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Locally recoverable codes from automorphism group of function fields of genus $g \geq 1$

Daniele Bartoli, Maria Montanucci, and Luciane Quoos

Abstract—A Locally Recoverable Code is a code such that the value of any single coordinate of a codeword can be recovered from the values of a small subset of other coordinates. When we have δ non-overlapping subsets of cardinality r_i that can be used to recover the missing coordinate we say that a linear code $\mathcal{C}$ with length n , dimension k , minimum distance d has $(r_1, \dots, r_\delta)$ -locality and denote by $[n, k, d; r_1, r_2, \dots, r_\delta]$. In this paper we provide a new upper bound for the minimum distance of these codes. Working with a finite number of subgroups of cardinality $r_i + 1$ of the automorphism group of a function field $\mathcal{F}|\mathbb{F}_q$ of genus $g \geq 1$ we propose a construction of $[n, k, d; r_1, r_2, \dots, r_\delta]$ -codes and apply the results to some well known families of function fields.

Index Terms—Codes for distributed storage; Locally Recoverable Codes; Hamming distance; covering maps; maximal curves.

I. INTRODUCTION

The study of Locally Recoverable codes (LRC for short) was motivated by the use of coding theory techniques applied to distributed and cloud storage systems. Local recovery techniques enable the repair of lost encoded data by a local procedure, that is by making use of small amount of data instead of all information contained in a codeword. Formally, a LRC code of length n is a code that produces an n -symbol codeword from k information symbols and, for any symbol of the codeword, there exist at most r other symbols such that the value of the symbol can be recovered from them. This value r is called the locality of the code. For example, a code of length $2k$ in which each coordinate is repeated twice, is an LRC code with locality $r = 1$. Generally the locality parameter satisfies $1 \leq r \leq k$ since the entire codeword can be found by accessing k symbols other than the erased symbol.

In distributed storage systems, erasure codes with locality r are preferred because a coordinate can be locally repaired by accessing at most r other coordinates. However, the local

repair may not be performed when some of the r coordinates are also erased. To overcome this problem, we can work with δ non-overlapping local repair sets of size no more than r_i for a coordinate. The formal definition of a Locally Recoverable Code is given in Definition II.1. We denote a linear code $\mathcal{C}$ with length n , dimension k , minimum distance d , and $(r_1, \dots, r_\delta)$ -locality by $[n, k, d; r_1, r_2, \dots, r_\delta]$.

In recent years, the study of locally repairable codes has attracted a lot of attention. Most of the results concern bounds on the minimum distance [9], [18], [27], [28] and construction of LRC codes [5], [11], [14]–[16], [19], [20], [22], [23], [26], [28].

The goal of this paper is twofold. First we present a new upper bound on the minimum distance of a $[n, k, d; r_1, r_2, \dots, r_\delta]$ -code. Second we present two constructions of LRCs over function fields $\mathcal{F}|\mathbb{F}_q$ of genus $g \geq 1$ using subgroups of their automorphism group. We apply these constructions on some well known families of curves with many rational points.

Let $\mathcal{C}$ be an $[n, k, d; r_1, \dots, r_\delta]$ -code. If $\delta = 1$ it was proved in 2012 that the minimum distance of the code is upper bounded by

$$d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2; \quad (1)$$

see [9]. This bound coincides with the classical Singleton bound when $r = k$. A code that achieves equality in (1) is called an optimal LRC code. In [19], [20], [26] the authors constructed optimal LRC codes using particular types of polynomials, cyclic codes, and elliptic curves respectively.

For general $\delta \geq 1$ and $r = r_1 = \dots = r_\delta$ the bound (1) was generalized in 2014 to

$$d \leq n - \sum_{i=0}^t \left\lceil \frac{k-1}{r^i} \right\rceil; \quad (2)$$

see [26].

Since in our paper we deal with general $[n, k, d; r_1, \dots, r_\delta]$ -codes, in order to check the quality of the constructed codes, exploring ideas from [28] we first generalize (1) to

$$d \leq n - k - \left\lceil \frac{(k-1)\delta + 1}{1 + \sum_{i=1}^{\delta} r_i} \right\rceil + 2; \quad (3)$$

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see Theorem III.2. In analogy with the definition of the singleton defect for linear codes, a notion of relative defect for $[n, k, d; r_1, \dots, r_\delta]$ -code is introduced using the bound in (3) as

$$\Delta(\mathcal{C}) = \frac{1}{n} \left(n - k - d + 2 - \left\lfloor \frac{(k-1)\delta + 1}{1 + \sum_{i=1}^{\delta} r_i} \right\rfloor \right). \quad (4)$$

Intuitively, the smaller the relative defect of a code, the better the code.

In this work we also provide two general constructions of LRC codes with locality $(r_1, \dots, r_\delta)$ over function fields $\mathcal{F}|\mathbb{F}_q$ of genus $g \geq 1$ using a finite number of subgroups of cardinality $r_i + 1$ of the automorphism group $\text{Aut}(\mathcal{F}|\mathbb{F}_q)$. We distinguish the cases of trivial intersection (Theorem IV.1) and nontrivial intersection (Theorem V.1) and we provide a generalization of the construction proposed in [15] for the rational function field.

It is well known that maximal function fields and function fields with many rational points provide algebraic geometry codes with “good” parameters; see e.g. [2]–[4], [13]. We build up several families of LRC codes applying Theorems IV.1 and V.1 to families of maximal function fields, such as the Hermitian [6], the Giulietti-Korchmáros [8], the generalized Hermitian [10], [17], and the Norm-Trace [24] ones. In particular, we obtain families of locally repairable codes with good relative parameters, since the relative defect $\Delta(\mathcal{C})$ as in (4) tends to zero when q goes to infinity; see Propositions IV.2, IV.4, IV.6, IV.8, and IV.9. Also, we compare the parameters of some of these LRC codes with the ones constructed in [11] obtained from fiber products of algebraic curves; see Remark IV.7.

The paper is organized as follows. In Section 2, we present some preliminaries on locally repairable codes, function fields, automorphism groups, and algebraic geometry codes. In Section 3 a new upper bound for the minimum distance d of a $[n, k, d; r_1, r_2, \dots, r_\delta]$ -code depending on the locality $(r_1, r_2, \dots, r_\delta)$ is provided; see Theorem III.2. This bound is used to define the relative defect δ of such code, a parameter that will be used to measure how good a code is. In Sections 3 and 4 we present two types of constructions of locally repairable codes via subgroups of the automorphism group of a function field. In Theorem IV.1 we deal with subgroups with trivial intersection, and in Theorem V.1 the case of nontrivial intersection is investigated. In the same sections we apply the constructions to some well known function fields with many rational places.

II. PRELIMINARIES

Consider an $[n, k, d]$ -code $\mathcal{C}$ over $\mathbb{F}_q$ and generator matrix $(g_1, \dots, g_n)$. For any subset $I \subset \{1, \dots, n\}$ let $\langle g_j : j \in I \rangle$ denote the $\mathbb{F}_q$ -subspace generated by the set $\{g_j : j \in I\}$.

The code $\mathcal{C} \subseteq \mathbb{F}_q^n$ is said to have locality $(r_1, r_2, \dots, r_\delta)$ if the value of every coordinate of a given codeword can be retrieved by accessing any of the δ disjoint recovering sets of cardinalities $r_1, r_2, \dots, r_\delta$. One error in the codeword can be corrected in a local way if no more than $\delta - 1$ erasures occur. A formal definition of a locally repairable code is given as follows.

Definition II.1. [1] The i -th coordinate, where $1 \leq i \leq n$, of an $[n, k, d]$ linear code $\mathcal{C}$ whose generator matrix is $(g_1, \dots, g_n)$ is said to have $(r_1, \dots, r_\delta)$ -locality if there exist pairwise disjoint repair sets $R_1^{(i)}, \dots, R_\delta^{(i)} \in \{1, \dots, n\} \setminus \{i\}$ such that for each $1 \leq j \leq \delta$

- i) $\#R_j^{(i)} = r_j$;
- ii) $g_i \in \langle g_\ell \mid \ell \in R_j^{(i)} \rangle$.

Note that $r_i = 1$ implies repetition and we only consider codes with $\delta \geq 1$ and $r_i \geq 2$. Additionally, we always assume $r_i < k$. We denote a linear code $\mathcal{C}$ with length n , dimension k , minimum distance d , and with $(r_1, \dots, r_\delta)$ -locality by

$$[n, k, d; r_1, r_2, \dots, r_\delta].$$

A set $I \subset \{1, \dots, n\}$ is called an *information set* for an $[n, k, d; r_1, r_2, \dots, r_\delta]$ -code if $\#I = \text{rank}(\langle g_j \mid j \in I \rangle) = k$.

We now fix some notation and collect some results on function fields and algebraic geometry codes; for more on this subject we refer to the book [25]. Given a function field $\mathcal{F}|\mathbb{F}_q$ and a function $z \in \mathcal{F}$ let $(z)_\infty$ be its pole divisor. For a divisor D in $\mathcal{F}$ the $\mathbb{F}_q$ -vector space

$$\mathcal{L}(D) = \{z \in \mathcal{F} : (z)_\infty \geq -D\} \cup \{0\}$$

is the Riemann-Roch space associated to D and its dimension is denoted by $\ell(D)$. Let

$$\begin{aligned} \text{Aut}(\mathcal{F}|\mathbb{F}_q) &= \{\sigma : \mathcal{F} \mapsto \mathcal{F} : \sigma \text{ is an automorphism} \\ &\text{of } \mathcal{F} \text{ and } \sigma(a) = a, \forall a \in \mathbb{F}_q\} \end{aligned}$$

be the automorphism group of the function field $\mathcal{F}|\mathbb{F}_q$. For a subgroup $\mathcal{H}$ of $\text{Aut}(\mathcal{F}|\mathbb{F}_q)$ we let $\mathcal{F}^\mathcal{H}$ denote the fixed field of $\mathcal{H}$. The group $\text{Aut}(\mathcal{F}|\mathbb{F}_q)$ acts on set of rational places of the function field in a natural way and, for any rational place $P \in \mathcal{F}$ and any subgroup $\mathcal{H}$ of $\text{Aut}(\mathcal{F}|\mathbb{F}_q)$, we let $P^\mathcal{H} = \{\sigma(P) : \sigma \in \mathcal{H}\}$ stand for the orbit of P under the action of $\mathcal{H}$. The orbit is either short or long provided $P^\mathcal{H} < \mathcal{H}$ or $P^\mathcal{H} = \mathcal{H}$ respectively.

Fixing $\mathcal{H}$ and $P \in \mathcal{F}^\mathcal{H}$ an $\mathbb{F}_q$ -rational place, the ramification of P in the Galois extension $\mathcal{F}|\mathcal{F}^\mathcal{H}$ is determined by the action of $\mathcal{H}$ on the orbit $P_1^\mathcal{H}$ where P_1 is any fixed $\mathbb{F}_q$ -rational place over P . In particular, P is totally ramified if and only if $P_1^\mathcal{H} = \{P_1\}$, and P is completely split if and only if $\#P_1^\mathcal{H} = \#\mathcal{H}$ and every place P_1 above P is $\mathbb{F}_q$ -rational.

Let G be a divisor on $\mathcal{F}$ and $P_1, P_2, \dots, P_n$ be pairwise distinct rational places on $\mathcal{F}$, with $P_i \notin \text{supp}(G)$ for all i .

Define $D = \sum_{i=1}^n P_i$. The linear algebraic geometry code $\mathcal{C}_{\mathcal{L}}(D, G)$ is defined as the image of the evaluation function

$$ev : \mathcal{L}(G) \rightarrow \mathbb{F}_{q^n}, f \mapsto (f(P_1), f(P_2), \dots, f(P_n)).$$

The $\mathcal{C}_{\mathcal{L}}(D, G)$ code has length n and the classical bound on the minimum distance d is

$$d \geq n - \deg(G). \quad (5)$$

III. A BOUND ON THE MINIMUM DISTANCE

Let $\mathcal{C}$ be an $[n, k, d; r_1, \dots, r_\delta]$ -code. Locally recoverable (LRC)-codes attaining equality in (1) are called *optimal*. Constructions of optimal LRC-codes on curves of genus one can be found in [19], and with minimum distance 3 and 4 were obtained using cyclic codes in [20].

In [28, Theorem 1] another upper bound on the minimum distance was derived for the case $\delta \geq 1$ and $r = r_1 = \dots = r_\delta$

$$d \leq n - k + 2 - \left\lfloor \frac{(k-1)\delta + 1}{(r-1)\delta + 1} \right\rfloor. \quad (6)$$

In what follows we generalize the ideas in [28] in order to provide a new bound on the minimum distance of a general $[n, k, d; r_1, \dots, r_\delta]$ -code. First we recall a well known lemma about the minimum distance of a code.

Lemma III.1. [21] *The minimum distance of any $[n, k, d]$ -code whose generator matrix is $(g_1, \dots, g_n)$ satisfies*

$$d = n - \max\{\#N : N \subset \{1, \dots, n\}, \text{rank}(\langle g_j \mid j \in N \rangle) < k\}.$$

Theorem III.2. *For an $[n, k, d; r_1, \dots, r_\delta]_q$ linear code, with $2 \leq r_1 \leq r_2 \leq \dots \leq r_\delta < k$,*

$$d \leq n - k + 2 - \left\lfloor \frac{(k-1)\delta + 1}{1 + \sum_{i=1}^{\delta} r_i} \right\rfloor. \quad (7)$$

Proof. Let $I \subset \{1, \dots, n\}$ be an information set such that each coordinate in I has $(r_1, \dots, r_\delta)$ -locality. For any $i \in I$ and $j = 1, \dots, \delta$ let $R_j^{(i)}$ be the repair set of cardinality $\#R_j^{(i)} = r_i$, and $1 \leq a \leq \delta$ denote by $N_a^{(i)}$ the set $\{i\} \cup R_1^{(i)} \cup \dots \cup R_a^{(i)}$. So,

$$\text{rank}(\langle g_j : j \in N_a^{(i)} \rangle) \leq 1 + \sum_{i=1}^a r_i. \quad (8)$$

Since I is an information set there exists $\{i_1, \dots, i_\ell\} \subset I$ such that

$$\text{rank}(\langle g_j : j \in \overline{N} \rangle) = k - 1,$$

where

$$\overline{N} = N_\delta^{(i_1)} \cup \dots \cup N_\delta^{(i_{\ell-1})} \cup R_1^{(i_\ell)} \cup \dots \cup R_\theta^{(i_\ell)} \cup \overline{R},$$

with $0 \leq \theta < \delta$, $\overline{R} \subset R_{\theta+1}^{(i_\ell)}$. Denote by X and Y the quantities $\sum_{i=1}^{\delta} r_i$ and $\sum_{i=1}^{\theta+1} r_i$ respectively. From (8) we have

$$\begin{aligned} k-1 &= \text{rank}(\langle g_j : j \in \overline{N} \rangle) \\ &\leq \sum_{r=1}^{\ell-1} \text{rank}(\langle g_j : j \in N_\delta^{(i_r)} \rangle) \\ &\quad + \sum_{r=1}^{\theta+1} \text{rank}(\langle g_j : j \in R_r^{(i_\ell)} \rangle) \\ &\leq (\ell-1)(1+X) + Y, \end{aligned}$$

that is

$$\ell-1 \geq \left\lceil \frac{k-1-Y}{1+X} \right\rceil.$$

Since $g_{i_r} \in \bigcap_{a=1}^{\delta} \langle g_j : j \in R_a^{(i_r)} \rangle$, the difference $\#N_a^{(i_r)} - \text{rank}(\langle g_j : j \in N_a^{(i_r)} \rangle)$ is at least a .

Therefore,

$$\begin{aligned} \#\overline{N} &\geq \text{rank}(\langle g_j : j \in \overline{N} \rangle) + (\ell-1)\delta + \theta \\ &\geq k-1 + \left\lceil \frac{k-1-Y}{1+X} \right\rceil \delta + \theta \\ &\geq k-1 + \frac{k-1-Y}{1+X} \delta + \theta \\ &= k-2 + \frac{(k-1)\delta + 1}{1+X} + \frac{(1+X)(1+\theta) - 1 - Y\delta}{1+X}. \end{aligned}$$

Consider now $(1+X)(1+\theta) - 1 - Y\delta$.

- Suppose first that $\theta = \delta - 1$. Then $(1+X)(1+\theta) - 1 - Y\delta = (1+X)\delta - 1 - Y\delta \geq \delta - 1$, since $X \geq Y$.
- If $\delta \geq \theta + 2$, then $(1+X)(1+\theta) - 1 - Y\delta =$

$$\begin{aligned} &X(\theta+1) - Y\delta + \theta \\ &= (\theta+1) \sum_{i=1}^{\theta+1} r_i + (\theta+1) \sum_{i=\theta+2}^{\delta} r_i - \delta \sum_{i=1}^{\theta+1} r_i + \theta \\ &\quad - (\delta - \theta - 1) \sum_{i=1}^{\theta+1} r_i + (\theta+1) \sum_{i=\theta+2}^{\delta} r_i + \theta \\ &\geq -(\delta - \theta - 1)(\theta+1)r_{\theta+1} \\ &\quad + (\theta+1)(\delta - \theta - 1)r_{\theta+2} + \theta \\ &\geq \theta. \end{aligned}$$

So, $(1+X)(1+\theta) - 1 - Y\delta \geq 0$. Since $\#\overline{N}$ is an integer,

$$\#\overline{N} \geq k-2 + \left\lceil \frac{(k-1)\delta + 1}{1+X} \right\rceil.$$

By Lemma III.1,

$$\begin{aligned} n &= d + \max\{\#N : N \subset \{1, \dots, n\}, \\ &\quad \text{rank}(\langle g_j : j \in N \rangle) < k\} \geq d + \#\overline{N} \\ &\geq d + k - 2 + \left\lceil \frac{(k-1)\delta + 1}{1+X} \right\rceil, \end{aligned}$$

and the claim follows. $\square$

Remark III.3. Note that for $\delta = 1$, Bound (7) is weaker than Bound (1).

In [15] the authors construct LRC codes over the rational function field $\mathbb{F}_q(x)$ (genus zero) dealing with subgroup of its automorphism group, and obtain codes with length $n \simeq q$ roughly and relative defect $\simeq 1/q$ according to Formula (4).

IV. GENERAL CONSTRUCTION FROM SUBGROUPS WITH TRIVIAL INTERSECTION

Many constructions of LRC codes from function fields arose in recent years, over the rational function field [14], [15], elliptic function fields [19] and over algebraic curves (see [22], [28]), fiber product of curves [11] and curves with separated variables [23]. In a variant of the different ways to construct LRC codes, we focus on codes from algebraic function fields of genus $g \geq 1$ using certain subgroups of the automorphism group of the underlying curve. In this section we deal with the case of subgroups with trivial intersection and in the next Section with non-trivial intersection subgroups.

Theorem IV.1. Let $\mathcal{F}|\mathbb{F}_q$ be a function field of genus g . Consider s subgroups $\mathcal{H}_i$ of the automorphism group of $\mathcal{F}|\mathbb{F}_q$, each of sizes $r_i + 1$, such that the group $\mathcal{G} \simeq \bigotimes_{i=1}^s \mathcal{H}_i$ is isomorphic to the internal direct product of $\mathcal{H}_1, \dots, \mathcal{H}_s$. Let $\mathcal{P}$ be a set of places in $\mathcal{F}$ lying over m rational places in the fixed field $\mathcal{F}^{\mathcal{G}}$ that are completely split in the extension $\mathcal{F}|\mathcal{F}^{\mathcal{G}}$. Define $n = m \prod_{i=1}^s (r_i + 1)$, that is, n to be the total number of places of $\mathcal{F}$ lying over the m selected rational places in $\mathcal{F}^{\mathcal{G}}$.

Suppose that there exists a place P_{∞} of $\mathcal{F}$ which is completely ramified in $\mathcal{F}|\mathcal{F}^{\mathcal{G}}$ and let $Q_{\infty}^{(i)}$ be the unique place in $\mathcal{F}^{\mathcal{H}_i}$ lying under P_{∞} . For $i = 1, \dots, s$ suppose further there exist functions z_i, w_i , such that

- i) $z_i \in \mathcal{F}^{\mathcal{H}_i}$, $\text{supp}((z_i)_{\infty}) = \{Q_{\infty}^{(i)}\}$;
- ii) $w_i \in \mathcal{F} \setminus \mathcal{F}^{\mathcal{H}_i}$, $\text{supp}((w_i)_{\infty}) = \{P_{\infty}\}$;
- iii) $w_i : P^{\mathcal{H}_i} \rightarrow \mathbb{F}_q$ is injective.

For each $i = 1, \dots, s$ let $t_i \geq 1$ be such that

$$V_i := \left\{ \sum_{\ell=0}^{r_i-1} \left(\sum_{j=0}^{t_i} a_{\ell j}^{(i)} z_i^j \right) w_i^{\ell} \in \mathcal{F} \mid a_{\ell j}^{(i)} \in \mathbb{F}_q \right\}$$

is contained in $\mathcal{L}((n-d)P_{\infty})$ for some $1 \leq d \leq n$ and let $V = \bigcap_{i=1}^s V_i \subset \mathcal{L}((n-d)P_{\infty})$. If $\dim_{\mathbb{F}_q}(V) > 0$, then there exists an

$$[n, \dim_{\mathbb{F}_q}(V), \geq d; r_1, \dots, r_s]\text{-recoverable code.}$$

Proof. Let $V = \bigcap_{i=1}^s V_i \subset \mathcal{L}((n-d)P_{\infty})$ and consider the following linear map

$$\begin{aligned} e_{\mathcal{P}} : V &\rightarrow \mathbb{F}_q^n \\ f &\mapsto e_{\mathcal{P}}(f) = (f(P_1), \dots, f(P_n)), \end{aligned}$$

where $\mathcal{P} = \{P_1, \dots, P_n\}$.

The linear code $e_{\mathcal{P}}(V)$ is contained in the algebraic geometry code $C_{\mathcal{L}}(\sum_i^n P_i, (n-d)P_{\infty})$. Then the minimum distance of the code $e_{\mathcal{P}}(V)$ is larger than or equal to the minimum distance of the linear code $C_{\mathcal{L}}(\sum_i^n P_i, (n-d)P_{\infty})$. By the bound on Equation (5), the code $C_{\mathcal{L}}(\sum_i^n P_i, (n-d)P_{\infty})$ has minimum distance at least d . By definition it has length n and dimension $\dim_{\mathbb{F}_q}(V)$.

Now, we deal with recoverability. First note that a rational place $Q \in \mathcal{F}^{\mathcal{G}}$ which is completely split in $\mathcal{F}|\mathcal{F}^{\mathcal{G}}$ satisfies $P^{\mathcal{H}_i} \cap P^{\mathcal{H}_j} = \{P\}$ for each $i \neq j \in \{1, \dots, s\}$, where P is an arbitrary place in $\mathcal{P}$ lying over Q . In fact, since Q is completely split, by the orbit stabilizer theorem, this means that the stabilizer in G of P is trivial. If $P^{\mathcal{H}_i}$ and $P^{\mathcal{H}_j}$ contain another common place $\tilde{P} \neq P$ then there exist $h_i \in \mathcal{H}_i$ and $h_j \in \mathcal{H}_j$ such that $\tilde{P} = h_i(P) = h_j(P)$ so that $h_i h_j^{-1}(P) = P$. Since $\mathcal{H}_i \cap \mathcal{H}_j$ is trivial, $h_i h_j^{-1}$ is a non-trivial element of $\mathcal{G}$, a contradiction. Consider a fixed place $P \in \mathcal{P}$ and an $f \in V$. We are going to show that $f(P)$ can be repaired by any of the s recoverable sets (depending on P)

$$R_i = \{f(P') : P' \in P^{\mathcal{H}_i} \setminus \{P\}\}, \text{ for}$$

of cardinality r_i for $i = 1, \dots, s$. Since $f \in V_i$ we can write $f = \sum_{\ell=0}^{r_i-1} \left(\sum_{j=0}^{t_i} a_{\ell j}^{(i)} z_i^j \right) w_i^{\ell}$, $a_{\ell j}^{(i)} \in \mathbb{F}_q$. Since $z_i \in \mathcal{F}^{\mathcal{H}_i}$ we have that $z_i(Q_1) = z_i(Q_2) = \bar{z}$ for any $Q_i, Q_j \in P^{\mathcal{H}_i}$, then for each $Q \in P^{\mathcal{H}_i}$ we have

$$\begin{aligned} f(Q) &= \sum_{\ell=0}^{r_i-1} \left(\sum_{j=0}^{t_i} a_{\ell j}^{(i)} z_i^j(Q) \right) w_i(Q)^{\ell} \\ &= \sum_{\ell=0}^{r_i-1} \left(\sum_{j=0}^{t_i} a_{\ell j}^{(i)} \bar{z}^j \right) w_i(Q)^{\ell} \\ &= \sum_{\ell=0}^{r_i-1} \gamma_{\ell, P}^{(i)} w_i(Q)^{\ell}, \end{aligned}$$

where $\gamma_{\ell, P}^{(i)} \in \mathbb{F}_q$ depends only on ℓ (and P clearly). So,

$$h(X) = \sum_{\ell=0}^{r_i-1} \gamma_{\ell, P}^{(i)} X^{\ell} \in \mathbb{F}_q[X]$$

is a polynomial of degree at most $r_i - 1$. Since the values $w_i(Q)$ with $Q \in P^{\mathcal{H}_i}$ are distinct we know exactly r_i different values of $h(X)$. By Lagrange interpolation we can recover $h(X)$ and then compute the missing value $h(w_i(P))$. By definition of $\mathcal{P}$, the recovering sets $P^{\mathcal{H}_i} \setminus \{P\}$ are pairwise disjoint. $\square$

In what follows we are going to construct examples of LRC codes over algebraic function fields of genus $g \geq 1$ based on Theorem IV.1. All the constructed codes comes from algebraic curves with many rational points.

A. Codes from the Hermitian Function Field I

Consider the Hermitian function field $\mathcal{F} = \mathbb{F}_{q^2}(x, y)$ with $y^{q+1} = x^q + x$. Choose a divisor $u \geq 2$ of $q+1$. Consider the following two subgroups of $\mathbb{F}_q$ -automorphisms of $\mathcal{F}$

$$\begin{aligned}\mathcal{H}_1 &:= \{(x, y) \mapsto (x + c, y) \mid c \in \mathbb{F}_{q^2}, c^q + c = 0\}, \\ \mathcal{H}_2 &:= \{(x, y) \mapsto (x, a^i y) \mid i = 0, \dots, u-1 \text{ and} \\ &\quad a \in \mathbb{F}_{q^2}^* \text{ of order } u \geq 2\}.\end{aligned}$$

It is easy to see that $\#\mathcal{H}_1 = q$, $\#\mathcal{H}_2 = u$, $\mathcal{F}^{\mathcal{H}_1} = \mathbb{F}_{q^2}(y)$ and $\mathcal{F}^{\mathcal{H}_2} = \mathbb{F}_{q^2}(x, y^u)$. The groups have trivial intersection and commute, then $\mathcal{G} = \mathcal{H}_1\mathcal{H}_2 = \mathcal{H}_1 \times \mathcal{H}_2$ and $\mathcal{F}^{\mathcal{G}} = \mathbb{F}_{q^2}(z := y^u) \subseteq \mathcal{F}^{\mathcal{H}_i}$, for $i = 1, 2$. Let P_∞ be the unique pole of x and $y \in \mathcal{F}$, then $(y)_\infty = qP_\infty$ and $(x)_\infty = (q+1)P_\infty$ in $\mathcal{F}$. Denote by $Q_\infty^i = P_\infty \cap \mathcal{F}^{\mathcal{H}_i}$, for $i = 1, 2$. In the extension $\mathcal{F} \mid \mathbb{F}_{q^2}(z)$, we have that $(q^2-1)/u$ places in $\mathcal{F}^{\mathcal{G}}$ are totally split, so $\#\mathcal{P} = q(q^2-1) = n$. For $t_i \geq 2, i = 1, 2$ consider the $\mathbb{F}_q$ -vector spaces

$$\begin{aligned}V_1 &:= \left\{ \sum_{\ell=0}^{q-2} \left(\sum_{j=0}^{t_1} a_{\ell,j} y^j \right) x^\ell : a_{\ell,j} \in \mathbb{F}_{q^2} \right\} \text{ and} \\ V_2 &:= \left\{ \sum_{\ell=0}^{u-2} \left(\sum_{j=0}^{t_2} a_{\ell,j} z^j \right) y^\ell : a_{\ell,j} \in \mathbb{F}_{q^2} \right\} \\ &= \left\{ \sum_{\ell=0}^{u-2} \sum_{j=0}^{t_2} a_{\ell,j} y^{(u-1)j+\ell} : a_{\ell,j} \in \mathbb{F}_{q^2} \right\}.\end{aligned}$$

Note that $\dim_{\mathbb{F}_{q^2}} V_1 = (q-1)(t_1+1)$ and $\dim_{\mathbb{F}_{q^2}} V_2 = (u-1)(t_2+1)$. A function in V_1 has pole only at the place P_∞ of order at most $q^2 - q - 2 + t_1q$, while a function in V_2 also has pole only at P_∞ of order at most $q((u-1)t_2 + u - 2)$.

Using the notations of the Theorem IV.1, we have $z_1 = y$, $z_2 = y^{u-1}$, $w_1 = x$ and $w_2 = y$. Now we choose the parameters:

- i) $t_1 = q-1$ and $\frac{2q^2-qu-2}{q(u-1)} \leq t_2 < \frac{q^2}{u-1} - 1$ so that

$$V_1, V_2 \subset \mathcal{L}(q((u-1)t_2 + u - 2)P_\infty),$$

$$\text{and } d = n - q(ut_2 + u - t_2 - 2) \geq 1.$$

Now,

$$V = V_1 \cap V_2 = \left\{ \sum_{j=0}^{q-1} a_j y^j : a_j \in \mathbb{F}_{q^2} \right\}$$

$$\text{and } \dim_{\mathbb{F}_{q^2}} V = q.$$

- ii) $u(t_2+1) = q+1, t_1 \geq 1, (t_1, t_2) \neq (1, 1)$, then

$$V_1, V_2 \subset \mathcal{L}((q^2 - q - 2 + t_1q)P_\infty),$$

$$\text{and } V = V_1 \cap V_2 = \left\{ \sum_{j=0}^{q-1} a_j y^j : a_j \in \mathbb{F}_{q^2} \right\}.$$

The two choices of parameters in the construction can be summarized as the following.

Proposition IV.2. *For every $1 < u$ a divisor of $q+1$ and $\frac{2q^2-qu-2}{q(u-1)} \leq t_2 < \frac{q^2}{u-1} - 1$ there exists a*

$[q(q^2-1), q, \geq q(q^2-1) - q(ut_2 + u - t_2 - 2); q-1, u-1]$ recoverable code $\mathcal{C}_1$.

For every $1 < u, u(t_2+1) = q+1, t_1 \geq 2$ and $(t_1, t_2) \neq (1, 1)$ there exists a

$[q(q^2-1), q, \geq q(q^2-1) - (q^2 - q - 2 + t_1q); q-1, u-1]$ recoverable code $\mathcal{C}_2$.

Remark IV.3. *The relative defect of the codes $\mathcal{C}_1$ and $\mathcal{C}_2$ (see Equation (4)) are*

$$\begin{aligned}\Delta(\mathcal{C}_1) &\leq \frac{1}{q(q^2-1)} \left(q(ut_2 + u - t_2 - 2) - q + 2 \right. \\ &\quad \left. - \left\lceil \frac{2q-1}{q+u-1} \right\rceil \right) = \frac{q(ut_2 + u - t_2 - 3) + 1}{q(q^2-1)},\end{aligned}$$

$$\begin{aligned}\Delta(\mathcal{C}_2) &\leq \frac{1}{q(q^2-1)} \left((q^2 - q - 2 + t_1q) - q + 2 \right. \\ &\quad \left. - \left\lceil \frac{2q-1}{q+u-1} \right\rceil \right) = \frac{q^2 - 2q + t_1q - 1}{q(q^2-1)} \simeq \frac{q+t_1}{q^2-1}.\end{aligned}$$

For the code $\mathcal{C}_1$, if t_2 is close to the lower bound $\frac{2q^2-qu-2}{q(u-1)}$, $\Delta(\mathcal{C}_1)$ is less or equal to $2/q$.

B. Codes from the Hermitian Function Field II

Consider the Hermitian function field $\mathcal{F} = \mathbb{F}_{q^2}(x, y)$ with $y^{q+1} = x^q + x$. Use the same notations as in Section IV-A. Consider now

$$\begin{aligned}V_1 &:= \left\{ \sum_{\ell=0}^{q-1} \left(\sum_{j=0}^{t_1 \leq u} a_{\ell,j} y^j \right) x^\ell : a_{\ell,j} \in \mathbb{F}_{q^2} \right\}, \\ V_2 &:= \left\{ \sum_{\ell=0}^{u-1} \left(\sum_{j=0}^{t_2 \leq q} a_{\ell,j} x^j \right) y^\ell : a_{\ell,j} \in \mathbb{F}_{q^2} \right\},\end{aligned}$$

of dimensions $\dim_{\mathbb{F}_{q^2}} V_1 = (t_1+1)q$ and $\dim_{\mathbb{F}_{q^2}} V_2 = (t_2+1)u$.

A function in V_1 has pole only at the place P_∞ of order at most $q^2 - q - 2 + t_1q$, while a function in V_2 also has pole only at P_∞ of order at most $q((u-1)t_2 + u - 2)$.

Using the notations of the Theorem IV.1 we have $z_1 = y$, $z_2 = x$, $w_1 = x$, $w_2 = y$. Also,

$$V = V_1 \cap V_2 = \left\{ \sum_{\ell=0}^{t_1} \left(\sum_{j=0}^{t_2} a_{\ell,j} x^j \right) y^\ell : a_{\ell,j} \in \mathbb{F}_{q^2} \right\}$$

and $\dim_{\mathbb{F}_{q^2}} V = (t_1+1)(t_2+1)$. For

$$M = \max\{t_1q + q^2 - 1, (u-1)q + t_2(q+1)\} \leq 2q^2 + q - 1$$

we have $V \subset \mathcal{L}(MP_\infty)$. In this case $n - d = M$ and $d \geq \max\{q^2 + t_1(q - t_2), (u - 1)q + t_2(q - t_1)\}$. We obtain the following proposition.

Proposition IV.4. *For every divisor $1 < u$ of $q + 1$, $1 \leq t_1 \leq u$, $1 \leq t_2 \leq q$ and*

$$M = \max\{t_1q + q^2 - 1, (u - 1)q + t_2(q + 1)\}$$

there exists a

$$[q(q^2 - 1), (t_1 + 1)(t_2 + 1), \geq q(q^2 - 1) - M; q - 1, u - 1]$$

recoverable code $\mathcal{C}$.

Remark IV.5. *The relative defect of this code satisfies*

$$\Delta(\mathcal{C}) \leq \frac{M - (t_1 + 1)(t_2 + 1) + 2 - \left\lceil \frac{2(t_1t_2 + t_1 + t_2) + 1}{q + u - 1} \right\rceil}{q(q^2 - 1)}.$$

Choosing $u = q + 1$, $t_1 = u$ and $t_2 = q$ we obtain $\Delta(\mathcal{C}) \leq \frac{q^2 - q - 1}{q(q^2 - 1)}$ and so the relative defect goes to zero as q goes to infinity.

C. Codes from the Giulietti-Korchmáros curve

Let $\mathcal{F} = \mathbb{F}_{q^6}(x, y, z)$ be the function field of the curve $\mathcal{GK}$ whose affine model is given by the complete intersection

$$\begin{cases} Z^{q^2 - q + 1} = Y^{q^2} - Y, \\ Y^{q + 1} = X^q + X. \end{cases}$$

This is a maximal curve over $\mathbb{F}_{q^6}$ and has $q^8 - q^6 + q^5 + 1$ rational places and only one place P_∞ at infinity. Moreover, P_∞ is the common pole of x, y and z with pole divisors

$$(x)_\infty = (q^3 + 1)P_\infty, (y)_\infty = (q^3 - q^2 + q)P_\infty, (z)_\infty = qP_\infty,$$

see [8]. In this case we consider three subgroups of the following types, let A be a subgroup of $\{a \in \mathbb{F}_{q^6} : a^q + a = 0\}$, $\eta, \omega \in \mathbb{F}_{q^6}$, with $\text{ord}(\eta) \mid q^3 + 1$, $\text{ord}(\omega) \mid q^2 - q + 1$, and $\gcd(\text{ord}(\eta), \text{ord}(\omega)) = 1$. Consider the following subgroups of $\text{Aut}(\mathcal{F}|\mathbb{F}_{q^6})$:

$$\begin{aligned} \mathcal{H}_1 &:= \{\sigma_a : (x, y, z) \mapsto (x + a, y, z) : a \in A\}; \\ \mathcal{H}_2 &:= \{\sigma_i : (x, y, z) \mapsto (x, \eta^{i(q^2 - q + 1)}y, \eta^i z) : \\ &\quad i = 0, \dots, \text{ord}(\eta) - 1\}; \\ \mathcal{H}_3 &:= \{\sigma_i : (x, y, z) \mapsto (x, y, \omega^i z) : \\ &\quad i = 0, \dots, \text{ord}(\omega) - 1\}, \end{aligned}$$

Then we have $\mathcal{G} = \mathcal{H}_1\mathcal{H}_2\mathcal{H}_3 \cong \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$ is a subgroup of $\text{Aut}(\mathcal{F}|\mathbb{F}_{q^6})$. The fixed fields satisfy $\mathbb{F}_{q^6}(y, z) = \mathcal{F}^{\mathcal{H}_1}$, $\mathbb{F}_{q^6}(x) \subseteq \mathcal{F}^{\mathcal{H}_2}$ and $\mathbb{F}_{q^6}(x, y) \subseteq \mathcal{F}^{\mathcal{H}_3}$. Now we notice that for all $P = (x, y, z)$ a rational place with $z \neq 0$ we have that $P^{\mathcal{H}_i} \cap P^{\mathcal{H}_j} = \{P\}$, $i, j \in \{1, 2, 3\}$. In fact, if $Q \in P^{\mathcal{H}_2} \cap P^{\mathcal{H}_3}$, since the unique possibility for $\eta^i = 1$ and $\omega^j = 1$ yields $i = j = 0$ from $\gcd(\text{ord}(\eta), \text{ord}(\omega)) = 1$,

we obtain $Q = P$. The other cases are trivial. Therefore, all the affine $\mathbb{F}_{q^6}$ -rational points of $\mathcal{GK}$ belong to long orbits with respect to $\mathcal{G}$ are those with $z \neq 0$. So, the cardinality of the set $\mathcal{P}$ as in Theorem IV.1 can be taken as $n = q^8 - q^6 + q^5 - q^3 = q^3(q^2 - 1)(q^3 + 1)$. Using the notations in Theorem IV.1 we consider

- i) $z_1 := z \in \mathcal{F}^{\mathcal{H}_1}$, $w_1 := x \in \mathcal{F} \setminus \mathcal{F}^{\mathcal{H}_1}$
- ii) $z_2 := x \in \mathcal{F}^{\mathcal{H}_2}$, $w_2 := z \in \mathcal{F} \setminus \mathcal{F}^{\mathcal{H}_2}$
- iii) $z_3 := x \in \mathcal{F}^{\mathcal{H}_3}$, $w_1 := z \in \mathcal{F} \setminus \mathcal{F}^{\mathcal{H}_3}$.

For any integers $t_i \geq 0$, $i = 1, 2, 3$, consider the following $\mathbb{F}_{q^6}$ -vector spaces

$$\begin{aligned} V_1 &:= \left\{ \sum_{\ell=0}^{\#A-2} \left(\sum_{j=0}^{t_1} a_{\ell,j} z^j \right) x^\ell : a_{\ell,j} \in \mathbb{F}_{q^6} \right\}, \\ V_2 &:= \left\{ \sum_{\ell=0}^{\text{ord}(\eta)-2} \left(\sum_{j=0}^{t_2} a_{\ell,j} x^j \right) z^\ell : a_{\ell,j} \in \mathbb{F}_{q^6} \right\}, \text{ and} \\ V_3 &:= \left\{ \sum_{\ell=0}^{\text{ord}(\omega)-2} \left(\sum_{j=0}^{t_3} a_{\ell,j} x^j \right) z^\ell : a_{\ell,j} \in \mathbb{F}_{q^6} \right\}. \end{aligned}$$

For $N_1 = \min\{\#A - 2, t_2, t_3\}$, $M_1 = \min\{t_1, \text{ord}(\eta) - 2, \text{ord}(\omega) - 2\}$ and

$$\begin{aligned} S &= \max\{t_2(q^3 + 1) + (\text{ord}(\eta) - 2)q, \\ &\quad (\#A - 2)(q^3 + 1) + t_1q, t_3(q^3 + 1) + (\text{ord}(\omega) - 2)q\}, \end{aligned}$$

we obtain that

$$V := V_1 \cap V_2 \cap V_3 = \left\{ \sum_{n_1=0}^{N_1} \sum_{m_1=0}^{M_1} a_{n_1, m_1} z^{m_1} x^{n_1} : a_{n_1, m_1} \in \mathbb{F}_{q^6} \right\}$$

is contained in $\mathcal{L}(SP_\infty)$ and it has dimension $\dim_{\mathbb{F}_{q^6}} V = (M_1 + 1)(N_1 + 1)$. Choosing t_i , $i = 1, 2, 3$ such that $q^8 - q^6 + q^5 - q^3 - S \geq 1$ we obtain the following proposition.

Proposition IV.6. *Let $q = p^\ell$, p prime. Then for any*

- i) $a = p^h$, $1 \leq h \leq \ell$,
- ii) $\text{ord}(\eta) \mid q^3 + 1$, $\text{ord}(\omega) \mid q^2 - q + 1$, with $\gcd(\text{ord}(\eta), \text{ord}(\omega)) = 1$,
- iii) $0 < t_1 < q^2(q^2 - 1)(q^3 + 1) - q^3 + 2q^2 - 1$, $0 < t_2 < q^3(q^2 - 1) - q$, $0 < t_3 < q^3(q^2 - 1) - 1$,
- iv) $N_1 = \min\{a - 2, t_2, t_3\}$, $M_1 = \min\{t_1, \text{ord}(\eta) - 2, \text{ord}(\omega) - 2\}$, and
- v) $S = \max\{(a - 2)(q^3 + 1) + t_1q, t_2(q^3 + 1) + (\text{ord}(\eta) - 2)q, t_3(q^3 + 1) + (\text{ord}(\omega) - 2)q\}$,

there exists a

$$[n, (M_1 + 1)(N_1 + 1), \geq n - S; a - 1, \text{ord}(\eta) - 1, \text{ord}(\omega) - 1]$$

recoverable code $\mathcal{C}_3$ over $\mathbb{F}_{q^6}$, where $n = q^8 - q^6 + q^5 - q^3$.

Its relative defect $\Delta(\mathcal{C}_3)$ is at most

$$\frac{S - (M_1 + 1)(N_1 + 1) + 2 - \left\lfloor \frac{3(M_1 N_1 + M_1 + N_1) + 1}{a + \text{ord}(\eta) + \text{ord}(\omega) - 2} \right\rfloor}{q^8 - q^6 + q^5 - q^3}.$$

Suppose that $(q^3 + 1)$ possesses two coprime divisors both close to $q\sqrt{q}$. Consider $q\sqrt{q} < t_1 < q^2$, $t_2, t_3 \simeq a$, so that $N_1 = a - 2$, $M_1 \simeq q\sqrt{q}$, $S \simeq N_1 q^3$ and therefore $\Delta(\mathcal{C}_3)$ is at most a value close to $N_1(q^3 - q\sqrt{q})/q^8$.

Remark IV.7. In [11] the authors consider fiber products of algebraic curves in order to construct LRC codes. In particular [11, Corollary 1] can be seen as a construction from trivially intersecting subgroups of the automorphism group of an algebraic curve. In what follows we choose codes with same length and compare the obtained relative defects.

In [11, Theorem 5.1] the authors construct codes on the generalized GK curve. For the special case of the curve GK with $\ell = q$, they obtain a

$$[n, (q^2 - q)(q^2 - 1), n - q^5 + 2q^3 - q^2 - q + 2; q - 1, q^2 - q]$$

recoverable code over $\mathbb{F}_{q^6}$, where $n = q^8 - q^6 + q^5 - q^3$.

The relative defect of such a code is $\frac{q^5 - q^4 - q^3 + 2q + 1}{n}$.

There are many choices for a, η, ω such that the code constructed from Proposition IV.6 has relative defect smaller than $\frac{q^5 - q^4 - q^3 + 2q + 1}{n}$. Suppose $3 \nmid (q + 1)$. Consider for instance $a = q$, $\text{ord}(\eta) = q + 1$, $\text{ord}(\omega) = q^2 - q + 1$, $t_1 = t_2 = t_3 = q - 1$. In this case, $N_1 = q - 2$, $M_1 = q - 1$ and $S = q^4 - q^2 - 1$. Note that S is the maximum $S = \max\{(a - 2)(q^3 + 1) + t_1 q, t_2(q^3 + 1) + (\text{ord}(\eta) - 2)q, t_3(q^3 + 1) + (\text{ord}(\omega) - 2)q\} = t_3(q^3 + 1) + (\text{ord}(\omega) - 2)q = (q - 1)(q^3 + 1) + (q^2 - q - 1)q = q^4 - q^2 - 1$, and hence there exists an

$$[n, \geq q^8 - q^6 + q^5 - q^4 - q^3 + q^2 + 1; q - 1, q, q^2 - q]$$

recoverable code $\mathcal{C}_3$ over $\mathbb{F}_{q^6}$, where $n = q^8 - q^6 + q^5 - q^3, q(q - 1)$, whose relative defect is at most

$$\frac{S - (M_1 + 1)(N_1 + 1) + 2 - \left\lfloor \frac{3(M_1 N_1 + M_1 + N_1) + 1}{a + \text{ord}(\eta) + \text{ord}(\omega) - 2} \right\rfloor}{n} \leq \frac{q^4 - 2q^2 + q}{n}.$$

Similar results can be obtained when $3 \mid (q + 1)$ by considering for instance $\text{ord}(\eta) = q + 1$ and $\text{ord}(\omega) = (q^2 - q + 1)/3$. Also in this case, the relative defect is roughly q^4/n .

D. Codes from the generalized Hermitian curve

Let q be odd. The curve $y^{q^\ell + 1} = x^q + x$ over $\mathbb{F}_{q^{2\ell}}$ with $\ell \geq 1$ odd has $q^{2\ell + 1} + 1$ rational points and genus $g = q^\ell(q - 1)/2$, see [10], [17]. This curve has only one rational point at infinity denoted by P_∞ , it is also the only pole of the functions x

and y with pole order $q^\ell + 1$ and q , respectively. Let $\mathcal{S}$ be the function field of the curve $y^{q^\ell + 1} = x^q + x$, and consider two subgroups of the automorphism group of $\mathcal{S}$, given by

$$\mathcal{H}_1 := \{(x, y) \mapsto (x + a, y) \mid a^q + a = 0 \text{ and } a \in \mathbb{F}_{q^{2\ell}}\},$$

and

$$\mathcal{H}_2 := \{(x, y) \mapsto (x, \lambda y) \mid \lambda \in \mathbb{F}_{q^{2\ell}} \text{ and } \lambda^{q^\ell + 1} = 1\},$$

so that $\#\mathcal{H}_1 = q$, $\#\mathcal{H}_2 = q^\ell + 1$ and the fixed fields are $\mathcal{S}^{\mathcal{H}_1} = \mathbb{F}_{q^{2\ell}}(y)$, $\mathcal{S}^{\mathcal{H}_2} = \mathbb{F}_{q^{2\ell}}(x)$, and $\mathcal{S}^{\mathcal{G}} = \mathbb{F}_{q^{2\ell}}(x^q + x) = \mathbb{F}_{q^{2\ell}}(y^{q^\ell + 1})$ by the direct product $\mathcal{G} \simeq \mathcal{H}_1 \times \mathcal{H}_2$. The number of rational places in $\mathcal{S}^{\mathcal{G}}$ that completely splits in the field extension $\mathcal{S}|\mathcal{S}^{\mathcal{G}}$ is $q^\ell - 1$, then we have $n = \#\mathcal{P} = q^{2\ell + 1} - q$. Let $0 \leq t_i, i = 1, 2$ and consider the vector spaces

$$V_1 := \left\{ \sum_{\ell=0}^{q-2} \left(\sum_{j=0}^{t_1} a_{\ell,j} y^j \right) x^\ell : a_{\ell,j} \in \mathbb{F}_{q^{2\ell}} \right\},$$

$$V_2 := \left\{ \sum_{\ell=0}^{q^\ell-1} \left(\sum_{j=0}^{t_2} a_{\ell,j} x^j \right) y^\ell : a_{\ell,j} \in \mathbb{F}_{q^{2\ell}} \right\},$$

Choose t_i such that for $M_1 = \max\{t_1, q^\ell - 1\}$, $M_2 = \max\{t_2, q - 2\}$ we have $S = M_1 q + M_2(q^\ell - 1) \leq q^{2\ell + 1} - q$. Then $V_i \subset \mathcal{L}(SP_\infty)$, for $i = 1, 2$, and for $m_1 = \min\{t_1, q^\ell - 1\}$, $m_2 = \min\{t_2, q - 2\}$ we obtain

$$V = V_1 \cap V_2 = \left\{ \sum_{\ell=0}^{m_1} \left(\sum_{j=0}^{m_2} a_{\ell,j} x^j \right) y^\ell : a_{\ell,j} \in \mathbb{F}_{q^{2\ell}} \right\}$$

and $\dim_{\mathbb{F}_{q^2}} V = (m_1 + 1)(m_2 + 1)$. The minimum distance of the code satisfies $d \geq q^{2\ell + 1} - q - S$. We have obtained the following proposition.

Proposition IV.8. Let q be odd. Consider $0 \leq t_i, i = 1, 2$ satisfying that

$$S = M_1 q + M_2(q^\ell - 1) \leq q^{2\ell + 1} - q$$

for $M_1 = \max\{t_1, q^\ell - 1\}$ and $M_2 = \max\{t_2, q - 2\}$. Fixing $m_1 = \min\{t_1, q^\ell - 1\}$ and $m_2 = \min\{t_2, q - 2\}$ there exists a

$$[q^{2\ell + 1} - q, (m_1 + 1)(m_2 + 1), \geq q^{2\ell + 1} - q - S; q - 1, q^\ell]$$

recoverable code $\mathcal{C}$ over $\mathbb{F}_{q^{2\ell}}$.

The codes arising from Proposition IV.8 have relative designed distance

$$\Delta(\mathcal{C}) \leq \frac{S - (m_1 + 1)(m_2 + 1) + 2 - \left\lfloor \frac{2(m_1 m_2 + m_1 + m_2) + 1}{q^\ell + q} \right\rfloor}{q^{2\ell + 1} - q}.$$

E. Codes from the Norm-Trace curve

The so called Norm-Trace curve is inspired on the surjective $\mathbb{F}_q$ -linear maps Trace and Norm from $\mathbb{F}_{q^\ell}$ to $\mathbb{F}_q$, and generalizes the Hermitian curve obtained for $\ell = 2$. This curve has been object of study in [7] and [24] for construction of codes. The Norm-Trace curve is defined over $\mathbb{F}_{q^\ell}$ by the affine equation:

$$y^{\frac{q^\ell-1}{q-1}} = x^{q^{\ell-1}} + x^{q^{\ell-2}} + \dots + x. \quad (9)$$

It has genus $g = \frac{1}{2}(q^{\ell-1} - 1)(\frac{q^\ell-1}{q-1} - 1)$, only one point at infinity P_∞ plus $q^{2\ell-1}$ affine rational points. The functions x and y have pole divisor $(x)_\infty = (\frac{q^\ell-1}{q-1})P_\infty$ and $(y)_\infty = (q^\ell - 1)P_\infty$. Now consider two subgroups of the automorphism group of the function field $\mathcal{N}$ of the curve (9), given by

$$\mathcal{H}_1 := \left\{ (x, y) \mapsto (x + a, y) \mid \sum_{i=0}^{\ell-1} a^{q^i} = 0, a \in \mathbb{F}_{q^\ell} \right\},$$

and

$$\mathcal{H}_2 := \left\{ (x, y) \mapsto (x, \lambda y) \mid \lambda \in \mathbb{F}_{q^\ell} \text{ and } \lambda^{\frac{q^\ell-1}{q-1}} = 1 \right\},$$

so that $\#\mathcal{H}_1 = q^{\ell-1}$, $\#\mathcal{H}_2 = \frac{q^\ell-1}{q-1}$ and the fixed fields are $\mathcal{N}^{\mathcal{H}_1} = \mathbb{F}_{q^\ell}(y)$, $\mathcal{N}^{\mathcal{H}_2} = \mathbb{F}_{q^\ell}(x)$ and $\mathcal{N}^{\mathcal{G}} = \mathbb{F}_{q^\ell}(x^{q^{\ell-1}} + x^{q^{\ell-2}} + \dots + x) = \mathbb{F}_{q^\ell}(y^{\frac{q^\ell-1}{q-1}})$ by the direct product $\mathcal{G} \simeq \mathcal{H}_1 \times \mathcal{H}_2$. The number of rational places in $\mathcal{N}^{\mathcal{G}}$ that completely splits in the field extension $\mathcal{N}|\mathcal{N}^{\mathcal{G}}$ is $q - 1$, then we have $n = \#\mathcal{P} = q^{\ell-1}(q^\ell - 1)$. Following a similar construction as in section IV-D we obtain the following.

Proposition IV.9. *Let $0 \leq t_i, i = 1, 2$ satisfying that*

$$S = M_1 q + M_2 (q^\ell - 1) \leq q^{\ell-1} (q^\ell - 1)$$

for $M_1 = \max\{t_1, \frac{q^\ell-1}{q-1} - 2\}$ and $M_2 = \max\{t_2, q^{\ell-1} - 2\}$.

Choosing $m_1 = \min\{t_1, \frac{q^\ell-1}{q-1} - 2\}$ and $m_2 = \min\{t_2, q^{\ell-1} - 2\}$ there exists a

$$[n_\ell, (m_1 + 1)(m_2 + 1), \geq n_\ell - S; \frac{q^\ell - q}{q - 1}, q^{\ell-1} - 1]$$

recoverable code $\mathcal{C}$ over $\mathbb{F}_{q^\ell}$, where $n_\ell = q^{\ell-1}(q^\ell - 1)$.

This codes have relative designed distance

$$\Delta(\mathcal{C}) \leq \frac{S - (m_1 + 1)(m_2 + 1) + 2 - \left\lceil \frac{2(m_1 m_2 + m_1 + m_2) + 1}{q^\ell + q} \right\rceil}{q^{\ell-1}(q^\ell - 1)}.$$

V. GENERAL CONSTRUCTION FROM SUBGROUPS WITH NON-TRIVIAL INTERSECTION

Here we consider non-trivial intersecting subgroups. This allows us to construct more examples.

Theorem V.1. *Let $\mathcal{F}|\mathbb{F}_q$ be a function field of genus g . Consider s non trivial subgroups $\mathcal{H}_i$ of the automorphism group of $\mathcal{F}|\mathbb{F}_q$ satisfying*

- (i) $\mathcal{G} \simeq \prod_{i=1}^s \mathcal{H}_i$ is a group; and let
- (ii) $\# \left(\mathcal{H}_i \setminus \left(\bigcup_{j < i} \mathcal{H}_j \right) \right) = r_i$.

Let $\mathcal{P}$ be a set of n places in $\mathcal{F}$ lying over rational places in the fixed field $\mathcal{F}^{\mathcal{G}}$ that are completely split in the extension $\mathcal{F}|\mathcal{F}^{\mathcal{G}}$. Suppose that there exists a place P_∞ of $\mathcal{F}$ which is totally ramified in the extension $\mathcal{F}|\mathcal{F}^{\mathcal{G}}$ and let $Q_\infty^{(i)}$ be the unique place in $\mathcal{F}^{\mathcal{H}_i}$ lying under P_∞ . Suppose there exists functions $z_i, w_i, i = 1, \dots, s$, such that

- (iii) $z_i \in \mathcal{F}^{\mathcal{H}_i}$, $\text{supp}((z_i)_\infty) = \{Q_\infty^{(i)}\}$;
- (iv) $w_i \in \mathcal{F} \setminus \mathcal{F}^{\mathcal{H}_i}$, $\text{supp}((w_i)_\infty) = \{P_\infty\}$;
- (v) $w_i : \mathcal{P}^{\mathcal{H}_i} \rightarrow \mathbb{F}_q$ is injective for each $i = 1, \dots, s$.

For each $i = 1, \dots, s$ let $t_i \geq 1$ be such that

$$V_i := \left\{ \sum_{\ell=0}^{r_i-1} \left(\sum_{j=0}^{t_i} a_{\ell j}^{(i)} z_j^j \right) w_i^\ell \in \mathcal{F} \mid a_{\ell j}^{(i)} \in \mathbb{F}_q \right\}$$

is contained in $\mathcal{L}((n-d)P_\infty)$ for some $1 \leq d \leq n$ and let $V = \bigcap_{i=1}^s V_i \subset \mathcal{L}((n-d)P_\infty)$. If $\dim_{\mathbb{F}_q}(V) > 0$, then there exists an

$$[n, \dim_{\mathbb{F}_q}(V), \geq d; r_1, r_2, \dots, r_s]\text{-recoverable code}.$$

Proof. The proof uses the same ideas as in Theorem IV.1. Let $V = \bigcap_{i=1}^s V_i \subset \mathcal{L}((n-d)P_\infty)$ and consider the following map

$$\begin{aligned} e_{\mathcal{P}} : V &\rightarrow \mathbb{F}_q^n \\ f &\mapsto e_{\mathcal{P}}(f) = (f(P_1), \dots, f(P_n)), \end{aligned}$$

where $\mathcal{P} = \{P_1, \dots, P_n\}$. The linear code $e_{\mathcal{P}}(V)$ is contained in the algebraic geometry code $C_{\mathcal{L}}(\sum_i^n P_i, (n-d)P_\infty)$, so it has minimum distance at least d . This follows from the fact that the minimum distance of $e_{\mathcal{P}}(V)$ is larger than or equal to the minimum distance of $C_{\mathcal{L}}(\sum_i^n P_i, (n-d)P_\infty)$. By the bound on Equation (5), the code $C_{\mathcal{L}}(\sum_i^n P_i, (n-d)P_\infty)$ has minimum distance at least d . By definition the code has length n and dimension $\dim_{\mathbb{F}_q}(V)$. Now, we deal with recoverability. Assume $P \in \mathcal{P}$ is a rational place, let $Q_i = P \cap \mathcal{F}^{\mathcal{H}_i}$ be the only place in $\mathcal{F}^{\mathcal{H}_i}$ lying under P , and $\mathcal{P}_i \subset \mathcal{P}$ be the set of places in $\mathcal{F}$ over Q_i for $i = 1, \dots, s$. Then $P \in \mathcal{P}_i$ and $\#\mathcal{P}_i = \#\mathcal{H}_i$ for $i = 1, \dots, s$ by definition of the set $\mathcal{P}$. Consider $\mathcal{Q}_i = \mathcal{P}_i \setminus \left(\bigcup_{j < i} \mathcal{P}_j \right)$ for any $i = 1, \dots, s$. Now, $\#\mathcal{Q}_i = r_i$ and the sets $\mathcal{Q}_i$ are pairwise disjoint. Using a similar approach as in Theorem IV.1, for an $f \in V_i$ one can show that $f(P)$ can be repaired by $\mathcal{Q}_i$. $\square$

We notice that with a slight change on hypothesis of Theorem V.1 we can construct codes with a different recoverability.

Corollary V.2. *With the same notation as in Theorem V.1, changing the hypothesis (ii) by*

(ii') *there exists $m \geq 1$ such that*

$$1 \leq \# \left(\mathcal{H}_i \cap \left(\bigcup_{j \neq i} \mathcal{H}_j \right) \right) \leq m,$$

with $\#\mathcal{H}_i = r_i + m$;

we also have the existence of an

$[n, \dim_{\mathbb{F}_q}(V), \geq d; r_1, r_2, \dots, r_s]$ -recoverable code.

Proof. The proof is similar to the one in Theorem V.1, we only need to deal with recoverability. Assume $P \in \mathcal{P}$ is a rational place, let $Q_i = P \cap \mathcal{F}^{\mathcal{H}_i}$ be the only place in $\mathcal{F}^{\mathcal{H}_i}$ lying under P , and $\mathcal{P}_i \subset \mathcal{P}$ be the set of places in $\mathcal{F}$ over Q_i for $i = 1, \dots, s$. Then $P \in \mathcal{P}_i$ and $\#\mathcal{P}_i = \#\mathcal{H}_i$ for $i = 1, \dots, s$ by definition of the set $\mathcal{P}$. From i) we also have $\# \left(\mathcal{H}_i \cap \left(\bigcup_{j \neq i} \mathcal{H}_j \right) \right) \leq m$. For any $f \in V$, using a similar approach as in Theorem IV.1 it we are going to show that $f(P)$ can be repaired by $\mathcal{P}_i \setminus \{P\}$ for any $i = 1, \dots, s$. Fixing i we have that $f \in V_i$ with $\#(\mathcal{P}_i \cap \left(\bigcup_{j \neq i} \mathcal{P}_j \right)) < m$ and $\#\mathcal{P}_i = m + r_i$, so we can choose a subset $\mathcal{Q}_i$ of $\mathcal{P}_i$ such that $\#\mathcal{Q}_i = \#\mathcal{P}_i - m = r_i$ and $\mathcal{Q}_i \cap \left(\bigcup_{j \neq i} \mathcal{P}_j \right) = \emptyset$. Then we can recover $f(P)$ from the set $\mathcal{Q}_i \subseteq \mathcal{P}_i$. We also have the repair sets $\mathcal{Q}_i$ are pairwise disjoint. $\square$

A. Codes from the generalized Hermitian curve

Following the same notation as in Subsection IV-D we work with the Generalized Hermitian curve $y^{q^\ell+1} = x^q + x$ over $\mathbb{F}_{q^{2\ell}}$ and its function field $\mathcal{S}$. In this case, let $\eta, \lambda \in \mathbb{F}_{q^{2\ell}}^*$ with $\text{ord}(\eta)$ and $\text{ord}(\lambda)$ divisors of $q^\ell + 1$ and $\gcd(\text{ord}(\eta), \text{ord}(\lambda)) = m > 1$. Consider two subgroups of the automorphism group of $\mathcal{S}$ given by

$$\mathcal{H}_1 := \{ \sigma_i : (x, y) \mapsto (x + a, \eta^i y) \mid a^q + a = 0 \text{ and } a \in \mathbb{F}_{q^{2\ell}}, i = 0, \dots, \text{ord}(\eta) - 1 \},$$

and

$$\mathcal{H}_2 := \{ \sigma_i : (x, y) \mapsto (x, \lambda^i y) \mid \lambda \in \mathbb{F}_{q^{2\ell}}, i = 0, \dots, \text{ord}(\lambda) - 1 \},$$

so that $\#\mathcal{H}_1 = q \cdot \text{ord}(\eta)$, $\#\mathcal{H}_2 = \text{ord}(\lambda)$, they commute and $\mathcal{G} = \mathcal{H}_1 \mathcal{H}_2$ has order $\frac{q \cdot \text{ord}(\eta) \text{ord}(\lambda)}{m}$. The fixed fields are

$$\mathcal{S}^{\mathcal{H}_1} = \mathbb{F}_{q^{2\ell}}(x^q + x, y^{\frac{q^\ell+1}{\text{ord}(\eta)}}), \quad \mathcal{S}^{\mathcal{H}_2} = \mathbb{F}_{q^{2\ell}}(x, y^{\frac{q^\ell+1}{\text{ord}(\lambda)}}),$$

and

$$\mathcal{S}^{\mathcal{G}} = \mathbb{F}_{q^{2\ell}}(x^q + x, y^{\frac{q^\ell+1}{m}})$$

where $\mathcal{G} \simeq \mathcal{H}_1 \times \mathcal{H}_2$ is the direct product. The number of rational places in $\mathcal{S}^{\mathcal{G}}$ that completely splits in the field extension $\mathcal{S}|\mathcal{S}^{\mathcal{G}}$ is $m(q^{2\ell} - 1)/(\text{ord}(\eta)\text{ord}(\lambda))$, then we have $n = \#\mathcal{P} = q^{2\ell+1} - q$. Let $0 \leq t_i, i = 1, 2$ and consider the vector spaces

$$V_1 := \left\{ \sum_{\ell=0}^{\text{ord}(\eta)-2} \left(\sum_{j=0}^{t_1} a_{\ell,j} y^{\frac{q^\ell+1}{\text{ord}(\eta)}} \right) x^\ell : a_{\ell,j} \in \mathbb{F}_{q^{2\ell}} \right\},$$

$$V_2 := \left\{ \sum_{\ell=0}^{\text{ord}(\lambda)-2} \left(\sum_{j=0}^{t_2} a_{\ell,j} x^j \right) y^\ell : a_{\ell,j} \in \mathbb{F}_{q^{2\ell}} \right\},$$

Choose t_i such that for $M_1 = \max\{t_1, \text{ord}(\lambda) - 2\}$, $M_2 = \max\{t_2, \text{ord}(\eta) - 2\}$ we have

$$S = \max \left\{ M_1 q \frac{q^\ell + 1}{\text{ord}(\eta)} + (q^\ell + 1)(\text{ord}(\eta) - 2), \right. \\ \left. M_2(q^\ell + 1) + q(\text{ord}(\lambda) - 2) \right\} \leq n.$$

Then $V_i \subset \mathcal{L}(SP_\infty)$, for $i = 1, 2$, and for $m_1 = \min\{t_1, \text{ord}(\lambda) - 2\}$, $m_2 = \min\{t_2, \text{ord}(\eta) - 2\}$ we obtain

$$V = V_1 \cap V_2 = \left\{ \sum_{\ell=0}^{m_1} \left(\sum_{j=0}^{m_2} a_{\ell,j} x^j \right) y^\ell : a_{\ell,j} \in \mathbb{F}_{q^{2\ell}} \right\}$$

and $\dim_{\mathbb{F}_{q^2}} V = (m_1 + 1)(m_2 + 1)$. The minimum distance of the code satisfies $d \geq q^{2\ell+1} - q - S$. We have obtained the following proposition.

Proposition V.3. *Let q be odd, $\eta, \lambda \in \mathbb{F}_{q^{2\ell}}^*$ with $\text{ord}(\eta)$ and $\text{ord}(\lambda)$ divisors of $q^\ell + 1$, $\gcd(\text{ord}(\eta), \text{ord}(\lambda)) = m > 1$ and $n = q^{2\ell+1} - q$. Choose t_i such that for $M_1 = \max\{t_1, \text{ord}(\lambda) - 2\}$, $M_2 = \max\{t_2, \text{ord}(\eta) - 2\}$ we have*

$$S = \max \left\{ M_1 q \frac{q^\ell + 1}{\text{ord}(\eta)} + (q^\ell + 1)(\text{ord}(\eta) - 2), \right. \\ \left. M_2(q^\ell + 1) + q(\text{ord}(\lambda) - 2) \right\} \leq n.$$

For $m_1 = \min\{t_1, \text{ord}(\lambda) - 2\}$, $m_2 = \min\{t_2, \text{ord}(\eta) - 2\}$ there exists a

$$[n, (m_1 + 1)(m_2 + 1), \geq n - S; q \cdot \text{ord}(\eta), \text{ord}(\lambda) - m]$$

recoverable code $\mathcal{C}$ over $\mathbb{F}_{q^{2\ell}}$.

The codes arising from Proposition V.3 have relative designed distance $\Delta(\mathcal{C})$ is at most

$$\frac{S - (m_1 + 1)(m_2 + 1) + 2 - \left\lfloor \frac{2(m_1 m_2 + m_1 + m_2) + 1}{q \cdot \text{ord}(\eta) + \text{ord}(\lambda) - m + 1} \right\rfloor}{q^{2\ell} - q}.$$

Remark V.4. *Every example proposed in the previous section can be used to apply the construction introduced in this section as well obtaining different parameters. Indeed it is*

sufficient to enlarge a little bit one of the two groups including elements from the second one. This is the case, for instance, when comparing examples arising from Propositions IV.8 and V.3. To this end, consider $\eta, \lambda \in \mathbb{F}_{q^{2\ell}}^*$, where $\text{ord}(\eta)$ divides $q^\ell + 1$, $\text{ord}(\lambda) = q^\ell + 1$ and the subgroups

$$\begin{aligned}\mathcal{H}_1 &= \{\sigma_i : (x, y) \mapsto (x + a, y) : a^q + a = 0, a \in \mathbb{F}_{q^{2\ell}}\}, \\ \mathcal{H}_2 &= \{\sigma_i : (x, y) \mapsto (x, \lambda^i y) : i = 0, \dots, q^\ell\}, \\ \mathcal{H}_3 &= \{\sigma_i : (x, y) \mapsto (x, \eta y) : i = 0, \dots, \text{ord}(\eta) - 1\}, \\ \tilde{\mathcal{H}}_1 &= \mathcal{H}_1 \times \mathcal{H}_3, \\ \tilde{\mathcal{H}}_2 &= \mathcal{H}_2.\end{aligned}$$

Note that $\mathcal{G} = \mathcal{H}_1 \times \mathcal{H}_2 = \tilde{\mathcal{H}}_1 \tilde{\mathcal{H}}_2$ and $n = q^{2\ell+1} - q$. Let $(t_1, t_2) = (q^\ell - 1, q - 2)$ and let $\eta \in \mathbb{F}_{q^{2\ell}}$ be such that $\text{ord}(\eta) \geq q - 2$. In this case, since $\text{ord}(\lambda) = q^\ell + 1$ the $\text{gcd}(\text{ord}(\lambda), \text{ord}(\eta)) = \text{ord}(\eta)$, Proposition V.3 provides an

$$[n, q^\ell(q - 1), \geq n - S^{(1)}; q \cdot \text{ord}(\eta), q^\ell + 1 - \text{ord}(\eta)]$$

recoverable code $\mathcal{C}_1$ over $\mathbb{F}_{q^{2\ell}}$, where

$$S^{(1)} = (q^{2\ell+1} - q)/\text{ord}(\eta) + (q^\ell + 1)(\text{ord}(\eta) - 2).$$

By Proposition IV.8 there exists an

$$[n, q^\ell(q - 1), \geq n - S^{(2)}; q - 1, q^\ell]$$

recoverable code $\mathcal{C}$ over $\mathbb{F}_{q^{2\ell}}$, where $S^{(2)} = 2q^{\ell+1} - 2q^\ell - 2q + 2$. This shows that recoverable codes with same length, dimension, but different minimum distance and recoverability can be obtained from similar settings.

Table I collects the parameters of the codes constructed in the previous sections. For the meaning of the symbols we refer to the corresponding proposition. Finally, in Table II we provide parameters of codes over $\mathbb{F}_{2^{12}}$ obtained for specific choices of subgroups in Propositions IV.2, IV.4, IV.6, IV.9.

Table I
PARAMETERS OF THE OBTAINED LOCALLY RECOVERABLE CODES.

	n	k	d	Rec.	Prop.
$\mathbb{F}_{q^2}$	$q(q^2 - 1)$	q	$\geq q(q^2 - 1) - q(ut_2 + u - t_2 - 2)$	$q - 1, u - 1$	IV.2
$\mathbb{F}_{q^2}$	$q(q^2 - 1)$	$(t_1 + 1) \cdot (t_2 + 1)$	$\geq q(q^2 - 1) - M$	$q - 1, u - 1$	IV.4
$\mathbb{F}_{q^6}$	$q^8 - q^6 + q^5 - q^3$	$(M_1 + 1) \cdot (N_1 + 1)$	$\geq q^8 - q^6 + q^5 - q^3 - S$	$a - 1, \text{ord}(\eta) - 1, \text{ord}(\omega) - 1$	IV.6
$\mathbb{F}_{q^{2\ell}}$	$q^{2\ell+1} - q$	$(m_1 + 1) \cdot (m_2 + 1)$	$\geq q^{2\ell+1} - q - S$	$q - 1, q^\ell$	IV.8
$\mathbb{F}_{q^\ell}$	$q^{\ell-1}(q^\ell - 1)$	$(m_1 + 1) \cdot (m_2 + 1)$	$\geq q^{\ell-1}(q^\ell - 1) - S$	$\frac{q^\ell - q}{q^{\ell-1} - 1}$	IV.9
$\mathbb{F}_{q^{2\ell}}$	$q^{2\ell+1} - q$	$(m_1 + 1) \cdot (m_2 + 1)$	$\geq q^{2\ell+1} - q - S$	$q \cdot \text{ord}(\eta), \text{ord}(\lambda) - m$	V.3

Table II
EXAMPLES OF THE LOCALLY RECOVERABLE CODES OVER $\mathbb{F}_{2^{12}}$.

Code $\mathcal{C}$	Parameters	$\Delta(\mathcal{C})$	Prop.
[262080, 64, ≥ 253952 ; 63, 4]	$u = 5, t_2 = 31$	0.03076	IV.2
[262080, 910, ≥ 257152 ; 63, 12]	$u = 13, t_1 = 13, t_2 = 64$	0.01524	IV.4
[62400, 12, ≥ 62226 ; 3, 4, 12]	$a = 4, t_1 = 3, t_2 = 2, t_3 = 2, \text{ord}(\eta) = 5, \text{ord}(\omega) = 13$	0.00259	IV.6
[4193280, 1395372, ≥ 2738 ; 1364, 1023]	$t_1 = 1363, t_2 = 1022, q = 4, \ell = 6$	0.66630	IV.9

Table III
EXAMPLES OF THE LOCALLY RECOVERABLE CODES OVER $\mathbb{F}_{3^6}$.

Code $\mathcal{C}$	Parameters	$\Delta(\mathcal{C})$	Prop.
[19656, 27, ≥ 18225 ; 26, 27]	$u = 28, t_2 = 1$	0.0714794	IV.2
[19656, 812, ≥ 18171 ; 26, 27]	$u = 28, t_1 = 28, t_2 = 27$	0.0327635	IV.4
[6048, 6, ≥ 6005 ; 2, 3, 6]	$a = 3, t_1 = 2, t_2 = t_3 = 1, \text{ord}(\eta) = 4, \text{ord}(\omega) = 7$	0.0325601	IV.6
[2184, 54, ≥ 2080 ; 2, 27]	$t_1 = 26, t_2 = 1, q = 3, \ell = 3$	0.0219780	IV.8
[58968, 7200, ≥ 655 ; 90, 80]	$t_1 = 89, t_2 = 79, q = 9, \ell = 3$	0.865385	IV.9
[2184, 81, ≥ 1582 ; 12, 24]	$t_1 = 26, t_2 = 2, q = 3, \ell = 3, \text{ord}(\eta) = 4, \text{ord}(\lambda) = 28$	0.237180	V.3

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