q$. The automorphism group $\mathrm{Aut}(\mathcal{H}_q)$ is extremely large with respect to the value $g(\mathcal{H}_q)$. Indeed it is known that the Hermitian curve is the unique curve of genus $g \geq 2$ up to isomorphisms admitting an automorphism group of order at least $16g^4$.

By a result commonly referred to as the Kleiman-Serre covering result, see [24] and [25, Proposition 6], a curve defined over $\mathbb{F}_{q^2}$ which is $\mathbb{F}_{q^2}$ -covered by an $\mathbb{F}_{q^2}$ -maximal curve is $\mathbb{F}_{q^2}$ -maximal as well. In particular, $\mathbb{F}_{q^2}$ -maximal curves can be obtained as Galois $\mathbb{F}_{q^2}$ -subcovers of an $\mathbb{F}_{q^2}$ -maximal curve $\mathcal{X}$, that is, as quotient curves $\mathcal{X}/G$ where G is a finite $\mathbb{F}_{q^2}$ -automorphism group of $\mathcal{X}$. Since $\mathrm{Aut}(\mathcal{H}_q)$ is large and has plenty of non-conjugated subgroups, it seems to be natural to construct maximal curves as Galois $\mathbb{F}_{q^2}$ -subcovers $\mathcal{H}_q/G$ of the Hermitian curve $\mathcal{H}_q$ with $G \leq \mathrm{PGU}(3, q)$. Indeed, most of the known maximal curves are constructed

¹This research was partially supported by Ministry for Education, University and Research of Italy (MIUR) and by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INdAM).

in this way; see for instance [12, 7, 13, 28, 29] and the references therein. The most significant cases treated in the literature are the following:

- G fixes an $\mathbb{F}_{q^2}$ -rational point of $\mathcal{H}_q$; see [3, 12, 1].
- G normalizes a Singer subgroup of $\mathcal{H}_q$ acting on three $\mathbb{F}_{q^6}$ -rational points of $\mathcal{H}_q$; see [12, 6].
- G has prime order; see [7].
- G fixes an $\mathbb{F}_{q^2}$ -rational point off $\mathcal{H}_q$; see [8, 29, 30].
- G fixes a self-polar triangle in $\text{PG}(2, q^2)$; see [8].

From these results, in order to obtain the complete list of genera of quotients $\mathcal{H}_q/G$ of the Hermitian curve $\mathcal{H}_q$, only the following cases for $G \leq \text{Aut}(\mathcal{H}_q)$ still have to be described and characterized completely:

- (1) G fixes a self-polar triangle in $\text{PG}(2, q^2)$;
- (2) G fixes a point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$, with $p = 2$ and $|G| = p^\ell d$ where $p^\ell \leq q$ and $d \mid (q - 1)$;
- (3) G fixes an $\mathbb{F}_{q^2}$ -rational point $P \notin \mathcal{H}_q$, with $q \not\equiv 1 \pmod{4}$;
- (4) G does not fix any point or triangle in $\text{PG}(2, q^6)$.

The structure of maximal subgroups of $\text{PGU}(3, q)$ fixing a point or a triangle in $\text{PG}(2, q^6)$ is known, see [27, 19]; yet, the structure of subgroups $\text{PGU}(3, q)$ fixing no points nor triangles is not explicitly described when $3 \mid (q + 1)$, that is, when $\text{PGU}(3, q) \neq \text{PSU}(3, q)$. Also, the groups $G \leq \text{PGU}(3, q)$ considered in the literature always have a fixed point or a fixed triangle in $\text{PG}(2, q^6)$ (with the single exception of [28, Proposition 5.1]).

In this paper we give the complete list of genera of quotients $\mathcal{H}_q/G$ when G does not fix any point or triangle in $\text{PG}(2, q^6)$. To this aim we also provide the list of such subgroups of $\text{PGU}(3, q)$ up to isomorphisms, and we give explicitly the list of maximal subgroups of $\text{PGU}(3, q)$, which was already provided in [27, 19] for $3 \nmid (q + 1)$.

The organization of the paper is the following. Section 2 provides preliminary results on the structure of $\text{PGU}(3, q)$ and $\text{PSU}(3, q)$, and on the Riemann-Hurwitz formula applied to quotients of the Hermitian curve. Section 3 contains a description of the maximal subgroups of $\text{PGU}(3, q)$ in relation to the maximal subgroups of $\text{PSU}(3, q)$. Also, Section 3 lists the subgroups of $\text{PGU}(3, q)$ acting on the plane without fixed points or triangles. Section 4 applies the results of Sections 2 and 3 to compute the genus of $\mathcal{H}_q/G$ whenever G does not fix any point or triangle; examples of new genera of maximal curves are collected in Table 2.

2. PRELIMINARY RESULTS

Throughout the paper, q will denote a power p^n of a prime p . Maximal subgroups of $\text{PSU}(3, q)$ were explicitly described in [27] and [19], see also [20, Theorem A.10].

The structure of maximal subgroups M of the automorphism group $\text{Aut}(\mathcal{H}_q) \cong \text{PGU}(3, q)$ of the Hermitian curve $\mathcal{H}_q$ which fix a point or a triangle in $\text{PG}(2, \mathbb{F}_{q^2})$ is known and the genera of quotients $\mathcal{H}_q/G$ with $G \leq M$ have been deeply investigated; see for instance [3, 6, 7, 8, 12, 26, 28, 29]. Such maximal subgroups are the following, up to conjugation.

- (I) The stabilizer M_1 of a point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$. The group M_1 has order $q^3(q^2 - 1)$ and is a semidirect product of its unique Sylow p -subgroup of order q^3 and a cyclic group C of order $q^2 - 1$. The group C fixes also another point $Q \neq P$ with $Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$.
- (II) The stabilizer M_2 of a pole-polar pair (P, ℓ) with respect to the unitary polarity associated to $\mathcal{H}_q$, with $P \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$ and $P \notin \ell$. The group M_2 has order $q(q + 1)^2(q - 1)$ and is a semidirect product isomorphic to $\text{SL}(2, q) \rtimes C_{q+1}$, where C_{q+1} is cyclic of order $q + 1$.
- (III) The stabilizer M_3 of a self-polar triangle $T = \{P_1, P_2, P_3\}$ with respect to the unitary polarity associated to $\mathcal{H}_q$, with $P_i \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$. The group M_3 has order $6(q + 1)^2$ and is a semidirect product of an abelian group $C_{q+1} \times C_{q+1}$ fixing T pointwise and a symmetric group S_3 acting faithfully on T .

- (IV) The stabilizer M_4 of a triangle $T = \{P_1, P_2, P_3\} \subset \mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$ which is invariant under the Frobenius automorphism $(X, Y, Z) \mapsto (X^{q^2}, Y^{q^2}, Z^{q^2})$. The group M_4 has order $3(q^2 - q + 1)$ and is a semidirect product $C_{q^2-q+1} \rtimes C_3$, where C_{q^2-q+1} is a Singer group acting semiregularly on $\text{PG}(2, q^2)$ and fixing T pointwise, and C_3 has a unique orbit on T .

For any $i \in \{1, 2, 3, 4\}$, the intersection $\text{PSU}(3, q) \cap M_i$ of $\text{PSU}(3, q)$ with the maximal subgroup M_i listed above is a maximal subgroup of $\text{PSU}(3, q)$ with index $\gcd(3, q + 1)$ in M_i ; see [27] and [19].

Theorem 2.1. ([27, 19]) *If p is odd, then the maximal subgroups of $\text{PSU}(3, q)$ not fixing a point nor a triangle are the following:*

- (i) *the Hessian groups of order 216 when $9 \mid (q + 1)$, and of order 72 when $3 \mid (q + 1)$ and $9 \nmid (q + 1)$;*
- (ii) *$\text{PGL}(2, q)$ preserving a conic;*
- (iii) *$\text{PSL}(2, 7)$ when $p = 7$ or -7 is not a square in $\mathbb{F}_q$;*
- (iv) *the alternating group A_6 when either $p = 3$ and n is even, or 5 is a square in $\mathbb{F}_q$ but $\mathbb{F}_q$ contains no primitive cube roots of unity;*
- (v) *a group of order 720 containing the alternating group A_6 as a normal subgroup when $p = 5$ and n is odd;*
- (vi) *the alternating group A_7 when $p = 5$ and n is odd;*
- (vii) *$\text{PSU}(3, p^m)$ where n/m is an odd prime different from 3;*
- (viii) *subgroups containing $\text{PSU}(3, p^m)$ as a normal subgroup of index 3, when $m \mid n$, $n/m = 3$, and $3 \mid (q + 1)$.*

If $p = 2$, then the maximal subgroups of $\text{PSU}(3, q)$ not fixing a point nor a triangle are the following:

- (vii') *$\text{PSU}(3, 2^m)$ where $m \mid n$ and n/m is an odd prime different from 3; when $m = 1$, this is the Hessian group of order 72;*
- (viii') *subgroups containing $\text{PSU}(3, 2^m)$ as a normal subgroup of index 3, where $n/m = 3$ and $3 \mid (q + 1)$. For $m = 1$ this is the Hessian group of order 216;*
- (ix') *a group of order 36, which exists as a maximal subgroup when $n = 1$.*

The following lemma recalls how an element of $\text{PGU}(3, q)$ of a given order acts on $\text{PG}(2, \overline{\mathbb{F}}_{q^2})$, and in particular on $\mathcal{H}_q(\mathbb{F}_{q^2})$; for the usual terminology about collineations of projective planes, see [21].

Lemma 2.2. ([29, Lemma 2.2]) *For a nontrivial element $\sigma \in \text{PGU}(3, q)$, one of the following cases holds.*

- (A) *$\text{ord}(\sigma) \mid (q + 1)$ and σ is a homology whose center P is a point off $\mathcal{H}_q$ and whose axis ℓ is a chord of $\mathcal{H}_q(\mathbb{F}_{q^2})$ such that (P, ℓ) is a pole-polar pair with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$.*
- (B) *$\text{ord}(\sigma)$ is coprime to p and σ fixes the vertices P_1, P_2, P_3 of a non-degenerate triangle T .*
- (B1) *The points P_1, P_2, P_3 are $\mathbb{F}_{q^2}$ -rational, $P_1, P_2, P_3 \notin \mathcal{H}_q$ and the triangle T is self-polar with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$. Also, $\text{ord}(\sigma) \mid (q + 1)$.*
- (B2) *The points P_1, P_2, P_3 are $\mathbb{F}_{q^2}$ -rational, $P_1 \notin \mathcal{H}_q$, $P_2, P_3 \in \mathcal{H}_q$. Also, $\text{ord}(\sigma) \mid (q^2 - 1)$ and $\text{ord}(\sigma) \nmid (q + 1)$.*
- (B3) *The points P_1, P_2, P_3 have coordinates in $\mathbb{F}_{q^6} \setminus \mathbb{F}_{q^2}$, $P_1, P_2, P_3 \in \mathcal{H}_q$. Also, $\text{ord}(\sigma) \mid (q^2 - q + 1)$.*
- (C) *$\text{ord}(\sigma) = p$ and σ is an elation whose center P is a point of $\mathcal{H}_q$ and whose axis ℓ is a tangent of $\mathcal{H}_q(\mathbb{F}_{q^2})$; here (P, ℓ) is a pole-polar pair with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$.*
- (D) *$\text{ord}(\sigma) = p$ with $p \neq 2$, or $\text{ord}(\sigma) = 4$ and $p = 2$. In this case σ fixes an $\mathbb{F}_{q^2}$ -rational point P , with $P \in \mathcal{H}_q$, and a line ℓ which is a tangent of $\mathcal{H}_q(\mathbb{F}_{q^2})$; here (P, ℓ) is a pole-polar pair with respect to the unitary polarity associated to $\mathcal{H}_q(\mathbb{F}_{q^2})$.*
- (E) *$p \mid \text{ord}(\sigma)$, $p^2 \nmid \text{ord}(\sigma)$, and $\text{ord}(\sigma) \neq p$. In this case σ fixes two $\mathbb{F}_{q^2}$ -rational points P, Q , with $P \in \mathcal{H}_q$, $Q \notin \mathcal{H}_q$.*

Throughout the paper, a nontrivial element of $\text{PGU}(3, q)$ is said to be of type (A), (B), (B1), (B2), (B3), (C), (D), or (E), as given in Lemma 2.2; G always stands for a subgroup of $\text{PGU}(3, q)$.

Every subgroup G of $\mathrm{PGU}(3, q)$ produces a quotient curve $\mathcal{H}_q/G$, and the cover $\mathcal{H}_q \rightarrow \mathcal{H}_q/G$ is a Galois cover defined over $\mathbb{F}_{q^2}$ where the degree of the different divisor Δ is given by the Riemann-Hurwitz formula [31, Theorem 3.4.13],

$$\Delta = (2g(\mathcal{H}_q) - 2) - |G|(2g(\mathcal{H}_q/G) - 2).$$

On the other hand, $\Delta = \sum_{\sigma \in G \setminus \{id\}} i(\sigma)$, where $i(\sigma) \geq 0$ is given by the Hilbert's different formula [31, Thm. 3.8.7], namely

$$(1) \quad i(\sigma) = \sum_{P \in \mathcal{H}_q(\mathbb{F}_{q^2})} v_P(\sigma(t) - t),$$

where t is a local parameter at P .

By analyzing the geometric properties of the elements $\sigma \in \mathrm{PGU}(3, q)$, it turns out that there are only few possibilities for $i(\sigma)$, as stated in the following theorem.

Theorem 2.3. ([29, Theorem 2.7]) *For a nontrivial element $\sigma \in \mathrm{PGU}(3, q)$ one of the following cases occurs.*

- (1) *If $\mathrm{ord}(\sigma) = 2$ and $2 \mid (q+1)$, then σ is of type (A) and $i(\sigma) = q+1$.*
- (2) *If $\mathrm{ord}(\sigma) = 3$, $3 \mid (q+1)$ and σ is of type (B3), then $i(\sigma) = 3$.*
- (3) *If $\mathrm{ord}(\sigma) \neq 2$, $\mathrm{ord}(\sigma) \mid (q+1)$ and σ is of type (A), then $i(\sigma) = q+1$.*
- (4) *If $\mathrm{ord}(\sigma) \neq 2$, $\mathrm{ord}(\sigma) \mid (q+1)$ and σ is of type (B1), then $i(\sigma) = 0$.*
- (5) *If $\mathrm{ord}(\sigma) \mid (q^2 - 1)$ and $\mathrm{ord}(\sigma) \nmid (q+1)$, then σ is of type (B2) and $i(\sigma) = 2$.*
- (6) *If $\mathrm{ord}(\sigma) \neq 3$ and $\mathrm{ord}(\sigma) \mid (q^2 - q + 1)$, then σ is of type (B3) and $i(\sigma) = 3$.*
- (7) *If $p = 2$ and $\mathrm{ord}(\sigma) = 4$, then σ is of type (D) and $i(\sigma) = 2$.*
- (8) *If $\mathrm{ord}(\sigma) = p$, $p \neq 2$ and σ is of type (D), then $i(\sigma) = 2$.*
- (9) *If $\mathrm{ord}(\sigma) = p$ and σ is of type (C), then $i(\sigma) = q+2$.*
- (10) *If $\mathrm{ord}(\sigma) \neq p$, $p \mid \mathrm{ord}(\sigma)$ and $\mathrm{ord}(\sigma) \neq 4$, then σ is of type (E) and $i(\sigma) = 1$.*

3. ON MAXIMAL SUBGROUPS OF $\mathrm{PGU}(3, q)$

Since we were not able to find references for the complete list of maximal subgroups of $\mathrm{PGU}(3, q)$, we determine it in this section when $3 \mid (q+1)$, that is, when $\mathrm{PGU}(3, q) \neq \mathrm{PSU}(3, q)$. The result is the following.

Theorem 3.1. *Let $q = p^n$ be a prime power such that $3 \mid (q+1)$. Then the following is the list of maximal subgroups of $\mathrm{PGU}(3, q)$.*

- (1) *The stabilizer M_1 of a point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$. The group M_1 has order $q^3(q^2 - 1)$ and is a semidirect product of its unique Sylow p -subgroup of order q^3 and a cyclic group C of order $q^2 - 1$. The group C fixes also another point $Q \neq P$ with $Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$.*
- (2) *The stabilizer M_2 of a pole-polar pair (P, ℓ) with respect to the unitary polarity associated to $\mathcal{H}_q$, with $P \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$ and $P \notin \ell$. The group M_2 has order $q(q+1)^2(q-1)$ and is a semidirect product isomorphic to $\mathrm{SL}(2, q) \rtimes C_{q+1}$, where C_{q+1} is cyclic of order $q+1$.*
- (3) *The stabilizer M_3 of a self-polar triangle $T = \{P_1, P_2, P_3\}$ with respect to the unitary polarity associated to $\mathcal{H}_q$, with $P_i \in \mathrm{PG}(2, q^2) \setminus \mathcal{H}_q$. The group M_3 has order $6(q+1)^2$ and is a semidirect product of an abelian group $C_{q+1} \times C_{q+1}$ fixing T pointwise and a symmetric group S_3 acting faithfully on T .*
- (4) *The stabilizer M_4 of a triangle $T = \{P_1, P_2, P_3\} \subset \mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$ which is invariant under the Frobenius automorphism $(X, Y, Z) \mapsto (X^{q^2}, Y^{q^2}, Z^{q^2})$. The group M_4 has order $3(q^2 - q + 1)$ and is a semidirect product $C_{q^2 - q + 1} \rtimes C_3$, where $C_{q^2 - q + 1}$ is a Singer group acting semiregularly on $\mathrm{PG}(2, q^2)$ and fixing T pointwise, and C_3 has a unique orbit on T .*
- (5) *The normal subgroup $\mathrm{PSU}(3, q)$ of index 3 in $\mathrm{PGU}(3, q)$.*
- (6) *The Hessian group $H_{216} \cong \mathrm{PGU}(3, 2)$, if p is odd and $9 \nmid (q+1)$.*
- (7) *$\mathrm{PGU}(3, p^m)$ where $m \mid n$ and n/m is an odd prime different from 3.*

In order to prove Theorem 3.1, we proceed with a case-by-case analysis on maximal subgroups M of $\text{PGU}(3, q)$ containing a maximal subgroup H of $\text{PSU}(3, q)$ not fixing a point nor a triangle, listed in Theorem 2.1.

Remark 3.2. *Let $3 \mid (q + 1)$, H be a maximal subgroup of $\text{PSU}(3, q)$, and M be a maximal subgroup of $\text{PGU}(3, q)$ containing H . Then the maximality of H in $\text{PSU}(3, q)$ implies either $M \cap \text{PSU}(3, q) = H$ or $M \cap \text{PSU}(3, q) = \text{PSU}(3, q)$. If $M \cap \text{PSU}(3, q) = H$, then*

$$3 \cdot |\text{PSU}(3, q)| = |\text{PGU}(3, q)| = |\text{PSU}(3, q)M| = |\text{PSU}(3, q)| \cdot |M| / |H|,$$

and hence $[M : H] = 3$. Also, H is normal in M since $\text{PSU}(3, q)$ is normal in $\text{PGU}(3, q)$.

3.1. Hessian groups.

Proposition 3.3. *Let $q = p^n$ be a power of an odd prime p such that $9 \mid (q + 1)$. Let $H \cong \text{PGU}(3, 2)$ be a maximal subgroup of $\text{PSU}(3, q)$ isomorphic to the Hessian group of order 216, and M be a maximal subgroup of $\text{PGU}(3, q)$ containing H . Then $M \cong \text{PSU}(3, q)$.*

Proof. Assume by contradiction $M \not\cong \text{PSU}(3, q)$. Then $M \cap \text{PSU}(3, q) = H$ and $|M| = 648$ by Remark 3.2. By direct checking with MAGMA [4], there are exactly 3 groups of order 648 containing a subgroup of order 216 isomorphic to $\text{PGU}(3, 2)$, namely $\text{SmallGroup}(648, i)$ with $i \in \{702, 756, 757\}$.

- $M \cong \text{SmallGroup}(648, 702)$. In this case M has a normal subgroup of order 3, say $\langle \alpha \rangle$; hence, M acts on the points fixed by α . By Lemma 2.2, α is either of type (A), or (B1), or (B3); hence, either M fixes a pole-polar pair (P, ℓ) with $P \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$, or M stabilizes a self-polar triangle $\{P_1, P_2, P_3\} \subseteq \text{PG}(2, q^2) \setminus \mathcal{H}_q$, or M stabilizes a Frobenius-invariant triangle $\{P_1, P_2, P_3\} \subseteq \mathcal{H}_q(\mathbb{F}_{q^6}) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$. Then M is properly contained in one of the maximal subgroups M_1, M_2, M_3, M_4 described in Theorem 3.1, a contradiction to the maximality of M .
- $M \cong \text{SmallGroup}(648, 756)$. In this case M has a normal subgroup of order 2, say $\langle \alpha \rangle$; hence, M acts on the points fixed by α . By Lemma 2.2, α is of type (A) and fixes a point $P \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$. Thus, M fixes P and is properly contained in the maximal subgroup M_2 of $\text{PGU}(3, q)$, a contradiction.
- $M \cong \text{SmallGroup}(648, 757)$. In this case M has a normal subgroup of order 2, and a contradiction follows as in the previous case.

Then $M \cap \text{PSU}(3, q) = \text{PSU}(3, q)$ and the claim is proved. □

Proposition 3.4. *Let $q = p^n$ be a power of an odd prime p such that $3 \mid (q + 1)$ and $9 \nmid (q + 1)$. Let $H \cong \text{PSU}(3, 2)$ be a maximal subgroup of $\text{PSU}(3, q)$ isomorphic to the Hessian group of order 72, and M be a maximal subgroup of $\text{PGU}(3, q)$ containing H . Then $M \cong \text{PGU}(3, 2)$ is isomorphic to the Hessian group of order 216.*

Proof. Arguing as in the proof of Theorem 9 in [27], one can explicitly construct a subgroup M of $\text{PGL}(3, q^2)$ containing H as a normal subgroup of index 3 and isomorphic to the Hessian group $\text{PGU}(3, 2)$ of order 216. By direct computation, the generators of M leave invariant the Hermitian curve $\mathcal{H}_q$ in its Fermat equation $X^{q+1} + Y^{q+1} + Z^{q+1} = 0$; hence, M is a subgroup of $\text{PGU}(3, q)$. Also, M is not contained in $\text{PSU}(3, q)$. Now we show that M is maximal in $\text{PGU}(3, q)$.

Suppose by contradiction there exists a proper subgroup M' of $\text{PGU}(3, q)$ containing M properly. By [27, Theorem 9], M is the unique tame subgroup of $\text{PGU}(3, q)$ such that M does not leave invariant a point, or a line, or a triangle and M contains homologies of order 3. Since $M \subseteq M'$, also M' does not leave invariant a point, or a line, or a triangle and M' contains homologies of order 3. Then M' contains p -elements, which are clearly elements of $\text{PSU}(3, q)$ because $p \nmid [\text{PGU}(3, q) : \text{PSU}(3, q)]$. From [27, Theorem 18], M' contains elements of type (C) because M' cannot contain only p -elements of type (D). From [27, Theorem 28], M' contains the entire $\text{PSU}(3, q)$; since M' is not contained in $\text{PSU}(3, q)$, this implies $M' = \text{PGU}(3, q)$, a contradiction. □

3.2. $\mathrm{PGL}(2, q)$ preserving a conic.

Proposition 3.5. *Let $q = p^n$ be a power of an odd prime p with $3 \mid (q + 1)$. Let H be a maximal subgroup of $\mathrm{PSU}(3, q)$ such that $H \cong \mathrm{PGL}(2, q)$ and H fixes a conic $\mathcal{C}$. Let M be a maximal subgroup of $\mathrm{PGU}(3, q)$ containing H . Then $M \cong \mathrm{PSU}(3, q)$.*

Proof. Assume by contradiction $M \not\cong \mathrm{PSU}(3, q)$. By Remark 3.2, $H = M \cap \mathrm{PSU}(3, q) \triangleleft M$ and $[M : H] = 3$. From [14, Remark 6.14], M is a semidirect product $M = \mathrm{PGL}(2, q) \rtimes C_3$ where $C_3 = \langle \alpha \rangle$ is a cyclic group of order 3 and $3 \mid n$. Let $I = \mathcal{H}_q \cap \mathcal{C}$ be the set of the $q + 1$ points of intersection between $\mathcal{H}_q$ and $\mathcal{C}$. The set I is the unique orbit of size $q + 1$ of H on $\mathcal{H}_q$; see [5, Lemma 3.1]. Since M normalizes H , M acts on the set of orbits of H on $\mathcal{H}_q$ having the same size; hence, M acts on I . By Lemma 2.2, α is either of type (A), or of type (B1), or of type (B3).

- (1) Assume that α is a homology, with center P and axis ℓ . Recall that $\mathcal{C}$ is irreducible, and let r be a secant line to $\mathcal{C}$ passing through P . Then the size of $(r \cap \mathcal{H}_q) \setminus \ell$ is 1 or 2, according to $r \cap \ell \in \mathcal{C}$ or $r \cap \ell \notin \mathcal{C}$, respectively. Since r and ℓ are fixed by α , we have that α acts on the 1 or 2 points of $(r \cap \mathcal{H}_q) \setminus \ell$. This is a contradiction to the action on the plane of α , which has long orbits of size 3 out of ℓ and P .
- (2) Assume that α is of type (B1). Since $o(\alpha) = 3$, this implies that $\alpha \in \mathrm{PSU}(3, q)$. In fact, we can use the Fermat model $X^{q+1} + Y^{q+1} + Z^{q+1} = 0$ of $\mathcal{H}_q$ and assume up to conjugation that α fixes the fundamental triangle, so that α is represented by a diagonal matrix $\mathrm{diag}(\lambda, \mu, 1)$ with $\lambda^3 = \mu^3 = 1$. As α is of type (B1), we have $\lambda \neq 1$, $\mu \neq 1$, $\mu \neq \lambda$. Then $\mu = \lambda^{-1}$ and $\det(\alpha) = 1$, so that $\alpha \in \mathrm{PSU}(3, q)$. This contradicts $M \not\subseteq \mathrm{PSU}(3, q)$.
- (3) Assume that α is of type (B3). By the Orbit-Stabilizer Theorem, the stabilizer M_P in M of a point $P \in I$ has order $3q(q - 1)$. From $P \in \mathcal{H}_q$ and [20, Lemma 11.44], we have that M_P is a semidirect product $M_P = M_P^1 \rtimes M_P^2$, where M_P^1 is the Sylow p -subgroup of M_P and M_P^2 is cyclic of order $3(q - 1)$. Analogously, the stabilizer H_P of P in H satisfies $H_P = H_P^1 \rtimes H_P^2$ with $|H_P^2| = q - 1$; up to conjugation, M_P^2 contains H_P^2 . Since $3 \nmid (q - 1)$, there exists an element $\beta \in M_P^2 \setminus H$ of order 3. As β fixes an $\mathbb{F}_{q^2}$ -rational point P of $\mathcal{H}_q$, β is a homology by Lemma 2.2. Then $M = H \rtimes \langle \beta \rangle$ and a contradiction is obtained as in Case (1).

□

3.3. $\mathrm{PSL}(2, 7)$ when $p = 7$ or $\sqrt{-7} \notin \mathbb{F}_q$.

Proposition 3.6. *Let $q = p^n$ be a power of an odd prime p such that $3 \mid (q + 1)$ and either $p = 7$ or $\sqrt{-7} \notin \mathbb{F}_q$. Let H be a maximal subgroup of $\mathrm{PSU}(3, q)$ with $H \cong \mathrm{PSL}(2, 7)$, and M be a maximal subgroup of $\mathrm{PGU}(3, q)$ containing H . Then $M \cong \mathrm{PSU}(3, q)$.*

Proof. Assume by contradiction $M \not\cong \mathrm{PSU}(3, q)$. By Remark 3.2, $H \triangleleft M$ and $[M : H] = 3$. From [14, Remark 6.14] and [34, Proposition 1.2 (i)] we obtain that $M = H \rtimes C_3$. Hence, H acts on the points fixed by $C_3 \triangleleft M$. By Lemma 2.2, this means that H is contained in one of the maximal subgroup M_2, M_3, M_4 described in Theorem 3.1, a contradiction to Theorem 2.1. □

3.4. The group $\mathrm{SmallGroup}(720, 765)$ when $p = 5$ and n is odd. From [27], $\mathrm{PSU}(3, q)$ has a maximal subgroup H of order 720 which contains a normal subgroup of order 360 isomorphic to the alternating group A_6 , when $p = 5$ and q is odd power of p . The following lemma gives H explicitly in the GAP notation.

Lemma 3.7. *Let q be an odd power of 5 and H be a subgroup of $\mathrm{PSU}(3, q)$ of order 720 containing A_6 . Then H is isomorphic to the subgroup of $\mathrm{P\Gamma L}(2, 9)$ named $\mathrm{SmallGroup}(720, 765)$.*

Proof. By direct checking, there are exactly 4 groups of order 720 containing a subgroup isomorphic to A_6 , namely the direct product $A_6 \times C_2$, the symmetric group S_6 , a semidirect product $\mathrm{SmallGroup}(720, 764) \cong A_6 \rtimes C_2$, and $\mathrm{SmallGroup}(720, 765)$.

- Suppose that $M \cong A_6 \times C_2$, say $C_2 = \langle \alpha \rangle$. Then by Lemma 2.2 α is of type (A). Since C_2 is normal in H , H fixes the center of α and is contained in the maximal subgroup M_2 of $\text{PGU}(3, q)$ described in Theorem 3.1. By the maximality of H in $\text{PSU}(3, q)$ we have $H = M_2 \cap \text{PSU}(3, q)$, which is a contradiction to $720 = |H| \neq |M_2 \cap \text{PSU}(3, q)| = q(q+1)^2(q-1)/3$.
- Suppose that $H \cong S_6$. Then H contains an elementary abelian subgroup of order 8. This is a contradiction to the fact that for odd q there are at most 3 involutions which commute pairwise; see also [23, Lemma 2.2 (viii)].
- Suppose that H is the semidirect product $\text{SmallGroup}(720, 764) \cong A_6 \rtimes C_2$. Then H contains a cyclic subgroup C_{10} of order 10; by Lemma 2.2, C_{10} is generated by an element of type (E), and the elements of order 5 in C_{10} are of type (C). Also, H contains a dihedral subgroup D_5 of order 10. If an involution normalizes an elation, then they commute, as they are both contained in a maximal subgroup of type M_2 and the involution is in the center of M_2 ; see [28] for the structure of M_2 . Hence, the elements of order 5 in D_5 are of type (D). But H has a unique conjugacy class of elements of order 5, so that we cannot have elements of type (C) and elements of type (D).

Then $H \cong \text{SmallGroup}(720, 765)$ and the claim is proved. $\square$

Proposition 3.8. *Let q be an odd power of 5, H be a maximal subgroup of $\text{PSU}(3, q)$ isomorphic to $\text{SmallGroup}(720, 765)$, and M be a maximal subgroup of $\text{PGU}(3, q)$ containing H . Then $M = \text{PSU}(3, q)$.*

Proof. Assume by contradiction $M \not\cong \text{PSU}(3, q)$. By Remark 3.2, $H = M \cap \text{PSU}(3, q) \triangleleft M$ and $|M| = 3|H| = 2160$. Also, $A_6 \triangleleft M$ since A_6 is characteristic in H . Hence, $M/\text{Ker}(\varphi)$ is isomorphic to an automorphism group of A_6 , where $\text{Ker}(\varphi)$ is the kernel of the action by conjugation of M on A_6 . As $A_6 \cong \text{PSL}(2, 9)$, $M/\text{Ker}(\varphi)$ is isomorphic to a subgroup of $\text{P}\Gamma\text{L}(2, 9)$. Since the largest subgroup of $\text{P}\Gamma\text{L}(2, 9)$ has order 720 and the order of $M/\text{Ker}(\varphi)$ is at least $|H| = 720$, we have $|\text{Ker}(\varphi)| = 3$. Thus, $M \setminus A_6$ contains an element α of order 3 commuting with A_6 elementwise. As $3 \mid (q+1)$, α is either of type (A), or of type (B1), or of type (B3), by Lemma 2.2. Hence, H is contained in one of the maximal subgroup M_2, M_3, M_4 described in Theorem 3.1, a contradiction to Theorem 2.1. $\square$

3.5. The alternating group A_6 when either $p = 3$ and n is even, or 5 is a square in $\mathbb{F}_q$ but $\mathbb{F}_q$ contains no primitive cube roots of unity.

Proposition 3.9. *Let $q = p^n$, where either $p = 3$ and n is even, or 5 is a square in $\mathbb{F}_q$ but $\mathbb{F}_q$ contains no primitive cube roots of unity. Let H be a maximal subgroup of $\text{PSU}(3, q)$ isomorphic to A_6 and M be a maximal subgroup of $\text{PGU}(3, q)$ containing H . Then $M = A_6$ if $p = 3$ and $M = \text{PSU}(3, q)$ otherwise.*

Proof. The claim is trivial for $p = 3$ by Theorem 2.1, as $\text{PGU}(3, q) = \text{PSU}(3, q)$. If $p \neq 3$ and $\mathbb{F}_q$ contains no primitive cube roots of unity, then $3 \mid (q+1)$. Assume by contradiction $M \not\cong \text{PSU}(3, q)$. By Remark 3.2, $H = M \cap \text{PSU}(3, q) \triangleleft M$ and $[M : H] = 3$. Arguing as in the proof of Proposition 3.8 it can be shown that $M/\text{Ker}(\varphi)$ is isomorphic to a subgroup of $\text{P}\Gamma\text{L}(2, 9)$, where φ is the action by conjugation of M on H ; H commutes elementwise with an element of order 3 in $M \setminus H$; H is contained either in M_2 or in M_3 or in M_4 as described in Theorem 3.1, a contradiction to Theorem 2.1. $\square$

3.6. The alternating group A_7 when $p = 5$ and n is odd.

Proposition 3.10. *Let q be an odd power of $p = 5$, H be a maximal subgroup of $\text{PSU}(3, q)$ isomorphic to A_7 and M be a maximal subgroup of $\text{PGU}(3, q)$ containing H . Then $M = \text{PSU}(3, q)$.*

Proof. Assume by contradiction $M \not\cong \text{PSU}(3, q)$. By Remark 3.2, $H = M \cap \text{PSU}(3, q) \triangleleft M$ and $[M : H] = 3$. Thus, $M/\text{Ker}(\varphi)$ is isomorphic to an automorphism group of H , where φ is the action by conjugation of M on H . Since $|\text{Aut}(A_7)| = 2|A_7|$ and $H \leq M$, we conclude that $\text{Ker}(\varphi)$ is cyclic of order 3. Hence, H commutes with an element α of order 3, and H acts on the points fixed by α . This is a contradiction to Theorem 2.1, according to which H cannot fix a point nor a triangle. $\square$

3.7. $\text{PSU}(3, p^m)$ where $m \mid n$ and n/m is odd. In this section $q = p^n$ can be even or odd. Let H be a subgroup of $\text{PSU}(3, q)$ isomorphic to $\text{PSU}(3, \bar{q})$, where $\bar{q} = p^m$, m divides n , and n/m is odd. We start with some lemmas on maximal subgroups of $\text{PSU}(3, q)$ containing H which we were not able to find clearly stated in [27, 19] nor elsewhere.

Lemma 3.11. *Let $q = p^n$ be a prime power and H be a subgroup of $\text{PSU}(3, q)$ isomorphic to $\text{PSU}(3, p^m)$, where $m \mid n$ and m/n is odd. Then $\text{PSU}(3, q)$ contains a unique conjugacy class of subgroups isomorphic to H . Also, H is not maximal in $\text{PSU}(3, q)$ unless n/m is prime.*

Proof. Consider the Hermitian curve $\mathcal{H}_q$ in its Norm-Trace equation $x^{q+1} = y^q + y$, let $P_\infty \in \mathcal{H}_q$ be the unique point at infinity of $\mathcal{H}_q$, and $\mathcal{A}(P_\infty)$ be the stabilizer of P_∞ in $\text{PGU}(3, q)$; we follow here the notation in [12]. From [12], $\mathcal{A}(P_\infty)$ has order $q^3(q^2 - 1)$ and is a semidirect product $\mathcal{A}(P_\infty) = \mathcal{A}_1(P_\infty) \rtimes \mathcal{C}(P_\infty)$ where $\mathcal{A}_1(P_\infty)$ is the unique Sylow p -subgroup of $\mathcal{A}(P_\infty)$ and $\mathcal{C}(P_\infty)$ is cyclic of order $q^2 - 1$. If $\sigma \in \mathcal{A}_1(P_\infty)$, then $\sigma(x) = x + b$ and $\sigma(y) = y + b^q x + c$, for some $b, c \in \mathbb{F}_{q^2}$ satisfying $c^q + c = b^{q+1}$; if $\sigma \in \mathcal{C}(P_\infty)$, then $\sigma(x) = ax$ and $\sigma(y) = a^{q+1}y$ with $a \in \mathbb{F}_{q^2}$. Also, $\text{PGU}(3, q) = \langle \mathcal{A}(P_\infty), w \rangle$, where w is the involution defined as $w(x) = \frac{x}{y}$, $w(y) = \frac{1}{y}$. The subgroup $\text{PSU}(3, q)$ is generated by $\text{PSU}(3, q) = \langle \text{PSU}(3, q) \cap \mathcal{A}(P_\infty), w \rangle$, where $\text{PSU}(3, q) \cap \mathcal{A}(P_\infty) = \mathcal{A}_1(P_\infty) \rtimes (\text{PSU}(3, q) \cap \mathcal{C}(P_\infty))$ and $|\text{PSU}(3, q) \cap \mathcal{C}(P_\infty)| = (q^2 - 1)/\gcd(3, q + 1)$. Let

$$K_1 = \{\sigma \in \mathcal{A}_1(P_\infty) \mid \sigma \text{ is defined over } \mathbb{F}_{p^{2m}}\}, \quad K_2 = \{\sigma \in \mathcal{C}(P_\infty) \cap \text{PSU}(3, q) \mid \sigma \text{ is defined over } \mathbb{F}_{p^{2m}}\},$$

and $\mathcal{A}_m(P_\infty) = \langle K_1, K_2 \rangle$. Clearly, $|K_2| = p^{2m} - 1$. To show that $|K_1| = p^{3m}$, let $b, c \in \mathbb{F}_{p^{2m}}$. By direct checking, the condition $c^q + c = b^{q+1}$ holds if and only if $c^{p^{m(r-2)}} + c = b^{p^{m(r-2)+1}}$, where $r = n/m$. If $r = 3$, the condition reads $c^{p^m} + c = b^{p^{m(r-2)+1}}$ and has p^{3m} solutions $(b, c) \in \mathbb{F}_{p^{2m}}$ by the properties of the norm and trace functions. If $r \geq 5$, then by writing $m(r - 2) = 2m + m(r - 4)$ we have that $c^q + c = b^{q+1}$ is equivalent to $c^{p^{m(r-4)}} + c = b^{p^{m(r-4)+1}}$; by induction, we obtain $|K_1| = p^{3m}$. Since K_2 normalizes K_1 , we have $\mathcal{A}_m(P_\infty) = K_1 \rtimes K_2$ and $|\mathcal{A}_m(P_\infty)| = p^{3m}(p^{2m} - 1)$. Clearly, the involution w is defined over $\mathbb{F}_{p^{2m}}$. Hence, the group $\langle \mathcal{A}_m(P_\infty), w \rangle$ is defined over $\mathbb{F}_{p^{2m}}$ and is isomorphic to $\text{PSU}(3, p^m)$, since it preserves the Hermitian curve $x^{p^{m+1}} = y^{p^m} + y$.

If $H \leq \text{PSU}(3, q)$ is isomorphic to $\text{PSU}(3, p^m)$, write $H = \langle S \rtimes C, \tilde{w} \rangle$, where $|S| = p^{3m}$, C is cyclic of order $p^{2m} - 1$, and $\tilde{w}$ is an involution. The group $S \rtimes C$ fixes a point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ as $\mathcal{H}_q$ has p -rank zero (see [20, Theorem 11.133]), and $\tilde{w}$ maps P to another point $Q \in \mathcal{H}_q(\mathbb{F}_{q^2})$. Since $\text{PSU}(3, q)$ is doubly transitive on $\mathcal{H}_q(\mathbb{F}_{q^2})$ and transitive on $\text{PG}(2, q^2) \setminus \mathcal{H}_q$, we have up to conjugation in $\text{PSU}(3, q)$ that $P = P_\infty$, Q is the other fixed point of w , and $\tilde{w} = w$, so that $H = \langle \mathcal{A}_m(P_\infty), w \rangle$. Hence, subgroups isomorphic to $\text{PSU}(3, p^m)$ are conjugated in $\text{PSU}(3, q)$ to the subgroup $\langle \mathcal{A}_m(P_\infty), w \rangle$ described above. This implies that $\text{PSU}(3, \bar{q})$ is not maximal in $\text{PSU}(3, q)$ unless n/m is prime; in fact, if $n/m = d_1 d_2$ with $d_1, d_2 > 1$, then there exists a proper subgroup $\text{PSU}(3, p^{m d_1})$ of $\text{PSU}(3, q)$ which contains $\text{PSU}(3, \bar{q})$ properly. $\square$

Lemma 3.12. *Let $q = p^n$ be a prime power and H be a subgroup of $\text{PSU}(3, q)$ isomorphic to $\text{PSU}(3, p^m)$, where $m \mid n$ and n/m is an odd prime.*

- If $n/m \neq 3$ or $3 \nmid (q + 1)$, then H is maximal in $\text{PSU}(3, q)$.
- If $n/m = 3$ and $3 \mid (q + 1)$, then a maximal subgroup M of $\text{PSU}(3, q)$ containing H is isomorphic to $\text{PGU}(3, p^m)$.

Proof. As shown in the proof of Lemma 3.11, we can assume up to conjugation that H consists of the $\mathbb{F}_{p^{2m}}$ -rational elements of $\text{PSU}(3, q)$. Let M be a maximal subgroup of $\text{PSU}(3, q)$ containing H . Since M does not leave invariant any point or triangle, we have by Theorem 2.1 that either $M \cong \text{PSU}(3, p^k)$, where $k \mid n$ and n/k is odd; or M contains $\text{PSU}(3, p^k)$ as a normal subgroup of index 3, where $k \mid n$, $3 \mid n/m$, and $3 \mid (q + 1)$. Hence, if the odd prime n/m is different from 3 or $3 \nmid (q + 1)$, then the second case cannot occur, and $M \cong H$ follows. Then we can suppose that $n/m = 3$ and $3 \mid (q + 1)$; this implies also $3 \mid (p^k + 1)$, so that $\text{PSU}(3, p^m) \neq \text{PGU}(3, p^m)$.

Let M be the subgroup of $\text{PGU}(3, q)$ made by the elements of $\text{PGU}(3, q)$ which are defined over $\mathbb{F}_{p^{2m}}$; note that $H \leq M$. By arguing as in the proof of Lemma 3.11, we obtain that M is isomorphic to $\text{PGU}(3, p^m)$, and hence $\text{PSU}(3, p^m)$ has index 3 in M . For any $\sigma \in M$, σ has a representative in $\text{GL}(3, p^{2m})$, whose determinant is in $\mathbb{F}_{p^{2m}}$ and hence is a cube in $\mathbb{F}_{q^2}$, because $\mathbb{F}_{q^2}$ is a cubic extension of $\mathbb{F}_{p^{2m}}$. Thus M is a subgroup of H , and satisfies the statement of this lemma.

Let $\tilde{M}$ be a maximal subgroup of $\text{PSU}(3, q)$ containing H as a normal subgroup of index 3; the claim follows if we prove that the case $\tilde{M} \not\cong \text{PGU}(3, p^m)$ cannot occur. If $\tilde{M} \not\cong \text{PGU}(3, p^m)$, then from [14, Remark 6.14] follows that $\tilde{M}$ is a semidirect product $H \rtimes C_3$. Let $C_3 = \langle \alpha \rangle$; clearly $\alpha \notin M \cong \text{PGU}(3, p^m)$, otherwise $\tilde{M}$ and M coincide. By Lemma 2.2, α is either of type (A), or of type (B1), or of type (B3).

- Suppose that α is of type (A) or (B1). As in Case (2) in the proof of Proposition 3.5, we can use the Fermat model of $\mathcal{H}_q$ and assume up to conjugation that α is a diagonal matrix of type $\text{diag}(\lambda, \mu, 1)$, where $\lambda^3 = \mu^3 = 1$; here, $\lambda = \mu$ or $\lambda \neq \mu$ according to α being of type (A) or (B1), respectively. In any case, α is defined over $\mathbb{F}_{p^{2m}}$ as $3 \mid (p^m + 1)$, and hence $\alpha \in M \cong \text{PGU}(3, p^m)$, a contradiction.
- Suppose that α is of type (B3), and let T be the $\mathbb{F}_{q^6}$ -rational triangle fixed pointwise by α . Then α is an element of the pointwise stabilizer of T in $\text{PSU}(3, q)$, which has order $(q^2 - q + 1)/3$. This is a contradiction to $o(\alpha) = 3 \nmid (q^2 - q + 1)/3$.

□

Proposition 3.13. *Let $q = p^n$ be a prime power and H be a subgroup of $\text{PSU}(3, q)$ isomorphic to $\text{PSU}(3, p^m)$, where $m \mid n$ and n/m is an odd prime. Let M be a maximal subgroup of $\text{PSU}(3, q)$ containing H , so that*

- *if $n/m \neq 3$ or $3 \nmid (q + 1)$, then $M \cong \text{PSU}(3, p^m)$;*
- *if $n/m = 3$ and $3 \mid (q + 1)$, then $M \cong \text{PGU}(3, p^m)$.*

Let $\tilde{M}$ be a maximal subgroup of $\text{PGU}(3, q)$ containing M .

- *If $n/m \neq 3$ or $3 \nmid (q + 1)$, then $\tilde{M} \cong \text{PGU}(3, p^m)$.*
- *If $n/m = 3$ and $3 \mid (q + 1)$, then $\tilde{M} \cong \text{PSU}(3, q)$.*

Proof. Assume without restriction that $\mathcal{H}_q$ has Norm-Trace equation $x^{q+1} = y^q + y$ and, up to conjugation, the elements of H are defined over $\mathbb{F}_{p^{2m}}$. The claim on M follows from Lemma 3.12. If $3 \nmid (q + 1)$, then $\text{PSU}(3, q) = \text{PGU}(3, q)$, and hence $\tilde{M} = M \cong \text{PGU}(3, p^m)$ by Lemma 3.12.

Now suppose that $n/m \neq 3$ and $3 \mid (q + 1)$. Then either $\tilde{M} \cap \text{PSU}(3, q) = M$, or $\tilde{M} = \text{PSU}(3, q)$. We prove the claim by providing a maximal subgroup $\hat{M}$ of $\text{PGU}(3, q)$ isomorphic to $\text{PGU}(3, p^m)$ and showing that the case $\tilde{M} \not\cong \hat{M}$ cannot occur. Arguing as in the proof of Lemma 3.11, let $\hat{M}$ be the subgroup of $\text{PGU}(3, q)$ made by the elements which are defined over $\mathbb{F}_{p^{2m}}$. Then $\hat{M} \cong \text{PGU}(3, q)$. Since $\hat{M}$ and $\text{PSU}(3, q)$ are maximal in $\text{PGU}(3, q)$, we have $\hat{M} \cap \text{PSU}(3, q) = M \cong \text{PSU}(3, p^m)$ and $[\hat{M} : M] = 3$. Let N be a maximal subgroup of $\text{PGU}(3, q)$ such that $\hat{M} \leq N$. Then $N \cap \text{PSU}(3, q) \neq \text{PSU}(3, q)$ by the maximality of $\text{PSU}(3, q)$ in $\text{PGU}(3, q)$; hence, $N \cap \text{PSU}(3, q) = M$ by the maximality of M in $\text{PSU}(3, q)$, and $[N : M] = 3$. Thus $|N| = |M|$ and $N = M$, proving that $\hat{M}$ is maximal in $\text{PGU}(3, q)$. Assume by contradiction that $\tilde{M} \not\cong \hat{M}$. Then by [14, Remark 6.14] $\tilde{M}$ is a semidirect product $H \rtimes C_3$, where $C_3 = \langle \alpha \rangle$ and $\alpha \notin \hat{M}$. By Lemma 2.2, α is either of type (A), or of type (B1), or of type (B3). If α is of type (A) or (B1), a contradiction follows as in the proof of Lemma 3.12. If α is of type (B3) and T is the $\mathbb{F}_{q^6}$ -rational triangle fixed by α , then α is the unique element of order 3 in the pointwise stabilizer C of T in $\text{PGU}(3, q)$, which has order $q^2 - q + 1$. Since $n/m \neq 3$, it is easily seen that $(p^{2m} - p^m + 1) \mid (q^2 - q + 1)$. Thus, the intersection $C \cap \hat{M} \cong \text{PGU}(3, p^m)$ has order $p^{2m} - p^m + 1$, which is divisible by 3; hence $\alpha \in \hat{M}$, a contradiction.

Finally, suppose that $n/m = 3$ and $3 \mid (q + 1)$. Assume by contradiction that $\tilde{M} \neq \text{PSU}(3, q)$, so that $\tilde{M} \cap \text{PSU}(3, q) = M$ and $|\tilde{M}| = 3|M| = 9|\text{PSU}(3, p^m)|$. Since M is normal in $\tilde{M}$, we have from [14, Remark 6.14] that $\tilde{M}$ is a semidirect product $\text{PGU}(3, p^m) \rtimes C_3$ and $3 \mid m$. Let $C_3 = \langle \alpha \rangle$, so that by Lemma 2.2 α is either of type (A), or of type (B1), or of type (B3). Arguing as above, $\alpha \notin \text{PSU}(3, q)$ implies that α is of

type (B3). Since the elements of type (A) cannot stabilize the $\mathbb{F}_{q^6}$ -rational triangle fixed pointwise by α , we have that α does not commute elementwise with $\text{PGU}(3, p^m)$. Hence, $\tilde{M}$ is a subgroup of the automorphism group $\text{PTU}(3, p^m) = \text{PGU}(3, p^m) \rtimes C_m$ of $\text{PGU}(3, p^m)$; here, C_m has order m and is generated by the Frobenius automorphism $\varphi : (X, Y, Z) \mapsto (X^p, Y^p, Z^p)$. Up to multiplying by an element of $\text{PGU}(3, p^m)$, we can assume that $\alpha \in C_m$. Thus, α commutes with those elements of $\text{PGU}(3, p^m)$ which are defined over $\mathbb{F}_p$; in particular, α commutes with the elements $\sigma : (x, y) \mapsto (ax, a^{q+1}y)$, $a \in \mathbb{F}_p$, which are of type (B2). This implies that α fixes a point $P \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$, a contradiction to α being of type (B3). $\square$

3.8. The group $\text{SmallGroup}(36, 9)$ when $q = 2$. From [19], $\text{PSU}(3, 2)$ has a maximal subgroup H of order 36. By direct checking, $H \cong \text{SmallGroup}(36, 9)$ in the GAP notation.

Proposition 3.14. *Let $q = 2^n$. Then $\text{PSU}(3, q)$ contains a subgroup isomorphic to $\text{SmallGroup}(36, 9)$ if and only if n is odd.*

Proof. If n is odd, the claim follows from Theorem 2.1 as $\text{PSU}(3, q)$ contains $\text{PSU}(3, 2)$. Conversely, suppose by contradiction that $\text{SmallGroup}(36, 9) \leq \text{PSU}(3, q)$ with n even. Then $\text{PSU}(3, q)$ contains an elementary abelian 3-subgroup E of order 9, whose nontrivial elements are of type (B2) from Lemma 2.2. Let $\sigma, \tau \in E \setminus \{id\}$ with $\tau \notin \langle \sigma \rangle$; then τ acts on the fixed points $\{P_1, P_2, P_3\}$ of σ , where $P_1 \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$ and $P_2, P_3 \in \mathcal{H}_q(\mathbb{F}_{q^2})$. By $E \leq \text{Aut}(\mathcal{H}_q)$ and the Orbit-Stabilizer Theorem, E fixes $\{P_1, P_2, P_3\}$ pointwise. Hence, E is cyclic by [20, Lemma 11.44 (c)], a contradiction. $\square$

Remark 3.15. *Let q be an odd power of 2. Then the only subgroups of $\text{PGU}(3, q)$ properly containing $\text{SmallGroup}(36, 9)$ are $\text{PSU}(3, 2^k)$ and $\text{PGU}(3, 2^k)$, where $k \mid n$.*

In fact, by direct checking, any degree 3 extension M of $\text{SmallGroup}(36, 9)$ contains a normal subgroup of order 3; if $M \leq \text{PGU}(3, q)$ then M would fix a point or a triangle and the same does $\text{SmallGroup}(36, 9)$, a contradiction. Then the claim follows from Theorem 2.1.

Remark 3.16. *When q is even, the only subgroups G of $\text{PSU}(3, q)$ which do not fix any point or triangle are $G \cong \text{PSU}(3, 2^k)$ when $k \mid n$, n is odd, and 3 divides n/k ; $G \cong \text{PGU}(3, 2^k)$ when $k \mid n$; and $G \cong \text{SmallGroup}(36, 9)$ when n is even.*

In fact, from Theorem 2.1, either $q = 2$ and $G \leq \text{SmallGroup}(36, 9)$; or $G \leq \text{PSU}(3, 2^k)$; or $G \leq \text{PGU}(3, 2^k)$. If $G < \text{SmallGroup}(36, 9)$, then G contains a cyclic normal subgroup, and hence G fixes a point or a triangle. If $G < \text{PSU}(3, 2^k)$ or $G < \text{PGU}(3, 2^k)$, apply inductively the same argument.

3.9. The subgroups of $\text{PGU}(3, q)$ without fixed points or triangles. We conclude Section 3 with the list of subgroups of $\text{PGU}(3, q)$ which do not fix a point or a triangle in $\text{PG}(2, q^6)$. This result can be deduced from Theorem 2.1 and the analysis ruled out in this section.

Theorem 3.17. *Let $q = p^n$ be a prime power. The subgroups of $\text{PGU}(3, q)$ which do not fix a point or a triangle are the following.*

- (1) $\text{PSU}(3, p^m)$, when $m \mid n$ and n/m is odd.
- (2) $\text{PGU}(3, p^m)$, when $m \mid n$ and n/m is odd.
- For $p > 2$, also the following subgroups.
- (3) The Hessian groups of order 216, 72, and 36; they are isomorphic to $\text{PGU}(3, 2)$, $\text{PSU}(3, 2)$, and $\text{SmallGroup}(36, 9)$, respectively.
- (4) $\text{PGL}(2, q)$ fixing a conic.
- (5) $\text{PSL}(2, 7)$, when $p = 7$ or -7 is not a square in $\mathbb{F}_q$.
- (6) The alternating group A_6 , when $p = 3$ and n is even, or 5 is a square in $\mathbb{F}_q$ and $\mathbb{F}_q$ contains no primitive cube roots of unity.
- (7) A group of order 720 containing the alternating group A_6 when $p = 5$ and n is odd, isomorphic to $\text{SmallGroup}(720, 765)$ in the GAP notation.

(8) The alternating group A_7 , when $p = 5$ and n is odd.

When q is an odd power of 2, also a group of order 36 isomorphic to $\text{SmallGroup}(36, 9)$ in the GAP notation.

4. GENERA OF $\mathcal{H}_q/G$ WHEN $G \leq \text{PGU}(3, q)$ HAS NO FIXED POINTS OR TRIANGLES IN $\text{PG}(2, \bar{\mathbb{F}}_{q^2})$

In this section we compute the genera of all quotients $\mathcal{H}_q/G$ such that $G \leq \text{PGU}(3, q)$ has no fixed points (hence also no fixed lines) nor fixed triangles in the plane $\text{PG}(2, \bar{\mathbb{F}}_{q^2})$. This is equivalent to require that G has no fixed points or triangles in $\text{PG}(2, q^6)$ (see Lemma 2.2). We proceed with a case-by-case analysis on maximal subgroups M of $\text{PSU}(3, q)$ and $\text{PGU}(3, q)$ containing G , as described in Theorem 2.1 and Section 3. In order to avoid the subgroups $G \leq \text{PGU}(3, q)$ which fix a point or a triangle, the following remark will be useful.

Remark 4.1. *If $G \leq \text{PGU}(3, q)$ has a cyclic normal subgroup C , then G fixes either a point or a triangle in $\text{PG}(2, q^6)$. In fact, G acts on the points fixed by a generator α of C ; hence, according to Lemma 2.2, G fixes either a point, if the type of α is (A) or (B2) or (C) or (D) or (E); or a triangle, otherwise.*

4.1. G is a subgroup of the Hessian group $H_{216} \cong \text{PGU}(3, 2)$.

Proposition 4.2. *Let $q = p^n$ be a power of an odd prime p with $3 \mid (q + 1)$, and H be a subgroup of $\text{PGU}(3, q)$ isomorphic to the Hessian group $H_{216} \cong \text{PGU}(3, 2)$. If G is a subgroup of H such that G has no fixed points or triangles in $\text{PG}(2, q^6)$, then the genus of $\mathcal{H}_q/G$ is one of the following:*

$$(2) \quad \frac{q^2 - 34q + 397 - 48\gamma - 54\delta}{432}, \quad \frac{q^2 - 10q + 133 - 54\delta}{144}, \quad \frac{q^2 - 10q + 61 - 18\delta}{72},$$

$$(3) \quad \text{where} \quad \gamma = \begin{cases} 0, & \text{if } 9 \mid (q + 1), \\ 3, & \text{if } 9 \nmid (q + 1), \end{cases} \quad \delta = \begin{cases} 0, & \text{if } 4 \mid (q + 1), \\ 2, & \text{if } 4 \mid (q - 1), \end{cases}$$

and $G \cong \text{PGU}(3, 2)$, $G \cong \text{PSU}(3, 2)$, $G \cong \text{SmallGroup}(36, 9)$, respectively. Conversely, if $\bar{g}$ is one of the integers in Equation (2), then there exists a subgroup G of H such that G has no fixed points or triangles in $\text{PG}(2, q^6)$ and $g(\mathcal{H}_q/G) = \bar{g}$.

Proof. By direct checking, $H \cong \text{PGU}(3, 2)$ has just 3 conjugacy classes of subgroups which do not admit cyclic normal subgroups; namely, they are H itself, $K \cong \text{PSU}(3, 2)$ (which is normal in H), and three groups $L_1, L_2, L_3 \cong \text{SmallGroup}(36, 9)$ of order 36 (which are normal in K). By Remark 4.1, we only have to consider the cases $G \in \{H, K, L_1\}$. To prove the claim, we will show that the first, second, and third value in Equation (2) are the genus of $\mathcal{H}_q/G$ with $G = H$, $G = K$, and $G = L_1$, respectively.

Case $G = H \cong \text{PGU}(3, 2)$. The group G contains

- a conjugacy class S_1 consisting of 9 elements of order 2;
- a conjugacy class S_2 consisting of 8 elements of order 3;
- a conjugacy class G_3 consisting of 12 subgroups of order 3; G_3 splits into 2 conjugacy classes S_3, S'_3 , each one consisting of 12 elements of order 3;
- a conjugacy class G_4 consisting of 24 subgroups of order 3; G_4 splits into 2 conjugacy classes S_4, S'_4 , each one consisting of 24 elements of order 3;
- a conjugacy class S_5 consisting of 54 elements of order 4;
- a conjugacy class G_6 consisting of 36 cyclic subgroups of order 6; G_6 splits into 2 conjugacy classes S_6, S'_6 , each one consisting of 36 elements of order 6.

We use Lemma 2.2 to determine the type of nontrivial elements $\sigma \in G$. If $\sigma \in S_1$, then σ is of type (A).

Let $\sigma \in S_5$. If $4 \mid (q - 1)$, then σ is of type (B2). If $4 \mid (q + 1)$, then σ is of type (B1). In fact, if σ is not of type (B1), then σ is of type (A) and S_5 contains only another element τ different from σ such that

$\tau^2 = \sigma^2$, namely $\tau = \sigma^{-1}$, because the homologies with given center form a cyclic group; but the number $|S_1| = 9$ of involutions in G is strictly smaller than $|S_5|$, a contradiction.

Let $\sigma \in S_6$. Recall that $6 \mid (q+1)$. Since a cyclic group of order 6 contains exactly 2 elements of order 6 and $|S_6|/2 > |C_1|$, the same argument used for S_5 shows that σ is of type (B1).

Let $\sigma \in S_2$. By direct checking, σ is contained in a subgroup $S_3 = \langle \sigma \rangle \rtimes C_2$ of H . Hence, an involution acts on the fixed points of σ , so that σ is not of type (B3), because an element of type (A) cannot fix a point which is not $\mathbb{F}_{q^2}$ -rational. The elements of S_2 , together with the identity, form an elementary abelian group $C_3 \times C_3$. Being conjugated, the elements of S_2 are all of the same type. If $C_3 \times C_3$ is generated by elements of type (A), then $C_3 \times C_3$ contains 2 elements of type (B1); in fact, using the Fermat model of $\mathcal{H}_q$ we have up to conjugation in $\text{PGU}(3, q)$ that $C_3 \times C_3 = \{\text{diag}(\lambda, \mu, 1) \mid \lambda^3 = \mu^3 = 1\}$, and $\text{diag}(\rho, \rho^{-1}, 1)$, $\text{diag}(\rho^{-1}, \rho, 1)$ are of type (B1), where ρ is a primitive cube root of unity. Therefore, the elements of S_2 are of type (B1).

Let $\sigma \in S_3$ or $\sigma \in S'_3$; since elements of S_3 and elements of S'_3 are inverse each other, they are of the same type. As $\langle \sigma \rangle$ is normalized by an involution, the same argument used for S_2 shows that σ is not of type (B3). We show that σ is of type (A); to this aim, assume by contradiction that the elements of S_3 and S'_3 are of type (B1). By direct checking, σ is contained in an elementary abelian 3 group $E = C_3 \times C_3$; E contains a subgroup $\bar{C}_3$ of order 3 whose nontrivial elements are in S_2 , while every element in $E \setminus \bar{C}_3$ is in S_3 or S'_3 . Let $\sigma' \in E \setminus \langle \sigma \rangle$, and denote by $T = \{P_1, P_2, P_3\}$ and $T' = \{P'_1, P'_2, P'_3\}$ the triangles fixed pointwise by σ and σ' , respectively. Then $T \neq T'$, since there are just 3 elements of order 3 and type (B1) which fix T pointwise. Also, T and T' have no vertex in common. In fact, σ acts on T' , because σ and σ' commute; if $T \neq T'$ and $P_1 = P'_1$ (as represented in the following picture), then σ acts on $\{P'_2, P'_3\}$ with long orbits, a contradiction to $o(\sigma) = 3$.

Therefore, the vertices of the triangles T_1, T_2, T_3, T_4 fixed pointwise by some nontrivial element of E are 12 distinct points altogether. By direct checking, the normalizer of E in G contains a subgroup $N \cong E \rtimes C_2$ of order 18. Let $C_2 = \langle \tau \rangle$. Since τ normalizes E , τ acts on T_1, T_2, T_3, T_4 . By direct checking, N contains elements of order 6. Hence, τ commutes with some nontrivial $\alpha \in E$, and thus τ fixes pointwise a triangle T_i , say $T_1 = \{P, Q, R\}$; we can assume that the center of the homology τ is P . Then the involution τ acts on the remaining 9 vertices, and thus fixes another point S , vertex of a triangle $\{S, U, V\}$. Since τ fixes S and S cannot be the center of τ , S is a point of the axis QR of τ , as represented in the following picture.

Since α acts on the line QR and α acts also on the triangle $\{S, U, V\}$, α should fix the point S , a contradiction. We have then shown that the elements in S_3 and S'_3 are of type (A).

Let $\sigma \in S_4$ or $\sigma \in S'_4$; the elements of S_4 and S'_4 are of the same type as they are inverse each other. Let $\gamma \in \{0, 3, q+1\}$ be such that $i(\sigma) = \gamma$. Then, by the Riemann-Hurwitz formula and Theorem 2.3,

$$q^2 - q - 2 = 216(2g(\mathcal{H}_q/G) - 2) + 9(q+1) + 24(q+1) + 48\gamma + 54\delta,$$

where $\delta = 0$ if $4 \mid (q+1)$, and $\delta = 2$ if $4 \mid (q-1)$. As $g(\mathcal{H}_q/G)$ is an integer, we have $(q+1)(q-26) \equiv 48\gamma \pmod{27}$. This implies $\gamma = 3$ if $9 \nmid (q+1)$, and $\gamma \in \{0, q+1\}$ if $9 \mid (q+1)$. We now suppose that $9 \mid (q+1)$ and show that $\gamma = 0$. By direct checking, H contains 8 elementary abelian 3-subgroups $E_1, \dots, E_8$ of order 9; for any $i = 1, \dots, 8$, L_i contains 2 elements of S_2 and 6 elements of $S_4 \cup S'_4$; for any $\tau \in S_2$, there exist exactly 2 indexes $i, j \in \{1, \dots, 8\}$ such that $\langle \tau \rangle \subset E_i$ and $\langle \tau \rangle \subset E_j$. Assume by contradiction that $\gamma = q+1$. Hence, the elements of $E_i \setminus \langle \tau \rangle$ and $E_j \setminus \langle \tau \rangle$ are of type (A). Then E_i and E_j are generated by elements of type (A), so that they fix pointwise two triangles T_i and T_j . Thus, τ is an element of type (B1) fixing both T_i and T_j pointwise; this implies $T_i = T_j$. This yields the contradiction $E_i = E_j$, because the pointwise stabilizer of T_i has the form $C_{q+1} \times C_{q+1}$ which has a unique elementary abelian 3-subgroup of order 9.

To sum up, the elements of S_1 , S_3 , and S'_3 are of type (A); the elements of S_2 , S_6 , and S'_6 are of type (B1); the elements of S_4 and S'_4 are of type (B3) or (B1) according to $9 \nmid (q+1)$ or $9 \mid (q+1)$, respectively; the elements of S_5 are of type (B1) or (B2) according to $4 \mid (q+1)$ and $4 \mid (q-1)$, respectively. By the Riemann-Hurwitz formula and Theorem 2.3,

$$g(\mathcal{H}_q/G) = \frac{q^2 - 34q + 397 - 54\delta}{432}, \quad \text{where } \delta = \begin{cases} 0, & \text{if } 4 \mid (q+1), \\ 2, & \text{if } 4 \mid (q-1). \end{cases}$$

Case $G = K \cong \text{PSU}(3, 2)$. The group G contains 9 elements of order 2, 54 elements of order 4, and 8 elements of order 3 contained in S_2 . By the Riemann-Hurwitz formula,

$$q^2 - q - 2 = 72(2g(\mathcal{H}_q/G) - 2) + 9(q+1) + 54\delta + 8 \cdot 0,$$

and hence

$$g(\mathcal{H}_q/G) = \frac{q^2 - 10q + 133 - 54\delta}{144}.$$

Case $G = L_1 \cong \text{SmallGroup}(36, 9)$. The group G contains 9 elements of order 2, 18 elements of order 4, and 8 elements of order 3 contained in S_2 . By the Riemann-Hurwitz formula,

$$g(\mathcal{H}_q/G) = \frac{q^2 - 10q + 61 - 18\delta}{72}.$$

Finally, we note that L_1 does not fix any point or triangle, and hence the same holds for K and H which contain L_1 . In fact, the group L_1 cannot fix any point $P \in \mathcal{H}_q$, since L_1 contains elements of type (B1). The group L_1 cannot fix any point $P \in \text{PG}(2, q^2) \setminus \mathcal{H}_q$, since L_1 contain an elementary abelian 3-subgroup $C_3 \times C_3$ whose nontrivial elements are of type (B1); so that if $\sigma_1, \sigma_2 \in (C_3 \times C_3) \setminus \{id\}$ and $\sigma_2 \notin \langle \sigma_1 \rangle$, then σ_2 acts without fixed points on the 3 points fixed by σ_1 . The group L_1 cannot fix any triangle $T \subset \mathcal{H}_q(\mathbb{F}_{q^6})$, since the stabilizer of T in $\text{PGU}(3, q)$ has odd order $3(q^2 - q + 1)$ unlike L_1 . The group L_1 cannot fix any triangle $T \subset \text{PG}(2, q^2) \setminus \mathcal{H}_q$, since the elements of order 4 should fix a vertex of T and interchange the other two vertexes of T ; so that their squares, that is the 9 involutions of L_1 , should fix T pointwise, a contradiction to the fact that there are exactly 3 involutions in $\text{PGU}(3, q)$ fixing T pointwise. $\square$

4.2. G is a subgroup of $\text{PGL}(2, q)$ preserving a conic. Recall that $\text{PGU}(3, q)$ contains a subgroup $H \cong \text{PGL}(2, q)$ preserving a conic $\mathcal{C}$, namely a Baer conic, i.e. the restriction of an irreducible conic to a Baer subplane of $\text{PG}(2, q^2)$; see [5] for a description of $\mathcal{C}$ and H .

Proposition 4.3. *Let $q = p^n$ be a power of an odd prime p , and $H \cong \text{PGL}(2, q)$ be a subgroup of $\text{PGU}(3, q)$ preserving an irreducible conic $\mathcal{C}$. If G is a subgroup of H such that G has no fixed points or triangles in $\text{PG}(2, q^6)$, then the genus of $\mathcal{H}_q/G$ is one of the following:*

$$(4) \quad \frac{q^2 - 16q + 103 - 24\gamma - 20\delta}{120},$$

when $p = 5$ or $5 \mid (q^2 - 1)$, with $G \cong A_5$ and

$$\delta = \begin{cases} 2, & \text{if either } p = 3 \text{ or } 3 \mid (q - 1), \\ 0, & \text{if } 3 \mid (q + 1), \end{cases} \quad \text{and} \quad \gamma = \begin{cases} 0, & \text{if } 5 \mid (q + 1), \\ 2, & \text{if } p = 5 \text{ or } 5 \mid (q - 1); \end{cases}$$

$$(5) \quad \frac{q^2 - q - 2 - \Delta}{\bar{q}(\bar{q} + 1)(\bar{q} - 1)} + 1,$$

where $q = \bar{q}^h$, $\bar{q} \neq 3$, $G \cong \text{PSL}(2, \bar{q})$, and

- $\Delta = +2(\bar{q} - 1)(\bar{q} + 1) + 2\frac{\bar{q}(\bar{q}+1)}{2}\left(\frac{\bar{q}-1}{2} - 2\right) + \frac{\bar{q}(\bar{q}+1)}{2}(q + 1) + \delta\frac{\bar{q}(\bar{q}-1)}{2}\left(\frac{\bar{q}+1}{2} - 1\right)$, if $\bar{q} \equiv 1 \pmod{4}$,
- $\Delta = +2(\bar{q} - 1)(\bar{q} + 1) + 2\frac{\bar{q}(\bar{q}+1)}{2}\left(\frac{\bar{q}-1}{2} - 1\right) + \frac{\bar{q}(\bar{q}-1)}{2}(q + 1) + \delta\frac{\bar{q}(\bar{q}-1)}{2}\left(\frac{\bar{q}+1}{2} - 2\right)$, if $\bar{q} \equiv 3 \pmod{4}$,

$$\text{with } \delta = \begin{cases} 2, & \text{if } h \text{ is even,} \\ 0, & \text{otherwise;} \end{cases}$$

$$(6) \quad \frac{q^2 - q - 2 - \Delta}{2\bar{q}(\bar{q} + 1)(\bar{q} - 1)} + 1,$$

where $q = \bar{q}^h$, $\bar{q} \neq 3$, $G \cong \text{PGL}(2, \bar{q})$, and

$$\Delta = 2(\bar{q} - 1)(\bar{q} + 1) + \frac{\bar{q}(\bar{q} + 1)}{2}(q + 1) + \frac{\bar{q}(\bar{q} - 1)}{2}(q + 1) + 2\frac{\bar{q}(\bar{q} + 1)}{2}(\bar{q} - 1 - 2) + \delta\frac{\bar{q}(\bar{q} - 1)}{2}(\bar{q} + 1 - 2)$$

and

$$\delta = \begin{cases} 2, & \text{if } h \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Conversely, if $\bar{g}$ is one of the integers in Equations (4) to (6), then there exists a subgroup G of H such that G has no fixed points or triangles in $\text{PG}(2, q^6)$ and $g(\mathcal{H}_q/G) = \bar{g}$.

Proof. If G has no fixed points or triangles, then the following holds: G has no cyclic normal subgroups, by Remark 4.1; G has no elementary abelian normal 2-subgroups of order 4; G has no normal p -subgroups. Then by [22, Hauptsatz 8.27] $G = K_1 \cong A_5$ with $p = 5$ or $5 \mid (q^2 - 1)$, or $G = K_2 \cong \text{PSL}(2, \bar{q})$, or $G = K_3 \cong \text{PGL}(2, \bar{q})$, with $q = \bar{q}^h$.

Elements α of order p in H are of type (D). In fact, suppose by contradiction that α is an elation. Then α fixes a point $P \in \mathcal{H}_q \cap \mathcal{C}$, where $\mathcal{C}$ is the fixed conic, and its tangent line pointwise; also, it acts with long orbits on the remaining q points belonging to each other line containing P . Since every of these lines intersects the conic $\mathcal{C}$ in either 1 or 2 points, and this intersection is preserved by σ , each line must be tangent to $\mathcal{C}$ at P , a contradiction.

Elements α of order dividing $q + 1$ and different from 2 are of type (B1). In fact, H contains a dihedral group $D = \langle \alpha \rangle \rtimes C_2$ and hence $\langle \alpha \rangle$ is normalized by an involution. If α is a homology, then α and the involution commute, a contradiction. Also, no elements of type (B3) are normalized by an involution.

Suppose that $G = K_1$. By the Riemann-Hurwitz formula and Theorem 2.3, the genus of $\mathcal{H}_q/G$ is given by Equation (4). Suppose that $G = K_2$ or $G = K_3$. The order statistics of G follows from the analysis of

$\text{PSL}(2, \bar{q})$ and $\text{PGL}(2, \bar{q})$ in [22, Chapter II.8]. Together with the Riemann-Hurwitz formula and Theorem 2.3, Equations (5) and (6) provide the genus of $\mathcal{H}_q/K_2$ and $\mathcal{H}_q/K_3$, respectively.

Finally, we show that K_1 , K_2 , and K_3 do not fix any point or triangle. The group K_1 contains 15 involutions which form 5 elementary abelian 2-groups $E_1, \dots, E_5$ of order 4, that intersect pairwise trivially. Hence, the triangles fixed pointwise by E_i and E_j are disjoint for $i \neq j$. This implies that K_1 cannot fix any point $P \in \mathcal{H}_q$ nor any self-polar triangle $T \subset \text{PG}(2, q^2) \setminus \mathcal{H}_q$; also, K_1 cannot fix any triangle $T \subset \mathcal{H}_q(\mathbb{F}_{q^6})$ as K_1 has even order. The groups K_2 and K_3 contain p -elements. The groups K_2 and K_3 cannot fix a point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$, since K_2 and K_3 contain elementary abelian 2-subgroups of order 4. The groups K_2 and K_3 cannot fix a point $P \in \text{PG}(2, q^2) \setminus \mathcal{H}_q(\mathbb{F}_{q^2})$ as they contain p -elements of type (D). The groups K_2 and K_3 cannot fix a triangle $T \subset \mathcal{H}_q(\mathbb{F}_{q^6})$ as they have even order. The groups K_2 and K_3 cannot fix a triangle $T \subset \text{PG}(2, q^2) \setminus \mathcal{H}_q$. Otherwise, they contain an abelian subgroup of index dividing 6, namely the pointwise stabilizer of T ; since an abelian subgroup has order at most $q + 1$, this yields $\bar{q} = 3$. $\square$

4.3. G is a subgroup of $\text{PSL}(2, 7)$ with $p = 7$ or $\sqrt{-7} \notin \mathbb{F}_q$.

Proposition 4.4. *Let $q = p^n$ be a power of an odd prime p , where either $p = 7$ or $\sqrt{-7} \notin \mathbb{F}_q$, and $H \leq \text{PGU}(3, q)$ be isomorphic to $\text{PSL}(2, 7)$. Then the genus of the quotient curve $\mathcal{H}_q/H$ is*

$$\frac{q^2 - 22q + 313 - 56\alpha - 48\beta - 42\gamma}{336},$$

where

$$\alpha = \begin{cases} 0, & \text{if } 3 \mid (q+1), \\ 2, & \text{otherwise;} \end{cases} \quad \beta = \begin{cases} 0, & \text{if } 7 \mid (q+1), \\ 3, & \text{if } 7 \mid (q^2 - q + 1), \\ 2, & \text{otherwise;} \end{cases} \quad \text{and } \gamma = \begin{cases} 0, & \text{if } 4 \mid (q+1), \\ 2, & \text{otherwise.} \end{cases}$$

Also, every proper subgroup of H fixes a point or a triangle.

Proof. The group H contains 21 elements of order 2, 56 elements of order 3, 48 elements of order 7, and 42 elements of order 4.

Case $p = 7$. We prove that elements of order 7 are of type (D); note that all elements $\sigma \in H$ of order 7 are of the same type, as they form a unique conjugacy class in H . Assume by contradiction that σ is of type (C). Since σ does not commute with any involution, σ acts with long orbits on the 21 centers $P_1, \dots, P_{21}$. Hence, $P_1, \dots, P_{21}$ are on three lines ℓ_1, ℓ_2, ℓ_3 containing the center P of σ , as represented in the following picture.

Consider an elementary abelian 2-subgroup E of order 4 in H , which fixes pointwise a self-polar triangle $T = \{P_1, P_2, P_3\}$ with $P_i \in \ell_1 \cup \ell_2 \cup \ell_3$. Thus, one of the following cases occurs, up to relabeling:

- (1) $P_1, P_2 \in \ell_3$ and $P_3 \in \ell_2$, or
- (2) $P_i \in \ell_i$ for $i = 1, 2, 3$.

Case (1)

Case (2)

Suppose that Case (2) occurs. Then there exists k such that $\sigma^k(P_1) = P_2$. This implies that $\sigma^k(T) = T$ and hence σ^k normalizes E , a contradiction to $7 \nmid |N_{\text{PGU}(3,q)}(E)|$. Hence Case (1) occurs. Since H contains 14 elementary abelian 2-subgroups of order 4 and just 21 involutions, there exists a self-polar triangle $T' = \{P'_1, P'_2, P'_3\} \neq T$ fixed by an elementary abelian 2-subgroup $E' \neq E$ such that $|E \cap E'| = 2$. Hence two vertices of T and T' coincide, say $P_1 = P'_1$ and $P_2 = P'_2$, while $P_3 \neq P'_3$ as $E \neq E'$. This is a contradiction, because P_3 is uniquely determined by its polar line P_2P_3 . This shows that the elements of order 7 in H are not elations, and hence are of type (D). The claim follows by the Riemann-Hurwitz formula and Theorem 2.3.

Case $p \neq 7$ and $\sqrt{-7} \notin \mathbb{F}_q$. The condition $\sqrt{-7} \notin \mathbb{F}_q$ implies that $p \equiv 2, 5, 6 \pmod{7}$ and n is odd. Note that H contains dihedral groups of order 6 and 8, proving that if $3 \mid (q+1)$ (resp. $4 \mid (q+1)$) then an element of order 3 (resp. of order 4) is of type (B1). The claim follows by the Riemann-Hurwitz formula and Theorem 2.3.

If K is a proper subgroup of H , then either K contains a cyclic normal subgroup of order 3 or 7, and hence K fixes a point or a triangle by Remark 4.1; or K contains a elementary abelian normal 2-subgroup of order 4, and hence K fixes a self-polar triangle. $\square$

Remark 4.5. All the congruences of p modulo 4 and 3 listed in Proposition 4.4 can occur. Table 1 provides a list of examples.

TABLE 1. Possible values for q in Proposition 4.4

$q \equiv_4$	$q \equiv_3$	$q \equiv_7$	Example for q
3	1	6	139
3	1	3	31
3	1	5	19
3	2	6	84
3	2	3	59
3	2	5	47
1	1	6	13
1	1	3	73
1	1	5	61
1	2	6	41
1	2	3	17
1	2	5	5

4.4. G is a subgroup of $\text{SmallGroup}(720, 765)$, when q is an odd power of 5.

Proposition 4.6. *Let q be an odd power of 5 and H be a subgroup of $\text{PGU}(3, q)$, $H \cong \text{SmallGroup}(720, 765)$. If G is a subgroup of H such that G has no fixed points or triangles in $\text{PG}(2, q^6)$, then the genus of $\mathcal{H}_q/G$ is one of the following:*

$$(7) \quad \frac{q^2 - 10q + 25}{72}, \quad \frac{q^2 - 16q + 55}{120}, \quad \frac{q^2 - 10q + 25}{144}, \quad \frac{q^2 - 46q + 205}{720}, \quad \frac{q^2 - 46q + 205}{1440}.$$

where $G \cong \text{SmallGroup}(36, 9)$, $G \cong A_5$, $G \cong \text{PSU}(3, 2)$, $G \cong A_6$, $G = H$, respectively. Conversely, if $\bar{g}$ is one of the integers in Equation (7), then there exists a subgroup G of H such that G has no fixed points or triangles in $\text{PG}(2, q^6)$ and $g(\mathcal{H}_q/G) = \bar{g}$.

Proof. By direct checking, either $G \cong \text{SmallGroup}(36, 9)$, $G \cong A_5$, $G \cong \text{PSU}(3, 2)$, $G \cong A_6$, $G = H$; or G contains a normal subgroup which is cyclic or elementary abelian of order 4, and hence G fixes a point or a triangle. If $G \cong \text{SmallGroup}(36, 9)$ or $G \cong \text{PSU}(3, 2)$, or $G \cong A_5$, then G has no fixed points or triangles and the genus of $\mathcal{H}_q/G$ is computed in Propositions 4.2 or 4.3, respectively.

Elements σ of order 3 are of type (B1); in fact, H contains dihedral subgroups of order 6 containing σ , implying that σ cannot be neither of type (B3) nor of type (A). Elements of order 5 are of type (D), because they are contained in dihedral groups of order 10. The genus of $\mathcal{H}_q/G$ for $G \cong A_6$ and $G = H$ can be computed by the Riemann-Hurwitz formula and Theorem 2.3.

Finally we note that, if $G = H$ or $G \cong A_6$, then G does not fix any point or triangle, because G contains a subgroup isomorphic to A_5 which fixes no points or triangles. $\square$

4.5. G is a subgroup of A_6 , when q is an even power of $p = 3$, or 5 is a square in $\mathbb{F}_q$ but $\mathbb{F}_q$ contains no primitive cube roots of unity.

Proposition 4.7. *Let $q = p^n$ be a power of an odd prime p , where either $p = 3$ and n is even, or $\sqrt{5} \in \mathbb{F}_q$ and $\mathbb{F}_q$ contains no primitive cube roots of unity. Let H be a maximal subgroup of $\text{PSU}(3, q)$ isomorphic to the alternating group A_6 . If G is a subgroup of H such that G has no fixed points or triangles in $\text{PG}(2, q^6)$, then the genus of $\mathcal{H}_q/G$ is one of the following:*

$$(8) \quad \frac{q^2 - 46q + 673 - 80\alpha - 90\beta - 144\gamma}{720}, \quad \frac{q^2 - 16q + 103 - 20\alpha - 24\gamma}{120},$$

$$(9) \quad \frac{q^2 - 10q + 61 - 18\beta}{72},$$

where

$$\alpha = \begin{cases} 2, & \text{if } p = 3, \\ 0, & \text{otherwise;} \end{cases} \quad \beta = \begin{cases} 2, & \text{if } p = 3 \text{ or } q \equiv 1 \pmod{4}, \\ 0, & \text{otherwise;} \end{cases} \quad \gamma = \begin{cases} 0, & \text{if } 5 \mid (q + 1), \\ 2, & \text{otherwise,} \end{cases}$$

and $G \cong A_6$, $G \cong A_5$, $G \cong \text{SmallGroup}(36, 9)$, respectively. Conversely, if $\bar{g}$ is one of the integers in Equation (8), or $p \neq 3$ and $\bar{g}$ is the integer in Equation (9), then there exists a subgroup G of H such that G has no fixed points or triangles and $g(\mathcal{H}_q/G) = \bar{g}$.

Proof. If G has a normal subgroup which is either cyclic or of order 4, then G fixes a point or a triangle. Hence, we can assume that $G \cong A_6$, or $G \cong A_5$, or $G \cong \text{SmallGroup}(36, 9)$. Also, if $p = 3$ and $G \cong \text{SmallGroup}(36, 9)$, then G fixes a point $P \in \mathcal{H}_q(\mathbb{F}_{q^2})$ because G has a normal Sylow 3-subgroup.

Elements of order 3 are either of type (D) or of type (B1), according to $p = 3$ or $p \neq 3$, because they are contained in dihedral subgroups of order 6. If $p = 5$ or $5 \mid (q + 1)$, then elements of order 5 are of type (D) or (B1), respectively, because they are contained in dihedral subgroups of order 10. If $4 \mid (q + 1)$, then

elements of order 4 are of type (B1), because they are contained in dihedral subgroups of order 8. Then Equation (8) follows from the Riemann-Hurwitz formula together with Theorem 2.3.

If $p \neq 3$ and $G \cong \text{SmallGroup}(36, 9)$, then by Proposition 4.2 G fixes no points nor triangles; if $p \neq 3$ and $G \cong A_5$ or $G \cong A_6$, then G fixes no points or triangles because G contains $\text{SmallGroup}(36, 9)$. If $p = 3$ and $G \cong A_5$, then G cannot fix any point or triangle, since G contains both elements of order 3 and elementary abelian 2-subgroups of order 4. If $p = 3$ and $G \cong A_6$, then G fixes no points nor triangles as G contains A_5 . $\square$

4.6. G is a subgroup of A_7 , when q is an odd power of $p = 5$.

Proposition 4.8. *Let q be an odd power of 5 and let H be a subgroup of $\text{PGU}(3, q)$ isomorphic to the alternating group A_7 . If G is a subgroup of H such that G has no fixed points or triangles in $\text{PG}(2, q^6)$, then the genus of $\mathcal{H}_q/G$ is one of the following:*

$$(10) \quad \frac{q^2 - 106q + 2665 - 720\beta}{5040}, \quad \frac{q^2 - 46q + 205}{720}, \quad \frac{q^2 - 22q + 229 - 48\beta}{336},$$

$$(11) \quad \frac{q^2 - 26q + 105}{240}, \quad \frac{q^2 - 16q + 55}{120}, \quad \frac{q^2 - 10q + 25}{72},$$

where

$$\beta = \begin{cases} 0, & \text{if } 7 \mid (q+1), \\ 3, & \text{otherwise,} \end{cases}$$

and $G \cong A_7$, $G \cong A_6$, $G \cong \text{PSL}(2, 7)$, $G \cong A_5 \rtimes C_2$, $G \cong A_5$, $G \cong \text{SmallGroup}(36, 9)$, respectively. Conversely, if $\bar{g}$ is one of the integers in Equations (10) and (11), then there exists a subgroup G of H such that G has no fixed points or triangles and $g(\mathcal{H}_q/G) = \bar{g}$.

Proof. By direct checking, either $G \cong A_7$, $G \cong A_6$, $G \cong \text{PSL}(2, 7)$, $G \cong A_5 \rtimes C_2$, $G \cong A_5$, $G \cong \text{SmallGroup}(36, 9)$, or G a normal subgroup which is cyclic or of order 4 and hence G fixes a point or a triangle. If $G \cong A_6$ or $G \cong \text{PSL}(2, 7)$ or $G \cong A_5$ or $G \cong \text{SmallGroup}(36, 9)$, then G has no fixed points or triangles and the genus of $\mathcal{H}_q/G$ is computed by Propositions 4.4, 4.7, and 4.2.

Elements of order 3 (resp. 5) are of type (B1) because they are contained in dihedral subgroups of order 6 (resp. 10). Elements of order 6 are of type (B1) because their squares are of type (B1). If $7 \mid (q+1)$, then elements of order 7 are of type (B1), since they are contained in semidirect products of order 21 but not in cyclic subgroups of order 21. Now Equations (10) and (11) follow from the Riemann-Hurwitz formula and Theorem 2.3.

If $G \cong A_5 \rtimes C_2$ or $G = H$, then G has no fixed points or triangles because G contains A_5 . $\square$

4.7. G is a subgroup of $\text{PGU}(3, p^m)$ with $m \mid n$ and n/m odd. Let G be a subgroup of $\text{PGU}(3, p^m)$ with $m \mid n$ and n/m odd such that G does not fix any point or triangle in $\text{PG}(2, q^6)$. By Theorem 3.1 and Proposition 3.14, either $G \cong \text{SmallGroup}(36, 9)$ when q is an odd power of 2; or $G \cong \text{PSU}(3, p^k)$ or $G \cong \text{PGU}(3, p^k)$ where $k \mid m$ and m/k is odd.

Proposition 4.9. *Let $q = p^n$ be a power of a prime p , and G be a subgroup of $\text{PGU}(3, q)$ such that either*

- (i) q is an odd power of 2 and $G = G_1 \cong \text{SmallGroup}(36, 9)$; or
- (ii) $G = G_2 \cong \text{PGU}(3, p^k)$ with $k \mid n$ and n/k odd; or
- (iii) $G = G_3 \cong \text{PSU}(3, p^k)$ with $3 \mid (q+1)$, $k \mid n$ and n/k odd.

Then

$$(12) \quad g(\mathcal{H}_q/G_1) = \frac{q^2 - 10q + 16}{72};$$

$$(13) \quad g(\mathcal{H}_q/G_2) = 1 + \frac{q^2 - q - 2 - \Delta}{2\bar{q}^3(\bar{q}^3 + 1)(\bar{q}^2 - 1)},$$

where

$$\Delta = (\bar{q} - 1)(\bar{q}^3 + 1) \cdot (q + 2) + (\bar{q}^3 - \bar{q})(\bar{q}^3 + 1) \cdot 2 + \bar{q}(\bar{q}^4 - \bar{q}^3 + \bar{q}^2) \cdot (q + 1) \\ + (\bar{q}^2 - \bar{q} - 2) \frac{(\bar{q}^3 + 1)\bar{q}^3}{2} \cdot 2 + (\bar{q} - 1)\bar{q}(\bar{q}^3 + 1)\bar{q}^2 \cdot 1 + (\bar{q}^2 - \bar{q}) \frac{\bar{q}^6 + \bar{q}^5 - \bar{q}^4 - \bar{q}^3}{3} \cdot \gamma,$$

with

$$\gamma = \begin{cases} 3, & \text{if } (\bar{q}^2 - \bar{q} + 1) \mid (q^2 - q + 1) \text{ and } \bar{q} \neq 2, \\ 0, & \text{if } (\bar{q}^2 - \bar{q} + 1) \mid (q + 1) \text{ and } \bar{q} \neq 2, \\ 3, & \text{if } \bar{q} = 2 \text{ and } 3 \mid (q + 1) \text{ and } 3 \nmid n, \\ 0, & \text{if } \bar{q} = 2 \text{ and } 3 \mid (q + 1) \text{ and } 3 \mid n; \end{cases}$$

$$(14) \quad g(\mathcal{H}_q/G_3) = \frac{3(q^2 - q - 2 - \Delta)}{2\bar{q}^3(\bar{q}^2 - 1)(\bar{q}^3 + 1)} + 1,$$

where

$$\Delta = (\bar{q} - 1)(\bar{q}^3 + 1) \cdot (q + 2) + (\bar{q}^3 - \bar{q})(\bar{q}^3 + 1) \cdot 2 + ((\bar{q} + 1)/3 - 1)(\bar{q}^4 - \bar{q}^3 + \bar{q}^2) \cdot (q + 1) \\ + ((\bar{q}^2 - 1)/3 - (\bar{q} + 1)/3) \frac{(\bar{q}^3 + 1)\bar{q}^3}{2} \cdot 2 + (\bar{q} - 1)((\bar{q} + 1)/3 - 1)(\bar{q}^3 + 1)\bar{q}^2 \cdot 1 + ((\bar{q}^2 - \bar{q} + 1)/3 - 1) \frac{\bar{q}^6 + \bar{q}^5 - \bar{q}^4 - \bar{q}^3}{3} \cdot \delta,$$

with

$$\delta = \begin{cases} 3, & \text{if } (\bar{q}^2 - \bar{q} + 1)/3 \mid (q^2 - q + 1), \\ 0, & \text{if } (\bar{q}^2 - \bar{q} + 1)/3 \mid (q + 1). \end{cases}$$

Proof. Suppose that $G = G_1$. Elements of order 3 are of type (B1), since they are contained in dihedral subgroups of order 6. Then Equation (12) follows from the Riemann-Hurwitz formula and Theorem 2.3.

Suppose that $G = G_2$. Then the genus of $\mathcal{H}_q/G_2$ has been computed in [28, Proposition 5.1], and Equation (13) follows.

Suppose that $G = G_3$. As pointed out in Lemma 3.11, we can assume up to conjugation in $\text{PGU}(3, q)$ that G is the subgroup of elements $\sigma \in \text{PSU}(3, q)$ such that σ is defined over $\mathbb{F}_{\bar{q}^2}$. First, we classify the elements of $\text{PSU}(3, \bar{q})$ seen as the automorphism group of a Hermitian curve $\mathcal{H}_{\bar{q}}$, using the order statistics of $\text{PSU}(3, \bar{q})$.

- (1) There are exactly $(\bar{q} - 1)(\bar{q}^3 + 1)$ elements of type (C). In fact, for each $P \in \mathcal{H}_{\bar{q}}(\mathbb{F}_{\bar{q}^2})$, there exist exactly $\bar{q} - 1$ elations in $\text{PSU}(3, \bar{q})$ with center P .
- (2) There are exactly $(\bar{q}^3 - \bar{q})(\bar{q}^3 + 1)$ elements of type (D). In fact, they are the p -elements of $\text{PSU}(3, \bar{q})$ which are not of type (C).
- (3) There are exactly $((\bar{q} + 1)/3 - 1)(\bar{q}^4 - \bar{q}^3 + \bar{q}^2)$ elements of type (A). In fact, for each $P \in \text{PG}(2, \bar{q}^2) \setminus \mathcal{H}_{\bar{q}}$ there exist exactly $(\bar{q} + 1)/3 - 1$ homologies in $\text{PSU}(3, \bar{q})$ with center P .
- (4) There are exactly $((\bar{q}^2 - 1)/3 - (\bar{q} + 1)/3) \frac{(\bar{q}^3 + 1)\bar{q}^3}{2}$ elements of type (B2). In fact, for each pair $\{P, Q\} \subset \mathcal{H}_{\bar{q}}(\mathbb{F}_{\bar{q}^2})$ there exist exactly $(\bar{q}^2 - 1)/3 - 1$ nontrivial elements of $\text{PSU}(3, \bar{q})$ fixing P and Q , and $(\bar{q} + 1)/3 - 1$ of them are homologies with axis PQ .
- (5) There are exactly $((\bar{q}^2 - \bar{q} + 1)/3 - 1) \frac{\bar{q}^6 + \bar{q}^5 - \bar{q}^4 - \bar{q}^3}{3}$ elements of type (B3). In fact, any point $P \in \mathcal{H}_{\bar{q}^2}(\mathbb{F}_{\bar{q}^6}) \setminus \mathcal{H}_{\bar{q}}(\mathbb{F}_{\bar{q}^2})$ determines a unique triangle $\{P, \Phi_{\bar{q}^2}(P), \Phi_{\bar{q}^2}^2(P)\} \subset \mathcal{H}_{\bar{q}}(\mathbb{F}_{\bar{q}^6})$ fixed by a Singer subgroup of order $(\bar{q}^2 - \bar{q} + 1)/3$; see [29].
- (6) There are exactly $(\bar{q} - 1)((\bar{q} + 1)/3 - 1)(\bar{q}^3 + 1)\bar{q}^2$ elements of type (E). In fact, consider a pair $\{P, Q\}$ with $P \in \mathcal{H}_{\bar{q}}(\mathbb{F}_{\bar{q}^2})$ and $Q \in \text{PG}(2, \bar{q}^2) \cap \ell_P$, where ℓ_P is the tangent line to $\mathcal{H}_{\bar{q}}$ at P . Any element of type (E) fixing P and Q is uniquely obtained as the product of an elation of center P and a homology of center Q ; thus there exist exactly $(\bar{q} - 1)((\bar{q} + 1)/3 - 1)$ such elements.
- (7) The remaining $(q^8 - 3q^7 + 8q^6 - 11q^5 + 9q^4 - 4q^3)/18 = q^3(q - 1)(q^2 - q + 1)(q^2 - q + 4)/18$ nontrivial elements are of type (B1).

Now we describe the elements in each class (1) – (7) according to their geometry with respect to $\mathcal{H}_q$.

- (i) The elements in class (1) are of type (C). In fact, let $\bar{S}$ be one of the $\bar{q}^3 + 1$ Sylow p -subgroups of $\text{PSU}(3, \bar{q})$ and S be the Sylow p -subgroup of $\text{PSU}(3, q)$ containing $\bar{S}$. Note that S is a trivial intersection set, since $\mathcal{H}_q$ has zero p -rank; see [20, Theorem 11.133]. Consider the explicit representation of S given in [12, Section 3], where $\mathcal{H}_q$ has norm-trace equation and S fixes the point at infinity. By direct computation, there are $\bar{q}$ elements of $\bar{S}$ in the center of S . Thus, $\bar{S}$ has exactly $\bar{q} - 1$ elements of type (C); see [12, Equation (2.12)].
- (ii) The elements in class (2) are of type (D). In fact, with $\bar{S}$ as in Case (i), the claim follows from Case (i) counting the remaining nontrivial elements of $\bar{S}$.
- (iii) Let $\sigma \in \text{PGU}(3, \bar{q})$ be in class (3). Then σ is contained in the pointwise stabilizer D of a self-polar triangle with respect to $\mathcal{H}_{\bar{q}}$ and D is an abelian group of order $(\bar{q} + 1)^2/3$; see [27]. Let $C_{q+1} \times C_{q+1} \leq \text{PGU}(3, q)$ be the pointwise stabilizer of a self-polar triangle $\mathcal{T}$ with respect to $\mathcal{H}_q$, such that $D \leq (C_{q+1} \times C_{q+1}) \cap \text{PSU}(3, q)$ which is an abelian group of order $(q + 1)^2/3$.

Up to conjugation, $\mathcal{H}_q$ has Fermat equation and $\mathcal{T}$ is the fundamental triangle, so that

$$D = \{(X, Y, Z) \mapsto (\lambda X, \mu Y, Z) \mid \lambda^{\bar{q}+1}, \mu^{\bar{q}+1} = 1\} \cap \text{PSU}(3, q).$$

By direct computation, D contains $3((\bar{q} + 1)/3 - 1)$ elements of type (A) and $(\bar{q}^2 - \bar{q} + 4)/3$ elements of type (B1).

Since $\text{PSU}(3, \bar{q})$ is transitive on $\text{PG}(2, \bar{q}^2) \setminus \mathcal{H}_q$, there are exactly $\frac{|\text{PSU}(3, \bar{q})|}{6|D|}$ self-polar triangles $\mathcal{T}'$ with respect to $\mathcal{H}_q$, whose pointwise stabilizer D' is conjugated to D under $\text{PSU}(3, \bar{q})$.

Note that D and D' intersect non-trivially if and only if $\mathcal{T}$ and $\mathcal{T}'$ have a vertex P in common; in this case, $(D \cap D') \setminus \{id\}$ is made by $(\bar{q} + 1)/3 - 1$ homologies with center P . The number of points in $\text{PG}(2, \bar{q}^2) \setminus \mathcal{H}_q$ lying on the polar of P is $\bar{q}^2 - \bar{q}$; hence, the number of D' which intersect D non-trivially is $((\bar{q} + 1)/3 - 1)(\bar{q} - 1)/2$.

Therefore, by direct computation, the number of elements of type (A) in $\text{PSU}(3, \bar{q})$ is exactly the number $((\bar{q} + 1)/3 - 1)(\bar{q}^4 - \bar{q}^3 + \bar{q}^2)$ of elements in class (3), and the remaining $(q^8 - 3q^7 + 8q^6 - 11q^5 + 9q^4 - 4q^3)/18$ elements of the subgroups of $\text{PSU}(3, \bar{q})$ conjugated to D are of type (B1) and in class (7).

- (iv) Let $\sigma \in \text{PSU}(3, \bar{q})$ be in class (4). Since the order $o(\sigma)$ of σ divides $\bar{q}^2 - 1$ but not $\bar{q} + 1$, we have that $o(\sigma)$ divides $q^2 - 1$ but not $q + 1$, as q is an odd power of $\bar{q}$. Therefore σ is of type (B2).
- (v) Let $\sigma \in \text{PSU}(3, \bar{q})$ be in class (6). Since the order of σ is $p \cdot d$ where $d > 1$ and $p \nmid d$, σ is of type (E).
- (vi) Let $\sigma \in \text{PSU}(3, \bar{q})$ be in class (5). By direct checking, the order $o(\sigma)$ of σ divides either $q^2 - q + 1$ or $q + 1$. Note that $(\bar{q}^2 - \bar{q} + 1)/3$ is not divisible by 3. If $(\bar{q}^2 - \bar{q} + 1) \mid (q^2 - q + 1)$ then every σ in class (5) is of type (B3), by [29].

If $(\bar{q}^2 - \bar{q} + 1) \mid (q + 1)$ then σ is either of type (A) or (B1). We note that σ is contained in the maximal subgroup (iv) of $\text{PSU}(3, \bar{q})$ in [27], which is a semidirect product $C_{(\bar{q}^2 - \bar{q} + 1)/3} \rtimes C_3$, where $\sigma \in C_{(\bar{q}^2 - \bar{q} + 1)/3}$ and C_3 does not commute with any subgroup of $C_{(\bar{q}^2 - \bar{q} + 1)/3}$. Therefore σ cannot be of type (A), because otherwise every element which normalizes σ should commute with σ . Then σ is of type (B1).

Now Equation (14) follows from the Riemann-Hurwitz formula and Theorem 2.3. $\square$

4.8. New genera of $\mathbb{F}_{q^2}$ -maximal curves. The results of Section 4 provide new genera of maximal curves over finite fields, with respect to the genera obtained in [1, 2, 3, 7, 8, 9, 10, 12, 15, 16, 17, 18, 26, 28, 30]. Table 2 collects some examples.

REFERENCES

- [1] Abdón, M., Quoos, L.: On the genera of subfields of the Hermitian function field, *Finite Fields Appl.* **10** (3), 271–284 (2004).

TABLE 2. New genera for maximal curves

F	new genera for F -maximal curves
$\mathbb{F}_{2^{14}}$	210
$\mathbb{F}_{2^{18}}$	3570
$\mathbb{F}_{2^{22}}$	9548, 57970
$\mathbb{F}_{3^8}$	44
$\mathbb{F}_{3^{12}}$	692, 4332, 14328
$\mathbb{F}_{3^{16}}$	59368, 357848
$\mathbb{F}_{3^{18}}$	19358
$\mathbb{F}_{5^6}$	39, 114, 200
$\mathbb{F}_{5^8}$	3172
$\mathbb{F}_{5^{10}}$	6682, 13364, 22360, 40352, 80964
$\mathbb{F}_{7^6}$	328
$\mathbb{F}_{7^8}$	17000, 47720
$\mathbb{F}_{11^4}$	106
$\mathbb{F}_{11^6}$	1286, 12211
$\mathbb{F}_{17^6}$	4776, 55488, 71517
$\mathbb{F}_{19^4}$	1038
$\mathbb{F}_{19^6}$	6706

- [2] Anbar, N., Bassa, A., Beelen, P.: A complete characterization of Galois subfields of the generalized Giulietti-Korchmáros function field, *Finite Fields Appl.* **49**, 132–143 (2018).
- [3] Bassa, A., Ma, L., Xing, C., Yeo, S.L.: Toward a characterization of subfields of the Deligne-Lusztig function fields, *J. Combin. Theory Ser. A* **120** (7), 1351–1371 (2013).
- [4] Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system. I. The user language, *J. Symbolic Comput.* **24** (3–4), 235–265 (1997).
- [5] Cossidente, A., Ebert, G.L.: Permutable polarities and a class of ovoids of the Hermitian surface, *European J. Combin.* **25** (7), 1059–1066 (2004).
- [6] Cossidente, A., Korchmáros, G., Torres, F.: On curves covered by the Hermitian curve, *J. Algebra* **216** (1), 56–76 (1999).
- [7] Cossidente, A., Korchmáros, G., Torres, F.: Curves of large genus covered by the Hermitian curve, *Comm. Algebra* **28** (10), 4707–4728 (2000).
- [8] Dalla Volta, F., Montanucci, M., Zini, G.: On the classification problem for the genera of quotients of the Hermitian curve, *Comm. Algebra* (2019), DOI 10.1080/00927872.2019.1601733.
- [9] Danisman, Y., Özdemir, M.: On the genus spectrum of maximal curves over finite fields, *J. Discr. Math. Sc. and Crypt.* **18** (5), 513–529 (2015).
- [10] Fanali, S., Giulietti, M.: Quotient curves of the GK curve, *Adv. Geom.* **12** (2), 239–268 (2012).
- [11] Garcia, A., Stichtenoth, H.: Algebraic function fields with many rational places, *IEEE Trans. Inf. Theory* **41**, 1548–1563 (1995).
- [12] Garcia, A., Stichtenoth, H., Xing, C.: On subfields of the Hermitian function field, *Compositio Math.* **120** (2), 137–170 (2000).
- [13] Giulietti, M., Hirschfeld, J.W.P., Korchmáros, G., Torres, F.: A family of curves covered by the Hermitian curve, *Sémin. Congr.* **21**, 63–78 (2010).
- [14] Giulietti, M., Korchmáros, G.: Algebraic curves with many automorphisms, *Adv. Math.* **349**, 162–211 (2019).
- [15] Giulietti, M., Korchmáros, G., Torres, F.: Quotient curves of the Suzuki curve, *Acta Arith.* **122** (3), 245–274 (2006).
- [16] Giulietti, M., Montanucci, M., Quoos, L., Zini, G.: On some Galois covers of the Suzuki and Ree curves, *J. Number Theory* **189**, 220–254 (2018).
- [17] Giulietti, M., Quoos, L., Zini, G.: Maximal curves from subcovers of the GK-curve, *J. Pure Appl. Algebra* **220**, 3372–3383 (2016).
- [18] Güneri, C., Özdemir, M., Stichtenoth, H.: The automorphism group of the generalized Giulietti-Korchmáros function field, *Adv. Geom.* **13**, 369–380 (2013).
- [19] Hartley, R.W.: Determination of the ternary collineation groups whose coefficients lie in the $GF(2^n)$, *Ann. of Math. Second Series* **27** (2), 140–158 (1925).
- [20] Hirschfeld, J.W.P., Korchmáros, G., Torres, F.: *Algebraic Curves over a Finite Field*. Princeton Series in Applied Mathematics, Princeton (2008).

- [21] Hughes, D.R., Piper, F.C.: *Projective Planes*. Graduate Text in Mathematics **6**, Springer, Berlin (1973).
- [22] Huppert, B.: *Endliche Gruppen. I*, Grundlehren der Mathematischen Wissenschaften **134**, Springer, Berlin (1967).
- [23] Kantor, W.M., O’Nan, M.E., Seitz, G.M.: 2-transitive groups in which the stabilizer of two points is cyclic, *J. Algebra* **21**, 17–50 (1972).
- [24] Kleiman, S.L.: Algebraic cycles and the Weil conjectures, in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. Pure Math. **3**, 359–386 (1968).
- [25] Lachaud, G.: Sommes d’Eisenstein et nombre de points de certaines courbes algébriques sur les corps finis, *C.R. Acad. Sci. Paris* 305, Série I, 729–732 (1987).
- [26] Ma, L., Xing, C.: On subfields of the Hermitian function fields involving the involution automorphism, *J. Number Theory* **198**, 293–317 (2019).
- [27] Mitchell, H.H.: Determination of the ordinary and modular ternary linear groups, *Trans. Amer. Math. Soc.* **12** (2), 207–242 (1911).
- [28] Montanucci, M., Zini, G.: On the spectrum of genera of quotients of the Hermitian curve, *Comm. Algebra* **46** (11), 4739–4776 (2018).
- [29] Montanucci, M., Zini, G.: Some Ree and Suzuki curves are not Galois covered by the Hermitian curve, *Finite Fields Appl.* **48**, 175–195 (2017).
- [30] Montanucci, M., Zini, G.: The complete list of genera of quotients of the $\mathbb{F}_{q^2}$ -maximal Hermitian curve for $q \equiv 1 \pmod{4}$, arXiv: 1806.04546.
- [31] Stichtenoth, H.: *Algebraic function fields and codes*, 2nd edn. Graduate Texts in Mathematics **254**. Springer, Berlin (2009).
- [32] van der Geer, G.: Curves over finite fields and codes, in: European Congress of Mathematics, vol. II (Barcelona 2000), *Progr. Math.* **202**, Birkhäuser, Basel, 225–238 (2001).
- [33] van der Geer, G.: Coding theory and algebraic curves over finite fields: a survey and questions, in: Applications of Algebraic Geometry in Coding Theory, Physics and Computation, NATO Sci. Ser. II Math. Phys. Chem. **36**, Kluwer, Dordrecht, 139–159 (2001).
- [34] Wall, C.T.C.: On the structure of finite groups with periodic cohomology. Lie groups: structure, actions, and representations, 381–413, *Progr. Math.* **306**, Birkhäuser/Springer, New York, (2013).

DEPARTMENT OF APPLIED MATHEMATICS AND COMPUTER SCIENCE, TECHNICAL UNIVERSITY OF DENMARK, ASMUSSENS ALLÉ,
BUILDING 303B, DK-2800 KONGENS LYNGBY, DENMARK
Email address: marimo@dtu.dk

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA, MILANO, ITALY
Email address: giovanni.zini@unimib.it