Innovative Design of Steel Girders in Cable-Supported Bridges
By application of numerical optimization methods

Baandrup, Mads

Publication date:
2019

Document Version
Publisher's PDF, also known as Version of record

Link back to DTU Orbit

Citation (APA):
Innovative Design of Steel Girders in Cable-Supported Bridges

By application of numerical optimization methods
Innovative Design of Steel Girders in Cable-Supported Bridges

- By application of numerical optimization methods

Mads Baandrup

Ph.D. Thesis

Department of Civil Engineering
Technical University of Denmark

2019
Supervisors:
Associate Professor Peter Noe Poulsen, DTU Civil Engineering
Associate Professor John Forbes Olesen, DTU Civil Engineering
Professor Ole Sigmund, DTU Mechanical Engineering
Technical Director Henrik Polk, COWI A/S

Assessment Committee:
Professor Matthew Gilbert, University of Sheffield, England
Professor Lars Damkilde, Aalborg University, Denmark
Professor Henrik Stang, DTU Civil Engineering, Denmark

Innovative Design of Steel Girders in Cable-Supported Bridges
-By application of numerical optimization methods

Copyright © 2019 by Mads Baandrup
Printed by DTU-Tryk
Department of Civil Engineering
Technical University of Denmark
Report: R-425
ISBN: 87-7877-524-8
Preface

This thesis is submitted as partial fulfillment of the requirements for the Danish Ph.D. degree. The study has taken place at COWI A/S, and the Department of Civil Engineering and Department of Mechanical Engineering at the Technical University of Denmark (DTU) between January 2017 and December 2019. Associate Professor Peter Noe Poulsen has been the principal supervisor at DTU, and Associate Professor John Forbes Olesen and Professor Ole Sigmund have been co-supervisors. Technical Director Henrik Polk has been the principal supervisor at COWI.

The industrial partner, COWI, is an international consulting group with a world-leading bridge engineering division. The company was founded in 1930 in Denmark and today employs around 7,000 people. The author was employed at COWI during the three years of study.

The project was funded partly by the COWI Foundation, grant C-131.02, and partly by Innovation Fund Denmark, grant 5189-00112B. Furthermore, a part of the work was funded through a PRACE (Partnership for Advanced Computing in Europe) grant, TopBridge, giving access to the Irene supercomputer.

The thesis is divided into three parts. In the first part, the research and findings are introduced and summarized. The second part comprises six appended papers, which present the research in greater detail. In the third part, supplementary material is collected. The thesis is presented in the format chosen by the Department of Civil Engineering at DTU.

Kongens Lyngby, the 31st of December 2019

Mads Baandrup
Preface to Published Version

The thesis was defended at a public defence on Tuesday the 26th of May 2020 at the Technical University of Denmark. Subsequently, the Ph.D. degree was awarded from the Technical University of Denmark.

Compared to the version submitted for assessment, few minor editorial corrections have been made. The status of Paper III has changed from submitted to published, and Paper IV has changed from accepted to published.

Kongens Lyngby, the 1st of June 2020

Mads Baandrup
Acknowledgements

The Ph.D. project was funded by the Department of Major Bridges International (COWI A/S), the COWI Foundation, and Innovation Fund Denmark. Furthermore, a part of the work was funded through a PRACE (Partnership for Advanced Computing in Europe) grant. I gratefully acknowledge all the funding of the project.

The presented work would not have been achieved without the help and support from a range of people, which I would like to take the opportunity to express my gratitude.

I gratefully acknowledge the support, guidance, and interesting discussions with my supervisors Associate Professor Peter Nøe Poulsen, Associate Professor John Forbes Olesen, Professor Ole Sigmund, and Technical Director Henrik Polk.

Furthermore, I would like to thank Niels Age, from the Department of Mechanical Engineering at DTU, for supervision and support during the work with PETSc and large-scale computing. Also, I would like to thank Jesper Warshow Sørensen, Søren Christen Christensen, Søren Lausten, and Simon Bjærre from COWI for assistance and discussions during the project. Moreover, I would like to thank my colleagues at COWI for their support and interest in the project.

Last, but not least, sincere thanks to my fellow Ph.D. and PostDoc colleagues at DTU Civil Engineering: Anders Bau Hansen, Asmus Skar, Daniel Vestergaard, Jesper Harrild Sørensen, Mads Emil Møller Andersen, Martin Jensen Meyland, Morten Andersen Herfelt, Sebastian Andersen, and Thomas Westergaard Jensen, for a great working environment.
Abstract

The main design principles for girders in cable-supported bridges have remained largely unchanged for the past 50 years and have reached a point where the potential for further development is limited. The design concept of closed steel box-girders with orthotropic stiffened decks has been used extensively in major cable-supported bridges due to its many advantages compared to the alternative of classic truss girders. However, the design concept is subject to substantial inherent fatigue issues. Furthermore, in future super-long bridges with spans beyond 3 km, the girder self-weight becomes a critical design factor preventing even longer bridge spans. Moreover, considering that the construction industry accounts for 39% of the world’s CO₂ emissions, attention must be broadened from the one-sided focus on construction costs to reducing material consumption.

To accommodate the challenges of decreasing self-weight significantly and reducing fatigue issues, it is anticipated that radical design changes will be required. With weak outlooks to new light-weight-high-strength materials, the identification of new, innovative, and more material-efficient design concepts using existing materials is needed.

In this thesis, entitled "Innovative design of steel girders in cable-supported bridges", three different methods of structural optimization are applied in search of innovative girder concepts. The main focus is on reducing self-weight and on identifying more efficient load-carrying principles and, thus, more material-efficient structures.

Initially, parametric optimization is applied to pursue possible weight savings to conventional girder design. As a basis, a multiscale finite element model with sophisticated boundary conditions is established. Subsequently, a simple parameter study is carried out, followed by a gradient-based optimization with constraints on fatigue and deformation. The main findings are possible weight savings in the range of 6%-14%, achieved by using thinner plates and narrower stiffening troughs. Despite the possible weight savings, the results are considered modest, and it is confirmed that the conventional design concept is limited in further development, without altering the structural concept.

Next, topology optimization of continuum structures is applied as the first step in search of innovative girder designs. The method is applied in giga-scale (2.1 billion finite elements) with a minimum of restrictions to identify a lower bound of the optimized designs. The highly detailed and intricate structure evolving is
significantly different from the conventional design, indicating more efficient load-carrying principles. Based on the main structural features of the optimized designs, a simple interpreted design is established from where an initial weight saving of 13% is achieved. After a subsequent simple parametric optimization to identify the full potential, a total weight saving in excess of 28% is gained while maintaining manufacturability.

Finally, large-scale truss optimization based on finite element limit analysis is applied with constraints on stresses as well as global and local stability. A significant weight saving of 45% is achieved with a truss girder considerably different compared to the conventional design. Notably, a torsion grid evolves along the circumference of the domain, as well as large members in the bottom to carry loads primarily in tension. However, the higher weight savings gained with the truss girder, compared to the interpreted design, are achieved at the expense of increasing structural complexity, and thus, aggravated constructibility. Finally, potential weight savings of up to 54% are observed through a simple parameter study.

The possible weight savings, identified from the various optimization studies, translate into total savings in material quantities of the entire bridge in the range of 16%-30%, and a reduction in CO$_2$ emissions in the range of 18%-30%.

The identified design principles and possible weight savings emphasize the potential of using significantly different girder design concepts. Hence, the potential weight reductions to be achieved may close the gap toward super-long cable-supported bridges and reduce material quantities, and thus reduce the environmental impact.
Det primære designkoncept for ståldragere i kabelbårne broer er igennem de seneste 50 år forblevet uændret og har nået et punkt, hvor potentialet ved videreudvikling er begrænset. Designkonceptet, bestående af lukkede kassedragere af stål med ortotropiske brodæk, har været bredt anvendt i store kabelbårne broer grundet dets mange fordele sammenlignet med de klassiske gitterdragere. På trods af den udbredte anvendelse er designkonceptet i høj grad udfordret af medfødte udmattelsesproblemer. Derudover er brodragernes egenvægt en kritisk designparameter i fremtidige superlange broer med hovedspænd på over 3 km, hvilket kan umuliggøre opførelsen af endnu længere broer. Yderligere taget i betragtning, at byggeindustrien udleder 39% af verdens kuldoxid, er det umuligt at fokus udvides fra det ensidige mål om anlægspris og hen imod et mål om at reducere materialeforbruget.

For at løse udfordringerne med at reducere egenvægten væsentligt og samtidig mindske udmattelsesproblemerne, forventes det, at betydeligt anderledes designkoncepter skal udvikles. Med ringe udsigter til nye letvægtsmaterialer af høj styrke, er det i stedet nødvendigt at identificere nye, innovative og mere materialeeffektive designkoncepter i kendte materialer, såsom stål.

I denne afhandling med titlen ”Innovativt design af ståldragere i kabelbårne broer” anvendes tre forskellige tilgange til strukturel optimering i undersøgelsen af innovative designkoncepter af brodragere. Det primære fokus er rettet mod at reducere egenvægten og ligeledes mod at identificere principper for effektiv lastføring og derved opnå konstruktioner med bedre materialeudnyttelse.


Herefter anvendes topologiopptimering af kontinuum-strukturer som det første skridt i forfølgelsen af innovative designs af brodragere. Metoden anvendes i gigaskala (2,1 milliarder elementer) og med et minimum af begrænsninger med det mål
at identificere en nedre løsning for de optimerede brodrager-designs. Den yderst de-
taljerede og komplicerede konstruktion, som opstår under optimeringen, adskiller sig i høj grad fra det konventionelle design. Ligeledes viser den optimerede struktur mere effektive principper for lastføring. Baseret på de grundlæggende strukturelle karakteristika i den optimerede drager, etableres et forenklet fortolket design, hvor en direkte vægtbesparelse på 13% opnås. Efterfølgende udføres en simpel parametrisk optimering for at identificere det fulde potentiale af designet, hvor en samlet vægtbesparelse på 28% opnås uden at bygbarheden forringes.

Til sidst anvendes stor-skala gitteroptimering baseret på finite element limit analysis med bibetingelser på spændinger såvel som global og lokal stabilitet. En betydelig vægtbesparelse på 45% opnås ved gitterdragereens signifikant anderledes udformning sammenlignet med det konventionelle design. Gitterdrageren består hovedsageligt af et vridningsgitter langs designområdets rand og af store elementer i bunden af drageren til primært at bære de påførte laster i træk. Imidlertid kommer den større vægtbesparelse, opnået for gitterdrageren sammenlignet med det fortolkede design, på bekostning af en forøget kompleksitet i det strukturelle design, og således af en forringet bygbarhed. Afslutningsvis findes potentielle vægtbesparelser på op til 54% igennem simple parameterstudier.

De mulige vægtbesparelser, fundet igennem de forskellige optimeringsstudier, svarer til samlede besparelser i materialemængder på 16%-30% for hele broen og en reducering i CO₂-udledning på 18%-30%.

De identificerede designprincipper og mulige vægtbesparelser understreger potentialet ved at anvende signifikant anderledes koncepter til design af brodrager. De opnåelige vægtbesparelser kan således eventuelt muliggøre opførelsen af superlange kabelbårne brøer og samtidig reducere materialeforbruget, og dermed mindske de miljømæssige indvirkninger.
## Contents

Preface iii  
Preface to Published Version v  
Acknowledgements vii  
Abstract ix  
Resumé xi

### I Introduction and Summary 1

1 Introduction 3  
1.1 Cable-Supported Bridges . . . . . . . . . . . . . . . . . . . . . . . 5  
1.2 Girders in Cable-Supported Bridges . . . . . . . . . . . . . . . . . 8  
1.3 Closed Box-Girders with Orthotropic Deck . . . . . . . . . . . . . 10  
1.4 Objectives and Structure of the Thesis . . . . . . . . . . . . . . . . 15  
1.5 Contribution to the Field of Research . . . . . . . . . . . . . . . . 19

2 Applied Methods 21  
2.1 Structural Optimization . . . . . . . . . . . . . . . . . . . . . . . 21  
2.2 Numerical Optimization Methods . . . . . . . . . . . . . . . . . . 29  
2.3 Numerical Methods for Structural Analysis . . . . . . . . . . . . . 41

3 State of the Art Bridge Design - The Osman Gazi Bridge 45  
3.1 The Bridge Girder . . . . . . . . . . . . . . . . . . . . . . . . . . . 46

4 Parametric Optimization of the Conventional Design Concept 51  
4.1 Multiscale Model of the Bridge Girder . . . . . . . . . . . . . . . . 52  
4.2 Parameter Studies . . . . . . . . . . . . . . . . . . . . . . . . . . . 56  
4.3 Gradient-Based Parametric Optimization . . . . . . . . . . . . . . 58  
4.4 Summary and Discussion . . . . . . . . . . . . . . . . . . . . . . . 62
# 5 Innovative Designs by Topology Optimization

- **5.1 Topology Optimization of Bridge Girders**
- **5.2 Interpretation and Quantitative Studies**
- **5.3 Summary and Discussion**

# 6 Innovative Designs by Truss Optimization

- **6.1 Truss Optimization Applying Finite Element Limit Analysis**
- **6.2 Truss Optimization of Bridge Girders**
- **6.3 Summary and Discussion**

# 7 Challenges toward Practical Application

- **7.1 Structural Robustness**
- **7.2 Manufacturing Challenges**
- **7.3 Construction Costs**
- **7.4 Interpretation of Optimized Structures**

# 8 Conclusions and Recommendations for Further Work

- **8.1 Conclusions**
- **8.2 Recommendations for Further Work**

# Bibliography

**II Appended Papers**

**Paper I**

"Optimization of orthotropic girders in cable supported bridges by parametric studies",
M. Baandrup, P.N. Poulsen, J.F. Olesen & H. Polk.
Conference proceeding in: *IABSE Congress 2019, New York City: The Evolving Metropolis, 2019*

**Paper II**

"Parametric Optimization of Orthotropic Girders in a Cable-Supported Bridge",
M. Baandrup, P.N. Poulsen, J.F. Olesen & H. Polk.
Published in: *Journal of Bridge Engineering, 2019*

**Paper III**

"Closing the gap towards super-long suspension bridges using computational morphogenesis",
M. Baandrup, O. Sigmund, H. Polk & N. Aage.
Published in: *Nature Communications, 2020*
Paper IV
"Truss optimization applying finite element limit analysis including global and local stability",
P.N. Poulsen, J.F. Olesen & M. Baandrup.
Published in: Structural and Multidisciplinary Optimization, 2020 . . . . . 163

Paper V
"Large-scale truss optimization including global and local stability",
M. Baandrup, J.F. Olesen & P.N. Poulsen.
Submitted to: Structural and Multidisciplinary Optimization, 2019 . . . . . 179

Paper VI
"Optimization of truss girders in cable-supported bridges including stability",
M. Baandrup, P.N. Poulsen, J.F. Olesen & H. Polk.
Submitted to: Journal of Bridge Engineering, 2019 . . . . . . . . . . . . . . 197

III Supplementary Material 215

Appendices 217
A Parameter Studies 219
B Parametric Optimization 223
C Topology Optimization of Bridge Girders 227
D Truss Optimization of Bridge Girders 249

Notes 275
I Critical Fatigue Details 277
Part I

Introduction and Summary
Chapter 1

Introduction

For thousands of years, bridges have made easy-access possible in places that would otherwise be impassable and thus facilitating fixed infrastructure links, enabling people to connect, trade, and exchange ideas. In the early days, stepping stones and fallen trees constituted the connecting link, and from the 13th century BCE, arch bridges of stones increased the distances possible to cross. Since the 16th century CE, suspended rope bridges made access across canyons and gorges possible, and in 1779 the first iron bridge was constructed in Shropshire, England, permitting an even larger free span, in this case of 30.6 m. However, it was not until the opening of the Union Bridge on the Scotland-England border in 1820 with a span of 137 m that the limits on possible free spans truly expanded. This significant increase in span length was solely due to the introduction of the suspension bridge technology, which evolved into cable-supported bridges.

Since then, cable-supported bridges have played a key role in infrastructure, facilitating fixed links on sites formerly separated by large distances or deep waters. Furthermore, they have become landmarks, such as the Brooklyn Bridge (Fig. 1.1) and the Golden Gate Bridge (Fig. 1.2).

Historically, the longest bridge spans double roughly every 50 years and will
Introduction

Figure 1.2: The Golden Gate Bridge, San Francisco, USA. The bridge opened in 1937 with a world-record-span of 1,280 m. Girder type: truss (photo from www.pixabay.com)

soon pass the 2-km-mark. Moreover, with eight out of the ten longest spans in history constructed within the past 15 years (and seven out of the ten within the past seven years!), the evolution of bridges is remarkable.

However, in the past 50 years, the main design principles of girders (the part of the bridge that carries the traffic) in cable-supported bridges have remained unchanged and have reached a point where the potential for further development is limited. The design concept of closed steel box-girders with orthotropic stiffened decks and transverse diaphragms has been used extensively due to its many advantages compared to the previously used classic truss girders. Such advantages include aerodynamic properties, reduced weight, and cheaper manufacturing. Despite these advantages, the design concept is subject to substantial inherent fatigue issues and will be challenged by self-weight in future super-long bridges with spans beyond 3 km. Hence, it is anticipated that radical design changes will soon be required.

Furthermore, considering that the construction industry accounts for 39% of the world’s CO₂ emissions (United Nations (2017)), attention must be drawn to material consumption and environmental impact, hence toward decreasing self-weight, rather than the one-sided focus on construction costs. With weak outlooks to new light-weight-high-strength materials, and having the fatigue problems in mind, the challenge to identify new and more material-efficient girder designs is present.

The main objective of this thesis is thus to identify new and innovative design concepts for steel girders in cable-supported bridges, with the focus on reducing self-weight while preserving other properties such as strength and durability. This is done by the application of different numerical methods to perform structural optimization.

In this introductory chapter, the background and motivation for the thesis are presented. Firstly, cable-supported bridges are described in general terms, followed by a brief introduction to different types of girders in cable-supported bridges. Secondly, the design concept of closed box-girders with orthotropic decks
is presented, including a review of the research conducted on the inherent fatigue issues. Briefly, alternative solutions to current design principles are reviewed and discussed. Throughout these sections, the challenges facing designers of the current and future girders are pointed out to motivate the present work. Subsequently, an outline of the objectives of the thesis is given, followed by a presentation of the thesis structure. Finally, the main contributions to the field of research are summarized.

1.1 Cable-Supported Bridges

In places where large free spans are required, e.g., due to deep waters, canyons, or intensive ship traffic, the only option is often a cable-supported bridge. Cable-supported bridges can be divided into two classes based on the different load-carrying principles: cable-stay bridges (Fig. 1.3) and suspension bridges (Fig. 1.4). Contrary to arch or beam bridges, the use of high-strength steel cables, makes cable-supported bridges with spans in the range of 200 m to 2,000 m possible (Gimsing and Georgakis (2012)).

Historically, cable-stay bridges have been constructed with free spans up to about 1,100 m (The Russky Bridge, Russia), and in general, they are not feasible to construct with spans beyond 1,500 m (Gimsing and Georgakis (2012)). Thus, for spans beyond 1,500 m, only suspension bridges are feasible, and since 1998, the Akashi Kaikyo Bridge in Japan holds the world record of the longest free span of 1,991 m.

In this thesis, span ranges are referred to by three different terms: long (> 1,000 m), very-long (> 2,000 m), and super-long (> 3,000 m). Long-span bridges have been constructed for almost a century since the opening in 1931 of the George

![Figure 1.3: Example of modern cable-stay bridge, the Øresund Bridge, connecting Denmark and Sweden. The bridge opened in 2000 with a span of 490 m. Girder type: truss (photo from www.pixabay.com)](image-url)
Introduction

**Figure 1.4**: Example of modern suspension bridge, the Great Belt Bridge, Denmark. The bridge opened in 1998 with a span of 1,624 m. Girder type: closed box-girder with orthotropic deck (photo from Sund & Bælt)

Washington Bridge in New York, USA, with a main span of 1,067 m. Additionally, very-long bridges are possible to construct with current design principles and technology. However, the first super-long bridge is still to be constructed, though with multiple design obstacles left to be overcome. Since the main challenges in the construction of future cable-supported bridges are found with the super-long bridges, and thus with suspension bridges, the focus of the remainder of this thesis will be on this bridge type.

One of the main challenges in super-long bridges is the self-weight of the girder. To illustrate this, a sketch of the load-carrying principles in a suspension bridge is shown in Fig. 1.5, where the nomenclature is also shown. The live-load, e.g., traffic loads, acts on the bridge girder and is transferred by hangers to the main cables. From the main cables, the loads are transferred partly into the anchorages (anchor blocks), and partly into the towers, and from there to the foundations. Hence, the live-load and the girder self-weight are imposed on all the remaining parts of the primary bridge structure. It should be noted here that for very-long suspension bridges, live-load is only a small fraction of the girder self-weight, typically in the range of 5%-20%.

Soon, with plans for bridges spanning beyond 3 km in Norway, Italy, and Indonesia, and spanning 5 km over the Strait of Gibraltar, self-weight will by far, be the governing load, exceeding 90% of the total loads (including live-load, wind load, and seismic action). In this regard, and with the load-path in mind, weight-minimization of the girder is a natural starting point. Thus, a reduced girder weight will have knock-on effects on the main cables, anchor blocks, towers, and tower foundations, as estimated in Table 1.1.

With currently available materials, a theoretical upper-limit of 5-km-spans (Lewis (2012)) is defined by the strength of the main cables. Thus, this is the limit where the main cables are capable of carrying nothing but themselves. To
Figure 1.5: Structural principle and nomenclature of a suspension bridge

Table 1.1: Estimated knock-on effects from the reduction of girder self-weight

<table>
<thead>
<tr>
<th>Bridge component</th>
<th>Assumption behind material saving</th>
<th>Percentage of girder weight-saving</th>
</tr>
</thead>
<tbody>
<tr>
<td>Main cables</td>
<td>Conservatively, the weight savings are equivalent to the total savings in the girder.</td>
<td>100%</td>
</tr>
<tr>
<td>Anchor blocks</td>
<td>The weight savings in the main cables translate directly to similar weight savings in the anchor blocks.</td>
<td>100%</td>
</tr>
<tr>
<td>Towers</td>
<td>Since the normal force in the towers (from the main cables) only contributes with 50% of the total forces (with the remaining 30% from bending moments and 20% from buckling), the weight reduction is only half of the reduction in the main cables.</td>
<td>50%</td>
</tr>
<tr>
<td>Tower foundations</td>
<td>The weight reduction in the tower foundations is equivalent to the reduction in the towers.</td>
<td>50%</td>
</tr>
</tbody>
</table>

*Girder weight-saving given in percentage of initial girder weight  
†The girder surface (e.g., asphalt) contributes to 20%-50% of the total weight transferred to the main cables. Hence, the percentage saved in structural steel of the girder needs to be adjusted before estimating knock-on effects

decrease the additional loads carried by the main cables, a reduction of the girder self-weight is needed and thus an apparent first step toward the construction of super-long bridges.

In addition to the focus on reducing girder self-weight, considering that the construction industry is currently responsible for 39% of the world’s CO₂ emissions (United Nations (2017)), attention should be broadened from construction costs to include reducing material consumption, and hence, self-weight. With weak outlooks to new light-weight-high-strength materials, the focus must be on identifying new and more material-efficient designs. This challenge is one of the main objectives of this thesis.
Previously, in Fairclough et al. (2018), the entire cable-supported bridge system was subject to optimization, in search of theoretically optimal forms of super-long bridges. However, in this thesis, the focus is entirely on the bridge girder, based partially on the above arguments. The bridge girder and the specific additional challenges regarding it are introduced and developed in the following sections.

### 1.2 Girders in Cable-Supported Bridges

In cable-supported bridges, either steel, concrete, or composite girders are used. Concrete and composite girders are particularly used in cable-stay bridges, due to the large compression forces in the deck. However, for long-span bridges, steel girders are used in both cable-stay and suspension bridges, due to their high strength-to-weight ratio. An overview of the typical application of girders for various bridge types and span lengths is shown in Fig. 1.6.

Many different types of steel girders exist, however, for long-span bridges, only two types are typically used: the classic truss girder and the closed box-girder with orthotropic deck. An example of a classic truss girder is shown in Fig. 1.7, where the girder of the Akashi Kaikyo Bridge, Japan, is shown. Similarly, an example of a closed box-girder with orthotropic deck is shown in Fig. 1.8, where the girder of the Osman Gazi Bridge, Turkey, is shown during installation.

Historically, truss girders were used in the majority of long suspension bridges until the 1960s (see e.g. Fig. 1.1 and 1.2), where the design concept of closed box-girders with orthotropic decks was introduced in the Severn Bridge, England. Since then, this design concept has been used in the majority of major suspension bridges, as well as numerous cable-stay bridges around the world.

In general, the benefits of closed box-girders are many compared to classic truss girders. They are lighter, and they require less maintenance since the interior of the girder is closed off from the environment. They are cheaper and simpler to

![Figure 1.6: Typical application of steel (green), concrete (blue), and composite (orange) girders for various bridge types and span lengths (reproduction of figure in Polk and Walker (2017))](image-url)
Introduction

Figure 1.7: Example of a classic truss girder, the Akashi Kaikyo Bridge, Japan. The bridge opened in 1998 with a world-record-span of 1,991 m (photo from Kansai Paint Co., Ltd.)

Figure 1.8: Example of a closed box-girder with orthotropic deck, during erection of the Osman Gazi Bridge, Turkey. The bridge opened in 2016 with a span of 1,550 m (photo from www.cranemarket.com)
Introduction

manufacture, due to the relative simplicity of the plate-based design. Moreover, they possess better aerodynamic properties due to the streamlined outer wind profile and the high torsional stiffness. However, they are less stiff, so when heavy rail-traffic is present, the truss girders are preferred, such as in the Øresund Bridge (Fig. 1.3).

Since closed box-girders primarily are used in major cable-supported bridges, and for the time being, is the only potential option in super-long bridges, classic truss girders will not be considered further in this thesis.

1.3 Closed Box-Girders with Orthotropic Deck

The design principle of a typical closed box-girder with orthotropic deck is shown in Fig. 1.9, where also the nomenclature is shown. Here, the outer skin plates are stiffened in the longitudinal direction by troughs and in the transverse direction by diaphragms spaced every 4-5 m. The diaphragms consist of either stiffened plate panels (as in Fig. 1.9 and in the left-hand side of Fig. 1.10) or a combination of plate panels and truss members (right-hand side of Fig. 1.10). The combination of skin plates and troughs makes up the orthotropic deck, with the name referring to how the deck stiffness is different in different directions.

The top part of the girder, being the orthotropic deck on which the traffic runs, is shown in Fig. 1.11, where also the upper part of the diaphragms are included. In the figure, the typical design parameters, or key dimensions, are shown. It should be noted that the design principle of the girder bottom and side panels is similar to the top part. The typical design parameters are: top plate thickness $t_{tp}$,
Figure 1.10: Typical diaphragm designs in closed box-girders. Left-hand side: stiffened plate panels. Right-hand side: combination of plate panels and truss members (figure from Liu et al. (2019))

Figure 1.11: Key dimensions of an orthotropic deck. Here shown for the top part (top skin plate with troughs, and upper part of the diaphragm), however, equivalent for the girder bottom part

trough thickness $t_{tr}$, diaphragm spacing $L_d$, trough width in top $w_{tr,t}$ and bottom $w_{tr,b}$, and trough height $h_{tr}$. It is usually preferred that the distance between the troughs in the top part is equal to the trough top width, for equal distribution of traffic load. However, this is not the case at the bottom and side of the girder.

In Table 1.2, the key dimensions of the orthotropic deck in a representative selection of bridge designs through time are summarized. From the table, it is clear how little the design has changed since the 1960s. The main design concept has remained unchanged, and even the key dimensions only vary slightly. This hints that the development of the design concept has stagnated. Even the most recent state-of-the-art design, the Osman Gazi Bridge, is very similar to the previous designs. Hence, to improve the design of bridge girders in cable-supported bridges, with the aim of reducing the self-weight, new and innovative design concepts are needed. The reader is referred to Polk and Walker (2017) for greater insight into the development of orthotropic girders through time. There, a similar conclusion on the limited possibilities in further design development was reached.

Henceforth in this thesis, the structural concept of the closed box-girder with orthotropic deck (Fig. 1.9) will be designated the conventional design concept.
Table 1.2: Key dimensions of the orthotropic deck, diaphragm spacing, and surface thickness (thk.) in former state-of-the-art suspension bridges

<table>
<thead>
<tr>
<th>Year</th>
<th>Bridge</th>
<th>Top plate thk.</th>
<th>Trough width</th>
<th>Trough depth</th>
<th>Trough thk.</th>
<th>Diaphragm spacing</th>
<th>Surface thk.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1966</td>
<td>Severn</td>
<td>12</td>
<td>—</td>
<td>—</td>
<td>6</td>
<td>4,600</td>
<td>38</td>
</tr>
<tr>
<td>1973</td>
<td>1st Bosporus</td>
<td>12</td>
<td>318</td>
<td>255</td>
<td>6</td>
<td>4,475</td>
<td>38</td>
</tr>
<tr>
<td>1981</td>
<td>Humber</td>
<td>12</td>
<td>286</td>
<td>260</td>
<td>6</td>
<td>4,525</td>
<td>40</td>
</tr>
<tr>
<td>1988</td>
<td>2nd Bosporus</td>
<td>14</td>
<td>327</td>
<td>274</td>
<td>8</td>
<td>4,480</td>
<td>38</td>
</tr>
<tr>
<td>1998</td>
<td>Great Belt</td>
<td>12</td>
<td>300</td>
<td>300</td>
<td>6</td>
<td>4,000</td>
<td>55</td>
</tr>
<tr>
<td>2016</td>
<td>Osman Gazi</td>
<td>14</td>
<td>300</td>
<td>360</td>
<td>7/8</td>
<td>5,000</td>
<td>60</td>
</tr>
</tbody>
</table>

1.3.1 Fatigue in Orthotropic Decks

One of the main challenges in the design of orthotropic decks is fatigue. Fatigue occurs over time as a consequence of, mainly heavy, traffic passing the bridge. The girder is thus subject to a repeated cyclic load, which over time, may weaken various parts of the deck, and in the worst case, result in loss of structural integrity.

With rapidly increasing traffic, both in weight and number of vehicles, the fatigue issues have become an increasing challenge to overcome. Since fatigue issues often are the design driver in this type of girder (Kozy and Connor (2010)), the topic is reviewed below.

To illustrate how the orthotropic deck has a predisposition to inherent fatigue problems, the load-carrying principle of live-load (e.g., a wheel load) is illustrated in Fig. 1.12. Firstly, the load (red arrow) is transferred from the top plate to the longitudinal troughs, then into the transverse diaphragms, which finally carry the load toward the hangers. Thus, the load is primarily carried in one direction at a time.

The transition from the top plate to troughs and from troughs to diaphragms introduces stress concentrations, and when subject to cyclic loading, eventually...
cracks may develop due to fatigue. Examples of fatigue cracks in various details of the orthotropic top deck are shown in Fig. 1.13.

Furthermore, the load path from the wheel to the hangers is inefficient due to the many changes in load direction. Hence, a more structurally efficient design ought to be obtainable and may lead to a reduction of fatigue issues.

The issue with fatigue cracks in orthotropic girders was first discovered in the 1970s, and since then, the area has been subject to significant research and development, see, e.g., Wolchuk (1990, 1999); Kolstein and Wardenier (1996); Fisher and Dexter (1997).

Since the introduction of orthotropic decks, the most significant design change has been the introduction of cutouts in the diaphragms around troughs (as shown in the inset in Fig. 1.11) to relax stresses in the welded connection. Furthermore, refined cutting and automatic welding techniques have been developed. However, the connection between the longitudinal deck plate and the transverse diaphragms still constitutes a challenge. Besides, the connection remains relatively costly to produce, also with the methods at hand today (Polk and Walker (2017)).

In recent years, the most applied method to study and improve the design in regard to fatigue has been parameter studies of typical design parameters (as given in Fig. 1.11). Some analytical studies have been made, such as the study of the top plate-to-trough detail in Backer et al. (2006). However, many numerical studies have been carried out, with particular focus on the diaphragm cutout and trough geometries (Connor (2004); De Corte and Van Bogaert (2007b); De Corte (2009); Fettahoglu (2016)), but also with focus on the cross beam geometry (Fettahoglu (2015)).

Besides the parameter studies, also minor design changes have been proposed. In Oh et al. (2011), followed up by Oh and Bae (2013), the installation of a bulkhead inside the troughs was suggested to reduce fatigue stresses. Furthermore, in Tang (2011) and Heng et al. (2018) the design and manufacturing of new U-shaped troughs were studied.

In addition to the studies mentioned above, improvement of the fatigue analysis methods has been investigated. This includes the use of fracture mechanics
techniques in De Corte and Van Bogaert (2007a) and Maljaars et al. (2018) and development of numerical calculation methods, see, e.g., Xiao et al. (2008); Aygün et al. (2012); Li et al. (2018). Finally, in Saifaldien-Shakir and Alemdar (2018) and Huang et al. (2019) numerical models were calibrated against experimental tests to improve the numerical capabilities in the analysis of fatigue.

Further studies of interest on fatigue in orthotropic decks include Battista et al. (2008); Qian and Abruzzese (2009); Kozy and Connor (2010); Song and Ding (2014); Zhang (2017); Liu et al. (2019). Additionally, for a more in-depth summary of the development in the fatigue design of orthotropic decks, the reader is referred to Polk and Walker (2017).

Despite the vast amount of research, the overall design concept has remained unchanged, and still today, fatigue issues are one of the main challenges when designing orthotropic bridge girders. Based on the many studies and different approaches to handle the fatigue problems, the conclusion is clear: the potentials in the further development of the conventional design concept seem very limited. This conclusion adds to the previously raised arguments on why new design concepts, significantly different from the current, are needed.

1.3.2 Alternative Design Concepts

Within the last decades, few alternative design concepts of bridge girders have been proposed. The focus has mainly been on alternatives to the orthotropic deck, with various proposals for sandwich deck solutions able to carry the surface loads isotropically with equal stiffness in both directions. The advantage of a sandwich deck solution is that the fatigue sensitive connections in the orthotropic deck do not occur, and the loads may be carried more directly due to the isotropic behavior. These concepts are briefly discussed below.

As an alternative solution to the orthotropic steel deck, a sandwich plate with a polyurethane core material between two steel plates (Fig. 1.14a) has been proposed in Battista et al. (2010). In 2003, this design was used for a small bridge with a span of 22.5 m (Shenley, Quebec). However, in Bjærre (2015), a detailed analysis showed that the core material had a low rigidity that leads to unacceptably large shear deformations, when used in cable-supported bridges. Despite this, it was shown that the concept of a sandwich plate has a hidden potential if used with a core material that possesses a stiffness equivalent to, e.g., a wood-based material. In De Corte (2011), a proposal of a steel-concrete sandwich deck was given (Fig. 1.14c), however, also, in this case, the shear capacity was low. Besides, the bonding between the outer plates and the core needed to be addressed. In Nielsen (2016), the sandwich deck concept was again studied experimentally, and also here, the bonding created problems. Furthermore, as an alternative to a solid core Chu et al. (2018) studied other structural concepts, in particular, truss cores (Fig. 1.14b). Finally, a sandwich deck of aluminum with a foam core has been proposed (Fig. 1.14d), see, e.g., Crupi et al. (2011) or metalfoam GmbH.

Despite the above proposals, alternatives to both the orthotropic deck and the
girder design, in general, have yet to be found. Thus, the search for new and innovative design concepts remains open.

1.4 Objectives and Structure of the Thesis

In the two following sections, the objectives of the thesis are presented, followed by an outline of the thesis. The latter should be considered as a reading guide to the remaining chapters in Part I, and how Part I is related to the appended papers in Part II and the supplementary material in Part III.

1.4.1 Objectives of the Thesis

Throughout the previous sections, a general introduction to cable-supported bridges and particularly to closed box-girders with orthotropic decks has been given. Along with the introduction, the various challenges to be faced in current and future super-long cable-supported bridges were identified. Below, the main points are
Introduction

summarized to compile motivations before the objectives of the present work are stated.

The main goals to be achieved going forward in the design of cable-supported bridges are:

- A reduction of girder self-weight
  ...
  ...to increase possible span-lengths achievable from knock-on effects and thus to close the gap toward super-long suspension bridges.
  ...
  ...to reduce material quantities in the entire bridge and thus reduce the environmental impact.

- An improvement of the fatigue strength in orthotropic decks
  ...
  ...to reduce maintenance and prolong the lifespan.

The past research on the above challenges has left a gap for investigation. This gap may be closed by the identification of entirely new and innovative design concepts, significantly different from the current practice.

The objectives of this thesis are thus:

1) To pursue any possible further weight savings in the conventional design concept of girders in cable-supported bridges.

2) To identify new and innovative girder design concepts,
  ...
  ...to decrease the girder self-weight significantly.
  ...
  ...to identify more efficient load-carrying principles and thus more material-efficient structures
  ...
  ...to reduce inherent fatigue issues.
  ...
  ...to replace a more than 60-year-old design concept, of which the development has stagnated.

To achieve the objectives, various numerical methods to perform structural optimization will be applied. Specifically, parametric optimization is used in the achievement of the first objective. To achieve the second objective, continuum topology optimization and truss-layout optimization are utilized. Here, the topology optimization method is applied with a minimum of restrictions to identify a lower bound in search of optimal bridge girder designs. Subsequently, the truss-layout optimization method is applied with constraints to account for yielding and stability phenomenons. Hence, the two different approaches to structural optimization are applied to broaden the investigation on new design concepts.

Structural optimization by numerical methods was chosen for the studies due to their capabilities of identifying material-efficient designs with the aim of reducing weight. Thus, the methods are considered highly beneficial toward achieving the thesis objectives.
In addition to the above main objectives, a secondary objective is to illustrate the potentials of applying numerical optimization methods in the civil engineering industry. In connection to this, the challenges toward practical application of the methods and results are sought.

1.4.2 Structure of the Thesis

The thesis is divided into three parts. In the first part, the research and findings are introduced and summarized. The second part comprises six papers, which presents the research in greater detail. In the third part, supplementary material is collected. This is the recommended thesis format at the Technical University of Denmark\(^1\) and the chosen format at the Department of Civil Engineering.

The first part can be read independently from the two latter parts. If a greater insight is of interest, the reader is referred to the appended papers in Part II. Repeatedly throughout Part I, references are given to the supplementary material in Part III. Primarily, the third part consists of additional data and results which were not included in the papers in Part II. Besides, Part III contains a note not created by the thesis author but provided for background information.

Part I consists of the chapters briefly outlined below:

- In Chapter 1, the work is motivated and the background presented.

- In Chapter 2, the applied methods and shared theory are introduced. Firstly, the principles of the three applied approaches to structural optimization are presented and discussed. Secondly, the mathematical background to the applied numerical optimization is given, followed by an introduction to common optimization algorithms. Finally, the two applied numerical methods (the finite element method and finite element limit analysis) in the structural analysis are introduced.

- In Chapter 3, the state-of-the-art-design of the Osman Gazi Bridge in Turkey is introduced. This specific bridge design, and particularly the girder design, is the starting point for the presented work. Hence, relevant information and details about the bridge are presented for subsequent common reference.

- In Chapters 4-6, the research and findings from the appended papers are presented in extended summaries. In the last section of each chapter, the main findings to take forward when reading the subsequent chapters are recapped and briefly discussed in the context of the entire thesis. A complete conclusion of each chapter is found in Chapter 8.

  - In Chapter 4, the conventional design concept is optimized by parametric optimization. A multiscale finite element model of a bridge girder

\(^1\)From the Ph.D. rulebook at DTU: "It is recommended that the thesis be based on scientific articles already published on the same scientific topic as the project (sub-theses)... A synopsis stating the relationship between the articles and summarizing the results is to be enclosed."
is established to compute fatigue stresses and displacement fields. Subsequently, the model is used in parameter studies to investigate the behavior of the girder, when typical design parameters are changed. Finally, a gradient-based parametric optimization is carried out in order to identify potential weight savings in the conventional design.

– In Chapter 5, topology optimization of continuum structures is applied to identify new and innovative designs of bridge girders. Based on the new insights gained from the highly detailed and intricate optimized structures, a simpler interpreted design is established to quantify the performance improvement. Finally, the interpreted design is subject to simple parametric optimization to identify the full potential.

– In Chapter 6, truss optimization is applied to identify innovative and light-weight girder designs. Truss optimization based on finite element limit analysis is applied and includes constraints on stresses as well as global and local stability. The general methods are first established, followed by an extension toward practical application, including large-scale methods. Finally, the application of the methods is shown by the optimized truss girders.

• In Chapter 7, the challenges toward practical application of both the numerical methods in general as well as the optimized bridge girders are addressed. The discussions include considerations on structural robustness, manufacturing challenges, and total construction costs. Furthermore, the potentials of the identified girder designs are reviewed in consideration of the challenges.

• In Chapter 8, conclusions are given, followed by recommendations for further work.

An overview of the chapters, with main sections, in Part I is given in Table 1.3, where also the connections to the appended papers in Part II are shown.

In addition to the appended papers, the following conference proceeding was published as part of the project:

1. ”Structural topology optimization of bridge girders in cable supported bridges”, M. Baandrup, N. Aage & O. Sigmund.
1.5 **Contribution to the Field of Research**

The novel contributions to the field of research are divided into two categories and summarized as follows:

- Contributions directly related to the thesis objectives:
  - Identification of significantly different bridge girder design concepts, revealing considerable achievable weight savings compared to the conventional design principles.
  - A systematic demonstration of the practical application of various structural optimization methods within the field of civil engineering, including discussions of the related challenges and potentials.

- Other relevant contributions:
  - Extension of giga-scale topology optimization to include finite element models exceeding 2 billion elements.
  - Newly proposed methods to include both global and local stability constraints in truss optimization, including the extension to large-scale models.
  - Identification of the importance of applying realistic material parameters in benchmark optimization examples when including constraints on global stability.
Introduction

– Identification of the large hidden potential in searching the solution space of concave optimization problems with multiple local minima to find significantly improved solutions.
Chapter 2

Applied Methods

In the present project, numerical optimization has been applied as the methodology to identify new and innovative bridge girder designs. In particular, three types of structural optimization methods were applied, including parametric optimization, truss optimization, and topology optimization of continuum structures. Furthermore, two numerical methods of structural analysis were applied: the finite element method (FEM) and finite element limit analysis (FELA).

In the present chapter, background knowledge and theory behind the applied methods are covered. Firstly, the principles of the three applied methods of structural optimization are presented and discussed. Secondly, the theoretical and mathematical background to numerical optimization is given, followed by an introduction to common optimization algorithms. Finally, the similarities and differences between the two applied methods of structural analysis are introduced.

The chapter aims at the reader with no particular prior knowledge on structural optimization by numerical methods, but who has an interest in the theory behind the applied methods. For the reader with in-depth knowledge in the field of structural optimization, the content of the chapter will mostly be well-known. Furthermore, the three main sections of the chapter may be read independently from each other.

2.1 Structural Optimization

Structural optimization is considered a special field within mathematical optimization and applied particularly within the fields of mechanical and civil engineering. Structural optimization by numerical methods was chosen for the present studies due to the capabilities of identifying material-efficient designs with the aim of reducing weight. Hence, the methods are considered highly beneficial toward achieving the thesis objectives.

A general introduction to the field is given here, followed by a presentation of the three methods applied, namely parametric optimization, truss optimization, and topology optimization of continuum structures. Subsequently, the key prop-
Applied Methods

Figure 2.1: Structural optimization methods, a) sizing optimization of a truss structure, b) shape optimization, and c) topology optimization (figure from Sigmund and Bendsøe (2003))

eerties of the three methods are briefly discussed in connection to the application in the present project.

The field of structural optimization was established by the publication of the seminal paper "The Limits of Economy of Material in Frame Structures" by A.G.M. Michell, Michell (1904). However, activity in the field first gained momentum some half-century later with the advent of electronic computers and soon after, numerical optimization methods. Today, the methods are subject to extensive research and applied as an important tool in the design process of structures in multiple industries.

Structural optimization can roughly be divided into three classes, as illustrated in Fig. 2.1. Sizing optimization is defined by a structure or domain which is fixed throughout the optimization, with the design variables being sizing parameters, e.g., truss member area or plate thicknesses. In shape optimization, the design variables define the shape of the domain, either the outer shape or inner holes. In shape optimization, the material is neither added or removed. Finally, in topology optimization of continuum structures, the design variables determine the number, location, and shape of holes, and the connectivity of the domain. However, structural optimization problems can be defined ambiguously in regards to the three classes. E.g., if the design variables are allowed to vanish during sizing optimization, the problem can be considered a special and more constrained case of topology optimization. In this thesis the term topology optimization solely refers to topology optimization of continuum structures (as shown in Fig. 2.1c and as presented in Section 2.1.3).

In structural optimization, the goal is often to minimize stresses, weight, or cost, or to maximize stiffness. Other goals could be maximizing the gap between critical eigen-frequencies or maximize the lowest eigen-frequency. Typical constraint functions are volume, stresses, deflection, stability, or manufacturing lim-
Partial differential equations often describe the physical behavior of the structures subject to numerical optimization. Numerical methods to carry out this task are discussed in Section 2.3.

The selection of commercial software for structural optimization is comprehensive, including TOSCA from Dassault Systems (for topology, size, shape, and bead optimization), Altair OptiStruct from Altair Engineering (for topology, composite, and multidisciplinary size and shape optimization), and COMSOL Optimization Module (for topology, size, and shape optimization), just to mention a few.

Despite the widely available software and the advancements made in the field of research, the industrial applications are yet very limited in some fields of engineering, e.g., in civil engineering. The limited use is typically related to practical considerations, e.g., production costs and constructibility. Other challenges may appear in structures only being optimal to a limited number of load cases, thus, too sensitive to other load cases, or the difficulty of including simple, practical considerations, e.g., stability constraints. A further discussion of the general challenges can be found in Ramm et al. (1994); Zhu et al. (2014); Koh et al. (2017).

In civil engineering, especially the challenge of the constructibility of the optimized designs is critical. However, the progressive development within additive manufacturing (3D printing) may soon eliminate this particular challenge, see e.g. Clausen (2016); Yuan et al. (2018); Lange and Feucht (2019); Vantyghem et al. (2019); Plocher and Panesar (2019); Buchanan and Gardner (2019). In Chapter 7, a continuation of the discussion is given in the context of the present work.

2.1.1 Parametric Optimization

Parametric optimization, or sizing optimization, is the most simple of the presented methods, as the main structural layout is fixed during the optimization. The design variables are thus defined as structural parameters such as geometric measures, plate thickness, cross-sectional areas of structural parts, and curvature radii. This method is mainly applied when the overall structural concept is fixed, and a more detailed optimization is sought.

In parametric optimization, the objective is often minimization of stress peaks, volume, or cost, which also is the case for the constraint functions. Examples of applied parametric optimization are found in Mrsa and Medic (1996); Peng et al. (2005); Ding et al. (2010). The method can be regarded as a direct application of the mathematical optimization algorithms and thus has an unlimited range of possibilities. Hence, this method is very general compared to the following two more specific methods.

2.1.2 Truss Optimization

Truss optimization is often based on the ground-structure approach. In this approach, a fixed grid (or mesh) of truss elements, connected at joints (nodes), constitute the design domain. Here, the cross-sectional area of each member is a design
variable. As the grid is fixed, a high number of elements is needed for sufficient design freedom to improve the solution quality.

The truss members are long slender and straight bars only subject to axial loads, hence, external loads are only applied to nodes. Although these assumptions seldom are met in actual structures, the field of truss optimization has been important to the field of structural optimization. This is mainly due to the simple structural system of bars and nodes. The problem size can easily be scaled with all complexities being present (e.g., advanced constraints, and non-linearity). The first paper on truss optimization dates back to 1964, where the problem was formulated as a Linear Program by Dorn et al. (1964), later followed up by Pedersen (1972) and Anderheggen and Knöpfel (1972). A general introduction to the field can be found in Bendsoe et al. (1994). Some more recent publications include Achtziger (1999a,b); Ben-Tal et al. (2000); Kočvara (2002); Stolpe and Svanberg (2003); Stolpe (2004).

Despite extensive research within the method, examples of practical applications are limited. The reason for this can, to a certain extent, be explained by the assumptions of the very simple structures, which rarely are met in reality.

In truss optimization, the objective is most often minimization of the potential energy (or compliance) of the structure with a constraint on the available amount of material, while fulfilling some force equilibrium. Furthermore, in the approach applied in the present work, constraints on yield stresses as well as global and local stability (buckling) can be imposed. In Fig. 2.2, the principle of truss optimization is illustrated with the objective of maximizing stiffness, showing results both with and without stability constraints.

Truss optimization can be regarded as sizing optimization or general topology optimization when the truss members are allowed to vanish (zero cross-sectional areas). If the node coordinates are included as design variables in addition to the member cross-sectional areas, the truss optimization problem is considered as a (general) topology optimization problem.
2.1.3 Topology Optimization

Topology optimization of continuum structures, here just referred to as topology optimization, is one of the most applied and well-know methods within the field of structural optimization. The method dates back to the 1980s (Bendsøe and Kikuchi (1988)) and has since been subject to extensive research. Today, the fundamental methods are applied in various industries and the extension into new fields of application continues, see, e.g., Sigmund (1997, 2001); Sigmund and Bendsøe (2003).

Topology optimization is a numerical procedure to redistribute material within a predetermined design domain, such that the resulting layout meets a prescribed set of performance targets for a given set of loads and boundary conditions. The approach enables unrestricted design freedom and can reveal new and interesting design concepts.

The simplest and most commonly used objective of topology optimization is a minimization of compliance, equivalent to maximizing the stiffness of the structure. Other typical objectives are a minimization of stresses or material volume. In density based topology optimization, the design domain is discretized into smaller elements with a density \( \rho \) assigned to each element. The densities, given in an interval from 0 (no material) to 1 (material), are the design variables defining the structural layout. In Fig. 2.3, the principle of topology optimization is illustrated on a continuum structure of a (half) simply supported beam. In the figure, the objective is minimization of the compliance.

![Figure 2.3: Example of topology optimization applied on a (half) MBB beam subject to a point load (figures from Baandrup (2015))](image-url)
Applied Methods

During the previous decades, the method has been applied within the automotive industry, Cavazzuti et al. (2010), and aircraft industry, Zhu et al. (2015). However, the practical application within the civil engineering industry is limited despite multiple papers demonstrating the potential for application within the field, as shown in Fig. 2.4. In Stromberg et al. (2011) and Stromberg et al. (2012) topology optimization was applied in the conceptual design of high-rise buildings (Fig. 2.4a). The optimal layout of reinforcement in concrete structures was studied with the method in Luo et al. (2013) and Amir and Sigmund (2013) (Fig. 2.4b). Since the structures evolving from topology optimization often have an organic and compelling design, the connection between architecture and structural design is an obvious field of interest, which was exemplified in Beghini et al. (2014) (Fig. 2.4c). Finally, the method was applied in the optimal design of bridge girders of various sizes and types in Briseghella et al. (2013); Kutylowski and Rasiak (2014); Sarkisian et al. (2018); Soroush et al. (2019), however, none of these investigations have been focused on cable-supported bridges.

The SIMP Method

In density based topology optimization, the optimization problem can either be formulated discrete, where \( \rho \) is either 0 or 1, or continuous, where \( \rho \) is defined continuously over the interval \([0; 1]\). To apply gradient-based optimization algorithms, the continuous formulation is needed. However, this choice introduces a problem with non-physical intermediate densities being neither void (0) or solid (1) material. To handle this, the SIMP (Solid Isotropic Material with Penalization) method was introduced in Bendsøe and Sigmund (1999). Since the SIMP method is one of the most used in topology optimization and applied in parts of the current work, a slightly more thorough introduction to the main concepts is given below.

In the SIMP method, the densities are penalized with a penalization factor \( p > 1 \), with the aim of enforcing almost binary designs (with \( \rho \) being either 0 or 1), thus, avoiding non-physical regions. With the penalization, non-physical intermediate densities are made unfavorable for the optimization algorithm. The stiffness \( E^* \) of each element is given as a function of the density with the penalization as

\[
E^*(\rho) = \rho^p E_{\text{solid}}, \quad 0 < \rho_{\text{min}} \leq \rho \leq 1, \quad p > 1
\]  

(2.1)

where \( E_{\text{solid}} \) is the modulus of elasticity, hence, the reference stiffness of the isotropic material, and \( \rho_{\text{min}} \in [10^{-6}; 10^{-4}] \) is an artificial minimum density to avoid singular and numerical unstable systems. In the case of no penalization, \( p = 1 \), the optimization problem is convex (as discussed later in Section 2.2.2), however, by the introduction of \( p > 1 \) the problem becomes concave. The effect of increasing penalty is shown in Fig. 2.5. A penalty factor of \( p = 3 \) or above is usually required to obtain a void/solid design (Sigmund and Bendsøe (2003)).

An alternative formulation to Eq. (2.1) is the modified SIMP method (Sigmund
Applied Methods

(a) Optimized structural high-rise design (figure from Beghini et al. (2014))

(b) Optimized reinforcement layout (figures from Amir and Sigmund (2013))

(c) Architectural building design (figure from Beghini et al. (2014))

Figure 2.4: Examples of application of topology optimization within the field of civil engineering

Figure 2.5: Effect on the stiffness from increasing penalty $p$ in the SIMP method
Figure 2.6: Example of checker-board patterns, when filtering techniques are not applied in the structure presented in Fig. 2.3 (figure from Baandrup (2015))

\[(2.2)\]

\[E^*(\rho) = E_{\text{void}} + \rho^p(E_{\text{solid}} - E_{\text{void}}), \quad 0 \leq \rho \leq 1, \quad p > 1\]

where \(E_{\text{void}} = E_{\text{solid}} \times \delta\), with \(\delta \in [10^{-6}; 10^{-4}]\), is a small artificial stiffness replacing \(\rho_{\text{min}}\) to avoid numerical problems. The key difference is how the design variable is no longer limited from below and thus, can span the full range from zero to one.

Despite the penalization factor relaxing the problem with non-physical intermediate densities, other well-known problems are introduced, such as checker-board patterns and mesh-dependent designs, see, e.g., Sigmund and Petersson (1998). The first problem (illustrated in Fig. 2.6) occurs from artificial numerical stiffness in the model. Hence, the patterns shown as checker-boards are numerical equal or stiffer than equivalent solid elements. The second problem of mesh-dependent designs leads to an undesirable design change when the domain resolution is changed.

Both undesirable effects are possible to reduce by the application of various filtering techniques or length-scale controls, see, e.g., Bourdin (2001); Sigmund (2007). The filters are applied to either the densities or the sensitivities (gradients) after each optimization iteration to smooth the boundaries of the structure. Density filters modify each density as a weighted average of the neighboring densities within some radius. Similarly, sensitivity filters are heuristically modifying each sensitivity as a weighted average of the neighboring sensitivities.

2.1.4 Discussion of Structural Optimization Methods

The three methods introduced, each representing different approaches to structural optimization, each possess distinct properties. In the following, these differences are discussed in the context of how the different methods are utilized in the present thesis.

In general, parametric optimization is often most applicable in a late stage of the design phase when the main structural layout is known. Thus, the method is beneficial with a more detail-oriented scope since the domain must be fixed. The clear benefit of the method is how the overall design concept often already is constructible. Hence, in the optimization problem, attention to manufacturing
and cost can often be disregarded since no significant design changes are expected to occur. However, this limit on the degree of design change is also a limit on how much the objective can change. In the present work, the parametric optimization is applied to the current design concept to identify further improvements. Thus, the purpose is in line with the above characteristics.

For less constrained optimization problems and thus larger design freedom, both the truss and topology optimization are beneficial. This indicates that the methodologies are best utilized in the initial design phase to identify overall design concepts. This is what the two methods are intended for in the present work.

Since the truss optimization is somehow constrained by the structural members being bar elements, the topology optimization of continuum models is the only method with unlimited design freedom. Despite the large design freedom, an exact material amount or exact stiffness must be defined to establish the optimization problem. Opposite, the formulations of truss optimization applied in the current work are not bounded by an exact material amount or an exact stiffness. Furthermore, due to the simple structural analysis in truss optimization, the inclusion of constraints on stresses as well as global and local stability is possible, also in large-scale models. Contrary, in large-scale topology optimization, such constraints are challenging and not yet possible to include. Finally, to handle large-dimension structures, such as bridge girders, topology optimization methods require access to high-performance computing, whereas truss optimization is manageable on modern desktop computers.

Regarding numerical optimization algorithms, both truss and topology optimization benefit from analytic gradients being available. This results in faster and numerically more stable solvers. In parametric optimization, the benefit from the option of freely combining input/output from different sources (such as "black-box" commercial software and open-source code) may soon become a disadvantage, since the use of finite-difference gradients will be required. Hence, numerical stability and convergence properties are often worsened.

The choice of optimization method is thus highly dependent on the scope of application. Each method, among many others than the presented, is suitable for specific tasks, and the choice should, therefore, be based on individual judgment.

2.2 Numerical Optimization Methods

In the following sections, the definitions of various fundamental mathematical optimization problems are presented together with important characteristics. In particular, the group of convex optimization problems, including the principle of duality, is covered in more detail. Subsequently, an introduction to the most relevant solution strategies (optimization algorithms) is given.
2.2.1 General Optimization Problems

The purpose of optimization is to identify better solutions to a given problem, ideally the best possible solution, under given limitations or constraints. In mathematical optimization, the value of the solution is defined by an objective function, also known as a cost or energy function. The goal of the optimization is to either increase (maximize) or decrease (minimize) the function, and thus achieve a better solution. The objective function is controlled by a set of input variables, which, during the optimization, are adjusted to change the value of the objective function. Often, multiple constraints are applied to the problem, and thus limiting the solution space.

A basic mathematical optimization problem thus consists of an objective function, \( f(x) \), controlled by a group of \( n \) design variables, \( x \in \mathbb{R}^n \). Furthermore, in most practical cases, the optimization problem is restricted by \( m \) constraint functions, \( c_j(x) \geq 0, \ j = 1, \ldots, m \), which must be satisfied. By tradition, optimization problems are formulated as minimization problems. However, a maximization problem, \( \max f(x) \), is easily transformed into a minimization problem, by minimizing the negative objective function, \( \min -f(x) \).

The general non-linear optimization problem with inequality constraints is defined as

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad c_j(x) \geq 0, \quad j = 1, \ldots, m \\
& \quad lb_i \leq x_i \leq ub_i, \quad i = 1, \ldots, n
\end{align*}
\]

The special box constraint (2.3c) defines lower bounds, \( lb_i \), and upper bounds, \( ub_i \), on the design variable \( x_i \). The box constraints can be considered a special case of the general inequality constraints (2.3b). Thus, for each design variable the box constraint can be replaced by two inequality constraints

\[
\begin{align*}
x_i - lb_i & \geq 0 \\
ub_i - x_i & \geq 0
\end{align*}
\]

In Problem (2.3) only inequality constraints are included. However, similarly to the box constraints, equality constraints can be given as two inequality constraints. As an example, the linear equality constraint \( Ax = b \) can be written as two inequality constraints

\[
\begin{align*}
Ax - b & \geq 0 \\
b - Ax & \geq 0
\end{align*}
\]

Optimality Conditions

For a point \( x^* \) to be a solution to the optimization problem (2.3), a set of first-order optimality conditions must be fulfilled (Nocedal and Wright (2006)). The
conditions, also called Karush-Kuhn-Tucker (KKT) conditions, are necessary conditions for the point \( x^* \) to be optimal, hence being either a minimum or stationary point for the objective function.

To establish the first-order optimality conditions, a function called the Lagrange function, or just the Lagrangian, which combines the optimization problem into one equation, is introduced. When the box constraints (2.3c) (and possible equality constraints) are handled as inequality constraints, the Lagrangian of Problem (2.3) is stated as

\[
L(x, \lambda) = f(x) - \sum_{j=1}^{m} \lambda_j c_j(x) \tag{2.6}
\]

where the vector \( \lambda \in \mathbb{R}^m \) is the Lagrange multipliers on the constraint functions.

From the partial derivative of the Lagrangian, the first-order optimality conditions for Problem (2.3) are stated as

\[
\nabla_x L(x, \lambda) = 0 = \nabla f(x) - \sum_{j=1}^{m} \lambda_j \nabla c_j(x) \tag{2.7a}
\]

\[
c_j(x) \geq 0, \quad j = 1, \ldots, m \tag{2.7b}
\]

\[
\lambda \geq 0 \tag{2.7c}
\]

\[
\lambda_j c_j(x) = 0, \quad j = 1, \ldots, m \tag{2.7d}
\]

where \( \nabla \) is a differential operator and \( \nabla_x \) is the partial derivative with respect to \( x \). Any point \((x^*, \lambda^*)\) fulfilling (2.7) is thus a solution to the Problem (2.3). The last condition (2.7d) is called the complimentary condition. If \( \lambda_i = 0 \) and \( c_i(x) = 0 \) the inequality constraint is weakly active. If \( \lambda_i > 0 \) and \( c_i(x) = 0 \) the inequality constraint is strongly active. In general, many optimization algorithms can be interpreted as methods for numerically solving the KKT system of equations and inequalities. Such optimization algorithms are discussed later in Section 2.2.3.

Classes of Optimization Problems

The general non-linear Problem (2.3) covers multiple sub-classes of optimization problems. The common ”hierarchy” of optimization problems, or programs, including the ones applied in the present work, are illustrated in Fig. 2.7. The figure shows how linear problems are a special case of quadratic problems, which again are a special case of semidefinite problems, and so on.

A major and important class of optimization problems is the convex problems, due to their numerous favorable properties. Therefore, in the following section, convex optimization problems are introduced.
2.2.2 Convex Optimization

Convex optimization covers a large group of optimization problems with special characteristics. A brief introduction to convex optimization, including some subclasses of problems, are presented in the following. However, the reader is referred to Boyd and Vandenberghe (2004) for an in-depth introduction to the field.

In many fields of engineering, the distinction between whether a problem is easy or hard to solve is whether or not the problem is linear or non-linear. However, in the field of mathematical optimization, the distinction between easy and hard problems is drawn between problems being convex and non-convex (concave) (Boyd and Vandenberghe (2004)). Hence, one of the characteristics of convex problems is that they are considered easy to solve, even with millions of variables. This is in particular due to another favorable property: that any local minima must be the global minimum. Thus, reliable optimization algorithms can very efficiently solve convex optimization problems and find the global minimum. On the other hand, concave and general non-linear optimization problems can be extremely difficult to solve, even with few variables. This is mainly due to the possible existence of multiple local minima and stationary points.

Whether the general optimization problem (2.3) is convex or non-convex depends on the convexity/concavity of the objective function, \( f(x) \), and constraint functions, \( c_j(x) \). Thus, if the objective function in (2.3) is convex and all constraint functions are concave (due to the direction of the inequality sign), then the optimization problem is convex.

In general, a function \( f(x) \) is convex if it satisfies the inequality

\[
 f(\alpha x_1 + (1 + \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2)
\]  

(2.8)

for all \((x_1, x_2) \in \mathbb{R}^n\) and \(\alpha \in [0; 1]\). Additionally, if the second derivative \(f''(x)\) is non-negative for all \(x\), the function \(f(x)\) is considered convex as well.

As an example, the function \(f(x) = ax^2 + b\) for \(a \geq 0\) is shown in Fig. 2.8. Since the second derivative \(f''(x) = 2a\) is non-negative for \(a \geq 0\), the function \(f(x)\) is convex.

Graphically, a convex set can be defined as a set where any points on a straight line between two points within the set, also are within the set, as illustrated in...
Applied Methods

Figure 2.8: Illustration of the convex function \( f(x) = ax^2 + b \), for \( a \geq 0 \)

Figure 2.9: Illustration of (a) convex and (b) concave functions and sets (grey domain). Hollow circle indicates the local minimum, and crosses indicate global minima

Fig. 2.9a. This is exactly what Eq. (2.8) states, where the right-hand side defines the straight line. If this is not the case, the function is concave, as illustrated in Fig. 2.9b.

In Fig. 2.9, optimum points are indicated by a hollow circle at the local minimum and crosses at global minima. From Fig. 2.9a, it is clear how point \( x_1 \) and \( x_2 \) end up at the same optimum, if moving down along the convex function. However, if point \( x_1 \) and \( x_2 \) move down along the concave function in Fig. 2.9b, they end up in two different optima where the optimality conditions are fulfilled. Hence, the concave function has multiple optima, whereas the convex function only has a single global optimum.

In the following, three convex optimization problems are introduced by increasing complexity. Subsequently, the concept of duality is presented.

Linear Optimization Problem

The simplest sub-class of convex optimization is when the objective function, as well as the constraint functions, are linear. When this is the case, the problem is referred to as a \textit{linear program} (LP). The general linear optimization problem
Applied Methods

with equality and inequality constraints is given as

\[
\begin{align*}
\min_{\mathbf{x} \in \mathbb{R}^n} & \quad \mathbf{c}^\top \mathbf{x} \\
\text{s.t.} & \quad \mathbf{A} \mathbf{x} = \mathbf{b} \\
& \quad \mathbf{C} \mathbf{x} \geq \mathbf{d}
\end{align*}
\] (2.9a)

where \( \mathbf{A} \in \mathbb{R}^{m_e \times n} \) and \( \mathbf{C} \in \mathbb{R}^{m_i \times n} \). The vector \( \mathbf{c} \in \mathbb{R}^n \) defines the cost function, hence the objective is a weighted sum of the design variables \( \mathbf{x} \).

Quadratic Optimization Problem

The next sub-class of convex optimization is referred to as a quadratic program (QP). In this class, the objective function is quadratic, and all constraint functions are linear. Thus, the general QP is given as

\[
\begin{align*}
\min_{\mathbf{x} \in \mathbb{R}^n} & \quad \frac{1}{2} \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{g}^\top \mathbf{x} \\
\text{s.t.} & \quad \mathbf{A}^\top \mathbf{x} = \mathbf{b} \\
& \quad \mathbf{C} \mathbf{x} \geq \mathbf{d}
\end{align*}
\] (2.10a)

where \( \mathbf{g} \in \mathbb{R}^n \) is a vector and \( \mathbf{H} \in \mathbb{R}^{n \times n} \) is a square matrix.

Semidefinite Optimization Problem

The final sub-class of convex optimization included here, is referred to as a semidefinite program (SDP). In this class, an additional semidefinite matrix constraint is added to the linear program (2.9), hence, given as

\[
\begin{align*}
\min_{\mathbf{x} \in \mathbb{R}^n} & \quad \mathbf{c}^\top \mathbf{x} \\
\text{s.t.} & \quad \mathbf{A} \mathbf{x} = \mathbf{b} \\
& \quad \mathbf{C} \mathbf{x} \geq \mathbf{d} \\
& \quad \mathbf{F}(\mathbf{x}) \succeq 0
\end{align*}
\] (2.11a)

where \( \succeq \) indicates \( \mathbf{F}(\mathbf{x}) \) being positive semidefinite. A symmetric matrix \( \mathbf{M} \) is said to be positive definite if it fulfils the criterion \( \mathbf{z}^\top \mathbf{M} \mathbf{z} \geq 0 \), \( \forall \mathbf{z} \in \mathbb{R}^n \). The matrix \( \mathbf{F}(\mathbf{x}) \) is defined as

\[
\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + \sum_{i=1}^{m_s} \mathbf{F}_i x_i
\] (2.12)

where \( \mathbf{F}_i \) are symmetric matrices and \( x_i \) is the \( i \)th element of \( \mathbf{x} \). The constraint (2.11d) is a so-called linear matrix inequality (LMI) defining a convex set of allowable vectors \( \mathbf{x} \).
Duality

A unique characteristic of convex optimization is the concept of duality. Every original convex problem, called the primal, can be formulated as a dual optimization problem with dual variables. The solution to the dual problem is always identical to the solution to the primal problem.

The Lagrangian (2.6), applied previously in the construction of the optimality conditions, is closely linked to the transition between the primal and dual problems. Thus, the general form of the dual problem is given as

\[
\begin{align*}
\min_{x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m} & -\mathcal{L}(x, \lambda) \\
\text{s.t.} & \nabla_x \mathcal{L}(x, \lambda) = 0 \\
& \lambda \geq 0
\end{align*}
\]

In this context, the Lagrange multipliers \( \lambda \) are also known as the dual variables, contrary to the primal variables \( x \). A general attribute, is how the number of dual variables is equivalent to the number of primal constraints \( m \). Similarly, the number of dual constraints is equivalent to the number of primal variables \( n \).

To exemplify, the primal semidefinite problem (2.11) is reformulated into its dual form. To handle the semidefinite constraint (2.11d) in the Lagrangian function, a semidefinite dual variable \( Y \) is introduced and \( F^*(Y) \) is defining the adjoint linear operator between \( F \) and \( Y \). To derive the differential of the Lagrangian function, a self-adjoint operator on \( Y \) is defined as

\[
\langle F(x), Y \rangle = \langle F^*(Y), x \rangle = x^\top F^*(Y)
\]

where the notation \( \langle \cdot, \cdot \rangle \) defines the inner matrix product. The Lagrangian of Problem (2.11) can thus be written as

\[
\mathcal{L}(x, y, z, Y) = c^\top x - y^\top (Ax - b) - z^\top (Cx - d) - \langle F(x), Y \rangle = x^\top (c - A^\top y - C^\top z - F^*(Y)) + b^\top y + d^\top z
\]

with the Lagrange multipliers \( y, z \), and \( Y \). The derivative of (2.15) with respect to \( x \) is given as

\[
\nabla \mathcal{L}_x(x, y, z, Y) = c - A^\top y - C^\top z - F^*(Y) = 0
\]

The dual problem is thus given as

\[
\begin{align*}
\min_{y, z, Y} & -b^\top y - d^\top z \\
\text{s.t.} & A^\top y + C^\top z + F^*(Y) = c \\
& z \geq 0 \\
& Y \succeq 0
\end{align*}
\]

with the dual variables \( y, z \), and \( Y \). Hence, the semidefinite part has changed from being defined explicitly on the constraint (2.11d) to be defined implicitly on the dual matrix variable \( Y \), (2.17d).
2.2.3 Optimization Algorithms

To find solutions to optimization problems, thus to find a point $x^*$ fulfilling the optimality conditions (2.7), various numerical optimization algorithms, also known as solvers, are available.

In the following, the concepts of three different optimization algorithms, all applied in the current work, are presented. The algorithms are sequential quadratic programming (SQP), the interior-point (IP) method, and the method of moving asymptotes (MMA). All three algorithms are gradient-based, hence utilizing either analytically or numerically computed gradients of the objective and constraint functions in search of solutions.

Multiple solver packages are available, where some of the most acknowledged and widely used include Artelys Knitro and IPOPT for LP, QP, and general non-linear problems, and MOSEK for LP, QP, and SDP. Furthermore, the MATLAB software package contains solvers for LP (linprog), QP (quadprog), and general non-linear problems (fmincon). All the mentioned solvers have the IP method implemented, and in addition Knitro and MATLAB have the SQP method implemented. The generic MMA algorithm for general non-linear problems is freely available for MATLAB, see Svanberg (1987).

Before the presentation of the three applied algorithms, brief introductions to finite-difference gradients and the basis for numerical solvers, Newton’s method, are given.

Finite-Difference Gradients

When analytical gradients are not available, which could be due to very complex optimization problems, non-smooth functions, or problems consisting of black-box components (such as commercial programs), finite-difference (FD) gradients can be computed. However, since FD gradients are approximations, the precision of the gradients are often worse than the equivalent analytical gradients. This may cause numerical problems, unstable solvers, and convergence problems. Furthermore, even with few design variables, the computational cost of computing FD gradients can be substantial. Despite the apparent disadvantages, the only option in many situations is the use of FD gradients.

The finite-difference gradient of a function $f(x)$ is computed as

$$f'(x) = \frac{f(x + \varepsilon) - f(x)}{\varepsilon} \approx \frac{\partial f(x)}{\partial x} \tag{2.18}$$

where $\varepsilon$ is a small perturbation relative to $f(x)$, e.g., $\varepsilon = 10^{-6} f(x)$.

Newton’s Method

The basis for many gradient-based optimization algorithms is Newton’s method, which is an iterative method for finding the roots of a differential function $f(x)$. The principle of the method is to find the roots by constructing a sequence $x^n$ from
an initial guess \( x^0 \) that converges toward some value \( x^* \) satisfying \( \nabla f(x^*) = 0 \).
The simplest form of Newton’s method, solving unconstrained problems, is shown in Algorithm 1.

Here, the third line is the derivative of the second-order Taylor expansion of \( f(x) \) around \( x^k \), from where the step calculated in line four is defined. The method uses curvature information, in form of the second derivative \( \nabla^2 f(x) \), to take a more direct route toward the solution and thus gains quadratic convergence.

A similar, but simpler algorithm is the steepest decent method, shown in Algorithm 2. This method only has linear convergence, as only the first derivative is used in the search of a solution.

**Algorithm 1** Newton’s method for unconstrained (non-linear) optimization problems

1: Initial guess \( x^0 \)
2: while \( \nabla f(x) \neq 0 \) do
3: \( \nabla^2 f(x) \cdot \Delta x + \nabla f(x) = 0 \)
4: \( \Delta x = -[\nabla^2 f(x)]^{-1} \nabla f(x) \)
5: \( x^{k+1} = x^k + \Delta x \)
6: end while

**Algorithm 2** Steepest decent method for unconstrained (non-linear) optimization problems

1: Initial guess \( x^0 \)
2: while \( \nabla f(x) \neq 0 \) do
3: \( \Delta x = -\nabla f(x) \)
4: \( x^{k+1} = x^k + \Delta x = x^k - \nabla f(x) \)
5: end while

**Sequential Quadratic Programming**

Sequential quadratic programming is a method equivalent to Newton’s method applicable to constrained optimization problems. The method is based on quadratic optimization problems but formulated to solve general non-linear optimization problems. The principle is to approximate the non-linear problem by a sequence of QP’s, which are "easy" to solve due to the convex property.

The basis of SQP is the solution of the general QP only with equality constraints, hence, Problem (2.10) without Constraint (2.10c). The Lagrangian of this QP is given as

\[
L(x, \lambda) = \frac{1}{2} x^\top H x + g^\top x - \lambda^\top (A^\top x - b)
\]  

(2.19)

The first-order optimality conditions are found from the derivatives of the La-
grangian, given as

\begin{align}
\nabla_x L(x, \lambda) &= Hx + g - A\lambda = 0 \quad (2.20a) \\
\nabla_\lambda L(x, \lambda) &= -(A^T x - b) = 0 \quad (2.20b) \\
\end{align}

Hence, the optimality conditions are a linear system of equations, which in matrix form can be written as

\[
\begin{bmatrix}
  H & -A \\
  -A^T & 0
\end{bmatrix}
\begin{bmatrix}
  x \\
  \lambda
\end{bmatrix}
= 
\begin{bmatrix}
  g \\
  b
\end{bmatrix}
\quad (2.21)
\]

Here, the matrix containing \(H\) and \(A\) is non-singular, symmetric, and indefinite. Equation (2.21) is easily solved by direct methods, e.g., by LU-factorization.

The general non-linear optimization problem (2.3) is solved by a sequence of QP’s established from the first-order optimality conditions (2.7). The QP’s can also be regarded as a linearization of Problem (2.3), and are given as

\begin{align}
\min_{p, l} & \quad \frac{1}{2} p^T Hp + [\nabla f(x^k)]^T p \\
\text{s.t.} & \quad [\nabla c_j(x^k)]^T p + c_j(x^k) \geq 0, \quad j = 1, \ldots, m \quad (2.22a, 2.22b)
\end{align}

where \(H = \nabla^2_{xx} L(x^k, \lambda^k)\), known as the Hessian, is the second derivative of the Lagrangian, and \(p\) and \(l\) are the primal and dual variables to Problem (2.22), respectively. The inequality constraints (2.22b) in the QP can be handled by active-set methods leading to sub-problems only with equality constraints, see, e.g., Nocedal and Wright (2006). In this case, it is seen that Problem (2.22) can be written on the form of (2.21) with \(g = \nabla f(x^k), A = \nabla c(x^k)\), and \(b = -c(x^k)\).

The SQP method is shown conceptually in Algorithm 3.

**Algorithm 3** Sequential Quadratic Programming for constrained non-linear optimization problems

1. Initial guess \(x^0, \lambda^0\)
2. while convergence is unsatisfied do
3. Evaluate \(f^k, \nabla f^k, \nabla^2_{xx} L^k, c^k, \) and \(\nabla c^k\)
4. Solve (2.22) to obtain \(p^k\) and \(l^k\)
5. Set \(x^{k+1} = x^k + p^k\) and \(\lambda^{k+1} = l^k\)
6. end while

SQP methods are most suitable when problems are not too large, thus with few design variables and few constraint functions, when functions and gradients can be evaluated with sufficiently high precision, and finally, when problems are smooth and well scaled.
Interior-Point Methods

Similar to the SQP algorithm, the interior-point method is based on Newton’s method and is applicable for solving constrained optimization problems. A large group of optimization problems is particularly suitable to be solved efficiently with the IP method, including LP’s, QP’s, and SDP’s, and in general convex problems.

The concept of the IP method is based on the earlier simplex method, developed to solve LP’s. The simplex method, first published by G. B. Dantzig (see, e.g., Dantzig (1998)), follows a simple scheme where it moves from vertex to vertex by updating the set of active constraints. The method works well in practice, however, the solver time increases exponentially with the problem size. In 1984, N. Karmarkar proposed an algorithm similar to the simplex method, also for linear programming, with only a polynomial increase of solver time, see Karmarkar (1984). This algorithm leads to a class of algorithms known as interior-point methods, which later were developed to handle convex problems while maintaining a polynomial increase of solver time. In Nesterov and Nemirovski (1988), it was shown that IP methods could solve general convex problems as efficiently as LP’s. The IP methods implemented in many state-of-the-art solvers today are based on the extension to Karmarkar’s algorithm suggested by Mehrotra (1992), called the predictor-corrector method.

The basic principle of the IP method is to transform the constrained problem into an unconstrained or a sequence of unconstrained problems. These unconstrained problems are then possible to solve by Newton’s method. The solution is approached from the interior of the feasible set, hence the name.

To cope with inequality constraints, a slack variable $s$ is introduced as

$$s \triangleq c(x) \geq 0$$

which implies

$$s - c(x) = 0$$

$$s \geq 0$$

The KKT conditions of (2.3) (without (2.3c)) thus becomes

$$\nabla_x L(x, \lambda) = 0 = \nabla f(x) - \sum_{j=1}^{m} \lambda_j \nabla c_j(x)$$

$$s_j - c_j(x) = 0, \quad j = 1, \ldots, m$$

$$\lambda \geq 0$$

$$s_j \geq 0, \quad j = 1, \ldots, m$$

$$\lambda_j s_j = 0, \quad j = 1, \ldots, m$$
which can be written as

\[ r_L = \nabla f(x) - \sum_{j=1}^{m} \lambda_j \nabla c_j(x) = 0 \] (2.26a)

\[ r_I = s_j - c_j(x) = 0, \quad j = 1, \ldots, m \] (2.26b)

\[ r_{SA} = S\Lambda e = 0 \] (2.26c)

\[ \lambda \geq 0 \] (2.26d)

\[ s \geq 0 \] (2.26e)

where the complimentary condition is given on matrix form with \( S \) and \( \Lambda \) being diagonal matrices containing \( s_j \) and \( \lambda_j \), respectively, as diagonal elements, and \( e = [1 \ldots 1]^\top \).

Solving (2.26) is equivalent to solve the following non-linear system of equations

\[ F(x, \lambda, s) = \begin{bmatrix} r_L \\ r_I \\ r_{SA} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0, \quad \text{such that} \quad (\lambda, s) \geq 0 \] (2.27)

This system can be solved by using Newton’s method. A modification of Newton’s method is known as the primal-dual interior-point method, which allows for larger steps along the search direction before any constraints are violated. Newton’s method is modified in two ways to allow for this. In the first way, the search direction is biased toward the interior of the non-negative orthant \((\lambda, s) \geq 0\), which allows moving further along the search direction before one of the components becomes negative. In the second way, the components of \((\lambda, s)\) are kept from moving too close to the boundary of the non-negative orthant. The previously mentioned predictor-corrector method is a further development of the primal-dual method. For a detailed introduction to both methods the reader is referred to Mehrotra (1992).

**Method of Moving Asymptotes**

The last optimization algorithm to be introduced here is the Method of Moving Asymptotes, proposed by Krister Svanberg in the 1980s, see Svanberg (1987). The general method is applicable for solving non-linear optimization problems, (2.3), and especially suitable for structural optimization with a large number of design variables and constraints, due to its efficiency and robustness.

The principle of the method is to solve a sequence of strictly convex and separable approximation sub-problems. These sub-problems are controlled by so-called “moving asymptotes”, which both stabilize and speed up the convergence of the general process. The approximations are first-order, and the curvature of the sub-problems is adapted with the moving asymptotes. The sub-problems are based on the sensitivity information at the current iteration point and solved with primal-dual IP methods.
2.3 Numerical Methods for Structural Analysis

To study the physical behavior of structures subject to internal and external actions, e.g., during structural optimization, various analytic and numerical methods are available. In the present work, two different methods for numerical structural analysis are applied: the Finite Element Method (FEM) and Finite Element Limit Analysis (FELA). Since FELA can be considered a special case of the general finite element method, the two methods share many characteristics, despite being based on different assumptions of material behavior. The shared principles of the two methods are briefly introduced below, before the individual characteristics, in connection to the application in the present work, are briefly discussed and summarized in the subsequent sections.

In structural analysis, the actions on a structure are often loads/forces from self-weight or live load (e.g., traffic on a bridge), dynamic effects from wind or earthquakes, or thermal effects from the change in temperature. The input to the analysis describes the geometry, material properties, loads, and support conditions. The analysis output is often displacements, stresses, or eigen-frequencies, used in the verification of given design criteria. The relation between the input and output, hence the physical behavior, can often be formulated as systems of partial differential equations, usually called the governing equations or state equations.

The principle idea of both methods is to discretize the structure, or domain, into a finite number of sub-domains called "elements". The elements are connected at points called "nodes", which together define the topology of the structure (called the "mesh"). The material properties are applied to the elements, whereas loads and support conditions are applied to the nodes. Dividing the structure into smaller homogeneous pieces makes it possible with numerical methods to solve the governing differential equations. Hence, each element represents some variation of the unknown variables, and the accuracy of the solution depends on how well the element variations approximate the exact solution. Thus, in most cases, only approximate solutions can be found. However, the quality of the solution can be improved by either using more elements or elements with more suitable properties for the problem at hand, e.g., higher-order elements.

Since the assumption of material behavior is one of the main differences of the two discussed methods, the stress-strain curves for four different material models are shown in Fig. 2.10 for common reference in the following sections.
Applied Methods

![Idealized stress-strain curves](image)

(a) Linear Elastic-Plastic  
(b) Elastic/Perfect-Plastic  
(c) Rigid/Linear Hardening  
(d) Rigid Perfect-Plastic

\[ \sigma = f_y \]

\[ \varepsilon \]

**Figure 2.10**: Idealized stress-strain curves. Stresses, \( \sigma \), are given as a function of strains, \( \varepsilon \), with an indication of the yield-stress \( f_y \)

### 2.3.1 Finite Element Method

FEM is the oldest and most commonly used of the two methods considered. Research on the method dates back to the 1950s and 1960s, and it soon became popular in numerous fields of application. A general and in-depth introduction to the method is available in Cook et al. (1989).

In the simplest form of FEM, used in the field of structural engineering, a linear elastic material behavior is assumed as well as small deformations. This behavior corresponds to the lower part of the curve in Fig. 2.10a from where also the load-displacement path can be determined. The mathematical problem is a linear system of equations defined as

\[ K \mathbf{u} = \mathbf{F} \]  
(2.28)

where \( K \) is the stiffness matrix, \( \mathbf{F} \) is a vector containing applied loads, and \( \mathbf{u} \) is a vector containing the unknown spatial coordinates, being the displacement field of the nodes. From the displacement field, strains and stresses can be computed.

Problems slightly more complex than the simplest form might include plasticity or large deformations. When this is the case, the problems become non-linear. Hence, the mathematical problem becomes a non-linear system of equations where the structural stiffness depends on the current state of stresses and deformations. To solve such a system, an iterative approach is necessary to increase the applied forces (or deformations) gradually. The iterative process is computationally heavy and may be numerically unstable. Additionally, for practical application, where...
the ultimate load-carrying capacity often is the primary goal of the analysis, the iterative approach is undesirable. Despite some potential challenges with non-linear FEM, one of the main advantages is how it is possible to capture the post-peak behavior, which may be relevant in some cases.

### 2.3.2 Finite Element Limit Analysis

FELA is a slightly younger method compared to FEM, with research initiating in the 1960s and 1970s, see Dorn et al. (1964); Pedersen (1972); Anderheggen and Knöpfel (1972). During the 1980s and 1990s several researchers contributed to the field, with focus on FELA for two-dimensional problems (see, e.g., Christiansen (1986); Sloan (1989); Damkilde and Høyer (1993); Krenk et al. (1994); Andersen and Christiansen (1995); Poulsen and Damkilde (2000)). Since the 2000s, the mature primal-dual interior-point methods became the main method of solving FELA problems (Krabbenhøft et al. (2007, 2008); Makrodimopoulos and Martin (2006, 2007)).

In FELA, convex optimization techniques are used to directly compute the upper or lower bound plastic collapse load (or limit load). Hence, the often numerical unstable and time-consuming iterative approach used to solve the non-linear systems in FEM is avoided. However, only the collapse mode is found, and no load-deflection path is available. The FELA problem may be formulated in either a kinematic or equilibrium form, equivalent to a primal and dual form of the optimization problem. In the method, rigid perfect-plastic material behavior is assumed, hence, the method is based on the classical plasticity theory. A rigid perfect-plastic material is a (theoretical) material that does not deform at stress levels below its yield point, applicable to model, e.g., steel. The material models are thus simplified, assuming a rigid behavior prior to yielding, as illustrated in Fig. 2.10d, and assuming small displacements.

The yield surface for rigid perfect-plastic materials can for a set of stresses \( \beta \), in general, be written as \( f(\beta) = 0 \), where the expression \( f \) is material- and model-dependent. Hence, any state of stress being sustained by the materiel should be within or on the boundary of the yield surface. The yield criterion can thus be given as

\[
    f_i(\beta) \leq 0
\]  

(2.29)

It is commonly assumed that the yield surface is convex in the stress space, which, e.g., is the case for the von Mises’ yield criterion.

The governing equations ensuring equilibrium in all nodal forces in the structure, hence, ensuring a statically admissible solution, are given as

\[
    H\beta = R
\]  

(2.30)

where \( H \) is the equilibrium matrix, \( R \) is a vector containing the applied loads, and \( \beta \) is a vector containing the stress variables.

From (2.29) and (2.30), the FELA optimization problem based on the lower bound method can be established. When applying the lower bound method, the
**Applied Methods**

The objective is to maximize the load-carrying capacity, while fulfilling the yield criteria and the equilibrium equations. Hence, the optimization problem is stated as

\[
\begin{align*}
\max_{\beta, \lambda} & \quad \lambda \\
\text{s.t.} & \quad H\beta = R\lambda \\
& \quad f_i(\beta) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\] (2.31a, b, c)

where \( \lambda \) is a load factor, representing the load-carrying capacity, and \( m \) is the number of yield criteria considered. Typically, the objective and constraint functions in (2.31) are convex, hence, the problem can be solved very efficiently by state-of-the-art solvers, as discussed in Section 2.2.3.

The method has been developed to handle various kinds of elements, such as beams and slabs (see, e.g., Damkilde and Høyer (1993); Krenk et al. (1994)), and trusses and plates (see, e.g., Damkilde (1995); Gilbert and Tyas (2003); Gilbert et al. (2014)). Particularly, in recent years, the method has been applied in the analysis of soil mechanics, Krabbenhøft et al. (2005); Akhlaghi (2006); Lyamin (2009), and in the analysis of reinforced concrete disks, slabs, and joints, Poulsen and Damkilde (2000); Herfelt et al. (2016, 2017, 2018); Herfelt (2017). Recently, the method was applied in the optimization of continuum structures, Fin et al. (2018); Herfelt et al. (2019).
Chapter 3

State of the Art Bridge Design - The Osman Gazi Bridge

The starting point for the search of new and innovative girder designs has, in all phases of the project, been the state-of-the-art Osman Gazi Bridge in Turkey. Hence, geometry, load cases, and various assumptions are all based on this specific bridge. In the following, relevant information and details about the bridge and girder are presented for subsequent common reference. Henceforth in this thesis, the structural concept of the presented girder will be designated the conventional design concept.

In July 2016, the 2,682-m-long Osman Gazi Bridge opened to traffic. By the opening, the main span of 1,550 m was the fourth-longest in the world, and the structural design, made by COWI A/S, was considered state-of-the-art. Since the present project was initiated around the time of the bridge opening, the design of the Osman Gazi Bridge was selected as the basis and starting point for the present project.

The overall structural layout of the suspension bridge including the main dimensions is shown in Fig. 3.1 and the finalized bridge is shown in Fig. 3.2. In Table 3.1 the general dimensions and material quantities are summarized.

![Figure 3.1: Structural layout and main dimensions of the Osman Gazi Bridge (figure from COWI)](image)
Table 3.1: General dimensions and material quantities of the Osman Gazi Bridge

<table>
<thead>
<tr>
<th>Component</th>
<th>Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Main span length</td>
<td>1,550 m</td>
</tr>
<tr>
<td>Side span length</td>
<td>556 m</td>
</tr>
<tr>
<td>Tower height</td>
<td>242 m</td>
</tr>
<tr>
<td>Girder, steel</td>
<td>34,000 ton</td>
</tr>
<tr>
<td>Main cables, steel</td>
<td>18,000 ton</td>
</tr>
<tr>
<td>Towers, steel</td>
<td>17,000 ton</td>
</tr>
<tr>
<td>Tower foundations, reinforced concrete</td>
<td>45,000 m³</td>
</tr>
<tr>
<td>Anchor blocks, reinforced concrete</td>
<td>130,000 m³</td>
</tr>
</tbody>
</table>

3.1 The Bridge Girder

The structural concept of the bridge girder with main dimensions is shown in Fig. 3.3. The girder is of the orthotropic closed box-girder type, as introduced in Section 1.3, with diaphragms consisting of plates and truss members. In Table 3.2 the general dimensions of the bridge girder are summarized (see Fig. 1.11 for nomenclature).

In the application of each of the three different structural optimization methods, a part of the bridge girder was subject to structural analysis. In the following paragraphs, common information, assumptions, and details directly related to the structural analysis are given.
Figure 3.3: The general girder cross-section of the Osman Gazi Bridge with main dimensions and indication of traffic lanes (figure from COWI)

Table 3.2: General dimensions of the bridge girder of the Osman Gazi Bridge

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height</td>
<td>$H$</td>
<td>4.75 m</td>
</tr>
<tr>
<td>Width, full</td>
<td>$W_{\text{full}}$</td>
<td>35.93 m</td>
</tr>
<tr>
<td>Width between hangers</td>
<td>$W$</td>
<td>30.10 m</td>
</tr>
<tr>
<td>Hanger distance</td>
<td>$L$</td>
<td>25 m</td>
</tr>
<tr>
<td>Diaphragm distance</td>
<td>$L_d$</td>
<td>5 m</td>
</tr>
<tr>
<td>Girder weight</td>
<td></td>
<td>12.8 ton/m</td>
</tr>
<tr>
<td>Surfacing</td>
<td></td>
<td>60 mm</td>
</tr>
<tr>
<td>Top plate thickness</td>
<td>$t_{tp}$</td>
<td>14 mm</td>
</tr>
<tr>
<td>Bottom plate thickness</td>
<td>$t_{bp}$</td>
<td>9 mm</td>
</tr>
<tr>
<td>Top plate trough thickness</td>
<td>$t_{tr}$</td>
<td>7/8 mm</td>
</tr>
<tr>
<td>Top plate trough width, top</td>
<td>$w_{tr,t}$</td>
<td>300 mm</td>
</tr>
<tr>
<td>Top plate trough width, bottom</td>
<td>$w_{tr,b}$</td>
<td>180 mm</td>
</tr>
<tr>
<td>Top plate trough height</td>
<td>$h_{tr}$</td>
<td>360 mm</td>
</tr>
<tr>
<td>Bottom plate trough thickness</td>
<td>$t_{tr}$</td>
<td>6 mm</td>
</tr>
<tr>
<td>Bottom plate trough width, top</td>
<td>$w_{tr,t}$</td>
<td>450 mm</td>
</tr>
<tr>
<td>Bottom plate trough width, bottom</td>
<td>$w_{tr,b}$</td>
<td>180 mm</td>
</tr>
<tr>
<td>Bottom plate trough height</td>
<td>$h_{tr}$</td>
<td>300 mm</td>
</tr>
</tbody>
</table>

Model Reduction Since a bridge girder is a continuous and periodic structure, a single section between two adjacent sets of hangers (see, e.g., Fig. 1.9 or 3.4) is representative for the entire span. Therefore, in most cases, when studying bridge girders, it is only necessary to include a single or a few sections in a sub-model.

In the present work, all models have been reduced from the entire bridge to a sub-model of just one or three girder sections. This is done both to reduce model complexity and computational effort.
Despite the repetitive nature of the girder, some variation occurs, e.g., varying plate thickness. In the present work, generic and average sections are studied, since the main scope is to identify new design concepts and not a specific design for a specific bridge.

**Support Conditions**  When reducing the model to, i.e., only a part of the girder, a common challenge is how to apply support conditions to the sub-model, since no exact boundary conditions exist.

In the sub-models, all applied loads are in equilibrium. Hence, boundary conditions are applied only to avoid rigid body motions and thus to avoid related singularities in the numerical system.

Previously, many proposals to apply reasonable boundary conditions to sub-models of bridge girders and orthotropic decks have been given, see, e.g., De Corte and Van Bogaert (2007b); Oh et al. (2011); Fettahoglu (2016). In the following chapters, different approaches to handle this challenge are proposed, dependent on the model and problem formulation.

**Load Cases**  To identify representative load cases to impose on the various sub-models during optimization, a global beam model of the Osman Gazi Bridge was used. The global beam model included self-weight, live loads, traffic loads, wind loads (static and dynamic), seismic loads, temperature loads, and accidental loads. The beam model was used in the detailed design of the bridge and implemented in the COWI in-house finite element software IBDAS (Integrated Bridge Design and Analysis Software).

IBDAS is based on 3D parametric solid modeling and provides procedures for fully integrated design and analysis of load-bearing structures. It contains special facilities for bridge design including specialized analyses such as automatic determination of extreme bridge traffic load effects, eigen-frequency analysis, time history analysis, and verification according to codes of practice.

The global beam model was used to identify the most critical section forces in the typical sections of the girder. The section forces were found from a linear and static ultimate limit state (ULS) analysis. In total 12 sets of section forces were identified equivalent to the maximum and minimum of the six section forces in a beam ($N_x$, $M_y$, $M_z$, $V_y$, $V_z$, and $M_t$). Each set of forces consists of the section forces acting on the girder ends of the sub-models, as shown in Fig. 3.4. In addition to the section forces, local uniformly distributed traffic load $p$ and hanger forces $P$ are included in each load case to ensure equilibrium between all applied loads.

Besides the 12 global load cases (LC 1-12), two local load cases have been included in the work presented. The local load cases consist only of uniformly distributed load $p = 5 \text{ kN/m}^2$ and hanger forces in equilibrium, thus, no global section forces. The load distribution in the two different load cases differ, as indicated in Fig. 3.5. In the first case, LC 13, the distributed load is applied to the entire top surface, whereas in LC 14, the load is applied only on one side of
Figure 3.4: Indication of global load cases, consisting of global section forces $N_x$, $M_y$, $M_z$, $V_y$, $V_z$, and $M_l$, local uniformly distributed load $p$, and hanger forces $P$. All loads applied are in equilibrium.

Figure 3.5: Indication of local load cases, consisting of distributed load $p$ and hanger forces $P$. All loads applied are in equilibrium.

the longitudinal center line, to give a skew distribution. In both load cases, hanger forces are applied to obtain equilibrium.

Material Parameters  The material used is steel with the general parameters: modulus of elasticity $E = 210$ GPa, Poisson’s ratio $\nu = 0.3$, yield stress $f_y = 335$ MPa, and density $\rho = 7,850$ kg/m$^3$.
State of the Art Bridge Design - The Osman Gazi Bridge
Chapter 4

Parametric Optimization of the Conventional Design Concept

The conventional design concept of a closed box-girder with orthotropic deck, as introduced in Section 1.3 and presented for the Osman Gazi Bridge in Section 3.1, has been subject to intensified research since the introduction in the 1960s. In recent decades the focus has mainly been on fatigue, as discussed in Section 1.3.1, and parameter studies of essential design parameters have been the main method of choice.

Here, as a starting point, the conventional design will be optimized by parametric studies. Hence, the current concept, concretized in the form of the Osman Gazi Bridge, is optimized to identify the potential regarding weight saving. The parametric optimization studies are carried out in two steps.

The first step consists of a parameter study to investigate girder behavior when typical design parameters are altered individually. Thus, this type of study is similar to many of the previous parametric studies mentioned in Section 1.3.1, however, increased focus is given to the support conditions. The parameter study was the main focus of the appended Paper I (Baandrup et al. (2019c)).

The second step is a gradient-based parametric optimization with the typical design parameters used as design variables and with the objectives of identifying potential weight and construction cost savings. The use of gradient-based optimization is novel in regard to orthotropic girders and civil engineering in general. This particular work was published in the appended Paper II (Baandrup et al. (2019d)).

Common for both steps is the use of an advanced multiscale finite element (FE) model of the Osman Gazi Bridge girder. The parametric FE-model is used to compute fatigue stresses and displacement fields. The focus of the following studies and thus of the FE-model is on local effects in the top part of the girder (the orthotropic deck), where many challenges often arise during the design process. Here, local effects include local deflection and buckling, as well as fatigue stresses. Since all effects are derived from local wheel loads, no global loads are imposed due
to their insignificance in regard to the local effects and top-deck design. The same model is used in both the parametric study and during the parametric optimization and was thus introduced in both Papers I and II, however, in greater detail in Paper II.

In the following sections, firstly, the main principles of the FE-model are outlined. Subsequently, the parameter studies are presented, followed by a summary of the parametric optimization. Finally, the main findings and learnings of the chapter are summarized and briefly discussed. The content of the following three sections are contained in Papers I and II, however, presented here in a summarized format.

4.1 Multiscale Model of the Bridge Girder

To carry out the studies in Papers I and II, a common multiscale FE-model with sophisticated displacement-based boundary conditions was established in the commercial software Abaqus (Abaqus (2016)). The model consisted of various models of different scale, from a global model of the entire bridge to a local sub-model of a single girder section (25 m), as shown in Fig. 4.1. The global model was originally established in Andersen and Hansen (2012) and since refined in Bjærre (2015) from where the present work departed.

Since only local effects were of interest in the studies, only a single typical section of the continuous girder was included, both to reduce model complexity and computational time. However, in the local model, no natural support conditions existed. Therefore, attention had to be given to ensure acceptable boundary conditions of this sub-model.

Previously, in multiple studies, where also only a minor part of the orthotropic girder was studied, the support conditions were subjected to an oversimplification. Hence, often the support conditions of the reduced model were either too flexible (e.g., in Oh et al. (2011) and Fettahoglu (2016)) or too stiff (e.g., in De Corte and Van Bogaert (2007b)). To address the challenge of applying realistic support conditions in the present work, the multiscale model with both a global and local level was established.

In the global beam model, five girder sections (125 m) were replaced with shell elements to compute displacement fields. Subsequently, these fields were transferred to the free edges of the local sub-model (indicated by dashed edges in Fig. 4.1). Thus, for each load case, the global model only needed to be solved once, whereafter, the local model with the imposed displacements could be solved independently during both the parametric studies and optimization. By this approach, a more realistic stiffness along the sub-model boundaries was ensured as no fixed and non-natural boundary conditions were applied contrary to what has often been seen in the literature.

The local model, shown in Fig. 4.2, consisted mainly of orthotropic shell and beam elements. The geometry of the elements was simplified (e.g., without mod-
Figure 4.1: Schematic overview of the multiscale model with main dimensions, terminology of sublevels, and an indication of displacement field boundaries (dashed edges) (figure from Baandrup et al. (2019d))

Figure 4.2: Detail levels of the local model with the major orthotropic shell/beam elements (general mesh size: $h_c = 1$ m) and the minor detailed shell part (general mesh size: $h_f = 0.1$ m), respectively (figure from Baandrup et al. (2019d))

eling troughs), but with stiffness properties equivalent to the complete orthotropic model assigned instead. However, a representative and minor detailed shell part included all structural details and a very fine mesh discretization. This part was used in the calculation of reliable fatigue stresses.
4.1.1 Parametric Model

The entire local model in Fig. 4.2 was made parametric, hence, adaptable in regard to a set of chosen design parameters. To do so, the local model was scripted in Python.

The design parameters, including the most common geometrical values of an orthotropic deck and most important dimensions regarding the local effects, are shown in Fig. 4.3.

![Figure 4.3: Design variables (figure from Baandrup et al. (2019d))]()
Since a single load case was superior in the contribution of stresses in these fatigue details, only vehicle 5A-H from the UK annex to EN1993-1-9 FLM4 (BSI (2008)) was included in the calculations. Descriptions of critical load positions and details on the fatigue stress calculations are left for the reader to find in Paper II.

The identification of the fatigue details, as well as critical stresses and loads, were based on experience from actual design work in COWI. This knowledge was summarized in a brief document prepared by COWI for the present project. The document is appended in Note I in the Supplementary Material.

**Eurocode Stiffness Requirement**

The Eurocode stiffness requirement (EN 1993-2-2007 Fig. C.4, CEN (2007)), related to the minimum stiffness of longitudinal stiffeners (top plate and troughs) relative to the diaphragm distance, was included in the studies. The requirement is included in the Eurocode to ensure sufficient stiffness in the orthotropic deck to prevent surface cracking. The requirement is reproduced in Fig. 4.5 and is in the following referred to as *EN stiffness requirement*.

**Figure 4.4:** Fatigue details near troughs and diaphragm. Stress directions indicated by arrows (figure from Baandrup et al. (2019d))

**Figure 4.5:** Eurocode stiffness requirement (stringers=troughs, and cross girders=diaphragms, hence $a = L_d$) (figure C.4 from CEN (2007))
Deflection Criterion

A simple deflection criterion was included, where a maximum acceptable relative vertical deflection between two adjacent diaphragms was given as $d/L_d \leq 1/400$. The deflection criterion is illustrated in Fig. 4.6.

![Diaphragm Diagram](image)

**Figure 4.6:** Deflection criterion $d/L_d \leq 1/400$ (figure from Baandrup et al. (2019d))

4.2 Parameter Studies

Initially, the established model was used for parameter studies. The aim of the studies was partly to study the behavior of orthotropic decks and partly to verify the performance of the FE-model before implementing it in an optimization framework.

In the appended Paper I, four parametric studies were presented, which are summarized in the following. However, parametric studies of all ten design parameters (Fig. 4.3) are contained in Appendix A, where also an extra load case for fatigue detail 3 (as discussed in Paper II) is included for all cases.

The four studies, shown in Fig. 4.7, were carried out on the following parameters: diaphragm distance, top plate thickness, trough thickness, and a scaling of the total trough stiffness (designated $\alpha$), thus, a simultaneous scaling of the parameters: $\alpha \times (t_{tp}, t_{tr}, w_{tr,t}, w_{tr,b}, h_{tr})$. The plots show the fatigue stress functions normalized with their maximum allowable stress, and the EN stiffness requirement normalized such that unity is the upper allowable limit. Hence, the design is acceptable for values below unity. Furthermore, the girder weight is plotted as a function of the design parameters, and a vertical solid line indicates the initial design values. Notice, the vertical axes vary to show all plotted functions clearly.

In Fig. 4.7a with changing diaphragm distance, the fatigue stresses in FD3 and FD4 near the top plate are close to constant, whereas the stresses in FD1 increase almost linearly with increasing diaphragm distance. Furthermore, striving for a weight reduction, the upper limit of the variable is near $L_d = 5.4$ m, where the EN stiffness requirement becomes critical.

In Fig. 4.7b, the three fatigue stress functions are affected significantly by the changing top plate thickness. Particularly stresses in FD3 and FD4 at the top
Figure 4.7: Parametric studies. Output is normalized with maximum allowable values, indicated by a horizontal dashed line, and a vertical solid line indicates the initial design values (figures from Baandrup et al. (2019c))
Parametric Optimization of the Conventional Design Concept

plate increase considerably by only a slight decrease in thickness. Thus, the initial top plate thickness is already near optimum when all other parameters are fixed.

In Fig. 4.7c, increasing trough thickness results in an almost linearly decrease in stresses in FD1 and FD4, contrary to the increasing stresses in FD3. In this case, when all other parameters are fixed, the trough thickness may be reduced to around 7.5 mm before stresses in FD4 become critical.

Finally, in Fig. 4.7d, the scaling of the trough stiffness shows a significant effect on the EN stiffness requirement with a very steep slope. Furthermore, the change in fatigue stresses differs from the other three cases, as all fatigue stress functions have parabola-like shapes. However, this behavior is very closely connected to the width and position of the local wheel load, which has a significant influence on the computed fatigue stresses.

Before proceeding to the actual optimization studies, some common conclusions from the parameter studies can be drawn. From both the four individually studies displayed and the additional in Appendix A, the diverse effects on the design from varying design parameters are shown. In addition to the insight from the studies, the smoothness of all curves and the absence of discontinuities verify the FE-model being suitable for gradient-based optimization. Furthermore, this observation supports the fairness in the various assumptions and model-simplifications.

In the above studies, the deflection criterion was not included since preliminary studies showed it never to be close to critical and thus out of interest compared to the other deformation criteria, the EN stiffness requirement.

4.3 Gradient-Based Parametric Optimization

Generally, in parameter studies, including the above, usually, only one or two design parameters are studied at a time. This is very similar to a typical design process in civil engineering since an engineer only can manage to study and assess the output of a few variables at a time. Hence, when changing several parameters concurrently, the sum of effects and consequences becomes too complex to comprehend. However, when only a few parameters are studied simultaneously, the full potential of the design is most likely not achieved.

It is thus of interest to perform numerical optimization of the conventional design concept, where multiple design parameters are changed simultaneously. Here, the FE-model will be used in a gradient-based parametric optimization to identify the full potential regarding weight saving as well as cost optimization.

Within the optimization framework, simultaneous optimization of several design variables is possible while fulfilling the given constraints on fatigue stresses and deformation. With the method, both general design trends may be identified as well as a specific optimized girder design.

In the following sections, firstly, the optimization framework is summarized, followed by a presentation of the primary results. The sections cover the main matters of the appended Paper II.
4.3.1 Optimization Framework

The basis for the optimization framework was a general nonlinear inequality-constrained optimization problem on the form of Eq. 2.3. Here, the design variables (x) were the design parameters (Fig. 4.3), defined with suitable lower and upper bounds. Two independent objective functions (f(x)) were defined for weight and price (construction costs) minimization, respectively. The constraint functions (c_j(x)) were given as the established fatigue stress functions, the deflection criterion, and the EN stiffness requirement.

The two different objective functions were included since they potentially can yield different solutions. The weight function was implicitly a function of the design variables. Opposite, the price function was explicit since it included welding and manufacturing costs besides differentiation on the steel prices.

A flowchart of the optimization framework, implemented in MATLAB (MATLAB (2016)), is shown in Fig. 4.8. In the flowchart, a stability check is indicated to be performed after optimization. This check was included since buckling is often a concern in thin-plated structures such as orthotropic girders. However, to reduce the computational effort, a buckling constraint was not included during optimization.

To solve the optimization problem, the nonlinear solver Knitro (Byrd et al. (2006)) was used with the sequential quadratic programming algorithm and finite-difference gradients, as described in Section 2.2.3.

Figure 4.8: Flowchart of the parametric optimization framework (figure from Baandrup et al. (2019d))
4.3.2 Results

In total, nine different optimization cases were studied as outlined in Table 4.1, where also the main results in the form of weight and price savings have been included in the last two columns. In the first eight cases, the objective was a minimization of weight, and in the last case, the objective was a minimization of price. Thus, the main focus was on weight minimization, since weight is a clear and unambiguous measure, and since the general project objective is reducing girder weight. The first three cases experienced an increasing complexity from a single free design variable to ten free design variables. In the five next cases, different lower bounds for plate thicknesses were studied together with a variation in constraint functions. The initial weight and price of the girder was 12.8 ton/m and 34,170 EUR/m, respectively.

In Appendix B, convergence plots are shown for all nine cases, containing the objective and constraint functions, as well as design variables, as a function of the iteration number. The plots show convergence toward local minima of all results.

In Case 1, the diaphragm distance $L_d$ was the only free design variable. During optimization the initial distance of $L_d = 5.00$ m increased to $L_d = 5.43$ m and thus a weight saving of only 0.4% was achieved. The increase in distance corresponds to what was observed in the parameter study in Fig. 4.7a.

Next, in Case 2 the four plate thicknesses $t_{tp}$, $t_{tr}$, $t_w$, and $t_f$ were included as design variables in addition to $L_d$. With the larger design freedom, the weight saving was increased to 5.7%, mainly achieved by a reduction in the trough thickness from $t_{tr} = 8.00$ mm to $t_{tr} = 6.00$ mm. Contrary to Case 1, the diaphragm distance decreased to $L_d = 4.96$ m in Case 2. This happened to maintain the fulfillment of the EN stiffness requirement, while the trough stiffness, and thus the deck stiffness, decreased. Furthermore, it should be noted how the decrease in diaphragm dis-

<table>
<thead>
<tr>
<th>Case</th>
<th>Objective</th>
<th>No. of design variables</th>
<th>Lower bound on plate thk.</th>
<th>EN stiff. req.</th>
<th>Weight saved</th>
<th>Price saved</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Weight</td>
<td>1</td>
<td>5 mm</td>
<td>Yes</td>
<td>0.4%</td>
<td>0.3%</td>
</tr>
<tr>
<td>2</td>
<td>Weight</td>
<td>5</td>
<td></td>
<td>No</td>
<td>5.7%</td>
<td>8.6%</td>
</tr>
<tr>
<td>3</td>
<td>Weight</td>
<td>10</td>
<td>5 mm</td>
<td>Yes</td>
<td>7.9%</td>
<td>11.8%</td>
</tr>
<tr>
<td>4</td>
<td>Weight</td>
<td>10</td>
<td>6 mm</td>
<td>No</td>
<td>8.9%</td>
<td>12.5%</td>
</tr>
<tr>
<td>5</td>
<td>Weight</td>
<td>10</td>
<td>6 mm</td>
<td>Yes</td>
<td>5.8%</td>
<td>8.7%</td>
</tr>
<tr>
<td>6</td>
<td>Weight</td>
<td>10</td>
<td>4 mm</td>
<td>No</td>
<td>6.7%</td>
<td>9.2%</td>
</tr>
<tr>
<td>7</td>
<td>Weight</td>
<td>10</td>
<td></td>
<td>Yes</td>
<td>8.4%</td>
<td>11.2%</td>
</tr>
<tr>
<td>8</td>
<td>Price</td>
<td>5 mm</td>
<td></td>
<td>Yes</td>
<td>13.8%</td>
<td>16.7%</td>
</tr>
<tr>
<td>9</td>
<td>Price</td>
<td>5 mm</td>
<td></td>
<td>Yes</td>
<td>7.6%</td>
<td>11.9%</td>
</tr>
</tbody>
</table>
Parametric Optimization of the Conventional Design Concept

Figure 4.9: Girder weight as function of lower bounds on plate thicknesses, with and without the EN stiffness requirement, respectively (reproduction of figure in Baandrup et al. (2019d), here, with minor designation error corrected)

...tance allowed the trough thickness to decrease below the 7.5 mm found as the limit in the parametric study in Fig. 4.7c. Hence, the results of this simple optimization problem clearly demonstrate the potential of applying gradient-based optimization, since the interaction of multiple variables may produce results that deviate from the results obtained when only a single parameter is taken into account at the time.

Finally, all ten design variables were included in Case 3, pushing the weight saving to 7.9%. Here, the significant contribution to the weight saving was found in more but narrower troughs with a thickness reaching the lower bound of $t_{tr} = 5.00$ mm. A similar study, but with the price function as the objective, was carried out in Case 9. Here, similar weight and price savings were seen, and it was observed that also, the changes in design parameters were similar. Thus, for the studied girder, no significant differences were observed from the different objective functions. The additional effects of welding and other manufacturing costs were thus not significant enough compared to the effect of the price of steel (basically a scaling of the weight) to have an impact on the results.

From the experience of the studies commented above, two main factors were constraining: the lower bound on plate thickness and the EN stiffness requirement (for documentation see Table 5 in Paper II or the constraint-function plots in Appendix B). Hence, a study of the variation in lower bounds and relaxation of the EN stiffness requirement was carried out in Cases 3-8 (C3-C8). The results are summarized in Fig. 4.9, where the girder weight is plotted as a function of the lower bounds for cases with and without the EN stiffness requirement.

Figure 4.9 shows that by removing the EN stiffness requirement, the weight savings were, in general, higher, with an increasing effect when the lower bound was reduced as well. The increased difference between Cases 7 and 8 was due to

---

1Remark: the similar figure in Paper II (Fig. 16), contains a minor error by interchanged designations of C3 and C5 as well as C4 and C6
the EN stiffness requirement constraint being active in Case 7, contrary to Cases 3 and 5 (see Paper II for details).

Case 5, with a minimum 6-mm trough thickness, was fully aligned with the requirements in the Eurocode EN 1993-2-2007 (CEN (2007)). In contrast, Case 8 was a less constrained problem. Thus, the range in possible weight savings was found from these two cases to 5.8%-13.8%.

4.4 Summary and Discussion

The main findings from the parametric optimization studies are that weight savings in the range 5.8%-13.8% are achievable, mainly by using thinner plates and narrower troughs. It is here noted that the initial design of the Osman Gazi Bridge was already (before optimization) considered state-of-the-art. In this view, the possible weight savings are quite significant and thus indicate the advantages of applying gradient-based parametric optimization. The benefits of numerical optimization are particularly noticeable when the problem complexity increases by the increasing number of design variables.

However, given all the various challenges in the conventional design concept, including self-weight and inherent fatigue issues, as discussed in Chapter 1, the weight savings are considered modest. Moreover, due to the use of a refined FE-model and a complex set of design variables, including both plate thicknesses as well as geometric measures, the possible weight savings are considered an upper bound for the conventional design.

Furthermore, based on the low trough thickness after optimization that indicates increasing stresses, it is of interest to discuss the choice of fatigue details. Hence, in further studies, it is relevant to include additional fatigue details near the troughs, such as the diaphragm-to-trough connection (e.g., FD2 in Note I). However, the inclusion of additional constraints, e.g., on new fatigue details, will only lead to similar or lower weight savings. This argument further supports the above assumption of an upper bound solution.

Based on the raised discussion, it is not considered worthwhile to perform further optimization studies, including additional fatigue constraints, of the conventional design.

From the above results and discussions, it may be concluded that the potential weight savings in the conventional design concept are limited and also not a solution to the inherent fatigue issues. Thus, the study showed very little room for improvement without altering the structural concept. Despite the study being of a specific bridge, the results and above conclusion are considered to be generally applicable.

New and alternative design concepts are thus needed to handle the various girder challenges. In the following two chapters, the possibilities for such innovative designs are studied through the application of two different numerical methods of structural optimization.
Chapter 5

Innovative Designs by Topology Optimization

In the previous chapter, it was concluded that the potential in any further development of the conventional design concept is limited. Hence, to overcome the various girder challenges, new and alternative structural concepts are needed.

In this chapter, topology optimization of continuum structures will be applied as a design tool to identify innovative design concepts of bridge girders. The goal is to reduce the self-weight significantly and to find more efficient load-carrying principles and, thus, more material-efficient designs. The method provides figuratively unrestricted design freedom contrary to the previous approach of parametric optimization, which covered only a minimal solution space.

Specifically, giga-scale topology optimization, with billions of degrees of freedom, is utilized to cover multiple length scales of the girder structure. Hence, the method can cover and reveal structural details of several meters down to a few millimeters in the same model. Furthermore, the method is applied with a minimum of restrictions, e.g., practical constraints, such as stability or constructability, are not imposed. The optimization problem is formulated with the objective of maximizing the structural stiffness for a given amount of material.

Due to the high degree of design freedom, the method is beneficial for identifying optimized structures with the sole purpose of carrying the applied loads most efficiently. For these reasons, the solution is considered a lower bound on optimal designs, and thus, the method is considered advantageous as the first step in search of innovative structural concepts.

Based on the insights gained from the highly detailed and intricate optimized structures evolving from topology optimization, a simpler interpreted design can be established. Subsequently, this design is used to quantify the performance improvement in comparison to the conventional design. Furthermore, the interpreted design is subject to a simple parametric optimization to determine the full potential.

In the following sections, firstly, the topology optimization studies are pre-
Innovative Designs by Topology Optimization

sented. Subsequently, interpretation and quantitative studies are covered. Finally, the main findings and learnings of the chapter are summarized and briefly discussed.

The work presented in this chapter is the content of the appended Paper III (Baandrup et al. (2020)), however, presented here in a summarized format.

5.1 Topology Optimization of Bridge Girders

A significant obstacle toward applying topology optimization to large structures, such as bridges girders, has been to cover multiple length scales. Therefore, giga-scale procedures with billions of elements are required to discretize the entire domain of just a single bridge girder section into finite elements with dimensions in the order of the conventional plate thicknesses. Until recently, the topology optimization method was limited to a few million elements. However, this limitation was overcome in Aage et al. (2017), where a full-scale Boeing 777 type aircraft wing was optimized using 1.1 billion elements.

The giga-scale procedure, first proposed by Aage et al. (2015), utilizes high-performance computing (HPC), allowing problems to be solved in parallel. The procedure was implemented within the PETSc toolkit (Portable, Extensible Toolkit for Scientific Computation)(Balay et al. (2016)), and the basis code is freely available at www.github.com/topopt.

In the following, the bridge girder model, which will be subject to optimization, is introduced, followed by the formulation of the optimization problem. Subsequently, the governing load cases are identified, and, finally, the main results of the studies are presented.

5.1.1 Bridge Girder Model

The bridge girder model subject to optimization is shown in Fig. 5.1 including an indication of dimensions, loads, and boundary conditions.

As discussed in Section 3.1, a single general section of the continuous and periodic girder is representative. However, to precisely impose boundary conditions and transfer loads, three sections were included in the three-dimensional continuum finite element model.

To narrow the scope and the number of studies, it was chosen to focus on the load-carrying internal structure of the primary girder domain, as shown in Fig. 5.1. Here, the walkways were neglected, and the outer wind profile was maintained. Hence, investigations of alternative outer shapes of the girder and domain, and possible new structures evolving from here were out of the scope. The purpose of the wind profile, defined by the inclined outer edges, is to improve the aerodynamic properties. By maintaining the original wind profile, the aerodynamic properties were, to some extent, considered unaltered since these properties were out of the scope.
Figure 5.1: Dimensions, loads, and boundary conditions of the three-section model of the bridge girder. Section forces were applied to the stiff end elements (blue) with $100 \times$ modulus of elasticity and density $\rho = 1$. Distributed load was applied to the solid top elements (green), and hanger forces $P$ to the solid hanger attachments (green), both with $\rho = 1$. The design domain is indicated by orange with a variable density $0 \leq \rho \leq 1$ (figure from Baandrup et al. (2020))

The top layer of elements, representing the road surface, and the hanger anchorages were fixed to be solid (indicated by green in Fig. 5.1), whereas material could freely be distributed in the remainder of the domain (indicated by orange in Fig. 5.1).

The entire end surface at one end of the model was fixed, and at the opposite end, global section forces were applied. Since the material distribution, and thus also the distribution of forces, in the optimized structure was unknown beforehand, the global section forces were applied to a stiff end surface (indicated by blue in Fig. 5.1). The stiffness of this surface was increased with $100 \times E_{\text{solid}}$ to ensure a smooth load transmission into the model. Additionally, distributed load $p$ was applied to the solid top surface, and hanger forces $P$ were applied at the solid hanger anchorages.

The boundary conditions described above and methods of applying global section forces were the reasons to model three sections instead of one. Thus, the purpose of the two outer sections was to transfer reaction forces and loads into the center section, which was subjected to optimization.

Since a bridge girder is symmetric about the longitudinal center axis and transversely symmetric about every hanger set, a symmetry condition was imposed, as indicated in Fig. 5.2. Hence, a quarter of the center section ($12.5 \, \text{m} \times 15.05 \, \text{m} \times 4.75 \, \text{m}$) was defined as the active design domain and mapped continuously to the rest of the model during the iterative optimization.

The model was subject to the 12 global and two local load cases introduced in Section 3.1. The loads in all 14 load cases are summarized in Section C.1 of Appendix C.
5.1.2 Optimization Problem

The objective of the optimization was to maximize the structural stiffness of the center section of the model for a given amount of material. This optimization problem may be regarded as equivalent to minimizing the volume for a given stiffness, hence equivalent to a weight minimization problem.

To pose the stiffness maximization problem as a minimization problem, the measure of compliance was used. Compliance is defined as the work done by external forces and is inversely proportional to structural stiffness. The optimization problem was formulated based on the SIMP method, as introduced in Section 2.1.3. The minimization problem was stated as

\[
\begin{align*}
\min_{\rho} & \quad \phi = \sum_{i=1}^{N_k} \alpha_i u_i^T K u_i \quad & \text{(Sum of compliance)} \quad (5.1a) \\
\text{s.t.} \quad & \quad K(\rho) u_i = F_i, \quad \forall i = 1, \ldots, N_k \quad & \text{(State equations)} \quad (5.1b) \\
& \quad V(\rho) - V^* - 1 \leq 0 \quad & \text{(Volume constraint)} \quad (5.1c) \\
& \quad 0 \leq \rho_e \leq 1, \quad \forall e = 1, \ldots, N_e \quad & \text{(Box constraints)} \quad (5.1d)
\end{align*}
\]

Here, \( \rho \) was a design variable vector containing the densities \( \rho_e \) for each of the \( N_e \) finite elements in the model. The objective function \( \phi \), (5.1a), was given as the sum of compliances of the \( N_k \) load cases, weighted by the assigned scaling factors \( \alpha_i \). Since only the stiffness of the center section was subject to optimization, the reduced stiffness matrix \( K \) only contained contributions from this section. The displacements were given in the vector \( u \), found as solution to the state equations in (5.1b), where \( F \) was the load vector.

The constraint (5.1b) ensured mechanical equilibrium through the state equations, which were solved using the finite element method, as introduced in Section 2.3.1. The stiffness matrix was given as a function of the design variables

\[
K(\rho) = \sum_{e=1}^{N_e} E(\rho_e) k_e^0
\]

(5.2)

where \( k_e^0 \) was the element stiffness matrix with unit modulus of elasticity. The actual modulus of elasticity \( E(\rho_e) \) was given as a function of the design variable.
Innovative Designs by Topology Optimization

by the modified SIMP approach in Eq. (2.2), as introduced in Section 2.1.3. In continuation hereof, each density was limited by box constraints (5.1d).

Finally, the volume constraint (5.1c) posed a limit on the amount of available material. Here, \( V(\rho) \) was the volume of the current structure and \( V^* \) was the maximum amount of available material, given by a volume fraction of the entire design domain.

During the optimization, a density filter was applied, as discussed briefly in Section 2.1.3. The optimization algorithm was the gradient-based method of moving asymptotes (MMA), as introduced in Section 2.2.3.

Giga-Scale Procedures

To solve the established optimization problem of the bridge girder model, giga-scale procedures were a necessity. Due to the multiple length scales from the huge domain size of 75 m down to plate thickness as small as 6 mm (in the conventional design), billions of finite elements would be required to discretize the domain into a sufficient fineness.

The domain enclosed by the outer dimensions in Fig. 5.1 was discretized into 2.1 billion elements (corresponding to a mesh of \( 4,384 \times 1,760 \times 272 \) elements) with a maximum element dimension of 17 mm. Although still above the desired minimum member size of 6 mm, this resolution was found to be sufficient to extract the design trends.

The allowable volume fraction was chosen as \( V^* = 3.0\% \), close to the typical volume fractions of 1.0%-1.5% in the conventional design (1.3% in the Osman Gazi Bridge). The larger volume fraction was chosen to ensure detailed results in connection with the corresponding larger element size in the model.

5.1.3 Identification of Governing Load Cases

Since the solver time of the optimization problem (5.1) scaled linearly with the number of included load cases and the giga-scale procedures required access to limited HPC facilities, a screening of the most governing load cases was carried out. Thus, from initial studies, the 14 load cases were reduced to the five most important and, thus, most influential on structural features.

In the screening, the optimization problem was solved for each load case with a coarse model discretization (50 million elements). Subsequently, the structural features were studied and compared, both qualitatively and quantitatively, before load cases 1, 5, 10, 13, and 14 were identified to be included in the final optimization problem. The structures optimized for each of the 14 load cases are found in Section C.2 of Appendix C.

After identification of the five governing load cases, the weights \( \alpha_i \) in the objective function (5.1a) were adjusted to increase the influence from the local load cases (LC 13 and 14). The optimized structures for three of the weight factor studies are found in Section C.2 of Appendix C (Fig. C.15-C.17).
5.1.4 Results

The final result of the topology-optimized bridge girder, discretized into 2.1 billion elements and optimized for the five governing load cases, is presented below. Evolving the optimized design required access to massive computational resources with run times of 85 hours on 16,000 CPUs.

Three sections of the optimized bridge girder design are shown in Fig. 5.3 with the top layer removed to reveal the optimized interior structures. Clearly, the design is very different from the conventional, as no perpendicular diaphragms or orthotropic decks are seen. Instead, a number of double-curved diaphragm-like panels and trusses are seen that transfer traffic loads directly to the hangers. Additionally, a longitudinal plate-like structure appears as an added support in the region of the hangers and the curved diaphragms (as shown in the close-up in the upper right-hand corner of Fig. 5.3). Furthermore, a decreased span length of the skin plates is possible due to an increased number of supports growing from the diaphragm-like panels.

Moreover, it can be noticed how the outer structure curves inwards between the hangers, and hence, does not utilize the entire domain. A consequence of this feature is a changed wind profile. To study the effect of imposing an unchangeable wind profile, a case with enforced outer skin plates is included in Fig. C.18 in Section C.2 of Appendix C. There, similar structural features in the interior of the domain are seen, despite the enforcement of the wind profile.
Figure 5.3: The result of the giga-scale topology optimization applied to the bridge girder model is shown after 400 steps of optimization using 2.1 billion design variables. Three full sections of the continuous girder are shown with the fixed top layer removed to reveal the internal details. A single section is shown in the lower left-hand side with overlaid interpreted curved diaphragms (red panels) (figure from Baandrup et al. (2020))

5.2 Interpretation and Quantitative Studies

The complexity of the optimized design in Fig. 5.3 makes construction infeasible due to cost and manufacturing considerations. Instead, the main structural features of the optimized design were interpreted into a simplified design. The purpose was partly to identify a feasible design, and partly to facilitate a quantitative comparison of the improvements compared to the conventional design, as a supplement to the qualitative study above.

An interpreted design was derived with a similar level of geometric and manufacturing complexity as the conventional. Therefore, initially, only the interior of the conventional design was adapted into an interpreted design. Fig. 5.4 provides the interpreted design (red) together with the conventional (blue). In the lower left-hand corner of Fig. 5.3, the interpreted design is shown overlaid on the topology-optimized design. Henceforth in this thesis, the red structure in Fig. 5.4 will be designated the interpreted design.
Figure 5.4: Three sections of the continuous girder are shown, including the conventional design concept (blue) and the interpreted girder design (red) in each of the two center sections, respectively. The fixed top layer is removed to reveal the internal details (figure from Baandrup et al. (2020)).

Table 5.1: Performance of the conventional and interpreted designs. Weighted compliance (5.1a) of the five governing load cases, and relative improvement compared to the conventional design.

<table>
<thead>
<tr>
<th>Design</th>
<th>Compliance [J]</th>
<th>Improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional design</td>
<td>264.6</td>
<td>—</td>
</tr>
<tr>
<td>Interpreted design</td>
<td>231.0</td>
<td>12.7%</td>
</tr>
<tr>
<td>Interpreted design - parametric optimization</td>
<td>189.6</td>
<td>28.4%</td>
</tr>
</tbody>
</table>

From the figures, it is seen that the number of diaphragms per section is increased from five to six, that four of the six diaphragms are curved toward the hangers, and that longitudinal panels to support the connection between hanger anchorage and diaphragms are added.

Subsequently, both the conventional design and the interpreted design were modeled with shell elements in the commercial software Abaqus (Abaqus (2016)). Here, models with similar dimensions, boundary conditions, and loads to Fig. 5.1 were created with identical material volumes for comparison.

To quantify the performance improvement, the weighted compliance (similar to the objective function (5.1a)) was computed for the five representative load cases. The numerical results are summarized in Table 5.1. Here, the interpreted design is seen to be 12.7% stiffer than the conventional. As mentioned previously, the change in compliance, or stiffness, can generally be translated to an equivalent...
change in volume, or weight. Hence, the stiffness increase may be seen as a weight reduction.

In addition to the interpretation and quantitative comparison above, an additional parametric sizing optimization was carried out on the interpreted design. Here, the Abaqus shell-model was used for parametric optimization with the plate thicknesses of the shell elements as design variables. The optimization problem was formulated similarly to Problem (5.1), however, with the box-constraints (5.1d) exchanged with lower bounds of 4 mm plate thickness. The non-linear optimization problem was solved by SQP methods with the use of finite-difference gradients, similarly to the studies in Section 4.3.

The result after parametric optimization is shown in the last row of Table 5.1. Here, a total weight saving of 28.4% compared to the conventional design is seen.

Knock-on Effects and Carbon Footprint

As discussed in Section 1.1, significant knock-on effects from the girder weight savings are achievable in the remaining bridge structure due to the load-carrying principles of suspension bridges. Here, the knock-on effects estimated in Table 1.1 in Section 1.1 were used together with the material quantities given in Table 3.1 in Chapter 3.

The 28.4% weight saving in structural steel in the girder (without walkways), translated into a reduction in loads transferred to the cables of 19.1% when adjusting for the contribution from walkways (about 11% of the self-weight) and surfacing (about 22% of the self-weight). The knock-on effects from the 19.1% reduction in transferred loads are given in material quantities in Table 5.2.

Furthermore, the table contains the estimated carbon footprints and savings for the total material quantities. To estimate the CO₂ emissions, the following values were used: 2,460 kgCO₂/ton for steel plates and 150 kgCO₂/ton for concrete (with a density of 2,400 kg/m³). Thus, a potential reduction of 43,000 tonnes of CO₂ is possible by the use of the parameter optimized interpreted design.

Table 5.2: Total quantities of steel, concrete, and CO₂ for the entire bridge. Savings are indicated relative to the conventional design.

<table>
<thead>
<tr>
<th>Design</th>
<th>Steel</th>
<th>Concrete</th>
<th>*CO₂</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Quantity</td>
<td>Saving</td>
<td>Quantity</td>
</tr>
<tr>
<td>Conventional design</td>
<td>69,000</td>
<td>—</td>
<td>175,000</td>
</tr>
<tr>
<td>Interpreted design</td>
<td>63,000</td>
<td>6,000</td>
<td>162,000</td>
</tr>
<tr>
<td>Interpreted design - para. opt.</td>
<td>56,000</td>
<td>13,000</td>
<td>146,000</td>
</tr>
</tbody>
</table>

*Calculations only take account of material quantities, hence disregarding construction methods
5.3 Summary and Discussion

The main findings of the topology optimization studies were partly the complex and highly detailed structures, and partly the achievable weight savings with the simple interpreted design.

From the complex optimized design, structural details very different from the conventional design were unveiled. Based on the insights from the pure topology optimization results, a simple interpreted design was established with a weight saving in excess of 28%. This saving should be seen in view of the maximum achievable weight saving of 13.4% in Chapter 4. Hence, the application of the giga-scale topology optimization procedure, followed by a quick interpretation and a simple parametric optimization, led to weight savings significantly larger than what was possible by modifying the conventional design in the previous chapter.

It is remarkable how this sequential approach of first applying topology optimization followed by a simplistic interpretation resulted in a simple, cost-efficient, and constructible design concept. The identified adaptions to the conventional design would hardly have been found without the topology optimization results. Thus, the results illustrate the benefits of applying topology optimization in the early phase of civil engineering design development, regardless of the complicated structures initially arising from the procedure.

Despite the significant weight savings achieved by the interpreted design with a similar level of geometric and manufacturing complexity as the conventional, the initial optimization problem was very simple. Hence, it was formulated without any constraints regarding practical considerations, such as stability or stresses. When studying the intricate design, particularly the neglect of stability constraints is visible in the very thin plates and truss members. Furthermore, as discussed, the question of constructibility is a challenge, which here was avoided by making an interpretation. However, it is of interest to include constraints on both stability, stresses, and constructibility directly in the initial optimization problem.

Since this is not yet possible in giga-scale topology optimization, the application of other structural optimization methods, such as truss-layout optimization, should be considered. Therefore, in the next chapter, truss structures are introduced in search of innovative designs to solve optimization problems taking account of various practical considerations.
Chapter 6

Innovative Designs by Truss Optimization

In the previous chapter, a first step was taken in search of new and innovative structural girder concepts. Here, the search is continued and approached differently with the application of truss optimization as the design tool. The goal is still to reduce self-weight significantly and to identify more material-efficient designs.

The simplicity of the truss structures allows for the formulation of more realistic optimization problems, taking account of various practical considerations. Notably, the possibility of including local and global stability is essential in order to avoid unstable structures.

In this chapter, truss optimization problems based on finite element limit analysis (FELA) are solved with constraints on stresses as well as local and global stability. Hence, some of the limitations in the pure topology optimization problem are here resolved.

The application of truss optimization should not be mistaken as a step backward to the classic truss girders (introduced in Section 1.2). Instead, new and lightweight structural concepts may unveil from the application of modern numerical optimization methods on simple truss structures.

Since no out-of-the-box truss optimization software exists with the required capabilities, a significant amount of work has been put into the development of such a framework suitable for practical application. This is done in multiple steps before the framework is applied in search of innovative bridge girder designs.

Firstly, this includes solving a general optimization problem capable of handling constraints on local and global stability. This challenge has previously been approached in many different ways, however, never before based on FELA. This particular work was published in the appended Paper IV (Poulsen et al. (2019))\(^1\).

\(^1\)In Paper IV, the author of the present thesis performed the implementations, analysis, and case studies, as well as writing the first draft of the paper. Furthermore, the author contributed significantly to the literature review, method development, new theoretical formulations, and interpretation of the results.
Secondly, the general optimization problem is extended toward practical application. This includes the possibility of solving large-scale problems while including the two challenging stability constraints. Furthermore, methods to both reduce truss complexity and to identify more optimal solutions among the multiple local minima of the concave problem are implemented. This specific work is the content of the appended Paper V (Baandrup et al. (2019a)).

Finally, the truss optimization framework is applied in the context of the thesis objectives. This application is the content of the appended Paper VI (Baandrup et al. (2019b)).

In all cases, to solve the optimization problems, the convex solver Mosek (MOSEK (2018)) was used. The solver is based on the interior-point algorithm, as introduced in Section 2.2.3. Furthermore, all parts were implemented in MATLAB (MATLAB (2016)).

In the following sections, firstly, the components of the FELA-based truss optimization framework are presented along with introductory examples. Subsequently, the optimization of bridge girders is carried out. Finally, the main findings and learnings of the chapter are summarized and briefly discussed. The content of the following two sections are contained in Paper IV, V, and VI, however, presented here in a summarized format.

6.1 Truss Optimization Applying Finite Element Limit Analysis

Trusses are simple and transparent structures that efficiently carry the applied loads to supports by means of simple structural principles (compression and tension). For these reasons, trusses have been used for centuries in all kinds of structures. Furthermore, because of their simplicity, trusses have been subject to numerical optimization for decades, as discussed in Section 2.1.2. However, truss-layout optimization often leads to lightweight structures prone to instabilities, which must be handled in order to obtain realistic structures.

As mentioned in Section 2.1.2, the first numerical studies of optimal truss structures took place in the 1960s. In Dorn et al. (1964), the truss optimization problem was formulated with linear programming, including equilibrium equations and yield-constraints under plastic assumptions. Here, it was shown that the solution to the optimization problem under plastic assumptions also was an optimal structure under elastic assumptions, when a single load configuration was considered.

A few years later, FELA was introduced according to the classic plasticity theory in Anderheggen and Knöpfel (1972). Here, the ultimate load of structures was modeled assuming a rigid-plastic material behavior. Concurrently, in Pedersen (1972), local stability was included in an early formulation to identify the optimal elastic layout of trusses, which was solved using linear programming.

As mentioned above, in practical application, it is essential both to include
local stability and to ensure an overall stable structure, thus to avoid global instability. However, as argued in, e.g., Zhou (1996) and Rozvany (1996), this is not trivial to do. Hence, many different approaches have been proposed, as discussed in the introduction of Paper IV (Poulsen et al. (2019)). Related to this, Evgrafov (2004) describes the so-called global stability singularity problem in elasticity-based concave formulations, where the optimal solution is disconnected from the interior of the solution domain. This illustrates the difficulties related to finding optimal solutions applying an elastic force distribution. Furthermore, it was stated in Tugilimana et al. (2018) that the problem was alleviated by stating a disaggregated form of the equilibrium as a two-step elastic relation. Recently, in Weldeyesus et al. (2019), a so-called relaxed formulation of the elastic truss optimization problem was presented, including global stability but disregarding kinematic compatibility and local stability. This formulation corresponds to a rigid-plastic force distribution and leads to a convex semidefinite problem.

In the following sections, the newly formulated FELA-based method to solve the truss optimization problem, including local and global stability, is introduced. Here, a rigid-plastic force distribution is assumed, while global stability is handled by the linear buckling problem assuming an elastic behavior, and thus, forming a convex problem. This may seem contradictory, but for a single load case, as stated above, it was previously shown that the solution to a plastic optimization problem is also a solution under elastic assumptions. Therefore, the behavior of the structure will be elastic until the critical load level, where it yields and buckles simultaneously, as discussed in Paper IV (Poulsen et al. (2019)).

When considering multiple load cases, members may yield or buckle before the maximum load. However, the distinctive load-displacement behavior of members that buckle is similar to members that yield, and in this respect, the model is valid. In Mikkelsen (2018), the overall behavior of truss structures optimized for multiple load cases was investigated, and here, a non-linear geometric analysis showed that the structure was able to carry the load after the first buckling and reach the intended load level subsequently. Hence, this study supports the use of an optimization method based on rigid-plastic behavior for multiple load cases.

6.1.1 General Optimization Problem

The principles of the general FELA-based optimization problem are presented together with a simple example for illustration. The content of this section is covered in detail in Paper IV (Poulsen et al. (2019)).

The truss optimization takes the ground-structure approach with an initial mesh of nodes connected by members, along with boundary conditions and node-based loads applied. The design variables are the cross-sectional areas of all members. The objective is to minimize the total volume of the structure while fulfilling four different types of constraints.

The truss-layout optimization problem with a specification of the objective and
Innovative Designs by Truss Optimization

constraint functions is given as the following semidefinite convex problem

\[
\begin{align*}
\min_{\beta, A} \quad & V = \sum_{e=1}^{N_e} a_e \ell_e = A^\top L \\
\text{s.t.} \quad & H \beta = R \\
& C/\beta - C_mA \leq 0 \\
& -I\beta - PA \leq q \\
& K_G(\beta_k) + K(A) \succeq 0 \quad \forall \ k = 1, \ldots, N_k
\end{align*}
\] (6.1a)

(6.1b) (6.1c) (6.1d) (6.1e)

where \(e\) refers to the \(N_e\) individual members and \(k\) refers to the \(N_k\) individual load cases.

The primary design variables are the member areas collected in vector \(A\), and the secondary design variables are the bar forces collected in vector \(\beta\). The objective function in (6.1a) defines the total truss volume \(V\) by a summation of the products of individual member areas \(a_e\) and lengths \(\ell_e\) (collected in vector \(L\)).

The first constraint (6.1b) defines the equilibrium equations. Here, the equilibrium matrix \(H\) ensures nodal equilibrium of internal (\(\beta\)) and external (load vector \(R\)) nodal forces for all unsupported nodes. Thus, a statically admissible solution is ensured.

The second constraint (6.1c), defining the yield condition, ensures all member stresses to be within the yield criterion. The effect of the constraint may be seen as a conversion of the bar forces to their absolute values (\(C\)), which are restricted by the yield stress, \(f_y\), of each bar (\(C_m\)). For a single element, \(e\), and a single load case, \(k\), the yield constraint is defined by two inequalities

\[
\begin{bmatrix}
1 \\
-1
\end{bmatrix} \beta_{e,k} - \begin{bmatrix}
f_y \\
f_y
\end{bmatrix} a_e \leq \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(6.2)

The final two constraints ensure restrictions against local and global instabilities. The principle difference between the two stability phenomenons is illustrated with the simple two-bar structure shown in Fig. 6.1. The bar geometries are defined by cross-sectional areas \(a\) and lengths \(l\). Both bars are simply supported and the structure is subject to a vertical load \(P\).

In an idealistic setting, only the vertical bar is needed to carry the load \(P\), since no horizontal forces exist. Hence, to fulfill a pure force equilibrium, the horizontal bar is irrelevant. However, in reality, the slightest imperfection in the system will cause an unstable structure if the horizontal bar does not exist (Fig. 6.1b). This phenomenon is referred to as global (in)stability.

Besides ensuring global stability, also the local stability of each bar must be secured. The phenomenon of local stability is identical to critical Euler buckling, as illustrated in Fig. 6.1c.

The slightly more complex stability constraints are introduced in the following. For the sake of simplifying the introduction, the global stability constraint is presented first.
Innovative Designs by Truss Optimization

Figure 6.1: Global and local instability illustrated by a two-bar structure subject to a vertical load $P$

Global Stability

The semidefinite global stability constraint \((6.1e)\) is formulated through the linear buckling problem. This problem is given as an eigenvalue problem, see, e.g., Cook et al. (1989), stated as

\[
(K + \lambda K_G)v = 0
\]

with eigenvalue $\lambda$ and corresponding eigenvector $v$. Here, $K$ is the global elastic stiffness matrix and $K_G$ is the geometric stiffness matrix. In the optimization problem, a priori, the load parameter/eigenvalue is $\lambda = 1$, and thus, the buckling condition is controlled by the design variables $A$ and $\beta$. A sufficient condition for this eigenvalue problem to be fulfilled, and thereby ensuring global stability, is to demand the summed stiffness matrix $K_S$ to be positive semidefinite

\[
K_S = K_G(\beta) + K(A) = S, \quad S \succeq 0
\]

where $K_G$ and $K$ are defined as functions of $\beta$ and $A$, respectively.

In the Mosek software, the semidefinite constraint is handled as a linear matrix inequality (LMI), as defined in Eq. (6.4). Here, $K_S$ is converted to a set of linear equality constraints using an implicit semidefinite slack variable $S$. The semidefinite requirement is thus moved from being explicit on $K_S$ to being implicit on $S$.

Local Stability

The restriction against local instability of each element is defined by constraint \((6.1d)\). The local buckling constraint is introduced through the critical Euler buckling load $P_{cr}$ for a simply supported member. Hence, the constraint for a single element, $e$, is given as

\[
-\beta_e \leq P_{cr,e} = \frac{\pi^2 \cdot EI_e}{L_e^2} \approx \alpha_e a_e^2
\]

where $E$ is the modulus of elasticity and $I_e$ is the second moment of area. The quadratic term with the constant $\alpha_e$ is an approximation of the exact term, to
make it a function of the design variable $a_e$. The constant $\alpha_e$ depends on the shape of the member cross-section. Onward, it is defined from the choice of a tube cross-section, which makes the approximation exact.

Despite the simplification in the above approximation, the constraint is concave and cannot be integrated directly into the otherwise convex optimization problem. The concavity is illustrated in Fig. 6.2. Here, the yield surface is given by the linear criterion $f_y a_e$. However, for members in compression, the quadratic stability function (6.5) is stricter. Hence, the feasible solution domain (gray hatch) becomes concave when including local stability.

To eliminate the concavity of Eq. (6.5), the quadratic constraint is linearized.
Innovative Designs by Truss Optimization

Figure 6.4: Flowchart to solve the general optimization problem (6.1)

through its tangent (Fig. 6.2) or secant (Fig. 6.3). The conservative tangent linearization is used for most of the compression members (when $\beta_e < -\varepsilon f_c^2/\alpha_e$, where $\varepsilon$ is a small number, e.g., $10^{-3}$). However, since the tangent implicitly imposes a minimum area on the member, the secant linearization is used for members with minimal compression forces. Thus, by supplementing with the (non-conservative) secant, member areas are allowed to vanish during the optimization.

The linearization of Eq. (6.5) is for all elements and load cases assembled to the matrix expression (6.1d), where $I$ is the identity matrix, $P$ is a matrix containing the parts linear in the design variable $a_e$, and $q$ is a vector containing the constant parts of the linearization.

Since the linearization requires a previous solution, $\beta_{i-1}$, it is updated in an iterative process, as illustrated in the flowchart given in Fig. 6.4. To find an initial solution $\beta_0$ to establish the local stability constraint, first problem (6.1) is solved without constraint (6.1d). Hereafter, the local stability constraint is updated iteratively until convergence. Hence, the overall concave problem, including global and local stability, is solved in an iterative process with convex sub-problems. The idea of handling the Euler stability criterion by an iterative process was previously discussed in Parkes (1975) as well as in He and Gilbert (2015).

Multiple Load Cases

Multiple load cases are, in general, handled by an expansion of the system of equations. Thus, the vectors $\beta$ and $R$ are expanded to include $N_k$ individual sets of the variables and loads. Similarly, the matrices in Problem (6.1) are expanded to apply the constraints for each load case.
Example: Planar Column Structure

To illustrate the general optimization problem (6.1), a vertically loaded, plane column is studied (see Fig. 6.5). The domain is rectangular, supported along the bottom edge, and loaded with a point load centered at the top. The parameters are given as $H = 4$ m, $W = 1.6$ m, $P = 10$ kN, $E = 210$ GPa, and $f_y = 235$ MPa.

The chosen $5 \times 6$-node mesh is shown in Fig. 6.6. All nodes are connected to the nearest and next-nearest nodes in all directions, giving a mesh of $N_e = 213$ members.

In total, three different versions of the general optimization problem has been studied, with increasing complexity: without stability constraints ((6.1) without (6.1d) and (6.1e)), with global stability ((6.1) without (6.1d)), and the complete problem (6.1) with global and local stability.

The results of the optimization study are shown in Fig. 6.7. In the figure, supported nodes are represented by solid circles while the load is applied at the white square. It must be stated that overlapping elements, e.g., two elements in the same plane extending from the same node to the closest and second closest node, respectively, also are plotted in the same plane. In the caption to each sub-figure, the total volume $V$ is given together with the number of elements $N_e$ in the final optimized structures above a given threshold of $10^{-2} \cdot \max(A)$. The threshold

---

**Figure 6.5**: Domain, boundary conditions, and loads of the column structure (figure from Poulsen et al. (2019))  
**Figure 6.6**: Initial ground structure of the column with $N_e = 213$ (figure from Poulsen et al. (2019))
Innovative Designs by Truss Optimization

Figure 6.7: Solution of the column structure (figure from Poulsen et al. (2019))

is chosen to remove elements with near zero-area, and given relative to the largest element area.

In Fig. 6.7a, the structure without any stability constraints is shown. Similar to Fig. 6.1b, the vertical load is carried directly to the support. However, in Fig. 6.7b, the effect of adding the global stability constraint is clear since the single-column structure now is supported by a thin web of elements. It should be noted how the volume only has increased slightly, despite the significant increase in the number of elements. Hence, the additional volume of material to constrain the structure from global instability is insignificant, however, crucial to the structural layout and integrity. Finally, in Fig. 6.7c, the inclusion of local stability is clearly visible in the increased thickness of the compressed center column. Here, it should be noted that the number of elements decreases since the material is better utilized in fewer but larger elements when constraining local stability. Furthermore, the volume increases significantly to retain local instabilities.

From the experience of additional studies included in Paper IV, as well as Paper V, the above observations can be considered general. Thus, similar behavior is seen despite other configurations of domain, loads, and support conditions.

The choice of using typical material parameters, $E$ and $f_y$, was deliberate despite the usual practice of using unity material parameters in optimization benchmark examples (Sigmund and Bendsoe (2003)). When using unity material parameters, the typical ratio of around 1:1000 between $E$ (GPa) and $f_y$ (MPa) is neglected. This ratio does not affect the results when optimizing without global stability constraints. However, from the studies presented in Paper IV, it was
Innovative Designs by Truss Optimization

proved that when including global stability constraints in the optimization problem, this ratio has a considerable impact on the final optimized structures. Hence, when doing structural optimization studies with global stability, the use of correct material parameters is highly recommended.

6.1.2 Toward Practical Application

The general optimization problem is now extended toward practical application. Firstly, it is desirable to allow for large-scale optimization, with either several load cases or a large number of elements to accommodate larger design freedom. Secondly, it is desirable to reduce the truss complexity, and thus implicitly cost, by reducing the number of elements in the final structures. Finally, it is of interest to handle the solution-space of multiple local minima owing to the overall concave nature of the general optimization problem. Therefore, methods to search the solution-space for more optimal structures are desirable.

The extensions are presented in the following, and subsequently, illustrated by three examples. The content of this section is covered in detail in Paper V (Baandrup et al. (2019a)).

Large-Scale Truss Optimization

When including global stability in the truss optimization problem, the challenges of solving large-scale problems amplify since the numerical problem size increases significantly.

Only recently, the combined problem of doing large-scale truss optimization, including stability constraints, has been approached. In Tugilimana et al. (2018), a large-scale structure of 3,351 members was optimized. In the paper, global stability was formulated as a semidefinite constraint, local stability was based on the critical Euler load, and the force equilibrium was formulated elastically. Similarly, in Weldeyesus et al. (2019), global stability was defined by a semidefinite constraint. However, local stability was disregarded. In the paper, a large-scale structure imitating 90,100 members was optimized, but the optimization problem being solved only contained 7,337 members.

In the present work, the convex problem (6.1) is transformed into its dual formulation. By solving the dual problem with the Mosek solver, the numerical problem size and memory consumption can be reduced considerably. Hence, the allowable magnitude of models that may be optimized increases correspondingly.

The main reason why the dual formulation is faster and less memory consuming is found in the global stability constraint (6.1e). By solving the dual, the linear equality constraints from the LMI in Eq. (6.4) vanish. This leaves only the \( N_k \) (now implicit) semidefinite constraints, which are treated as conic constraints.

To transform the primal problem (6.1) into its dual, the general technique in Eq. (2.14)-(2.17), presented in Section 2.2.2, is applied. The complete transformation is left for the reader to find in Paper V.
Innovative Designs by Truss Optimization

The dual optimization problem of Problem (6.1) is given as

$$\begin{align*}
\min_{y, z, t, Y} & \quad -R^\top y + q^\top t \\
\text{s.t.} & \quad \begin{bmatrix} H^\top & 0 \\ 0 & -C_m^\top \\ C^\top -C_m^\top \\ I^T \\ P^T \end{bmatrix} z + \begin{bmatrix} I \\ 0 \end{bmatrix} t + K^*_S(Y) = \begin{bmatrix} 0 \\ L \end{bmatrix} \\
& \quad z \geq 0 \\
& \quad t \geq 0 \\
& \quad Y_k \succeq 0 \quad \forall k = 1, \ldots, N_k
\end{align*}$$

(6.6) with the dual vector variables $y, z,$ and $t$. Additionally, $K^*_S$ is a self-adjoint linear operator on the semidefinite dual matrix variable $Y$. Thus, the semidefinite part has changed from being defined explicitly on the constraint (6.1e) to being defined implicitly on $Y_k$, (6.6e).

An interesting feature of the dual problem (6.6), is how the dual variables can be interpreted as physical quantities. Hence, the dual variables $y$ and $z$ are representing the nodal displacements (collapse mechanism) and the plastic strains, respectively, see, e.g., Krenk et al. (1994). Similarly to $y$, the dual variable $t$ can be interpreted as nodal displacements but in connection to local stability. Finally, the critical eigenmode in regard to global stability can be found from $Y$, given as $\varphi = \pm \sqrt{\text{diag}(Y)}$.

An example of the performance increase by the dual formulation compared to the primal is shown in Fig. 6.8. Here, the solver time is plotted as a function of an increasing number of elements $N_e$. The basis of the study is the plane cantilever beam studied in the following section (see Fig. 6.11). Here, the optimization problem is studied without constraints on local stability.

In the study, the maximum model size for the primal formulation is found to $N_e = 12,684$ with a solver time of 803 minutes. For the dual formulation, the limit is found to $N_e = 34,496$ with a solver time of 2,259 minutes (not shown in Fig. 6.8). These upper limits are, of course, hardware-dependent, and for the study, a desktop with an Intel Core i7-6500U CPU 2.59 GHz and 32 GB RAM was used.

Reduction of Truss Complexity

With the increasing number of possible elements, the complexity of the trusses is also expected to increase, leading to an increase in the number of joints and construction costs. Two heuristic methods to reduce truss complexity are formulated by imposing penalties on small member areas and short members, respectively.

Both penalty methods are introduced into the primal objective function (6.1a), by adjusting the weighting of all members. Consequently, the penalties discourage the use of short and thin members in the final structures, however, allowing them if needed.

Department of Civil Engineering - Technical University of Denmark
Two heuristic methods are proposed to penalize short and thin members, however allowing them if necessary. The penalty functions are applied and updated in the dual solution of the optimization problem. The penalty function, $p_{\ell,e}$, on short members is shown in Fig. 6.9, where $\ell,0$ is the true length of member $e$. The penalty varies linearly from the longest members, not being penalized, to the shortest, being penalize with $p_{\ell,max}$. In present studies $p_{\ell,max} = 2$ was found to be adequate. Similar methods to penalize short members has previously been considered in Parkes (1975) and He and Gilbert (2015).

The penalty function, $p_{\alpha,e}$, on small area members is shown in Fig. 6.10, where only the penalty is applied to members with areas below $f_y/\alpha_e$. The penalty function is updated iteratively since it is based on the cross-sectional area from the previous iteration. The iterative process takes place concurrently with the

---

**Figure 6.8:** Solver times as a function of increasing number of elements, of the optimization problem with global stability, shown for the primal and dual formulations. Markers indicate data points, connecting lines are only shown for clarity (figure from Baandrup et al. (2019a))

**Figure 6.9:** Penalty function of short members (figure from Baandrup et al. (2019a))

**Figure 6.10:** Penalty function of small area members (figure from Baandrup et al. (2019a))
Innovative Designs by Truss Optimization

process shown in the flowchart in Fig. 6.4. In present studies, $p_{a,\text{max}} = 1$ was found to be adequate.

Searching the Concave Solution-Space

By the introduction of the local stability constraint (6.1d), the overall optimization problem becomes concave. As discussed and illustrated in Section 2.2.2, concave problems have a solution-space of multiple local minima, contrary convex problems with only a single global optimum.

Experience showed that the final solution to Problem (6.1) in some cases was highly dependent on the initial convex solution ($\beta_0$ in Fig. 6.4), since $P$ and $q$ are based on this. Therefore, a heuristic strategy to search the concave solution-space of local minima was proposed to find more optimal solutions.

The strategy is to generate $N_j$ "random" initial solutions, designated $\beta_{0,j}$, which will result in $N_j$ different designs. The $N_j$ initial solutions are found by solving a modified version of Problem (6.1). The first modification is to replace the length vector in the objective function with a random vector of equal dimension, to force new solutions to be generated. However, this modification results in structures with very different volumes than the original. To avoid this, an extra constraint is added to the optimization problem ensuring the volume being close to the original. Subsequently, the "random" solutions, $\beta_{0,j}$, are used in place of $\beta_0$ in the loop in Fig. 6.4. Due to the concave nature of the overall optimization problem, these "random" solutions may lead to final structures with lower volumes than the structures found directly from solution $\beta_0$.

Examples of Practical Application

Three examples are presented to illustrate the effects of the extensions toward practical application. For all structures, similar material parameters to the column structure in the previous section are used. Also, the results are visualized similarly, however, with a threshold on member areas of $10^{-4} \cdot \max(A)$.

Cantilever Beam  The first example is a plane cantilever beam as shown in Fig. 6.11. The domain is rectangular, supported along the left edge, and loaded with a point load at the upper right corner. The parameters are given as $L = 5$ m and $P = 100$ kN. The chosen $5 \times 3$-node mesh is shown in Fig. 6.12. All nodes are connected to the nearest and next-nearest nodes in all directions, giving a mesh of $N_e = 78$ members.

In Fig. 6.13, the optimized structures from four different optimization problems are shown with increasing complexity of the optimization problem. Here, it is seen that the structures in Fig. 6.13a and 6.13b are very similar, despite the inclusion of global stability in the second case. However, this is closely related to the coarseness of the mesh. When adding the local stability constraint, Fig. 6.13c, the change is significant in both structural layout, number of elements, and volume. As
Innovative Designs by Truss Optimization

**Figure 6.11:** Domain, loads, and boundary conditions of the plane cantilever beam (figure from Baandrup et al. (2019a))

**Figure 6.12:** Initial ground structure of the plane cantilever beam with $N_e = 78$ (figure from Baandrup et al. (2019a))

(a) Without stability, $V = 170.2 \times 10^{-4}$ m$^3$, $N_e = 37$

(b) With global stability, $V = 170.2 \times 10^{-4}$ m$^3$, $N_e = 39$

(c) With global and local stability, $V = 232.8 \times 10^{-4}$ m$^3$, $N_e = 11$

(d) With global and local stability, and penalty functions, $V = 232.8 \times 10^{-4}$ m$^3$, $N_e = 5$

**Figure 6.13:** Solutions of the plane cantilever beam. Color indication: blue=compression, red=tension (figure partly from Baandrup et al. (2019a))
Innovative Designs by Truss Optimization

(a) Histogram of 100 solution searches. Final volume shown for optimization problem with global and local stability, and penalty functions

(b) Best final solution, with global and local stability, and penalty functions, \( V = 174.3 \times 10^{-4} \text{ m}^3 \), \( N_e = 11 \). Color indication: blue=compression, red=tension

Figure 6.14: Plane cantilever beam by solution-space search (figure (a) from Baandrup et al. (2019a))

observed for the column structure, the number of elements decreases considerably but with an increased total volume. Finally, when introducing the penalty functions, Fig. 6.13d, the number of elements is reduced further but without changing the volume.

Next, the strategy for searching the concave solution-space is applied with \( N_j = 100 \). Fig. 6.14a provides the distribution of the 100 solutions. Here, it is clear that the majority are similar in volume to the first solution in Fig. 6.13d. However, a better solution is found with a final volume only slightly larger than the volume for the purely material optimized structure in Fig. 6.13a. This final solution is shown in Fig. 6.14b. When comparing Fig. 6.13d and 6.14b the reason for the significant reduction in volume is clear. The two large compression members in the first solution are in the second bisected in order to reduce the buckling length and thus to reduce the member area required to ensure local stability. Consequently, an increase in the number of elements from 5 to 11 is seen, since additional supports are required to ensure global stability.

The plane cantilever beam is now extended to a three-dimensional cantilever beam, as shown in Fig. 6.15. The line load is given as \( p = 20 \text{ kN/m} \). A mesh of \( 9 \times 5 \times 5 \) nodes is used with all nodes connected to all other nodes, thus, containing a total of 25,200 elements.

The final result, which is the best solution after \( N_j = 20 \) solution-space searches, is shown in Fig. 6.16. Despite the initial mesh of 25,200 elements, the final structure only contains 59.
Innovative Designs by Truss Optimization

Figure 6.15: Domain, loads, and boundary conditions of the 3D cantilever beam (figure from Baandrup et al. (2019a))

Figure 6.16: Solution of the 3D cantilever beam with global and local stability, and penalty functions, \( V = 234.9 \times 10^{-4} \text{ m}^3 \), \( N_e = 59 \). Color indication: blue=compression, red=tension (figure from Baandrup et al. (2019a))

Large-Scale Roof Structure  The large-scale capabilities are demonstrated by optimization of a roof structure subject to two load cases, as shown in Fig. 6.17. The domain is a box supported along two adjacent end surfaces. Two load cases are applied: a downwards uniformly distributed load \( p = 5 \text{ kN/m}^2 \) over the entire top surface and a downwards point load \( P = 1 \text{ MN} \) on the top face at the free corner. The dimensions are \( W = 50 \text{ m} \) and \( H = 5 \text{ m} \), and a mesh of 32,640 elements was used. The optimization problem was solved on a node with 256 GB RAM by a duration of 259.4 hours (10.8 days). The result is shown in Fig. 6.18. After optimization, the structure contains only 278 members.

For a single load case, the roof structure was in Paper V solved with 52,326 elements (by a duration of 323.8 hours (13.5 days)).

Figure 6.17: Domain, loads, and boundary conditions of the roof structure. Grey-scale hatch indicate area of downwards distributed load \( p \) (figure from Baandrup et al. (2019a))
Figure 6.18: Roof structure subject to two load cases, solution with global and local stability, and penalty functions, $V = 7.26 \text{ m}^3$, $N_e = 278$ (figure from Baandrup et al. (2019a))

6.2 Truss Optimization of Bridge Girders

The framework developed for FELA-based truss optimization is here utilized as a design tool to identify innovative design concepts of bridge girders.

The girder optimization is carried out based on the same assumptions as for the topology optimization in the preceding chapter and as introduced in Chapter 3. With the different approach of using truss structures, and thus the opportunity to include additional relevant constraints, further insight into alternative design concepts will be uncovered. Hence, with the inclusion of constraints on stresses as well as global and local stability, the structural layout of a simplified bridge girder model is optimized to reduce self-weight.

In the following sections, firstly, the girder model is presented, along with various assumptions. Concurrently, the arrangement of the model for truss optimization is discussed. Finally, the main results of the studies are summarized. The presented studies are covered in detail in Paper VI (Baandrup et al. (2019b)).

6.2.1 Bridge Girder Model

The bridge girder model subject to optimization was a single section, as shown in Fig. 6.19 including the main dimensions. Similar to the topology optimization model, the outer geometry of the domain was defined by the wind profile. Since the section was part of a continuous girder, all loads applied were in equilibrium. Therefore, the boundary conditions, imposed by the six fixed degrees of freedom shown in the figure, were applied only to prevent rigid body movement and to
Innovative Designs by Truss Optimization

establish a non-singular system.

Since the girder was modeled with truss elements, the traffic load was assumed to be carried by a top plate. Hence, the basis of the optimization was an assumption of a continuous top plate spanning up to 3 m in two directions and thus distributing the surface loads to the underlying trusses. The specific top plate design was not considered. However, it could be constructed as a sandwich element with steel plate skins similar to the ones discussed in Section 1.3.2. In the present work, an equivalent steel plate thickness of \( t_{tp} = 20 \) mm was selected. For comparison, the equivalent plate thickness in the Osman Gazi Bridge was 26 mm for the orthotropic deck (top plate and troughs) spanning 5 m.

The model was subject to the 12 global and two local load cases introduced in Section 3.1. Contrary to the topology optimization, all 14 load cases were included simultaneously in the truss optimization study. The loads were applied as indicated in Fig. 6.20. Here, the global section forces were applied to each end of the single girder section. Additionally, the distributed load \( p \) was applied to the top surface, and hanger forces \( P \) at the location of the hanger anchorages. The loads in all 14 load cases are summarized in Section D.1 of Appendix D.

Contrary to the topology optimization model in the preceding chapter, only a single section of the truss girder was needed. The reason for this was partly that six degrees of freedom was sufficient to avoid a non-singular system. In contrast, the giga-scale topology optimization model required the entire end surface fixed to avoid numerical instabilities. Furthermore, the global section forces were applied differently in the truss model, as explained below. Thus, no additional girder section was needed for load transmission.

The ground-structure mesh of the model is shown in Fig. 6.21. The mesh was constructed based on a \( 9 \times 11 \times 3 \)-node grid, where all nodes were connected to the nearest and next-nearest nodes in all directions. A total of \( N_e = 6,665 \) members were contained in the mesh.

The optimization problem to be solved was established from the general problem (6.1). However, three additional constraint functions were included to define the bridge model partly. The first additional constraint handled the assumption about the top plate. The second additional constraint handled a symmetry con-

\[
\begin{align*}
W &= 30 \text{ m} \\
H &= 4.75 \text{ m} \\
L &= 25 \text{ m} \\
u_x &= u_y = u_z = 0 \\
u_x &= u_y = u_z = 0
\end{align*}
\]

**Figure 6.19:** Domain of the single girder section with indication of dimensions and boundary conditions (figure from Baandrup et al. (2019b))
Figure 6.20: Global section forces applied to the end surfaces, local distributed load \( p \) (indicated by downward arrows), and hanger forces \( P \) (figure from Baandrup et al. (2019b))

Figure 6.21: Initial ground-structure of the bridge girder section, \( N_e = 6,665 \) (figure from Baandrup et al. (2019b))
Innovative Designs by Truss Optimization

dition applied similarly to the topology optimization model. Finally, the third additional constraint was used to impose the global section forces in the model, which also required a reduction of the equilibrium equations. The complete optimization problem was thus given as

$$\min_{\beta, A} \quad A^\top L$$

subject to:

$$H_R\beta = R_R$$

(Reduced Equilibrium)

$$C\beta - C_m A \leq 0$$

(6.7c)

$$-I\beta - PA \leq q$$

(6.7d)

$$DA = A_0$$

(Top Plate)

$$SA = 0$$

(Symmetry Mapping)

$$G\beta = G_0$$

(Global Section Forces)

$$K_G(\beta_k) + K(A) \succeq 0 \quad \forall k = 1, \ldots, N_k$$

(6.7h)

The optimization problem is shown in its primal formulation. However, to include all 14 load cases simultaneously, the dual formulation of the problem was solved. The derivation of the dual formulation is shown in Section D.2 of Appendix D.

In addition to solving the dual problem, the strategy to reduce truss complexity was applied. The solution-space search was not applied due to computational limitations. However, due to the many load cases and additional constraints included, which both narrowed the solution space, it was considered fair to assume the gain from applying this strategy to be modest.

The principles behind the three additional constraints are introduced below.

**Top Plate**

The assumed top plate was in the truss model implemented with a fixed truss grid in the upper layer of the girder, as indicated in Fig. 6.22. Since the top plate was fixed, only the underlying structure was subject to optimization.

The truss members in the fixed top plate were assigned fixed cross-sectional areas, such that the strength was equivalent to a steel plate with a thickness of $t_{tp} = 20$ mm. The assignment of fixed areas was introduced by the linear equality.

![Continuous top plate](image1)  ![Truss top plate](image2)

**Figure 6.22:** Assumption of a continuous top plate and the equivalent modeled truss top plate (reproduction of figure from Baandrup et al. (2019b))

92 Department of Civil Engineering - Technical University of Denmark
constraint (6.7e), where $A_0$ contained the fixed areas and $D$ was an index matrix to identify the relevant members. Since the truss members were assigned fixed cross-sectional areas, they were discarded as design variables.

**Symmetry Mapping**

The symmetry mapping was defined from the two symmetry lines shown in Fig. 6.23, imposing symmetry along the longitudinal center line and the transverse center line of each section.

![Symmetry mapping of the structure indicated by truss members with equal area $a_k = a_l = a_m = a_n$ (figure from Baandrup et al. (2019b))](image)

Figure 6.23: Symmetry mapping of the structure indicated by truss members with equal area $a_k = a_l = a_m = a_n$ (figure from Baandrup et al. (2019b))

The symmetry was introduced into the optimization problem by the linear equality constraint (6.7f), where $S$ was an index matrix, mapping the symmetry between the regions $k$, $l$, $m$, and $n$ in Fig. 6.23.

**Global Section Forces**

Since the top plate and hanger positions were fixed, both the distributed load $p$ and the hanger forces $P$ were applied to the model through the load vector $R$. However, this was not possible to do for the global section forces, since the material distribution in the optimized structure was unknown beforehand. Hence, it was not possible to apply the section forces to specific nodes on the end surface through $R$.

Instead, the section forces were defined as summations of node forces on the end surfaces, making the distribution unrestricted and given as a result of the optimization. Examples of summation for one end surface with $N_{end}$ nodes is shown for three of the six section forces in Fig. 6.24. Here, $F_{x,i}$, $F_{y,i}$, and $F_{z,i}$ are the node forces in node $i$, and $y_i$ and $z_i$ are the distances to the end surface center $(c_y, c_z)$.

The summations were linear functions of the design variables in $\beta$. They were introduced into the optimization problem by the linear equality constraint (6.7g), where $G$ was a section force equilibrium matrix and $G_0$ was a vector containing the assigned section force values.

When global section forces were applied through (6.7g), the equilibrium matrix $H$ and load vector $R$ were reduced to avoid contradicting equilibrium constraints.
Innovative Designs by Truss Optimization

\[ N_x = \sum_{i=1}^{N_{\text{end}}} F_{x,i} \]
\[ M_y = \sum_{i=1}^{N_{\text{end}}} F_{x,i} \cdot z_i \]
\[ M_t = \sum_{i=1}^{N_{\text{end}}} -F_{y,i} \cdot z_i + \sum_{i=1}^{N_{\text{end}}} F_{z,i} \cdot y_i \]

**Figure 6.24**: Contribution of node forces \( F \) from node \( i \) to the global section forces, with distance \((y_i, z_i)\) to the center of the end surface \((c_y, c_z)\). Examples of summation are shown for \( N_x, M_y, \) and \( M_t \) (figure partly from Baandrup et al. (2019b))

on the end surfaces. The reduced general equilibrium constraint was marked with subscript \( R \).

### 6.2.2 Results

The bridge girder optimized for all 14 load cases is presented in the following. Subsequently, a study of varying top plate thickness is presented.

Similar to the previously presented truss optimization results, a threshold was applied to illustrate the results. Here, two threshold values were applied to identify the main structure \((N_e, \text{Main} \mid a_e > 0.1 \max(A))\) and the detailed structure \((N_e, \text{Det} \mid a_e > 0.01 \max(A))\), respectively.

To report the weight of the structures, the unit **weight per meter girder** was used. From the objective function of volume \( V \) [m\(^3\)], the reported unit was given as \( W = V \cdot \rho/25 \text{ m} \) [ton/m], where \( \rho \) is the density of steel. For comparison, the weight of the Osman Gazi Bridge (without walkways) is 11.13 ton/m, where the top part (top plate and troughs) accounts for 5.25 ton/m (47.2%).

In addition to the main result presented here, further results are included in Section D.3 of Appendix D. The appendix contains structures optimized for all 14 load cases individually. Moreover, a study of the combined load cases 1, 5, 10, 13, and 14, equivalent to the topology optimization study, is included.

A single section of the optimized girder, visualized with the detailed structure of \( N_{e,\text{Det}} = 472 \) elements, is shown in Fig. 6.25. Three sections of the continuous girder are shown in Fig. 6.26. Here, for clarity, only the main structure with \( N_{e,\text{Main}} = 344 \) elements per section is shown. To enhance the design concept, notable members are highlighted by color. The weight of the optimized girder was found to 6.09 ton/m.

The structural design is governed by the large torsion and bending moments to be carried. Besides a torsion grid along the circumference of the domain, large longitudinal members in the lower part of the domain are introduced. Furthermore, the optimized design is very different from the conventional. Not only because truss
Innovative Designs by Truss Optimization

Figure 6.25: Bridge girder result optimized for 14 load cases (LC 1-14), $W = 6.09$ ton/m. Perspective view of detailed structure, $N_{e,Det.} = 472$. Boundary of domain indicated by thin black lines (figure from Baandrup et al. (2019b))

structures were studied, but also due to the overall structural concept. Thus, in general, loads are carried more directly to the hangers, and transferred to the heavy bottom grid, to be carried by efficient tension members. In this way, the material is better utilized compared to the conventional design, where loads are carried to the hangers via the perpendicular diaphragms.

Notably, two essential design principles were found. The first is the largest bottom members spanning across two sections (highlighted in red), hence from the hanger in one side, though the neighboring section, to the hanger on the opposite side. Thus, to efficiently support bending and torsional moments, the interaction between the sections reaches further than just the nearest neighboring sections. Here it should be noted that this principle emerges from the optimization, even though the mesh (Fig. 6.21) allows a span across just a single section. The second design principle is how the outer structure curves inwards between the hangers (highlighted in blue), and thus, does not utilize the entire domain. Interestingly, a similar result was seen from the topology optimization.

The total weight of 6.09 ton/m is equivalent to a weight saving of 45.3% compared to the conventional design of the Osman Gazi Bridge. This significant reduction in weight indicates a considerable undiscovered potential in search of alternative design concepts of bridge girders.

In the weight measurement of the optimized design, only truss members were included, thus, the contribution from connections were neglected. Furthermore, construction costs were not considered.

Knock-on Effects and Carbon Footprint

Similar to the methods in the previous chapter, the knock-on effects and carbon footprints are estimated, based on the weight reduction of 45.3%.

The 45.3% weight saving in structural steel in the girder (without walkways),
Innovative Designs by Truss Optimization

(a) Perspective and section view. Boundary of domain indicated by thin black lines

(b) Top view

Figure 6.26: Bridge girder result optimized for 14 load cases (LC 1-14), three spans, $W = 6.09 \text{ ton/m}$. Main structure without top plate, $N_{e,\text{Main}} = 344$ per section (figure from Baandrup et al. (2019b))

translates into a reduction in loads transferred to the cables of 30.5% after adjustment of walkways and surfacing. The knock-on effects from this reduction are given in material quantities in Table 6.1. A potential reduction of 69,000 tonnes of CO$_2$ is possible by use of the light-weight truss girder.

Influence of Top Plate Thickness

For the optimized structure above, the ratio between the top plate weight of 4.03 ton/m and the total weight of 6.09 ton/m is 66.2%. The comparable ratio of the
**Table 6.1:** Total quantities of steel, concrete, and CO$_2$ for the entire bridge. Savings indicated relative to the conventional design.

<table>
<thead>
<tr>
<th>Design</th>
<th>Steel</th>
<th>Concrete</th>
<th>*CO$_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Quantity</td>
<td>Saving</td>
<td>Quantity</td>
</tr>
<tr>
<td>Conventional design</td>
<td>69,000</td>
<td>—</td>
<td>175,000</td>
</tr>
<tr>
<td>Optimized truss structure</td>
<td>48,000</td>
<td>21,000</td>
<td>128,000</td>
</tr>
</tbody>
</table>

*Calculations only take account of material quantities, hence disregarding construction methods.

The conventional design is 47.2%. This difference indicates a conservative and too large choice of the equivalent top plate thickness of 20 mm. Hence, further potential weight savings may be achievable by reducing the plate thickness.

To investigate the influence of the chosen top plate thickness, a parametric study was carried out. In the study, the fixed equivalent plate thickness was varied from 10 mm to 25 mm by 5 mm intervals.

The results are summarized in Table 6.2, where the Osman Gazi Bridge is included for reference. The structures for Cases 1-4 are included in Section D.3 of Appendix D for completeness, where also a case with $t_{tp} = 5$ mm is included.

In Case 1 with $t_{tp} = 10$ mm, the ratio of 49.4% is close to the conventional design, and a weight saving of 63.2% was achieved. However, an equivalent top plate of 10 mm is most likely not capable of spanning the assumed 3 m while subject to traffic loads. With a thicker equivalent top plate of $t_{tp} = 15$ mm, the total weight of 5.11 ton/m is equivalent to a weight saving of 54.1%.

**Table 6.2:** Study of varying top plate thickness $t_{tp}$ for the Osman Gazi Bridge and four truss girder cases. Weight savings are given relative to the total weight of the Osman Gazi Bridge (table from Baandrup et al. (2019b)).

<table>
<thead>
<tr>
<th>Case</th>
<th>$t_{tp}$</th>
<th>Total weight</th>
<th>Weight saving</th>
<th>Weight of top plate</th>
<th>Weight of lower part</th>
<th>Ratio between top plate and total weight</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[mm]</td>
<td>[ton/m]</td>
<td>[—]</td>
<td>[ton/m]</td>
<td>[ton/m]</td>
<td>[—]</td>
</tr>
<tr>
<td>OGB$^*$</td>
<td>26$^+$</td>
<td>11.13$^b$</td>
<td>—</td>
<td>5.25</td>
<td>5.88</td>
<td>47.2%</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>4.09</td>
<td>63.2%</td>
<td>2.02</td>
<td>2.07</td>
<td>49.4%</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>5.11</td>
<td>54.1%</td>
<td>3.03</td>
<td>2.08</td>
<td>59.3%</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>6.09</td>
<td>45.3%</td>
<td>4.03</td>
<td>2.06</td>
<td>66.2%</td>
</tr>
<tr>
<td>4</td>
<td>25</td>
<td>7.09</td>
<td>35.3%</td>
<td>5.04</td>
<td>2.05</td>
<td>71.1%</td>
</tr>
</tbody>
</table>

$^*$Osman Gazi Bridge

$^b$Equivalent plate thickness of the orthotropic top deck (top plate and troughs)

$^+$Without walkways
6.3 Summary and Discussion

The main findings of the truss optimization studies applied in search of new bridge girder designs were that a weight saving of 45.3% could be achieved with potential further weight savings of up to 54.1% by using a thinner top plate. The significant weight savings were achieved with optimized truss girders, considerably different compared to the conventional design concept. In the optimized designs, the loads were carried more directly toward the hangers, and mainly by the heavy bottom grid to utilize efficient tension members. Thus, the substantial weight reduction and the new design principles stress the potential in alternative design concepts.

The achievable weight savings from applying truss optimization are significantly higher compared to the 28.4% achieved for the interpreted design in the previous chapter. This is despite the inclusion of constraints on stresses as well as global and local stability, leading to a more restricted solution space. However, the 28.4% saving was achieved from a crude interpretation and not from the original topology-optimized design. Thus, if the weight reduction (or stiffness increase) was evaluated for the pure topology optimization result, even larger potential weight savings might be quantified. This would, though, be at the expense of designs not being constrained against stability issues and yielding, which were taken care of in the optimized truss designs.

In continuation of the discussion above, in general, several challenges are left to be handled toward practical application of both the topology- and truss-optimized designs. The challenges are, to a great extent, related to the structural complexity of the designs, and include, among others, the already-mentioned constructibility and final construction costs. However, in the present work, these challenges were deprioritized since the girder self-weight is crucial to the construction of super-long suspension bridges, compared to, e.g., the construction costs, as discussed in Chapter 1. Despite the reduced focus, these practical matters are essential to consider in the next step toward application of the designs. Hence, in the next chapter, such challenges are discussed, and the potentials of the identified new design concepts are reviewed.
Chapter 7

Challenges toward Practical Application

In the three preceding chapters, three different methods of structural optimization have been applied in search of innovative design concepts of steel girders in cable-supported bridges: parametric optimization, topology optimization, and truss optimization. By the use of parametric optimization, the conventional design was optimized with a result of modest weight savings. However, from the two latter methods, significantly different designs were identified with substantially achievable weight savings. However, these advancements were mostly gained at the expense of a considerable increase in structural complexity, which introduces various challenges to handle toward practical application.

Concurrently with the previous presentations of results, some of the challenges were briefly pointed out. In this chapter, a more complete and combined discussion on the topic is given. Hence, the potentials of the identified designs are reviewed, considering the present challenges.

Simultaneously with the application of the various optimization methods on the specific bridge case, the general possibilities of using numerical optimization methods in civil engineering have been illustrated. Additionally, several of the challenges toward practical application of the bridge girder designs are considered to be general in structural optimization, where some were briefly discussed in Section 2.1.

In the following sections, the general challenges are discussed concurrently with the specific challenges regarding the practical application of the optimized bridge girder designs found in the present studies. Thus, the challenges to be handled on a general level are exemplified by the results of the bridge girder subjected to optimization.

Notably, three main challenges are identified and discussed: structural robustness, manufacturing (or constructibility), and construction costs. The order in which the challenges are introduced below is, to a great extent, the order of how they should be addressed. Hence, if the structural robustness of a design is un-
Challenges toward Practical Application

acceptable, it does not make sense to discuss manufacturing hereof. Similarly, if manufacturing is infeasible, it does not make sense to consider possible construction costs.

One possible approach to accommodate all three challenges is to make interpretations of the optimized structures, e.g., as demonstrated in Section 5.2. This approach is discussed briefly at the end of this chapter.

In general, the identified girder designs are considered a first iteration. Thus, besides handling the challenges discussed here, follow-up studies must be conducted. Such studies involve verification against all relevant load cases, dynamics, buckling, and a thorough fatigue assessment.

The content of the present chapter is partly based on input from discussions with bridge engineers during workshops, based on presentations of the present project and results. The workshops took place during the last two years of the project and were conducted in COWI bridge departments in Denmark, the United Kingdom, and Canada.

7.1 Structural Robustness

The first challenge to consider is structural robustness. The term refers here to the robustness against, e.g., buckling, fatigue, and other load cases than those included during the optimization.

The issues often arise from simple optimization problems, where some factors relevant to robustness have been disregarded. An example, discussed further below, is neglecting constraints on stability, leading to slender designs, which may be prone to buckling. Hence, the optimization algorithms benefit from the disregarded constraints toward the goal of minimizing the objective function.

Indeed, it is of interest to include all relevant constraints to ensure robust designs evolving directly from the optimization. However, this is rarely possible due to limitations in either computational resources, implementation, or the theoretical foundation. Thus, the formulation of simple optimization problems may be the only option when applying numerical optimization methods.

Bridge Girder Designs

The final design after the parametric optimization in Section 4.3 is largely considered robust since no significant changes occurred to the overall geometry. However, since the fatigue assessment during the optimization was simplified, the final design may not be sufficient against all fatigue issues. Particularly, the reduced number of fatigue details (the three considered most critical in the original design) is of interest to readdress. As mentioned in Section 4.4, the low trough thickness may indicate fatigue issues near the troughs. Hence, to ensure the structural robustness of the design after parametric optimization, it is relevant to include additional fatigue details in potential further studies.
The main challenge concerning structural robustness of the pure topology-optimized design in Section 5.1 is the absence of stability constraints in the optimization problem. For large-scale topology optimization, it is not yet possible to include such constraints. However, for small problems, various procedures are being developed, see, e.g., Ferrari et al. (2018) where topology optimization with a lower bound on the critical load factor, as computed by linearized buckling analysis, is reviewed. The results of the study can be seen in Fig. 7.1.

From the topology-optimized structure (shown in Fig. 5.3), many thin plates are seen as well as very slender truss members. Thus, buckling will inevitably be a problem without further post-processing of the structure. One option is to make interpretations of the design, as done in Section 5.2, and as discussed further at the end of this chapter.

The interpreted design (Fig. 5.4) is, to a certain extent, more robust due to the similarities with the conventional girder design. However, the varying span lengths of the top deck, introduced by the curved diaphragms, is a challenge to overcome to avoid unacceptable high stresses.

The structural robustness of the optimized truss girder identified in Section 6.2 is dependent on the assumption of a top plate spanning up to 3 m in two directions to distribute traffic load to the underlying truss. The specific design of this top plate was not considered. However, the significant weight savings in sight is an apparent motivation for continued research on this matter.

Figure 7.1: Topology optimization including a lower bound on the fundamental buckling load factor $P_c$ (figure from Ferrari et al. (2018))
7.2 Manufacturing Challenges

The second challenge to handle is manufacturing, also referred to as constructibility. Here, these terms refer to the feasibility of manufacturing the designs with currently available construction techniques. Only the constructibility itself is considered, with no thoughts on the total costs, which are addressed in the next section.

For decades, manufacturing has been one of the key challenges in structural optimization, and particularly within civil engineering, it still is. Hence, this is usually a point of criticism, since the often complex structures evolving from numerical optimization are either very difficult or maybe even impossible to manufacture. Additionally, in civil engineering, the usually large size of the structural components increases this challenge further.

One approach to accommodate the challenge is to rationalize the optimized designs to encourage simpler structures. However, this will mostly be at the expense of an aggravated objective value. One example of rationalization was given in Section 6.1.2, where penalty functions were introduced to modify the objective function. Another method was recently proposed in Fairclough et al. (2019), where rationalization constraints were introduced gradually, e.g., by removing complex connections or introducing standardizing components. The results of the study can be seen in Fig. 7.2.

![Figure 7.2: Rationalization of a beam truss structure. Here, the self-weight decreases as the structural complexity increases (reproduction of figure in Fairclough et al. (2019))](image-url)
Another option is to rely on new manufacturing methods. Thus, to be able to fabricate the highly detailed structures, alternative construction techniques may be required. Here, the progressive development within additive manufacturing (3D printing) is very suitable, which particularly is the case for topology optimization due to the unrestricted design freedom.

In recent years, the coupling of the two fields has been subject to extensive research, see, e.g., Clausen (2016); Zegard and Paulino (2016); Lange and Feucht (2019); Vantygheem et al. (2019); Plocher and Panesar (2019); Liu et al. (2019). However, additive manufacturing is still today, mainly feasible for smaller structures, and less so for the large-scale components often used in civil engineering.

The method is though advancing for larger structures. One example is given in Yuan et al. (2018), where a 14-m-long pedestrian bridge was printed in plastic. Another example is the Dutch company MX3D, which printed a 10-m-long steel bridge in 2018. The bridge can be seen in Fig. 7.3.

For a general overview of 3D printing in construction, the reader is referred to Buchanan and Gardner (2019).

**Bridge Girder Designs**

During the parametric optimization, no significant changes occurred to the overall geometry, as mentioned above. Hence, no particular manufacturing challenges are
Challenges toward Practical Application

considered in this case.

The complexity of the topology-optimized designs makes construction infeasible with current manufacturing methods, as discussed in Section 5.2. However, the results are highly beneficial as inspiration for new design concepts. Thus, the interpreted design (Fig. 5.4) is closer to being constructible due to the geometric complexity resembling the conventional design. However, as mentioned in the previous section, the challenge of uneven span lengths of the top deck must be addressed.

Possible manufacturing challenges regarding the optimized truss girder include a large number of slender members. These members exist to retain global stability, however, manufacturing hereof may be difficult. Furthermore, the joints connecting a large number of members may account for another fabrication challenge. Thus, similar to the topology-optimized designs, post-processing is necessary toward practical application.

7.3 Construction Costs

Finally, the total construction costs are addressed, hence, the compiled cost of materials and fabrication (e.g., labor and energy). This aspect is important to consider when working with structural optimization since the manufacturing costs easily outweigh the savings in material costs. This is particularly relevant in civil engineering, where material costs are often low compared to manufacturing costs. Moreover, in civil engineering, the selection criterion of bid evaluations is mostly total construction costs, and never carbon footprint, and thus never solely material quantities.

Despite the apparent benefits of including construction costs directly into the optimization problem, e.g., by defining an objective function of total construction costs, this is not trivial. Firstly, it is complicated to estimate the fabrication costs of the optimized designs evolving from, e.g., topology or truss optimization. Secondly, if it were possible to estimate, it would be challenging to make a general quantification needed to establish a continuous objective function. This has only been investigated in a few works, including Asadpoure et al. (2015), where fabrication costs were incorporated into topology optimization of discrete structures and lattices. Contrary to the often very complex or unknown construction-costs functions, the weight (or volume) is, in most cases, easily computed, why this is often the objective of choice.

In continuation of the preceding discussions, the possibility of new manufacturing methods is also relevant when considering costs. Thus, with the significant manufacturing challenges to be handled initially, the construction costs of complex optimized structures should not be quantified based on conventional fabrication techniques.
Bridge Girder Designs

Concerning cable-supported bridges, the question of construction costs is particularly interesting. As mentioned multiple times, for super-long suspension bridges, self-weight, and thus material quantities, is the decisive factor compared to construction costs. Hence, in this particular case, when it becomes a question of constructibility, self-weight is governing, and total construction costs are thus less relevant. Nevertheless, the challenge is still of interest to address briefly.

During the parametric optimization studies, it was, in fact, possible to establish a function estimating the total girder price. Thus, due to the fixed overall geometry, it was possible to quantify welding and manufacturing of the different steel plates. In this case, no distinct differences were though observed between the weight and price functions. Nevertheless, this observation is interesting since the negligence of construction costs often is subject to criticism. In the case of gradient-based parametric optimization, the difference in the result was negligible.

The quantification of the construction costs of both the topology-optimized girder and the truss girder is problematic due to the manufacturing challenges pointed out above. Hence, construction costs are not considered further for these structures.

7.4 Interpretation of Optimized Structures

As mentioned in the introduction of this chapter, a possible approach to accommodate all of the three discussed challenges is an interpretation of the optimized structures. However, the interpretation of the optimization results will, by nature, be less optimal than the starting point. Nevertheless, this approach is highly beneficial for the use of structural optimization within civil engineering.

The benefits of interpretation were demonstrated for the topology-optimized design in Section 5.2. Here, the main features of the complex optimized girder were interpreted into a simple and feasible design concept. Similarly, the main structural features of the truss girder, identified and discussed in Section 6.2.2, may be interpreted to establish a feasible girder design. However, this task is left for future work.

By interpretation, the possible challenges of structural robustness can be reduced, since factors not possible to include during initial optimization may be included during the subsequent interpretation phase. Furthermore, both manufacturing and construction costs challenges may be accommodated by interpreted designs that are constructible by well-known techniques. Thus, the main features of otherwise infeasible structures may be interpreted into simple, feasible designs with a similar level of geometric and manufacturing complexity as well-known civil engineering principles.

By the use of interpretation, numerical optimization is, in general, mainly beneficial in the initial design phase, where the main structural concept is established.
Challenges toward Practical Application
Chapter 8

Conclusions and Recommendations for Further Work

In the present work, three different methods of numerical optimization were applied with the aim of identifying innovative designs of steel girders in cable-supported bridges.

The focus was on reducing the girder self-weight, partly to increase possible span-lengths. Additionally, a decrease in girder self-weight will lead to reduced material quantities in the entire bridge, and thus, lowering the environmental impact. Furthermore, the goal was to identify more efficient load-carrying principles and, thus, more material-efficient structures.

In addition to the main objectives, restated above, a secondary objective was to illustrate the potentials of applying numerical optimization methods in the civil engineering industry.

In the first section below, the main conclusions of the work are presented. Subsequently, recommendations for further work are briefly discussed.

8.1 Conclusions

The main conclusions to each of the primary chapters, Chapters 4-7, are presented in the following. Finally, concluding remarks on the objectives of the thesis are given.

Parametric Optimization of the Conventional Design Concept

In Chapter 4, parametric optimization was applied to pursue possible weight savings in the conventional girder design concept. Here, a multiscale FE-model with sophisticated boundary conditions of the Osman Gazi Bridge girder was established to form the basis of the studies.
Conclusions and Recommendations for Further Work

Firstly, the FE-model was used in parameter studies to investigate the behavior of the orthotropic girder, when typical design parameters were altered individually. From the studies, the diverse effects on the design from varying design parameters were observed. Furthermore, the studies verified that the FE-model was suitable for gradient-based optimization due to all functions being smooth and continuous.

Secondly, the FE-model was the basis of gradient-based parametric optimization. Here, the primary objective was to minimize weight while fulfilling constraints, including fatigue stresses, a deflection criterion, and a Eurocode stiffness requirement. The established optimization framework facilitated the optimization of up to 10 design variables simultaneously.

From the parametric optimization studies, weight savings in the range of 5.8%-13.8% were achieved, mainly by using thinner plates and narrower troughs. Here, the maximum weight saving with all current Eurocode requirements fulfilled was found to 5.8%.

The achievable weight savings indicated the advantages of applying gradient-based parametric optimization. The benefits of the numerical optimization were particularly noticeable when the problem complexity increased by the increasing number of design variables. Here, the optimization algorithm was capable of utilizing the high complexity of multiple variables toward a better design. With the increasing use of parametric designs in the civil engineering industry, an extension to the demonstrated parametric optimization is apparent.

Despite the potentials of applying parametric optimization, and the achieved weight savings, the results were considered modest, given the various other inherent challenges in the conventional design concept. Furthermore, the results were considered an upper bound for the conventional design, and it was concluded that this design concept was limited in further development, without altering the structural concept. Although the study was of a specific bridge, the results and the above conclusions are considered to be generally applicable.

Innovative Designs by Topology Optimization

In Chapter 5, topology optimization based on continuum structures was applied as the first step in search of innovative girder designs. The goal was to reduce the self-weight significantly and to identify more efficient load-carrying principles.

Initially, the unrestricted design freedom from giga-scale procedures was utilized together with a minimum of constraints to identify a lower bound of the optimized designs. From the optimization of the girder model discretized into 2.1 billion finite elements, a highly complex structure evolved. The design was significantly different from the conventional, as no perpendicular diaphragms or orthotropic decks were seen. Instead, a number of double-curved diaphragm-like panels and trusses were observed to transfer traffic loads directly to the hangers. Additionally, a longitudinal plate-like structure appeared as an added support in the region of the hangers and the curved diaphragms.

Subsequently, based on the new insights gained from the intriguing, but com-
plex and infeasible optimized structures, a simpler and feasible interpreted design was established. Hence, the interpreted design was derived with a level of geometric and manufacturing complexity resembling the conventional. With the interpreted design, a weight saving of 12.7% was achieved, compared to the conventional design.

Finally, the interpreted design was subject to simple parametric optimization to identify the full potential. After the parametric optimization, a total weight saving of 28.4% was achieved. Thus, the application of the giga-scale topology optimization procedure, followed by a quick interpretation and a simple parametric optimization, led to weight savings significantly larger than what was possible without altering the structural concept in Chapter 4.

The identified adaptions to the conventional design would hardly have been found without the topology optimization results. Hence, the results illustrate the benefits of applying topology optimization in the early phase of civil engineering design development, regardless of the complicated structures initially arising from the procedure.

The 28.4% saving in girder self-weight translated into a total reduction of material quantities in the entire bridge of 16%-19% and a reduction in CO$_2$ emissions of 18%.

Innovative Designs by Truss Optimization

In Chapter 6, truss-layout optimization was used as a design tool in the continued search of innovative girder designs. The simplicity of the truss structures allowed for the formulation of optimization problems with constraints on stresses as well as global and local stability. Similar to the use of topology optimization, the goal was to reduce the self-weight significantly and to identify more efficient load-carrying principles.

Initially, a truss optimization framework based on finite element limit analysis was developed, including the constraints mentioned above. In the proposed method, the global stability constraint was solved by the introduction of a semidefinite constraint, whereas local stability was handled through the critical Euler buckling load.

Subsequently, the framework was extended toward practical application that included large-scale methods, reduction of truss complexity, and methods to search the concave solution-space. Here, the large-scale methods were able to solve a truss structure of 32,640 elements while subject to two load cases. Furthermore, a large hidden potential in searching the solution space of concave optimization problems with multiple local minima to find significantly better solutions was identified.

Finally, the truss optimization framework was applied in the context of the thesis objectives. Since truss elements were used, the basis of the optimization was an assumption of a continuous top plate able to carry the surface loads to the underlying truss.

A significant weight saving of 45.3% was achieved with an optimized truss
Conclusions and Recommendations for Further Work

girder significantly different compared to the conventional design concept. From the design, new and efficient load-carrying principles were observed. Here, the main structure consisted of a torsion grid along the circumference of the domain, as well as large members in the bottom to carry bending moments. Furthermore, load paths were generally oriented more directly toward the hanger attachment and going through the bottom part of the girder to benefit from tension members.

Additionally, the possibility of further weight savings up to 54.1% was observed through a simple parameter study of the assumed top plate thickness.

The 45.3% saving in girder self-weight translated into a total reduction of material quantities in the entire bridge of 27%-30% and a reduction in CO₂ emissions of 30%.

Challenges toward Practical Application

In Chapter 7, the challenges toward the practical application of both the numerical methods in general as well as the optimized bridge girders were addressed. Particularly three challenges were considered, including structural robustness, manufacturing, and total construction costs.

The challenge of structural robustness was observed as a result of optimization problems that did not include all relevant constraint functions. However, this is often not possible due to limitations in either computational resources, implementation, or the theoretical foundation. Such challenges were observed in all parts of the presented studies. Notably, the negligence of stability during topology optimization was apparent in thin plates and slender truss members.

Manufacturing difficulties and increased total construction costs are challenges arising from often very intricate designs evolved during structural optimization. Usually, both challenges are points of criticism toward the practical application of numerical optimization, particularly within civil engineering. Often the pure optimized structures are, in fact, infeasible due to cost and manufacturing considerations. However, with the progressive and promising development within additive manufacturing, the challenges of both fabrication and total construction costs may be solved or minimized.

Besides additive manufacturing, a possible approach to accommodate all three challenges was found in the interpretation of the intricate optimized designs. This method was already demonstrated in Section 5.2 based on the topology optimization results, as discussed above. The method benefits of extracting the main structural features of the optimized designs from where simple and feasible designs are found.

Concluding Remarks on the Objectives of the Thesis

Possible further weight savings in the conventional girder design concept were identified through the application of numerical parametric optimization. The most
Conclusions and Recommendations for Further Work

A significant achievable weight saving of 13.8% was considered an upper bound, stressing the limitations in further development.

Innovative and significantly different design concepts were identified through the use of topology optimization and truss-layout optimization, respectively. The self-weight of the identified designs was considerably below the conventional design, with savings in the range of 28.4%-45.3%. These savings translated into total savings in material quantities in the range of 16%-30%, and a reduction in CO$_2$ emissions in the range of 18%-30%.

Furthermore, from the identified designs, more efficient load-carrying principles were observed, and thus more material-efficient structures.

Finally, a systematic demonstration of the practical application of various structural optimization methods within the field of civil engineering, including discussion of the related challenges, was given.

Despite the apparent challenges still to be addressed, the newly identified design principles and possible weight savings emphasize the potential of using significantly different design concepts. Thus, the considerable weight savings to be achieved may close the gap toward super-long cable-supported bridges and reduce material quantities, and thus reduce the environmental impact.

8.2 Recommendations for Further Work

Recommendations for further work are given in the following. The recommendations are based on the research presented in the thesis and given as suggestions on possible next steps toward the main goals to be achieved in the design of future cable-supported bridges.

- In the formulation of the topology optimization problem, it is desirable to include constraints on stability, as previously discussed. However, since this is not yet possible for large-scale topology optimization, new theoretical formulations are needed to be developed.

- It is of interest to interpret the optimized truss girders to accommodate the present challenges toward application. Hence, similar to the interpretation of the topology optimization results, it is relevant to establish a feasible design based on the structural features of the optimized truss girders. Additionally, to implement the truss girders, a specific design of the assumed top plate must be identified.

- In connection with the above suggestions, in general, it is relevant to perform further detail-oriented optimization studies of the interpreted designs with a focus on constructibility and total costs. Moreover, a detailed assessment of the fatigue performance of the identified design concepts is relevant to carry out.
Conclusions and Recommendations for Further Work

- A natural next step in the truss-based optimization is the inclusion of node coordinates as design variables, as studied in Vestergaard (2019). From such an extension, larger design freedom is gained, and thus, better solutions with lower self-weight and reduced structural complexity. Furthermore, with the larger design freedom, the number of elements in the ground mesh may be decreased without reducing the quality of the solutions. With the introduction of node coordinates as design variables, the base problem becomes concave. However, the concavity can be handled through a linearization similar to how the concave local stability constraint was implemented. Furthermore, also the developed methods for searching the concave solution space are valuable in this regard.

- Finally, one of the fatigue issues in the conventional design may be reduced by the utilization of structural optimization methods. Thus, to accommodate the significant fatigue issues related to the trough cutouts in the diaphragms, it is of interest to apply topology optimization methods. By this approach, new and better cutout shapes may be identified to reduce the fatigue stresses.
Bibliography


BIBLIOGRAPHY


BIBLIOGRAPHY


BIBLIOGRAPHY


BIBLIOGRAPHY


Part II

Appended Papers
"Optimization of orthotropic girders in cable supported bridges by parametric studies"

M. Baandrup, P.N. Poulsen, J.F. Olesen & H. Polk

Conference proceeding in: *IABSE Congress 2019, New York City: The Evolving Metropolis, 2019*
Optimization of orthotropic girders in cable supported bridges by parametric studies

Mads Baandrup  
Ph.d. student  
Department of Civil Engineering, Technical University of Denmark  
Department of Major Bridges International, COWI A/S  
2800 Kgs. Lyngby, Denmark  
mjba@byg.dtu.dk  

Mr. Baandrup is M.Sc. in civil engineering with specialization in structures and optimization. He is an industrial Ph.D. student working on the optimization of bridge girders.

Peter Noe Poulsen  
Associate Professor  
Department of Civil Engineering, Technical University of Denmark  
2800 Kgs. Lyngby, Denmark  
pnp@byg.dtu.dk  

Mr. Poulsen’s field of research is within computational structural engineering.

John Forbes Olesen  
Associate Professor  
Department of Civil Engineering, Technical University of Denmark  
2800 Kgs. Lyngby, Denmark  
jof@byg.dtu.dk  

Mr. Olesen’s field of research is within computational structural engineering.

Henrik Polk  
Technical Director  
Department of Major Bridges International, COWI A/S  
2800 Kgs. Lyngby, Denmark  
hpo@cowi.com  

Mr. Polk is Technical Director and has over 30 years of experience working with orthotropic steel girders.

Contact: mjba@byg.dtu.dk

1 Abstract

For the last six decades closed-box orthotropic steel girders have been widely used in cable supported bridges due to their simple but useful structural concepts. Several numerical parametric studies were previously carried out in order to investigate inherent fatigue stress problems and in general, to improve the bridge girder designs. However, often such studies have been carried out with over-simplified finite element models, especially where boundary conditions have been challenging. In the present work, an advanced multi-scale FE model of a suspension bridge is established with sophisticated boundary conditions applied to a local parametric sub-model of a bridge girder. Thus, the model accommodates realistic support conditions. With this sub-model, a parametric study of the usual design parameters is carried out with focus on fatigue and a Eurocode stiffness requirement. The study reveals trends and correlations for the varying design variables. Finally, the parametric sub-model is utilized in an automatic gradient-based optimization of multiple design variables simultaneously with the goal of minimizing weight. The methods allow bridge engineers to push material utilization to its limits by giving new insight into the effect of changing design parameters.

Keywords: Orthotropic bridge girders; Cable supported bridges; Optimization, Parametric Study.
2 Introduction

Since the 1960s [1] the orthotropic steel girder design has been of great use in large cable supported bridges. Over the recent decades several numerical parametric studies were carried out to study and optimize the design concept [2]–[4]. However, the support conditions are often too simplified, and either too flexible [2], [4], or too stiff [3]. In the present work, the challenge of applying realistic support conditions to the studied continuous bridge girder was addressed by the use of an advanced multi-scale FE model, where displacement fields were transferred from a full global model to a simplified local sub-model. The sub-model was subsequently used in four parametric studies of the usual design parameters and a gradient-based parametric optimization.

3 Methods

An advanced multi-scale FE model of the Ozman Gazi suspension bridge (Turkey) was developed. All dimensions and other input were based on the detailed design done by COWI A/S. The bridge deck is a closed steel mono-box girder with orthotropic deck and truss diaphragms. The model is similar to the one used in [5], where all details are available. The software Abaqus was used for modeling and the parametric model was scripted in Python.

Figure 1 Schematic overview of multi-scale model with terminology of sub-levels. Top view: global model with main dimensions. Middle view: local model with main dimension and indication of displacement field boundaries (dashed edges). Bottom view: detail levels of local model with respectively the major orthotropic shell/beam elements (general mesh size: $h_c = 1$ m) and minor detailed shell part (general mesh size: $h_f = 0.1$ m) [5]
3.1 Multi-scale FE model

The multi-scale model is seen in Figure 1. A global beam model with five sections of shell elements was created to perform initial calculations (see [6] for details). During the parametric studies and optimization, only a local model of a single section (25 m) of the girder was used in order to reduce the model size and thus calculation time. As the local model was cut out from a continuous girder, the boundary conditions were a challenge. To accommodate realistic support conditions on the local model, the displacement fields from the global model were applied to the cut edges (see Figure 1 middle view), hence no dof’s were fixed, contrary to what is often the case. After applying the “global” displacement fields, the local model could be solved with the same internal loads, and the “global” solution could be recovered. The sub-model feature in Abaqus enabled the above solution method.

The local model contained orthotropic shell/beam elements, and a minor detailed part where fatigue stresses were calculated (see Figure 1 bottom view).

3.1.1 Parametric model

The geometry of the minor detailed shell part was the target of the study - see Figure 2 top view for terminology - and was thus made parametric with the usual design variables shown in Figure 2 middle view and listed in Table 1.

<table>
<thead>
<tr>
<th>Design parameter</th>
<th>Symbol</th>
<th>Initial value</th>
<th>Final value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diaphragm distance</td>
<td>( L_d )</td>
<td>5,000</td>
<td>5,000</td>
</tr>
<tr>
<td>Top plate thickness</td>
<td>( t_{tp} )</td>
<td>14</td>
<td>13.9</td>
</tr>
<tr>
<td>Trough thickness</td>
<td>( t_{tr} )</td>
<td>8</td>
<td>6.0</td>
</tr>
<tr>
<td>Trough height</td>
<td>( h_{tr} )</td>
<td>360</td>
<td>367.4</td>
</tr>
<tr>
<td>Trough width top</td>
<td>( w_{tr,t} )</td>
<td>300</td>
<td>300.0</td>
</tr>
<tr>
<td>Trough width bottom</td>
<td>( w_{tr,b} )</td>
<td>180</td>
<td>174.4</td>
</tr>
<tr>
<td>DB web height</td>
<td>( h_w )</td>
<td>1,000</td>
<td>1000.4</td>
</tr>
<tr>
<td>DB web thickness</td>
<td>( t_w )</td>
<td>14</td>
<td>12.9</td>
</tr>
<tr>
<td>DB flange width</td>
<td>( w_f )</td>
<td>250</td>
<td>182.8</td>
</tr>
<tr>
<td>DB flange thickness</td>
<td>( t_f )</td>
<td>12</td>
<td>6.0</td>
</tr>
</tbody>
</table>

3.2 Output functions

The local parametric model was used to calculate different output utilized in the following studies.

3.2.1 Fatigue details

Three critical fatigue details were identified based on the experience from the detailed design work. The details are seen in Figure 2 bottom view, where FD1 is the diaphragm cut-out, FD3 is the trough to top plate welding, and FD4 is the diaphragm to top plate welding. The fatigue stresses were calculated according to the hot spot method, and the fatigue stress criteria were simplified to a maximum allowable stress for each detail; FD1: 65 MPa, FD3: 40 MPa, and FD4: 38 MPa. As a single load case was superior in the contribution of fatigue stresses, only vehicle 5A-H from the UK annex to EN1993-1-9 FLM4 was included in the calculations. Load positions and fatigue stress calculations are similar to the methods in [5], where detailed information can be found.

3.2.2 Eurocode stiffness requirement

The Eurocode stiffness requirement (EN1993-2-2007 Fig. C.4) to the minimum stiffness of...
longitudinal stiffeners relative to the diaphragm distance was included in the studies, and designated EN stiff. req. in the following.

3.3 Optimization framework

To perform gradient-based optimization, an optimization framework was developed in Matlab, where the basis was a general non-linear inequality constrained optimization problem given as

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad f(x) \\
\text{s.t.} & \quad c_j(x) \leq 0, \quad j = 1 \ldots m \\
& \quad l_i \leq x_i \leq u_i, \quad i = 1 \ldots n
\end{align*}
\]  

(1)

where \( f(x) \) is the objective function (girder weight pr. meter), \( c_j(x) \) the non-linear constraint functions (fatigue stresses and EN stiff. req.), and \( x \) the design variables with lower and upper bounds; \( l_i \) and \( u_i \) on variable \( x_i \).

The optimization was based on numerical finite difference gradients and sequential quadratic programming (SQP) (solved with the commercial solver KNITRO). The optimization methods are similar to the ones in [5].

4 Results and discussion

The local model was applied in four parametric studies followed by an automatic gradient-based parametric optimization.

4.1 Parametric studies

Four parametric studies were carried out on the following parameters; diaphragm distance \( l_d \), top plate thickness \( t_{tp} \), trough thickness \( t_{tr} \), and a scaling of the total trough stiffness, hence a simultaneous scaling of the parameters \( t_{tp}, t_{tr}, w_{tr,t}, w_{tr,b}, h_{tr} \). The results of the four parametric studies are seen in Figure 3. The three fatigue stress functions are normalized with their maximum allowable stresses and the Eurocode stiffness requirement is normalized such that unity is the upper allowable limit. Thus, values below one are acceptable in the design. Furthermore, the girder weight is plotted as function of the variables.

In the first case the effect of a change in the diaphragm distance is studied. The fatigue stresses in Detail 3 and 4 are, as expected, close to constant, whereas stresses in Detail 1 increase almost linearly with increasing diaphragm distance. Additionally, the EN stiff. req. approaches the limit rapidly, and
passes near $L_d = 5.4 \text{ m}$, which thus is the upper limit when minimizing the weight by only changing $L_d$.

In the second case all three fatigue stresses are affected significantly by the changing top plate thickness. In particular Detail 3 and 4 near the top plate increase rapidly just by a minor decrease in thickness. The minimum thickness is thus already found, if all other parameters were fixed.

In the third case the stresses in Detail 1 and 4 decrease almost linearly with increasing trough thickness, contrary Detail 3, where stresses increase. The trough thickness can be reduced to 7.5 mm before FD4 is passed, if all other parameters were fixed.

In the last case all parameters defining the trough stiffness are changed simultaneously. The effect on the EN stiff. req. is significant with a very steep slope. Also, the change in fatigue stresses differs from the previous cases, as all three effects follow parabola-like curves. However, this behavior is very closely connected to the width and position of the local wheel load in vehicle 5A-H (see [5]).

All four parametric studies show the diverse effects on the design from varying design parameters. However, for all cases the possible weight reduction is limited, when only changing the parameters individually.

### 4.2 Parametric optimization

A gradient-based parametric optimization, with all 10 design variables being optimized simultaneously, was carried out. The objective was to minimize the girder weight under the constraints of the three fatigue stress functions and the EN stiff. req. The result of the iterative optimization process is seen in Figure 4, and the design parameter values after optimization are seen in the last column of Table 1. It is noted that only $t_{tr}$ and $t_r$ reached their lower bounds of 6 mm, whereas the rest of the parameters settled within the bounds as defined in Eq. (1). The lower bound of 6 mm was chosen for all plate thicknesses due to Eurocode requirements. Additionally, it is seen in the middle plot that the optimization result was constrained by the fatigue details 1 and 4 reaching their maximum stresses.

![Figure 4 Optimization results of 10 design variables with minimization of weight as objective function, and fatigue and Eurocode stiffness requirement as constraint functions. The constraint functions are shown on the form of Eq. (1), hence with a maximum allowable limit of zero. The design variables are shown normalized to their initial values.](image)

From the initial weight of 12.8 ton/m a weight reduction of 5.8%, equivalent to 0.74 ton/m or 1,993 ton in total for the 2,682 m long bridge girder, was achieved. The weight reduction was mainly achieved from decreased trough thickness and less from the other parameters directly. However, the change in the other parameters allowed the trough thickness to exceed the 7.5 mm found as the minimum in the parametric study. Thus, this outcome indicates the potential of applying gradient-based optimization in the design process.
5 Conclusions

A multi-scale FE model with sophisticated boundary conditions was developed to accommodate realistic support conditions. The model was applied in four parametric studies, which showed the diverse effects on the design from variations in different design parameters. Finally, a gradient-based optimization was carried out, where a weight reduction of 5.8% was achieved mainly by decreasing trough thickness. It is believed, that the insights from these studies enable bridge engineers to better understand and push the designs to their limits.

6 Acknowledgement

The project is financed by COWI Foundation grant C-131.02 and Innovation Fund Denmark grant 5189-00112B.

7 References


"Parametric Optimization of Orthotropic Girders in a Cable-Supported Bridge"

M. Baandrup, P.N. Poulsen, J.F. Olesen & H. Polk

Published in: Journal of Bridge Engineering, 2019
Parametric Optimization of Orthotropic Girders in a Cable-Supported Bridge

Mads Baandrup1; Peter Noe Poulsen2; John Forbes Olesen3; and Henrik Polk4

Abstract: In the last six decades, closed-box orthotropic steel girders have been widely used in cable-supported bridges. Several parametric studies were previously carried out to reduce inherent fatigue stress problems and to generally improve bridge girder designs. However, in most cases, only one or two parameters were studied simultaneously; hence, the full potential of orthotropic girders is not achieved. In the present work, a multiscale finite-element (FE) model of a suspension bridge is established with sophisticated boundary conditions applied to a local parametric submodel of a bridge girder. With this local model an automated gradient-based parametric optimization is carried out with the goal of minimizing the weight and price of the girder. It is possible to simultaneously optimize several design variables and fulfill constraint functions on fatigue stresses, deformation, and buckling. The results show potential weight savings of 6%–14% and price savings of 9%–17%, mainly found by using thinner plates and narrower troughs. Besides the explicit savings, the results indicate the potential for applying gradient-based optimization in civil engineering designs. DOI: 10.1061/(ASCE)BE.1943-5592.0001499, © 2019 American Society of Civil Engineers.

Introduction

Since the 1960s (Wolchuk and Harris 1959), the orthotropic steel girder design has been of great use in large cable-supported bridges. During the last six decades, the design concept has proven to have many advantages compared to the alternative of using truss girders but also many challenges, in particular fatigue problems (Wolchuk 1990; Fisher and Dexter 1997). The design concept of the closed steel box with orthotropic plates has been under continuous development and subject to extensive research in search of a more optimal design (Wolchuk 1999). The search for an optimized design is continued in this study, where a gradient-based approach was applied in a parametric optimization of the usual design variables.

Previously, several parametric studies of orthotropic decks were completed. In Oh et al. (2011), followed up by Oh and Bae (2013), a parametric study of the influence of cross-beam height, thickness, and shear area and deck plate thickness with a focus on fatigue stresses in the cutout around troughs was carried out. The main findings were that increasing the height, thickness, and shear area of the cross-beam was able to reduce the maximum stresses, whereas the thickness of the deck plate was not closely related to the stresses in the cutout. Similarly, Connor (2004) studied the influence of the cutout geometry on stresses at the trough-to-diaphragm connections. Here a parametric study of certain variables in the cutout showed that the recommendations in the 2002 AASHTO LRFD Bridge Design Specifications produced increased stresses in fatigue-sensitive regions (AASHTO 2002). Furthermore, several other parametric studies have been conducted (De Backer et al. 2006; De Corte and Van Bogaert 2007; De Corte 2009; Tang 2011; Fettahoglu 2015, 2016) that showed possible improvements to the orthotropic steel deck.

A common feature in all of these studies is that only one or two design parameters were studied at a time. Such an approach is very similar to the usual design process carried out by engineers when designing orthotropic steel girders, which is a natural consequence of the increasing complexity arising when several design parameters are changed simultaneously. The sum of effects and consequences on, for example, stresses in fatigue details and structural stiffness is often too complex to comprehend. Thus, it is of great interest to study the effect on design when optimizing multiple design variables simultaneously, which is possible when applying automatic gradient-based optimization methods. Previous examples of gradient-based parametric optimization are seen in Mrsa and Medic (1996), Peng et al. (2005), and Ding et al. (2010).

In the present study, a gradient-based parametric optimization of an orthotropic bridge girder in a cable-supported bridge was conducted. The basis of the optimization was Turkey’s 2,682-m-long Osman Gazi Bridge, opened in July 2016. The aims of the study were mainly to optimize the design parameters of the bridge girder and subsequently to identify trends in the various design variables leading to more optimal structures. A tertiary goal was to prove the concept and possibilities of applying gradient-based optimization in the design of large civil engineering structures—for example, cable-supported bridges.

The model subjected to optimization was a single section of a closed orthotropic steel bridge girder, for which the usual design parameters of the top deck were defined as design variables. The basis of the optimization was a suitable finite-element (FE) model, where the detail level varied from coarse discretized orthotropic shell and beam elements to a finer discretized shell model from where fatigue stresses were calculated.

Since only a single section of the entire bridge girder was modeled, attention was given to ensure acceptable boundary conditions for this local model. To ensure correct stiffness along the boundaries of this single-section model, a multiscale model was established. Hence, a global beam-shell model of the entire cable-supported bridge was modeled and subsequently analyzed for each load case. Afterward, displacement fields from this global model were applied...
to the boundary of the local model, onto which the optimization was subsequently performed. This method reduced the model size in the optimization but ensured the correct stiffness along the boundaries of the local model.

The gradient-based parametric optimization was carried out within an optimization framework developed for the present study. Two independent objective functions were defined: weight and price minimization. The choice of design variables (e.g., plate thicknesses, diaphragm distance, trough geometry) and constraints (fatigue, deformation, and buckling) may be varied and combined as desired by the user.

The implemented optimization framework and FE model were used in nine different optimization cases. In the first eight cases, the objective was minimization of the weight; in the last case, the objective was minimization of price. The main focus was on weight minimization because weight is a clear and unambiguous measure, and reducing girder weight in cable-supported bridges is a general goal. In the first three cases, the complexity was increased from a single free design variable to 10 free design variables. In the next five load cases, different lower boundaries for plate thicknesses were studied together with a variation in constraint functions. The effect of minimizing price was then compared to the results for minimizing weight.

**Methods**

The basis for the optimization framework was the general nonlinear inequality constrained optimization problem given as

\[
\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad c_j(x) \leq 0, \quad j = 1 \ldots m
\]

\[
lb_j \leq x_j \leq ub_j, \quad i = 1 \ldots n
\]

where \( f(x) \) = nonlinear objective function; \( c_j(x) \) = nonlinear constraint functions; and \( x \) = design variable vector. Each design variable \( x_j \) had lower \( lb_j \) and upper \( ub_j \) bounds. In this study, the different elements of the general optimization problem (Eq. 1) were defined and tied to the actual optimization problem being solved.

An optimization framework was developed to facilitate the gradient-based parametric optimization of the bridge girder. The flowchart in Fig. 1 gives a general overview of the framework, which consists of two main parts: (a) the optimization environment itself and (b) the studied FE model (bold enhancement).

The optimization environment was implemented in MATLAB (2016), and the problem was solved with the nonlinear solver KNITRO (Byrd et al. 2006), applying the sequential quadratic programming (SQP) algorithm. The optimization algorithm was gradient-based, but because several of the constraint functions were based on the FE model, no analytic gradients were available. The gradients of the objective and constraint functions with respect to the design variables were thus computed numerically by the finite difference method. The design variables, objective and constraint functions, and stability check are all described in detail in the final part of this section.

The local FE model, which served as the basis for constraint functions and the stability check, was modeled in the commercial software Abaqus (2016). The model was scripted in Python (Nagar 2018) to allow for altering design variables and to allow it to be called from the MATLAB optimization framework. The local model was of a single section of a bridge girder, with dimensions based on the detailed design of the Osman Gazi Bridge (Turkey). Designed by COWI A/S, the Osman Gazi Bridge opened to traffic in July 2016 possessing the world’s fourth-longest main span of 1,550 m and a total length of 2,682 m. It is a state-of-the-art design within large cable-supported bridges. The bridge deck is a closed steel monoblock girder with orthotropic deck and truss diaphragms. The FE model is described in detail in the following subsection.

Despite the study being based on a specific bridge, the results in regard to trends and correlations between the various design parameters are considered to be generally valid and the methods generally applicable. However, the exact results with respect to minimization of weight and price are only valid for the specific bridge.

**Finite-Element Model**

The optimization focused only on the top part of the girder because many challenges often arise when designing this part. Further, only local effects (such as fatigue, local deformation, and buckling) were included in the optimization problem, and the basis for the optimization was a local FE model of a single section of the bridge girder, which further reduced the problem size handled during the iterative optimization process.

This local model was an individual FE model but was posed as a lower level in a multiscale model, which was established to ensure correct boundary conditions for the local model. The multiscale model and the detail levels and boundary conditions of the local model are described in the following subsections.

**Multiscale Model**

When reducing a continuous deck girder to a local model of only one section, special attention must be given to the boundary conditions of this section. The solution to this challenge was to establish a multiscale model and utilize the Abaqus submodel feature. A schematic overview of the multiscale model with main dimensions and terminology of sublevels is shown in Fig. 2.

The multiscale model consisted of two models with different detail levels: global and local. The global model was mainly made from beam elements, but five sections (in total 125 m) of the bridge girder were replaced by orthotropic shell-beam elements (for de-
The orthotropic shells were introduced to reduce a complete shell model consisting of the top plate and troughs to a single shell element with stiffness properties equivalent to the complete model. Similarly, beam parts in the diaphragms were reduced to orthotropic beam elements with equivalent stiffness properties. This method of model reduction was previously proven applicable in Bjærre (2015).

The local model corresponded to the center section of the global orthotropic shell elements. Detail levels of this model are described in the next subsection.

The global model was used solely to calculate admissible boundary conditions for the local model and was not included in the optimization itself. The principles applied for determining the boundary conditions on the local model are described following the next subsection. The material applied to the entire multiscale model was steel with elasticity modulus $E = 210$ GPa and Poisson’s ratio $\nu = 0.3$.

**Detail Levels of Local Model**

The local model was established with two detail levels as shown in Fig. 3.

The major part of the local model consisted of orthotropic shell and beam elements similar to the global orthotropic shell-beam elements. A minor part of the upper orthotropic deck and diaphragm beam in the local model were modeled as a detailed shell part.

For the detailed shell part, all plates (top plate, troughs, diaphragm beam with cutouts, and diaphragm beam flange) were modeled for a $5 \times 5$-m part of the girder. This detailed part was used to apply a finer mesh so reliable fatigue stresses could be calculated. The detailed part was tied at all free edges as a slave connection to the master edges of the orthotropic elements, such that all displacement fields were transferred correctly.

The orthotropic shell-beam elements were meshed with a relatively coarse mesh with element size $h_c = 1$ m, and the detailed part was in general meshed with a finer mesh of $h_f = 0.1$ m (see Fig. 3).

The terminology of an orthotropic deck (similar to the detailed shell part) used throughout the study is given in Fig. 4. The top plate and longitudinal trough, together, are named the **top deck**, and the diaphragm beam (DB) web and DB flange, together, are called the **diaphragm beam**.

**Boundary Conditions on Local Model**

The global model was established to calculate correct boundary conditions for the local model, which were applied by the submodel feature in Abaqus. The principle of the submodel feature was to apply displacement fields from the complete global model to the boundaries (cut edges) of the local model. Hence no fixed boundary conditions were applied to the local model.

For each load case (applied with the same configuration on both the global and the local models), the displacement fields of the global model were computed once. The displacement fields along the edges of the global orthotropic shells (indicated by dashed lines)

---

**Fig. 2.** Schematic overview of multiscale model with main dimensions, terminology of sublevels, and indication of displacement field boundaries (dashed edges).

**Fig. 3.** Detail levels of local model with, respectively, the major orthotropic shell-beam elements (general mesh size: $h_c = 1$ m) and minor detailed shell part (general mesh size: $h_f = 0.1$ m).

**Fig. 4.** Terminology of an orthotropic deck.
in Fig. 2) were then applied to the edges of the local model (also indicated by dashed lines in Fig. 2).

This method allowed for using a reduced model, thus decreasing the calculation time for the optimization process while maintaining precise displacements along the boundaries and furnishing the correct stiffness of the model.

**Optimization Framework**

The fundamental parts of the optimization framework illustrated in Fig. 1 are specified in the following subsections.

**Design Parameters**

The usual design parameters of the top deck and diaphragm beams, shown in Fig. 5, were chosen as design variables. Table 1 provides the initial values of the parameters used for the optimization (which are equivalent to the detailed design of the Osman Gazi Bridge) and the initial choice of lower and upper bounds for all design variables.

**Parametric FE Local Model**

To allow the design variables to change during optimization, the entire local FE model was made parametric. This way it could automatically be regenerated with the updated parameters. Various parts of the model were influenced by the updated and changing geometry, which had to be handled accordingly:

- The position of the wheel loads followed the geometry of the detailed part; hence, the position above where the fatigue stresses were calculated was consistent, as indicated in Fig. 8.
- The size of the detailed part varied when the geometry of the troughs changed; however, the position of the detailed part in the local model was kept consistent.
- The orthotropic shell and beam elements were assigned a customized stiffness matrix computed from the parametric design variables.

**Objective Functions**

Two different objective functions were implemented. The first function computed the total wt/m of the girder; hence, only the material amount was minimized and not necessarily the total price. The objective function in (Eq. 1) computing the wt/m is given as

$$f_1(x) = \rho_1 \left( A_{tp} + A_{tr} + V_D t_D \over L_d \right)$$

where \(\rho_1\) is density of steel; \(A_{tp}\) is cross-section area of all outer skin plates; \(A_{tr}\) is cross-section area of all troughs; and \(V_D = \text{total volume of a single diaphragm. The weight function is implicitly a function of the design variables } x\).

To handle a minimization of the total price of the bridge girder, a second objective function was implemented, including a differentiation on the steel price besides welding and manufacturing costs. The steel and welding prices were estimated by experienced bridge design engineers. However, exact prices were not crucial in the scope of optimization but only the ratio between the different prices. This objective function is given as

$$f_2(x) = P_1(x) + P_2(x)$$

where \(P_1(x) = \text{price per meter of the top plate, troughs under the top plate, and the diaphragm beams; and } P_2(x) = 20,163 \text{ EUR/m, a fixed price for the rest of the girder per meter (with an estimated steel price of 2.7 EUR/kg, including manufacturing costs). Price function } P_1(x) \text{ is given implicitly as a function of the design variables } x \text{ as}

$$P_1(x) = \rho_1 \left[ A_{tp} P_{tp} + A_{tr} P_{tr} + V_D P_{DB} t_D \over L_d \right]$$

$$+ P_m \left( A_{tr} P_{tr} - A_{tp} P_{tp} + V_D P_{DB} t_D \over L_d \right)$$

$$+ P_m \left( V_D P_{tr} - V_D P_{tp} + V_D P_{DB} t_D \over L_d \right)$$

where \(A_{tp}\) is cross-section area of the top plate; \(P_{tp}\) is an estimated steel price on automated welding; \(P_{tr}\) is total cross-section area of the longitudinal weldings between troughs and top plate; \(P_{DB}\) is an estimated price on automated welding; \(A_{tr}\) is total cross-section area of the longitudinal weldings in the top plates;

---

**Table 1. Design parameters and initial values**

<table>
<thead>
<tr>
<th>Design parameter</th>
<th>Symbol</th>
<th>Initial value (mm)</th>
<th>Lower bound (mm)</th>
<th>Upper bound (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diaphragm distance</td>
<td>(L_d)</td>
<td>5,000</td>
<td>4,170</td>
<td>6,250</td>
</tr>
<tr>
<td>Top plate thickness</td>
<td>(t_{tp})</td>
<td>14</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>Trough thickness</td>
<td>(t_{tr})</td>
<td>8</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>Trough height</td>
<td>(h_t)</td>
<td>360</td>
<td>100</td>
<td>600</td>
</tr>
<tr>
<td>Trough width top</td>
<td>(w_{tr,t})</td>
<td>300</td>
<td>200</td>
<td>400</td>
</tr>
<tr>
<td>Trough width bottom</td>
<td>(w_{tr,b})</td>
<td>180</td>
<td>50</td>
<td>300</td>
</tr>
<tr>
<td>DB web height</td>
<td>(b_w)</td>
<td>1,000</td>
<td>500</td>
<td>1,500</td>
</tr>
<tr>
<td>DB web thickness</td>
<td>(t_w)</td>
<td>14</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>DB flange width</td>
<td>(w_f)</td>
<td>250</td>
<td>50</td>
<td>500</td>
</tr>
<tr>
<td>DB flange thickness</td>
<td>(t_f)</td>
<td>12</td>
<td>5</td>
<td>20</td>
</tr>
</tbody>
</table>
Fatigue details

**Constraint Functions**
Several different constraint functions were implemented to ensure that the optimized designs were not subject to fatigue or buckling issues and fulfilled stiffness and deformation requirements. The different types of constraint functions, several of which were computed from the local FE model, are elaborated on here.

**Fatigue.** Mitigation of fatigue issues is one of the main design drivers for orthotropic steel bridge girders. The computation and analysis of fatigue stresses in the general design process of a bridge girder is a comprehensive and detail-oriented task. Therefore, several simplifications were introduced to include fatigue as a constraint function in the optimization process.

The number of fatigue details (FDs) was reduced to three of the most important for the specific bridge (see Fig. 6), which were identified in Bjærre (2015) and in the experience of COWI A/S through its actual design work. The choice of limited fatigue details was reasoned to ensure acceptable computation times during the optimization. Information about the fatigue details is given in Table 2.

Bjærre (2015) showed that for these three details, the major contribution to fatigue damage came from the vehicle Type 5A–H truck in the Eurocode 1993/1/9 UK Annex fatigue load model 4 (FLM4) (which the fatigue design of the Osman Gazi Bridge was based on) (BSI 2008). Thus, this truck’s load was the only load configuration considered for the fatigue details. The configuration of 5A–H with axle loads is shown in Fig. 7.

For the three fatigue details in Table 2, the largest stress variations occur for different axle positions (see column 2 in Table 3). For FD1 and FD4, a single truck position (load cases LC1 and LC7, respectively) was included, whereas for FD3, two truck positions (load cases LC5 and LC6) were included. The position of the axles for the four load cases on the detailed shell part is shown in Fig. 8. In the figure, a reference point is indicated; during optimization, the detailed shell part and the axle positions were defined relative to this point, and the fatigue stresses were measured near this point.

The usual Palmgren-Miner summation was reduced to critical target stresses for each fatigue detail. Hence, the constraint functions for the three fatigue details were established in such a way that calculated fatigue stresses in the details were acceptable if below the target. The target stresses provided in Table 3 were identified by COWI A/S during the design work.

Around the fatigue details, the mesh was discretized according to XIII-1823-07 IFW Recommendations for fatigue design of welded joints and components 2008 (section 2.2.3.4 on type A hot spots) (Hobbacher 2008). The fatigue stresses at the weldings were calculated by the hot spot method, given as

\[ \sigma_r = 1.5\sigma_{0.5r} - 0.5\sigma_{1.5t} \]  

where \( \sigma_{0.5r} \) and \( \sigma_{1.5t} \) are the stresses calculated, respectively, at 0.5t and 1.5t from the connection; and \( t \) = plate thickness. Consequently, the mesh around the fatigue details was discretized to the fineness shown in Figs. 9–11 for fatigue details FD1, FD3, and FD4, respectively. In the figures, the relevant stress fields are shown.

The fatigue performance and resistance details were dependent on the changing design parameters, which should be considered when studying the results. However, the magnitude of change in design variables was considered to be acceptable in relation to the use of a constant target stress.

---

**Table 2. Fatigue details**

<table>
<thead>
<tr>
<th>Identifier</th>
<th>Location</th>
<th>Detail category</th>
<th>Detail in EN 1993-1-9</th>
<th>Stress direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>FD1</td>
<td>Diaphragms cutout</td>
<td>125</td>
<td>Table 8.1 detail 5</td>
<td>Maximum principal stress along cutout</td>
</tr>
<tr>
<td>FD3</td>
<td>Trough to top plate welding</td>
<td>71</td>
<td>Table 8.8 detail 7</td>
<td>Transverse direct stress in underside of top plate</td>
</tr>
<tr>
<td>FD4</td>
<td>Top plate to diaphragm welding</td>
<td>71</td>
<td>Table 8.4 detail 8</td>
<td>Longitudinal direct stress in underside of top plate</td>
</tr>
</tbody>
</table>

---

**Fig. 7.** 5A–H truck configuration with indication of axle loads.
Eurocode EN Stiffness Requirement. The minimum stiffness requirement to longitudinal stiffeners (troughs) shown in Fig. C.4 of Eurocode EN 1993-2-2007 Annex C.1.2.2 was introduced as a constraint function in the optimization (CEN 2007). The requirement describes the upper limit on the ratio between the diaphragm distance $L_d$ and the moment of inertia $I_B$ of the troughs, including the top plate. The constraint function, based on the empirical graph in Eurocode EN, is given as

$$c_{\text{stiff}} = \frac{L_d}{C_{0}} - \frac{2.8939}{9000I_B} - 1 \leq 0$$

(6)

where $L_d$ and $I_B$ are given in SI units. The constraint is designated EN stiffness requirement in the results discussion.

Deflection Criterion. In addition to the stiffness requirement from the Eurocode, a simple deflection criterion was included. The maximum acceptable relative vertical deflection between two adjacent diaphragms was given as 1/400. The resulting deflection criterion is shown in Fig. 12, and the constraint function is given as

$$c_{\text{def}} = \frac{d}{L_d} \leq 1/400 \Rightarrow c_{\text{def}} = \frac{d}{L_d} - 1/400 \leq 0$$

(7)

The load case giving the maximum deflection is the case where three rear axles of truck 5A-H are positioned in between two diaphragms. This constraint is designated deflection criteria in the results discussion.

Buckling. Because buckling is often a concern in thin-plated structures such as orthotropic steel girders, a constraint against buckling was made for the four load cases from the fatigue details and for the fifth load case from the deflection criterion. However, initial studies carried out before the optimization showed that buckling would not be an issue for these load cases; thus, the buckling constraint was not included during the optimization but was carried out as a check of the final design after optimization to ensure that stability was not an issue.

Results

The developed optimization framework was applied for nine cases, as outlined in Table 4 (Columns 1–5). In Cases 1–8, the objective was minimization of weight; in Case 9, the objective was minimization of price. The initial weight and price of the girder, with dimensions given in Table 1, were 12.81 ton/m and 34,170 EUR/m, respectively. In the first three cases, the complexity of the optimization was increased from one design variable to 5 and 10 design variables, respectively, after which 10 design variables were used in the remaining cases. The lower bound on the plate thicknesses varied for some of the cases, where the point of reference was 5 mm, as indicated in Table 1.

For all optimization cases, the four fatigue load cases and the deflection criterion were included. The inclusion of the EN stiffness requirement varied from case to case because it appears to be very constraining on the optimization and thus is interesting to leave out in some cases. The constraint function limits, initial values, and values after optimization for all cases are summarized in Table 5. As the table provides, all constraints were fulfilled (below the maximum limit) for all cases; further, FD1, FD4, and EN stiffness re-
Table 4. Studied optimization cases, including weight and price reductions after optimization

<table>
<thead>
<tr>
<th>Case</th>
<th>Objective</th>
<th>Number of design variables</th>
<th>Lower bound on plate thickness (mm)</th>
<th>EN stiffness requirement</th>
<th>Weight saved (%)</th>
<th>Price saved (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Weight</td>
<td>1</td>
<td>5</td>
<td>Yes</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>Weight</td>
<td>5</td>
<td>5</td>
<td>Yes</td>
<td>5.7</td>
<td>8.6</td>
</tr>
<tr>
<td>3</td>
<td>Weight</td>
<td>10</td>
<td>5</td>
<td>Yes</td>
<td>7.9</td>
<td>11.8</td>
</tr>
<tr>
<td>4</td>
<td>Weight</td>
<td>10</td>
<td>5</td>
<td>No</td>
<td>8.9</td>
<td>12.5</td>
</tr>
<tr>
<td>5</td>
<td>Weight</td>
<td>10</td>
<td>6</td>
<td>Yes</td>
<td>5.8</td>
<td>8.7</td>
</tr>
<tr>
<td>6</td>
<td>Weight</td>
<td>10</td>
<td>6</td>
<td>No</td>
<td>6.7</td>
<td>9.2</td>
</tr>
<tr>
<td>7</td>
<td>Weight</td>
<td>10</td>
<td>4</td>
<td>Yes</td>
<td>8.4</td>
<td>11.2</td>
</tr>
<tr>
<td>8</td>
<td>Weight</td>
<td>10</td>
<td>4</td>
<td>No</td>
<td>13.8</td>
<td>16.7</td>
</tr>
<tr>
<td>9</td>
<td>Price</td>
<td>10</td>
<td>5</td>
<td>Yes</td>
<td>7.6</td>
<td>11.9</td>
</tr>
</tbody>
</table>

Note: Objective function is in bold.
quirement were the constraints limiting the optimization, whereas the deflection criteria in all cases was far from its limit. The EN stiffness requirement was clearly the more strict of the two deformation criteria. The data in Table 5 is further elaborated on in the following subsections.

As provided in Table 5, the fatigue constraints never reached their maximum limits exactly (as the EN stiffness requirement did) but only came very close. The reason for this lay in the coupling of the Abaqus FE model and its optimization in MATLAB and in the use of finite difference gradients. The nonsmooth steps taken by the optimization algorithm to generate the gradients resulted in nonsmooth responses from the FE model. The nonsmooth responses from the FE model were mainly due to remeshing. It was thus expected that the fatigue constraints could not reach their limits exactly.

For all optimization cases, the start guess was the original design; however, optimization with random start guesses was carried out to ensure that the same final values were obtained. Even though this is the case for all presented results, it is no guarantee for global minimum. Furthermore, a buckling check was carried out for all final results to ensure that there were no stability issues.

The main results for all cases are summarized in Table 4 (columns 6 and 7), where both the savings in weight and price are shown relative to the initial design. These results are elaborated on in the following subsections, where the results for all cases are presented and discussed.

**Minimization of Girder Weight**

The first eight cases had the objective function of minimization of the girder’s weight per meter. This is the more simple of the two objective functions. In Cases 1–3, the complexity increased with the number of free design variables. In Cases 3–8, two effects were studied: (1) the influence of different lower bounds on plate thicknesses and (2) the effect of the EN stiffness requirement.

**Case 1: One Design Variable**

In Case 1, the diaphragm distance $L_d$ was the only free design variable. The objective function, constraint functions, and design variable as a function of iteration number are shown in Fig. 13. In the figure, the constraint functions are all normalized with their initial values (row 2 in Table 5) and defined on the form in Eq. (1); hence, function values below 0 are acceptable.

The initial diaphragm distance was $L_d = 5$ m, and the optimized distance was found to be $L_d = 5.43$ m, which gave a savings of 0.4% in weight (0.05 ton/m). This was the optimal distance when all other parameters were fixed to their initial values. The reason for the relatively low weight reduction was found in the relatively low contribution from the diaphragms to the total amount of steel in the girder.

When studying the constraint functions (row 3 in Table 5), it was found that the maximum limit on the Eurocode stiffness requirement was reached and hence acted as the stopping criterion for the optimization algorithm. Furthermore, all fatigue stresses increased slightly, but in general they were not affected by the changing diaphragm distance.

**Case 2: Five Design Variables**

The complexity of the optimization problem was increased slightly in Case 2, where the free design variables were the diaphragm distance $L_d$ and the four plate thicknesses $t_{fp}$, $t_{fr}$, $t_{ts}$, and $t_p$. The objective function, constraint functions, and design variables as a function of iteration number are shown in Fig. 14. In the figure, the design

### Table 5. Constraint function limits, initial values, and values after optimization

<table>
<thead>
<tr>
<th>Case</th>
<th>FD1–LC1 (MPa)</th>
<th>FD3–LC5 (MPa)</th>
<th>FD3–LC6 (MPa)</th>
<th>FD4–LC7 (MPa)</th>
<th>Deflection criteria</th>
<th>EN stiffness requirement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum limit</td>
<td>65.00</td>
<td>40.00</td>
<td>40.00</td>
<td>38.00</td>
<td>1/400</td>
<td>0.000</td>
</tr>
<tr>
<td>Initial value</td>
<td>57.87</td>
<td>32.80</td>
<td>36.59</td>
<td>37.67</td>
<td>1/5,058</td>
<td>-0.170</td>
</tr>
<tr>
<td>Case 1</td>
<td>59.87</td>
<td>33.14</td>
<td>36.73</td>
<td>37.73</td>
<td>1/4,494</td>
<td>0.000</td>
</tr>
<tr>
<td>Case 2</td>
<td>64.99</td>
<td>30.93</td>
<td>35.59</td>
<td>37.91</td>
<td>1/4,314</td>
<td>0.000</td>
</tr>
<tr>
<td>Case 3</td>
<td>64.56</td>
<td>29.90</td>
<td>35.10</td>
<td>37.49</td>
<td>1/4,065</td>
<td>-0.002</td>
</tr>
<tr>
<td>Case 4</td>
<td>64.16</td>
<td>29.96</td>
<td>35.16</td>
<td>37.91</td>
<td>1/3,493</td>
<td></td>
</tr>
<tr>
<td>Case 5</td>
<td>64.94</td>
<td>30.96</td>
<td>36.03</td>
<td>37.90</td>
<td>1/4,172</td>
<td>-0.014</td>
</tr>
<tr>
<td>Case 6</td>
<td>64.98</td>
<td>31.42</td>
<td>36.29</td>
<td>37.79</td>
<td>1/3,894</td>
<td>—</td>
</tr>
<tr>
<td>Case 7</td>
<td>63.70</td>
<td>27.38</td>
<td>32.61</td>
<td>37.97</td>
<td>1/4,125</td>
<td>0.000</td>
</tr>
<tr>
<td>Case 8</td>
<td>63.83</td>
<td>27.94</td>
<td>34.17</td>
<td>37.98</td>
<td>1/1,966</td>
<td>—</td>
</tr>
<tr>
<td>Case 9</td>
<td>64.73</td>
<td>29.17</td>
<td>34.54</td>
<td>37.84</td>
<td>1/4,075</td>
<td>0.000</td>
</tr>
</tbody>
</table>

© ASCE 04019118-8 J. Bridge Eng.

J. Bridge Eng., 2019, 24(12): 04019118

Fig. 13. Optimization results for Case 1 (one design variable).
variables are all normalized with respect to their initial values (column 3 in Table 1), and the constraint functions are also normalized.

The weight savings was found to be 5.7%, equivalent to 0.72 ton/m—in total 1,931 tons for the 2,682-m-long bridge girder. The knock-on effects on cables, towers, and substructure, as discussed previously, are also assumed to be significant; these effects are inherently present for all the optimization cases.

The initial and final values of the five design variables are given in Table 6.

Table 6. Initial and final design variables for Case 2 (five design variables)

<table>
<thead>
<tr>
<th>Case</th>
<th>$L_d$ (m)</th>
<th>$t_p$ (mm)</th>
<th>$t_t$ (mm)</th>
<th>$t_l$ (mm)</th>
<th>$t_f$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>5</td>
<td>14</td>
<td>8</td>
<td>14</td>
<td>12</td>
</tr>
<tr>
<td>Case 2</td>
<td>4.96</td>
<td>14.0</td>
<td>6.0</td>
<td>13.2</td>
<td>5.0</td>
</tr>
</tbody>
</table>

When comparing the final value of the diaphragm distance according to the Case 1 result, the change from the initial value is now in the opposite direction. Hence, when adding more design variables, it was more beneficial for the optimization algorithm to reduce the diaphragm distance to allow for a reduction in plate thicknesses. The reduction of the plate thicknesses was clearly more beneficial than the increase in diaphragm distance for the objective function. This finding indicates the benefits of using optimization tools in the design process when the number of unknowns increases.

When looking at the constraint functions (row 4 in Table 5), the maximum limit on the Eurocode stiffness requirement was again the stopping criterion for the optimization. However, the fatigue stress limits for FD1 and FD4 were almost reached and thus were close to being active stopping constraints. Interestingly, the fatigue stresses for FD3 (for both load cases) decreased from their initial values.

Cases 3–8: Ten Design Variables

The complexity of the problem was increased fully when all 10 design variables were free in the optimization problem in Cases 3–8. The initial and final design variables for Cases 3–8 are summarized in Table 7.

In general, major changes were seen in reduced trough thickness $t_l$, and flange thickness $t_f$. Minor changes were seen in $h_m$, $w_{tr,tf}$, $L_d$, and $w_f$ and design parameters $L_d$, $t_p$, $w_{tr,tf}$, and $h$, experienced only slight changes. The largest contribution to weight savings came from reduced trough thickness, because the troughs account for a much larger part of the deck than the diaphragm beam flanges. The reduction of trough thickness was seen to influence the other design variables during the optimization and hence was governing the design.

In the following discussion, Case 3 is presented similarly to Cases 1 and 2, followed by the studies including Cases 3–8.

Case 3. The objective function, constraint functions, and design variables as a function of iteration number are shown in Fig. 15. In the figure, constraint functions and design variables are normalized. The weight savings was found to be 7.9%, equivalent to 1.0 ton/m—in total 2,682 tons for the entire girder.

Compared with the five design variables in Case 2, slight changes were seen in Case 3 as a result of the higher degree of freedom due to the increase in the number of free design variables. The top plate thickness $t_p$ was reduced slightly, and the diaphragm distance $L_d$ was increased. Furthermore, both the trough and flange thicknesses ($t_t$ and $t_f$) were reduced to the lower bound of 5 mm. Finally, both the top width $w_{tr,tf}$ and bottom width $w_{tr,tf}$ of the troughs were reduced (hence, the number of troughs was increased), whereas the trough height $h_t$ was increased to compensate for the loss of moment of inertia. Hence, the trend was more, but narrower, troughs in the orthotropic deck plate when weight reduction was the goal. This trend and the reduced plate thicknesses were the reasons for the relatively large weight savings.

When studying the constraint functions, as seen in Table 5, row 5, no constraint limits were reached, although the EN stiffness requirement was very close (similar to Cases 1 and 2). Instead, the limiting factors were the lower bounds on the trough and flange thicknesses.

Variation in lower bounds and relaxation of EN stiffness requirement. From the results of Cases 1–3, it was clear that the most limiting constraint function was the EN stiffness requirement; in Case 3, the lower bound on plate thicknesses was the limiting factor. The effect of these two aspects is considered in more detail here.

In Cases 3 and 7, the lower bound on the plate thicknesses was increased to 6 mm and reduced to 4 mm, respectively. Case 5, with minimum 6-mm trough thickness, was thus fully aligned with the requirements in the Eurocode EN 1993-2:2007. In contrast, Case 7 allowed the optimization to further utilize even smaller plate thicknesses.

Cases 4, 6, and 8 were all similar to Cases 3, 5, and 7, respectively, but without the EN stiffness requirement constraint. The EN stiffness requirement is included in the Eurocode to ensure enough stiffness in the orthotropic deck to prevent surface cracks and to
ensure a limit on vertical deflections. However, it is more strict than the simple deflection criterion; thus, Cases 4, 6, and 8 show how the design would evolve if the EN stiffness requirement was either disregarded or relaxed.

The main results for Cases 3–8 are summarized in Fig. 16, where Cases 3, 5, and 7 (with EN stiffness requirement) and Cases 4, 6, and 8 (without EN stiffness requirement) are grouped, respectively. The relative weight savings are provided in Table 4.

Fig. 16 shows that by removing the EN stiffness requirement, the weight savings were in general higher, with an increasing effect when the lower bound was reduced as well. The increased difference between Cases 7 and 8 was due to the EN stiffness requirement constraint being active in Case 7, contrary to Cases 3 and 5. Furthermore, the gain in reducing the lower bound from 5 to 4 mm was higher when the EN stiffness requirement was disregarded.

When considering the results shown in Tables 4 and 7, several additional conclusions can be deducted. Case 5 showed that the maximum weight savings with all Eurocode requirements fulfilled was 5.8% (equivalent to a price savings of 8.7%). In this case it was mainly the trough and flange thicknesses that were reduced, whereas most other parameters only experienced minor changes. In Case 7 the lower bound on plate thicknesses were not reached for any of the plates; hence, the maximum weight savings including the EN stiffness requirement was 8.4% (equivalent to a price savings of 11.2%). By disregarding the requirement of a minimum 6-mm trough thickness, an additional weight savings of 2.6% (percentage points) was achievable. Note that Case 7 was the only case (of all nine) where no variables reached lower or upper bounds.

In Cases 4, 6, and 8, the effect of disregarding the EN stiffness requirement constraint was clearly seen in the change of trough height $h_t$ compared to Cases 3, 5, and 7, respectively. The trough height was reduced significantly in all three cases because no compensation in total trough stiffness was required. This reduction fully governed the additional weight savings in Cases 4 and 6, and partly in Case 8. With the final design in the three cases, the EN stiffness requirement was exceeded by 13.7% in Case 6, 25.8% in Case 4, and 205.3% in Case 8. The limit of the deflection criterion for the three cases was far from being reached (see column 6 in Table 5), even though deck stiffness was reduced considerably due to the weight savings, further indicating that the Eurocode stiffness requirement is very strict.

Finally, some additional notes about Case 8 should be emphasized. Because Case 8 had both very low bounds on plate thickness and no EN stiffness requirement constraint, it was considered to be an academic example to indicate potential savings if the Eurocode requirements were reassessed. The potential weight and price savings of, respectively, 13.8% and 16.7%, were significantly higher than for any other case. Case 8’s changes in design parameters are also notable. The diaphragm distance was increased to $L_d = 5.76$ m, and the top plate thickness was reduced to $t_p = 12.3$ mm, together with significant reduction in all trough dimensions. The trend to-

### Table 7. Initial and final design variables for Cases 3–8 (10 design variables)

<table>
<thead>
<tr>
<th>Case</th>
<th>$L_d$ (m)</th>
<th>$t_p$ (mm)</th>
<th>$t_w$ (mm)</th>
<th>$h_t$ (mm)</th>
<th>$w_{tw}$ (mm)</th>
<th>$w_{tr}$ (mm)</th>
<th>$h_c$ (mm)</th>
<th>$t_w$ (mm)</th>
<th>$w_f$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>5.00</td>
<td>14.0</td>
<td>8.0</td>
<td>360.0</td>
<td>300.0</td>
<td>180.0</td>
<td>1,000.0</td>
<td>14.0</td>
<td>250.0</td>
</tr>
<tr>
<td>Case 3</td>
<td>5.02</td>
<td>13.9</td>
<td>5.0</td>
<td>390.9</td>
<td>296.1</td>
<td>166.2</td>
<td>1,007.2</td>
<td>12.7</td>
<td>238.7</td>
</tr>
<tr>
<td>Case 4</td>
<td>5.01</td>
<td>13.8</td>
<td>5.0</td>
<td>354.6</td>
<td>294.2</td>
<td>167.4</td>
<td>1,004.5</td>
<td>12.9</td>
<td>244.7</td>
</tr>
<tr>
<td>Case 5</td>
<td>5.00</td>
<td>13.9</td>
<td>6.0</td>
<td>367.4</td>
<td>300.0</td>
<td>174.4</td>
<td>1,000.4</td>
<td>12.9</td>
<td>182.8</td>
</tr>
<tr>
<td>Case 6</td>
<td>5.00</td>
<td>13.6</td>
<td>6.0</td>
<td>349.2</td>
<td>292.9</td>
<td>169.7</td>
<td>979.4</td>
<td>12.7</td>
<td>243.9</td>
</tr>
<tr>
<td>Case 7</td>
<td>4.99</td>
<td>14.4</td>
<td>4.2</td>
<td>396.0</td>
<td>300.3</td>
<td>203.5</td>
<td>998.0</td>
<td>13.9</td>
<td>242.3</td>
</tr>
<tr>
<td>Case 8</td>
<td>5.76</td>
<td>12.3</td>
<td>4.0</td>
<td>311.2</td>
<td>250.3</td>
<td>153.8</td>
<td>986.5</td>
<td>14.6</td>
<td>77.9</td>
</tr>
</tbody>
</table>

### Fig. 15. Optimization results for Case 3 (10 design variables).

### Fig. 16. Girder weight as function of lower bounds on plate thicknesses, respectively with and without the EN stiffness requirement.
ward more but narrower and thinner troughs was thus similar to several of the other cases provided in Table 7.

**Minimization of Girder Price**

The final case, Case 9, was similar to Case 3, but with the objective of minimizing girder price per meter. The price function includes, besides a differentiation on the price of steel, welding and manufacturing costs. The price reduction obtained was 11.9%, compared to 11.8% for Case 3; the final design variables are given in Table 8.

As Table 8 provides, the changes in design variables were very similar to the changes in Case 3. Because both the price savings and the variable changes were very similar, it can be concluded that changing the objective from minimizing weight to minimizing price did not influence the results significantly. The additional effects of welding and other manufacturing costs were thus not large enough compared to the effect of the price of steel (basically a scaling of the weight) to have an impact on the results.

**Conclusions**

A multiscale FE model with sophisticated boundary conditions of a single section of an orthotropic bridge girder was established to form the basis of an automated gradient-based parametric optimization. The optimization framework facilitated simultaneous optimization of up to 10 design parameters with the goal of minimizing either the weight or the price of the girder. Concurrently with the optimization, constraint functions including fatigue stresses, EN stiffness requirements, and deflection criterion were fulfilled, ensuring acceptable final designs.

In the nine optimization cases, potential weight savings were found in the range of 5.8%–13.8% and price savings in the range of 8.7%–16.7%. The maximum weight savings with all current Eurocode requirements fulfilled was found to be 5.8% (8.7% price reduction), equivalent to 1,990 tons of steel for the entire girder. Note that the initial design was already (before optimization) considered to be state-of-the-art.

The complexity of the optimization cases was stepwise increased from a single design variable to five and ultimately 10 free variables. When increasing the complexity of the problem, it became apparent how the optimization algorithm was capable of utilizing the additional design variables toward a better design in regard to the objective function. These results indicate the benefits of using optimization tools in the design process when the number of unknowns, and thus the complexity of the problem, increases.

From the initial optimization cases it was apparent that two limiting factors were governing the optimizations: the Eurocode stiffness requirement and the lower bound on plate thicknesses. By disregarding the EN stiffness requirement, potential weight savings of 6.7% were achievable; by lowering the bound on plate thickness from 6 to 4 mm, potential weight savings of 8.4% were found. A combination of these two relaxations showed theoretical weight savings of up to 13.8% (16.7% price reduction) without any other constraints being exceeded. These results indicate potential additional weight and price savings if the Eurocode requirements are reassessed.

In general, the trends in the designs were toward more but narrower and thinner troughs, with an increased trough height to compensate the lost total trough stiffness. The reduction of trough thickness seemed to influence the other design variables during optimization and hence govern the design. With the change in objective function from weight to price, similar trends were seen, concluding with no significant differences in output between the two objective functions.

Furthermore, the study demonstrated how gradient-based optimization tools in principle can be utilized in design tasks of civil engineering structures to identify both trends and actual weight or price savings. Compared to previous parameter studies of single variables, the high complexity of multiple parameters was easily managed by the gradient-based optimization algorithm, thus enabling the full utilization of orthotropic bridge girders.

**Data Availability Statement**

Some or all data, models, or code generated or used during the study are available from the corresponding author by request. This includes the multiscale Abaqus FE model and the MATLAB optimization framework.

**Acknowledgments**

The presented study is part of an industrial Ph.D. project with the title “Innovative Design of Steel Bridge Girders in Cable-Supported Bridges” and is carried out in cooperation with COWI A/S DTU Civil Engineering and DTU Mechanical Engineering. The project is supported financially by the COWI Foundation grant C-131.02 and Innovation Fund Denmark Grant 5189-00112B.

**References**


---

Table 8. Initial and final design variables for Case 9 (10 design variables)

<table>
<thead>
<tr>
<th>Case</th>
<th>Ld (m)</th>
<th>tps (mm)</th>
<th>tps (mm)</th>
<th>hw (mm)</th>
<th>wtr (mm)</th>
<th>wtp (mm)</th>
<th>htr (mm)</th>
<th>tsp (mm)</th>
<th>wtr (mm)</th>
<th>t (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>5.00</td>
<td>14.0</td>
<td>8.0</td>
<td>360.0</td>
<td>300.0</td>
<td>180.0</td>
<td>1,000.0</td>
<td>14.0</td>
<td>250.0</td>
<td>12.0</td>
</tr>
<tr>
<td>Case 9</td>
<td>5.00</td>
<td>14.1</td>
<td>5.0</td>
<td>381.6</td>
<td>300.6</td>
<td>186.6</td>
<td>1,006.0</td>
<td>13.1</td>
<td>241.4</td>
<td>6.3</td>
</tr>
</tbody>
</table>


Paper III

"Closing the gap towards super-long suspension bridges using computational morphogenesis"

M. Baandrup, O. Sigmund, H. Polk & N. Aage

Published in: Nature Communications, 2020
Girder design for suspension bridges has remained largely unchanged for the past 60 years. However, for future super-long bridges, aiming at record-breaking spans beyond 3 km, the girder weight is a limiting factor. Here we report on a design concept, inspired by computational morphogenesis procedures, demonstrating possible weight savings in excess of 28 percent while maintaining manufacturability. Although morphogenesis procedures are rarely used in civil engineering, often due to complicated designs, we demonstrate that even a crude extraction of the main features of the optimized design, followed by a simple parametric optimization, results in hitherto unseen weight reductions. We expect that further studies of the proposed design, as well as applications to other structures, will lead to even greater weight savings and reductions in carbon footprint in a construction industry, currently responsible for 39 percent of the world’s CO₂ emissions.
Since the opening of the Union Bridge on the Scotland–England border in 1820, suspension bridges have played a key role in civil infrastructure, facilitating fixed links on sites formerly separated by large distances or deep waters, and have become world-known landmarks such as the Brooklyn and Golden Gate Bridges. With the longest bridge spans doubling approximately every 50 years, soon passing 2 km, and with eight out of the ten longest spans in history constructed within the past 15 years, the evolution of bridges has been significant. However, since the 1950s, the conventional design concept for bridge girders, the orthotropic closed steel box-girder (Fig. 1 bottom left), has remained largely unchanged1–3, despite significant design challenges due to inherent fatigue issues4,5 (the phenomenon of failure from repeated cyclic loading). The simple and manufacturing friendly plate-based concept consists of outer skin plates stiffened longitudinally by troughs and transversely by diaphragms every 4–5 m, built with either trusses or solid plates (“conventional design” shown in Fig. 2 and indicated by blue in Fig. 1). Consequently, loads are carried indirectly in a non-natural zig-zag fashion, from bridge deck to hangers, causing stress concentrations and eventually, fatigue problems.

Soon, with plans for bridges spanning beyond 3 km in Norway, Italy, and Indonesia, and approaching the theoretical limit6 of 5 km over the Strait of Gibraltar, self-weight will by far, be the governing load, exceeding 90% of the total bridge loads (including traffic, wind, and seismic). Furthermore, considering that the construction industry accounts for 39% of the world’s CO2 emissions7, attention must be shifted from construction cost to reducing material consumption and hence, self-weight. With weak hopes for new mass-manufacturable, light-weight-high-strength construction materials, and keeping the fatigue problems in mind, the focus must be turned to the identification of new and material-efficient bridge designs.

Previously, research focus has concentrated on reducing fatigue problems4–10, while very little attention has been given on identifying new design concepts11, where no works focus particularly on weight or CO2 emissions reductions. Furthermore, a recent gradient-based parametric optimization study showed very little room for improvement without altering the geometrical concept12. Hence, to pave the way for super-long suspension bridges and to reduce environmental impact, new and innovative design concepts are needed.

We seek to reinvent the more than 60-year-old girder design concept. With a focus on redesigning and decreasing the girder weight, knock-on effects will ensure similar weight savings in the remaining bridge structure (main cables, towers, foundations, and anchor blocks)6. Our study illustrates the benefits of applying a morphogenesis procedure with unrestricted design freedom in the early phase of civil engineering design development, and how a simplistic interpretation of the highly detailed optimized geometry, results in a simple, cost-efficient and constructible design concept. It is expected that similar conclusions can be drawn for applications to other structural problems, particularly within civil engineering.

Results

Topology optimization of bridge girder. To allow for greatest possible design freedom, a giga-scale computational morphogenesis procedure13,14 is used to identify the new design concept. The original concept, dating back to 198815, known as topology optimization, identifies structural solutions with unrestricted design freedom. The iterative numerical procedure redistributes material within a predetermined design domain to optimize a set of performance targets for a given set of loads and boundary conditions16. During the past few decades, the approach has become the preferred design tool in automotive17 and aerospace18 industries, but less so in civil engineering19, though with a few studies of high-rise buildings20, concrete reinforcement21, as well as bridges22–24. The limited number of applications in civil engineering.

Fig. 1 Conventional and interpreted girder design. In the backgound, the Osman Gazi Bridge is shown with an indication of the main span of 1550 m. Insert in the lower left shows a conventional girder section of the Osman Gazi Bridge with truss diaphragms. In the upper right insert, three sections of the continuous girder are shown with solid diaphragms. The conventional design (blue) and the interpreted design (red), extracted from the optimization result, are shown in each of the two center sections, respectively. Here, the fixed top layer is removed to reveal the internal details.
engineering is partly due to a conservative industry, owing to often high-cost/high-risk projects, and partly due to the many challenges in structural optimization\cite{16,17}, such as including cost\cite{18} and reducing complexity\cite{19}. Another obstacle is the low volume fractions typically encountered in civil engineering structures (often in the order of a few percent) requiring very fine design resolutions. Finally, it has been a challenge to cover multiple length scales. Bridge girders with spans of kilometers and widths in the order of 20–40 m are built from plates with thicknesses as low as 6 mm. Hence, to discretize the entire periodic girder domain (Supplementary Fig. 1) into finite elements (voxels) with dimensions in the order of the plate thicknesses, and allowing designs with very low volume fractions, requires billions of elements and thus giga-voxel resolution procedures. Until recently, computational morphology methods were limited to a few million elements, however, this limitation was overcome when the methodology was applied to the design of a full-scale Boeing 777 type aircraft wing using 1.1 billion voxels\cite{20}.

Here, the giga-scale morphogenesis procedure is further extended and applied to a model of a single 25-m-long girder section, with outer geometry and dimensions as in Fig. 2 (see “Methods” for details). Due to periodicity, a single periodic section is representative, however, to precisely impose boundary conditions and transfer loads, two adjacent sections are included in the three-dimensional continuum model (Supplementary Fig. 1). The focus of the study is the load-carrying internal structure of the main domain; hence, the outer wind profile is maintained while walkways are neglected. The dimensions of the full model are 75 m × 30.1 m × 4.75 m, with a design domain of 12.5 m × 15.05 m × 4.75 m (a quarter of one section). During the optimization process, the design domain is mapped to the rest of the model, ensuring that symmetry is maintained along the two center section axes (Supplementary Fig. 1). The top layer of elements, representing the road surface, and the hanger anchorages are fixed to be solid, whereas material can freely be distributed in the remainder of the domain (Supplementary Fig. 1). Through extensive numerical studies, involving global dynamic and aero-structural modeling of the entire bridge, five representative static load cases are chosen for the optimization procedure. These, including the geometry and dimensions, are all based on the actual design of Turkey’s 2682-m-long Osman Gazi Bridge, opened in July 2016 (Fig. 1).

The objective of the optimization is to maximize the stiffness of the center section for a given amount of material. The full model is discretized into 2.1 billion elements (corresponding to a mesh of 4384 × 1760 × 272 elements) with a maximum element size of 17 mm. Although still above the desired minimum member size of 6 mm, this resolution is found to be sufficient to extract the design trends. If a coarser design resolution had been utilized, identification of the important details of the novel design would not have been possible. The giga-voxel resolution comes at a high cost, however, meaning that it requires access to massive computational resources with run times of up to 85 h on 16,000 CPU cores.

Three sections of the optimized bridge girder design are shown in Fig. 3, with the top layer removed to reveal the resulting internal structures. Apart from the thin fixed top layer and fixed hanger anchorage, no a priori assumptions were made on the geometry. Clearly, the design is very different from the conventional, as no perpendicular diaphragms or orthotropic skin plates are seen. Instead, a number of double-curved diaphragm-like panels and trusses have appeared as result of the morphogenesis process. These unconventional features transfer traffic loads and self-weight directly to the hangers instead of through the inefficient conventional zig-zag patterns. Additionally, a longitudinal plate-like structure appears in the region of the hangers and the curved diaphragms as added support. Furthermore, a decreased span length of the skin plates is possible due to an increased number of supports growing from the diaphragm-like panels. However, the complexity of the optimized design prohibits a direct application in the construction industry due to cost and manufacturing considerations.

Interpretation of optimized design. A simplified girder model may be derived by interpreting the main structural features of the optimized design. To facilitate meaningful comparisons to the conventional design, and to convince structural engineers about feasibility, we strive for a derived model at similar level of geometric and manufacturing complexity. Figure 1 shows the best interpreted design together with the conventional layout, and in the lower left-hand side of Fig. 3, the interpreted design is shown overlaid on the optimized design. For simplicity, only diaphragms built from solid plates are considered, however, the principles are equally valid for truss diaphragms. From the rendering of the interpreted design, it is seen that the number of diaphragms per section is increased from five to six, that four of the six diaphragms are curved towards the hangers, and that longitudinal...
panels to support the connection between hanger anchorage and diaphragms have been added. Although the changes appear limited, it is important to recognize that even these seemingly small topological modifications could not have been obtained by classical shape optimization without prior knowledge of the morphogenesis outcome. This confirms the need for the giga-scale resolution free-form design methodology. The interpreted model is subsequently imported and analyzed using a commercial finite element product, revealing a significant 12.7% weight saving compared to the conventional design—just based on a quick interpretation (see Methods for the analysis behind these predictions and Supplementary Table 1).

Parametric optimization. From a subsequent sizing optimization of the interpreted design, where the design variables are the thicknesses of the plates (see Fig. 1 and Supplementary Fig. 2), the total weight saving reaches 28.4% (see "Methods" for details). In comparison, a weight saving of 13.8% was achieved in ref. 12, where a similar, but more refined parametric sizing optimization was carried out on the conventional design. Owing to the detail and complexity of the parametric model in ref. 12, the reported 13.8% change provides an upper bound for the conventional design. Hence, the application of the giga-scale morphogenesis procedure, followed by a quick interpretation and a simple parametric optimization, leads to weight savings significantly larger than what was possible by modifying the conventional design through simple parametric modifications.

Discussion
Although neither the giga-scale optimization nor the parametric model included fatigue in the optimization formulations, we remark that the variation between the largest stresses in the new design is insignificant and absolute values are not bigger than in the conventional design; hence, fatigue problems are not worse than in the conventional design (see Supplementary Table 1).

The interpreted design is preliminary; hence, follow-up studies, including all load cases, dynamics, fatigue, buckling, etc. must be carried out. However, time and cost of these additional studies are trivial when weight becomes the decisive factor. Particularly dynamics, covering e.g. wind vibrations, is of interest due to the significant reduction in girder self-weight. However, we believe that such a detailed redesign process will not change the overall conclusions and insight gained from the presented study. Also, the curved diaphragms leave uneven spans of skin plates, which must be handled, e.g., by redistributing trough and skin plate material from decreased to increased spans. Furthermore, the constructability of the interpreted design must be considered, e.g., by studying the effect of curved diaphragms on construction costs. Despite the slightly more complex design, due to the curved diaphragms, the changes are considered moderate and are repeated between sections. With modern digital production tools and considering recent developments in comparable parametric high-riser designs, we do not believe that these points will be game changers.

The achieved 28.4% weight savings in the girder, equivalent to 8200 tonnes of steel, generate total savings of 13,000 tonnes of steel and 19,000 m³ of concrete in the entire bridge due to knock-on effects (see "Methods" for the analysis behind these estimations and Supplementary Table 2). These savings translate into a reduction of 43,000 tonnes of CO₂ emissions, corresponding to 358 million km of car-driving equivalent to 9000 times around the globe. The overall impact of the rather simple adaptions to the conventional design, identified from the results of the giga-scale morphogenesis procedure applied to the bridge girder model is shown after 400 steps of optimization using 2.1 billion design variables. Three sections of the continuous girder with the fixed top layer removed reveal the internal details. A single section is shown in the lower left-hand side with overlaid interpreted curved diaphragms (red panels).
morphogenesis procedure, is thus significant and may help to close the gap towards the construction and environmental impact of future's super-long suspension bridges. The achieved savings in weight and CO₂ emissions indicate a large potential in applying the demonstrated methods to other civil engineering structures.

Methods

General

First, the finite element model of the bridge girder, subject to optimization, is described, and subsequently, the details of the applied topology optimization methods are introduced. Finally, the methods of interpretation, parametric optimization, and weight-saving estimations are described.

Bridge girder model

The optimization studies are based on Turkey’s 2682-m-long Osman Gazi Bridge (Fig. 1), which at the opening in July 2016 possessed the World’s fourth-longest main span of 1550 m. The COWI-made bridge design, including the orthotropic closed steel box-girder, is considered state-of-the-art, and hence, a suitable basis for optimization and identification of new and innovative bridge girder designs. Geometry, dimensions, and loads are in the following, based on this bridge design.

As a bridge girder is a repetitive periodic structure, a single section, defined by the span between two adjacent sets of hangers (25 m), is representative. However, to impose boundary conditions and global loads, three sections (75 m × 30.1 m × 4.75 m) of the bridge girder are modeled. Hence, a three-dimensional continuum model is established as the design domain, see Supplementary Fig. 1. The domain does not include walkways but maintains the outer shapes defining the aerodynamic wind profile, as aerodynamic optimization is not included in the present work. However, aerodynamic loads are taken into account through the applied loads discussed below. The thin top layer of elements, representing the road surface, and in the form of hanger forces, are modeled. Hence, a quarter of the center domain is mapped to the adjacent sections, as discussed previously. The global load cases were extracted from an envelope of load combinations taking into account the amount of available material, with V being the volume of the current structure and V the maximum amount of available material and the third defines the box constraints on the design variables. To avoid well known, but undesirable effects inherent in topology optimization, such as checkerboards and mesh-dependencies, a filtering of the densities is performed after each optimization iteration. Here, the density filter from ref. 21, an image processing type convolution filter, is applied. Hence, the filter modifies each density as a weighted average of the adjacent densities, given by a filter radius. The filter is thus a method to smoothen the boundaries of the structure. A filter radius of 1.5 times the maximum dimension of a mesh-element is used. Finally, a fully parallelized version of the gradient-based method of moving asymptotes (MMA1430) is used as the optimization algorithm.

Sensitivity analysis

Since the objective function in problem (4) is the compliance of only the center section of the girder, the sensitivity analysis becomes somewhat more complex since the problem is no longer self-adjoint. The objective function without the scaling factor a is given as

Φ = \frac{1}{\lambda} \sum_{i}^{N} u_i \epsilon_i

Here K is a stiffness matrix only containing contributions from the center section of the model. With this formulation, only the compliance of this section is minimized. Undesirable effects from the adjacent sections (including effects from the boundary conditions and stiff end surface) are thus avoided. To ensure the periodic structure, the center domain is mapped to the adjacent sections, as discussed previously. However, when introducing K, the optimization problem is no longer self-adjoint, and to find the sensitivities, an adjoint problem must be solved. The sensitivities of Eq. (5) differentiated with respect to the design variable ρi are given by

\frac{\partial \Phi}{\partial \rho_i} = \sum_{n}^{N} \frac{\partial}{\partial \rho_i} \sum_{n}^{N} \frac{\partial}{\partial u_n} \frac{\partial u_n}{\partial \rho_i}

Here A denote the solutions to the adjoint problems

K A = -2K u

Finally, the derivatives of the two stiffness matrices are given as

\frac{\partial K}{\partial \rho} = \sum_{n}^{N} \frac{\partial p_n}{\partial \rho} (E_{void} - E_{solid}) K

\frac{\partial K}{\partial \rho} = \sum_{n}^{N} \frac{\partial p_n}{\partial \rho} (E_{void} - E_{solid}) K

Here n is the number of elements in the center section, equivalent to the contributions to K.
Giga-scale procedure. To solve the established optimization problem of the described bridge girder model, giga-scale procedures are a necessity. Due to the multiple length scales from the huge domain size of 75 m down to plate thickness as small as 6 mm (in the conventional design), billions of finite elements are required to discretize the domain into a sufficient fineness. The giga-scale morphogenesis procedures presented in ref. 13, based on the PETSC31–33 framework suitable for large-scale parallel computing, are thus extended to handle the given model and optimization problem. Hence, the entire domain enclosed by the outer dimensions in Supplementary Fig. 1 is discretized into 2.1 billion elements (corresponding to a mesh of 4384 × 1760 × 272 elements) with a maximum element size of 17 mm. Despite the element size being about three times larger than the smallest plate thickness in the conventional design, the fineness of the mesh is capable of revealing intriguing and detailed new structural features. The allowable volume fraction is chosen as \( V = 0.30 \), close to the typical volume fractions of 1.0–1.5% in the conventional design (1.3% in the Osman Gazi Bridge). The slightly larger volume fraction is chosen to ensure detailed results, in connection with the corresponding larger element size in the model. The purpose of the optimization should therefore be clear to the reader: to give qualitative insight into how an optimized girder design could look, and thus not to show a final optimal structure.

An exact volume fraction is thus not crucial for the study in question. Considering the quite simple final result of our study, i.e. exchange original five straight spars with six curved ones, one could reasonably object that an expensive supercomputing procedure, as suggested, is overkill and a simple shape optimization procedure could have been sufficient. However, this outcome could not have been predicted without it. A priori, it was not known that this diaphragm layout and placement was key to the weight reduction. A change from five to six diaphragms would require more than a simple shape optimization process, and the bending of each of the diaphragms would have required knowledge of the required parameterization.

Ensuring structural details. Since the optimization problem (4) is non-convex, due to the stiffness-penalization, any gradient-based solution method will end up in a local minimum. To ensure high-quality designs (strong local minima), and to allow for a smooth and fast convergence, a continuation strategy is applied for the penalization parameter in the SIMP interpolation. In the strategy, similar to the one in ref. 14, the penalization parameter is slowly and smoothly raised in steps of 0.25 from 1 to 3 over a total of 400 design cycles.

Computations. To solve the giga-scale optimization problem requires a massive amount of computational resources. The presented results were generated on the Joliet-Curie Supercomputer in France. The optimal number of processors capable of solving the problem was found by numerical studies and memory requirements to be around 16,000. Notably, the symmetry mapping required heavy memory use. During each of the 400 design cycles, ten PDEs (five load cases and five adjoint problems) with around 8.3 billion degrees of freedom are solved with an average solver time of 35 s per equation. The total run time, including IO and checkpointing for restarts, was 85 h.

Interpretation. Since the results of the morphogenesis procedure are mainly qualitative, this case study is performed to determine an estimated figure of weight savings. Hence, based on the insight from the main structural features of the results in Fig. 3, the interior of the conventional design is adapted into an interpreted design. This is identified by visual examination of the results in Fig. 3, and found as the best among a screening of multiple initial interpretations. For simplicity, only diaphragms of solid plates are considered. Here, the number of diaphragms per section is increased from five to six, four of the six diaphragms are curved towards the hangers, and longitudinal panels to support the connection between hanger anchorage and diaphragms are added (called “hanger steel plates”), see Fig. 1. Subsequently, both the conventional design (Figs. 1 and 2) and the interpreted design (Fig. 4) are modeled with shell elements in the commercial finite element software Abaqus (by SimuliaTM). The models include skin plates with troughs, walkway plates with troughs, and transverse diaphragm plate panels. The models have identical material volumes, and the plate thickness of the different parts are summarized in Supplementary Table 4 (see Supplementary Fig. 2 for designation). The shell models represent the three girder sections previously described with similar support conditions, loads, and material parameters.

To quantify the performance improvement, the weighted compliance of the center sections (3), is computed for the five load cases used in the initial topology optimization. The numerical results are summarized in Supplementary Table 1. In column two, the weighted compliance of the five load cases is shown. In column three, the improvement is shown, hence the interpreted design is seen to be 12.7% stiffer than the conventional for the five load cases. The change in compliance, or stiffness, can generally be translated to an equivalent change in volume, or weight. Hence, the stiffness increase may be seen as a weight reduction.

To ensure that the five chosen load cases are representative for the original 14, the weighted compliance of all 14 load cases is computed as well for the shell models. The results are seen in column four of the table. Since the improvement, seen in column five, is similar to the figures in column three, the five load cases are concluded indeed to be representative. Furthermore, the average of the maximum von Mises stress in each of the five load cases is included in column six, with the change shown in column seven. Since the change in stresses is small, fatigue problems are not considered to increase in the improved design.

Parametric optimization. In addition to the interpretation and quantitative comparison above, additional parametric optimization is carried out on the interpreted design. The Abaqus shell model of the interpreted design is used for the parametric optimization, which is carried out in the commercial software CSight (by SimuliaTM). The objective of the optimization is to minimize the compliance in the center section of the model without increasing the amount of material, hence similar to the topology optimization problem, Eq. (4). However, the design variables are now the plate thickness of the shell elements, as indicated in Supplementary Fig. 2 and Supplementary Table 4. A lower bound of 4 mm plate thickness is imposed, similar to the studies in ref. 12. The non-linear optimization problem is solved by sequential quadratic programming with the use of finite difference gradients.

The results of the parametric optimization are summarized in Supplementary Table 1 and Supplementary Table 4. From the first of the two tables, it is seen that the parametric optimization leads to a total weight saving of 28.4% compared to the conventional design. Besides, when studying the weighted compliance of the 14 cases and the von Mises stress, the design after parametric optimization, similar conclusions as above are reached. In Supplementary Table 4, it is seen that all troughs reach the lower limit of 4 mm, similar to the trend seen in ref. 15. Furthermore, it is seen that the thickness of the inclined web plate, located next to the hangers, increases significantly. However, this can be explained from the simplicity of the shell model, since the, in reality, structural complex high-stress region near the hangers, is modeled with few plates. Hence, during optimization, the thickness of the inclined web plate increases to reduce the compliance near the hangers.

Estimation of knock-on effects and CO₂ emission savings. Due to the load-carrying principles of suspension bridges, all weight savings in the bridge girder move up the load path. The potential knock-on effects are here estimated based on conservative, but fair assumptions. The material quantities in the Osman Gazi Bridge are 33,600 tonnes of girder steel (with walkways), 18,000 tonnes of main tower steel, 17,000 tonnes of tower steel, 45,000 m³ of tower foundation concrete, and 130,000 m³ of anchor block concrete.

The 28.4% (8,200 tonnes) weight saving in structural steel in the girder (without walkways), translates into a reduction in loads transferred to the cables of 19.1% when adjusting for the contribution from walkways (about 11% of self-weight) and 60 mm surfacing (about 22% of self-weight). Conservatively, the weight saving in the main cables is equivalent to the 19.1% reduction, when minor effects from cable self-weight are disregarded. Further, the main cable saving translates directly to the anchor block saving. However, in the towers, the saving is halved to 9.55% since the normal force (from the main cable) only contributes with roughly 50% of all tower forces (with the remaining 30% from bending moments and 20% from buckling). Finally, the saving in the tower foundations is equivalent to 9.55%. From the above estimated knock-on savings, the total steel and concrete quantities can be found, which are summarized in Supplementary Table 2, along with the savings in the interpreted design before and after parametric optimization.

To estimate the CO₂ emissions, the following values (from the ICE Inventory of carbon and energy v3.0 Database) are used; 2460 kgCO₂e per ton for steel plates (with density 7850 kg per m³) and 150 kgCO₂e per ton for concrete (with density 2400 kg per m³). The emissions are only calculated based on the material quantities, hence disregarding construction methods. The results are summarized in Supplementary Table 2.

Data availability
The results and data sets presented in this work, i.e. the optimized designs in the form of STL files as well the Abaqus models including loads, are freely available online from https://doi.org/10.11853/DU.6b92806.

Code availability
The basic C++ MPI code used for the giga-scale optimization is publicly available at https://github.com/topopt/TopOpt_in_PETSc (ref. 14).

Received: 21 January 2020; Accepted: 14 May 2020; Published online: 01 June 2020

References
Supplementary Information

Title of manuscript: Closing the gap towards super-long suspension bridges using computational morphogenesis

Authors: Baandrup et al.
Supplementary Information Items

Supplementary Figure 1 | FE model of the bridge girder. Dimensions, loads, and boundary conditions of the three-section model of a bridge girder. Section forces were applied to the stiff end elements (blue) with $100 \times$ modulus of elasticity and density $\rho = 1$. Distributed load was applied to the solid top elements (green) and hanger forces $P$ to the solid hanger attachments (green), both with $\rho = 1$. Design domain indicated by orange, $0 \leq \rho \leq 1$. Mapping of the active design domain A to the remaining model is indicated on the inserted illustration by graded color. Here, dashed lines indicate section borders, and dotted lines indicate symmetry lines.

Supplementary Figure 2 | Design variables in parametric optimization. Indication of design variables during parametric optimization (steel plate thicknesses) on a single girder section.
**Supplementary Table 1 | Performance of the conventional and interpreted designs.** Weighted compliance of the five load cases (LC) included during the optimization and the in total 14 load cases, respectively. Furthermore, average of the maximum von Mises stresses for the five load cases is shown. Relative improvements and changes are compared to the conventional design.

<table>
<thead>
<tr>
<th>Design</th>
<th>Weighted compliance 5 LC [ J ]</th>
<th>Improvement</th>
<th>Weighted compliance 14 LC [ J ]</th>
<th>Improvement</th>
<th>Average of max von Mises stress [ MPa ]</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional design</td>
<td>264.6</td>
<td>-</td>
<td>655.6</td>
<td>-</td>
<td>225.4</td>
<td>-</td>
</tr>
<tr>
<td>Interpreted design</td>
<td>231.0</td>
<td>12.7%</td>
<td>574.8</td>
<td>12.3%</td>
<td>248.6</td>
<td>10.3%</td>
</tr>
<tr>
<td>Interpreted design - parametric optimization</td>
<td>189.6</td>
<td>28.4%</td>
<td>473.9</td>
<td>27.7%</td>
<td>199.2</td>
<td>-11.6%</td>
</tr>
</tbody>
</table>

**Supplementary Table 2 | Total quantities of steel, concrete, and CO₂ for the entire bridge.** Estimated material quantities calculated by knock-on effects from the savings in girder self-weight. The CO₂ calculations only take account of material quantities, hence disregarding construction methods.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional design</td>
<td>69,000</td>
<td>-</td>
<td>175,000</td>
<td>-</td>
<td>233,000</td>
<td>-</td>
</tr>
<tr>
<td>Interpreted design</td>
<td>63,000</td>
<td>6,000</td>
<td>162,000</td>
<td>13,000</td>
<td>213,000</td>
<td>20,000</td>
</tr>
<tr>
<td>Interpreted design - parametric optimization</td>
<td>56,000</td>
<td>13,000</td>
<td>146,000</td>
<td>29,000</td>
<td>190,000</td>
<td>43,000</td>
</tr>
</tbody>
</table>

**Supplementary Table 3 | Loads applied to the FE model.** Load case 1-12: global, load case 13-14: local. *Only distributed load on half of the top surface and average hanger force due to skew load.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.3</td>
<td>65.6</td>
<td>8.5</td>
<td>0.2</td>
<td>0.6</td>
<td>-23.1</td>
<td>-8.8</td>
<td>2.3</td>
</tr>
<tr>
<td>2</td>
<td>-16.6</td>
<td>-99.6</td>
<td>52.2</td>
<td>-0.1</td>
<td>0.5</td>
<td>47.8</td>
<td>-12.4</td>
<td>3.7</td>
</tr>
<tr>
<td>3</td>
<td>-2.1</td>
<td>-247.3</td>
<td>-18.7</td>
<td>-0.2</td>
<td>1.4</td>
<td>-26.4</td>
<td>-12.5</td>
<td>3.5</td>
</tr>
<tr>
<td>4</td>
<td>4.7</td>
<td>169.3</td>
<td>6.0</td>
<td>0.0</td>
<td>0.5</td>
<td>-34.3</td>
<td>-7.5</td>
<td>2.3</td>
</tr>
<tr>
<td>5</td>
<td>0.1</td>
<td>28.8</td>
<td>-818.2</td>
<td>4.6</td>
<td>1.1</td>
<td>-10.0</td>
<td>-7.8</td>
<td>2.3</td>
</tr>
<tr>
<td>6</td>
<td>-1.3</td>
<td>-20.8</td>
<td>403.0</td>
<td>5.4</td>
<td>0.3</td>
<td>24.4</td>
<td>-8.8</td>
<td>2.5</td>
</tr>
<tr>
<td>7</td>
<td>0.1</td>
<td>-13.1</td>
<td>-344.2</td>
<td>-5.3</td>
<td>0.3</td>
<td>-7.9</td>
<td>-7.8</td>
<td>2.3</td>
</tr>
<tr>
<td>8</td>
<td>-1.3</td>
<td>-20.8</td>
<td>403.0</td>
<td>5.4</td>
<td>0.3</td>
<td>24.4</td>
<td>-8.8</td>
<td>2.5</td>
</tr>
<tr>
<td>9</td>
<td>6.1</td>
<td>-36.7</td>
<td>-5.9</td>
<td>-0.1</td>
<td>1.3</td>
<td>55.3</td>
<td>-9.2</td>
<td>3.0</td>
</tr>
<tr>
<td>10</td>
<td>-13.4</td>
<td>-125.1</td>
<td>-25.6</td>
<td>0.0</td>
<td>-0.7</td>
<td>47.4</td>
<td>-11.6</td>
<td>2.5</td>
</tr>
<tr>
<td>11</td>
<td>-0.1</td>
<td>-13.5</td>
<td>226.5</td>
<td>0.2</td>
<td>1.0</td>
<td>-170.8</td>
<td>-11.4</td>
<td>3.4</td>
</tr>
<tr>
<td>12</td>
<td>-2.2</td>
<td>-18.4</td>
<td>-235.8</td>
<td>-0.2</td>
<td>1.0</td>
<td>165.5</td>
<td>-11.0</td>
<td>3.2</td>
</tr>
<tr>
<td>13</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-5.0</td>
<td>1.6</td>
</tr>
<tr>
<td>14*</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-5.0</td>
<td>0.8</td>
</tr>
<tr>
<td>Design</td>
<td>Top plate</td>
<td>Bottom plate</td>
<td>Inclined web plate</td>
<td>Nose plate</td>
<td>Top plate troughs 1</td>
<td>Top plate troughs 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------------------------------</td>
<td>-----------</td>
<td>--------------</td>
<td>--------------------</td>
<td>------------</td>
<td>---------------------</td>
<td>---------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conventional design</td>
<td>14.0</td>
<td>9.0</td>
<td>12.0</td>
<td>10.0</td>
<td>7.0</td>
<td>8.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Interpreted design</td>
<td>14.0</td>
<td>9.0</td>
<td>12.0</td>
<td>10.0</td>
<td>7.0</td>
<td>8.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Interpreted design - parametric optimization</td>
<td>12.0</td>
<td>14.5</td>
<td>25.4</td>
<td>4.9</td>
<td>4.0</td>
<td>4.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Design</th>
<th>Bottom plate troughs</th>
<th>Diaphragms, group 1</th>
<th>Diaphragms, group 2</th>
<th>Diaphragms, group 3</th>
<th>Diaphragms, group 4</th>
<th>Hanger steel plate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional design</td>
<td>6.0</td>
<td>10.0</td>
<td>10.0</td>
<td>10.0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Interpreted design</td>
<td>6.0</td>
<td>6.9</td>
<td>6.9</td>
<td>6.9</td>
<td>6.9</td>
<td>8.3</td>
</tr>
<tr>
<td>Interpreted design - parametric optimization</td>
<td>4.0</td>
<td>8.3</td>
<td>6.3</td>
<td>5.9</td>
<td>6.3</td>
<td>8.8</td>
</tr>
</tbody>
</table>
"Truss optimization applying finite element limit analysis including global and local stability"

P.N. Poulsen, J.F. Olesen & M. Baandrup

Published in: Structural and Multidisciplinary Optimization, 2020
Truss optimization applying finite element limit analysis including global and local stability

Peter Noe Poulsen¹ · John Forbes Olesen¹ · Mads Baandrup¹,²

Received: 7 September 2019 / Revised: 30 October 2019 / Accepted: 26 November 2019
© Springer-Verlag GmbH Germany, part of Springer Nature 2020

Abstract
For practical applications of optimized truss structures, it is essential to include global and local stability in order to obtain stable and realistic structures. The challenge of including both global and local stability has previously been approached in many ways. However, these proposals often lead to ill-conditioned optimization problems, with convergence issues due to the concavity of the problem. In this paper, a new method for handling both global and local stability in truss optimization is presented. The proposed method is based on the finite element limit analysis method. Initially, the global stability problem is solved by a convex semidefinite constraint, and subsequently, the concave local stability problem is included through an iterative process, where the local stability constraints are linearized and solved by a convex sub-problem. This step-wise approach diminishes convergence issues due to the concavity of the problem. The proposed method is demonstrated through three different applications showing significant effects of including global and local stability in the optimized designs, while at the same time demonstrating the validity and potential of the proposed method.

Keywords Truss topology optimization · Global stability · Convex problem · Limit analysis

1 Introduction
Truss structures are by nature, simple and transparent structures, which efficiently carry the applied loads to the supports. For these reasons, trusses have been applied for centuries in all sorts of structures. Because of this efficiency, and their apparent simplicity, trusses have been subject to numerical optimization for decades. Truss topology optimization often leads to lightweight structures prone to instabilities which must be dealt with in order to obtain realistic structures.

The first numerical investigations of optimal truss structures were performed more than 50 years ago in Dorn et al. (1964) introducing a ground structure, with the cross sectional areas as continuous design variables. The problem was formulated as a linear programming (LP) problem including the equilibrium equations and minimising the weight, under the assumption that the material consumption was linearly related to the numerical size of the normal forces. This assumption was based on utilizing the material to its yield stress. It was shown that the result of this optimization under plastic conditions was also an optimal structure under elastic conditions when considering a single load configuration.

A few years later, a numerical method for modeling the ultimate load of structures assuming a rigid-plastic material behaviour was formulated, Anderheggen and Knöpfel (1972). The finite element limit analysis (FELA), by both the lower bound approach and the upper bound approach, was introduced according to the classic plasticity theory.

Pedersen (1972) included local stability in an early formulation for obtaining the elastic optimal layout of trusses. The formulation included position of the joints and restriction on displacements and was solved using LP in an iterative procedure.

In practical applications of optimized trusses, it is crucial to obtain an overall stable structure. However, several
challenges arise when implementing this requirement, as discussed in (Zhou 1996; Rozvany 1996), and many different approaches to overcome them have been proposed. One suggestion to secure an overall stable structure (as supporting bars may approach zero) was the introduction of so-called chains to account for the change in effective buckling lengths (Achtziger 1999a; 1999b). A more direct way to handle the global stability phenomenon is by including the interaction between the individual members. In (Ben-Tal et al. 2000; Kočvara 2002), the global stability was handled, setting up the linear buckling problem. Due to an elastic distribution of forces, the formulation resulted in a non-convex semidefinite programming problem. Other ways of handling the global stability have been investigated, see e.g. Descamps and Filomeno Coelho (2014) where it was handled by a perturbation of the nodal coordinates, and (Guo et al. 2005; Torii et al. 2015; Changizi and Jalalpour 2017) where the constraint was based on predetermined overall buckling modes for the structure, found by eigenvalue analysis. The problem has also been treated applying beam elements in Torii et al. (2015); Changizi and Jalalpour (2017); Madah and Amir (2017) eliminating the local buckling at the cost of a larger global stability problem.

Finding the optimal solution in a non-convex problem is not a simple task, and the search may involve many different attempts and much computational effort. Related to this search for the optimal solution (Evgrafov 2004) describes the so-called global stability singularity problem where the optimal solution is not connected to the interior of the solution domain. This shows the difficulties related to finding optimal solutions applying an elastic distribution of forces. It has been stated in Tugilimana et al. (2018) that the problem was alleviated by stating a disaggregated form of the equilibrium as a two-step elastic relation. Very recently, Weldeyesus et al. (2019) presented a so-called relaxed formulation of the elastic optimal truss layout, including global stability but disregarding the kinematic compatibility and local stability. This, in fact, corresponds to a rigid plastic force distribution and results in a convex semidefinite problem.

It may seem contradictory to apply both a rigid plastic force distribution and a linear buckling problem assuming an elastic behaviour but for a single load configuration, it was previously shown that the result of a plastic optimization is also an optimal structure under elastic conditions and therefore the behaviour of the structure will be elastic until the critical load level where it yields and buckles simultaneously. When optimizing the structure considering multiple load cases, bars may yield or buckle before the maximum load. The characteristic load-displacement behaviour of bars that buckle is similar to bars that yield and in this respect, the model is still valid. The overall behaviour of truss structures that has been optimized for multiple load cases has been investigated, see Mikkelsen (2018), and it was shown by a non-linear geometric analysis that the structure was able to carry load after the first buckling and reach the intended load level.

Since the introduction of plastic optimization, by application of FELA, the method has been developed to handle various kinds of elements, such as beams and slabs (Damkilde and Hoyer 1993; Krenk 1994). Particularly, in recent years, the method has been applied in the analysis of soil mechanics, (Krabbenhoft et al. 2005; Akhlaghi 2006; Lyamin 2009), and in the analysis of reinforced concrete disks, slabs and joints, (Poulsen and Damkilde 2000; Herfelt et al. 2016; 2017; Herfelt 2017; Herfelt et al. 2018). Recently, the method was applied in the optimization of continuum structures (Fin et al. 2018; Herfelt et al. 2019), but so far it has not been applied in truss optimization including global and local stability.

In this paper, a method based on FELA is proposed to solve the problem of truss optimization including global and local stability.

Firstly, the truss optimization problem is solved considering only global stability. The global stability is formulated through the linear buckling problem introducing a semidefinite constraint, preserving the convexity of the problem. Secondly, the local buckling constraint is introduced through the critical Euler buckling load, making the overall problem non-convex. However, the problem can be solved step-wise by a linearization of the local buckling constraint. Hence, the overall concave problem, including global and local stability, is solved in an iterative process with convex sub-problems.

To demonstrate the validity of the proposed method, three plane cases are studied: a simple two-bar structure, a tip-loaded cantilever beam and a vertically loaded column. The study of the stability for a simple two-bar structure allows for comparison against an analytical solution. The cantilever beam study exhibits the effect of increasing mesh density and convergence properties, as well as demonstrates the capability of handling multiple load cases. Additionally, the effects on the design of first adding the global stability constraint and subsequently, the local buckling constraint are shown. The column study gives insight into the importance of using realistic material parameters when including stability in truss optimization. Finally, the effects on the truss layout from increasing column slenderness are studied.

2.2 Truss layout optimization

The structures studied in this paper are trusses defined by \(N\) nodal points in a 2-dimensional space \(d = 2\). The number of supports is denoted \(N_s\) and the number of equilibrium...
2.1 Finite element limit analysis

The finite element limit analysis is a method assuming perfect plastic materials. The material models are simplified assuming a rigid behaviour prior to plasticity. The method is based on the finite element method, as all variables, equations and inequalities are either node or element based. However, the load-deflection path is not determined, as no deformations occur before yielding. Unlimited deformation capacity is assumed, only the collapse mode is found, and small displacements are assumed. The yield criterion is a convex surface in the stress space. The mathematical problem of FELA is formulated as a convex optimization problem which can be solved very efficiently by state-of-the-art solvers. The reader is referred to (Anderheggen and Knöpfel 1972; Damkilde and Høyer 1993; Poulsen and Damkilde 2000; Krabbenhøft et al. 2008) for an in-depth introduction to the method.

The pure volume optimization formulation contains two parts. The first part, the equilibrium equations, ensures equilibrium in all nodal forces in the structure, hence ensures a statically admissible solution. The second part, the yield condition, ensures all element stresses to be within the yield criterion.

The bar element with a constant normal force, $N_\ell = \beta_\ell$, together with the element nodal forces, is shown in Fig. 1.

The element nodal forces, with respect to the local element axes, can be written as

$$
\frac{F_1^\ell}{F_2^\ell} = \begin{bmatrix} \beta_\ell \cdot [-1] \\ 0 \\ 1 \\ 0 \end{bmatrix}
\quad \Leftrightarrow \quad q_e = h_e^\top \beta_\ell
$$

where $q_e$ is the element nodal force vector, $h_e^\top$ is the element equilibrium matrix, and $\beta_\ell$ is the element stress vector.

The global equilibrium matrix $H[Ne \times Ne]$ is assembled in the standard way

$$
H = \sum_{e=1}^{Ne} H_e \quad \text{with} \quad H_e(\mathbf{d}_e, \mathbf{d}_\beta) = T_e^\top \mathbf{h}_e
$$

where $\mathbf{d}_e^{[2d]}$ is an index vector mapping from the local to the global equation numbers, $\mathbf{d}_\beta^{[2d]}$ is an index vector mapping from the local to the global stress parameters and $T_e^{[2d \times 2d]}$ is the transformation matrix from local to global coordinates. The equilibrium equations can now be written as

$$
H \beta = R
$$

where $\beta^{[Ne \times 1]}$ is a vector collecting all $\beta_\ell$ and $R^{[Ne \times 1]}$ is a vector containing external nodal loads.

The yield condition for a bar element and thereby the restrictions on the normal force can be given by the tension and compression normal capacities, $N_{y,\ell}^+$ and $N_{y,\ell}^-$, and expressed by the yield stress and the cross sectional area, $f_y$ and $a_e$

$$
\begin{bmatrix} 1 \\ -1 \end{bmatrix} \beta_\ell \leq \begin{bmatrix} N_{y,\ell}^+ \\ N_{y,\ell}^- \end{bmatrix} \quad \Leftrightarrow \quad c_e \beta_\ell \leq c_{m,e} a_e
$$

where $c_e$ and $c_{m,e}$ are restriction matrices for the element and $a_e$ contains the cross sectional area for the element. The restriction matrices can be assembled similarly to $H$ by use of $C_e(\mathbf{d}_e, \mathbf{d}_\beta) = c_e$ and $C_{m,e}(\mathbf{d}_e) = c_{m,e}$, where $\mathbf{d}_e$ is an index vector mapping from the local to the global restriction numbers. In matrix notation, the yield conditions for the system can be given as

$$
C_\beta - C_m A \leq 0
$$

where $C_{2Ne \times N_1}$ transforms the bar forces and $C_{m}^{[2Ne \times N_1]}$ contains the yield stresses.

Thus, the linear optimization problem (LP) for pure volume optimization is given as

$$
\min_{\beta, A} \quad A^\top L \quad \text{s.t.} \quad H \beta = R \quad C_\beta - C_m A \leq 0
$$

The problem with linear equality and inequality constraints is convex, and very efficient convex solvers may be utilized.

For $N_k$ multiple load cases, the equilibrium equations must be set up for each load case with separate stress parameters. This can be handled by replacing the matrices and vectors in the optimization problem with a collection of
sub-matrices and sub-vectors as shown for the equilibrium equations

\[
H = \begin{bmatrix}
H_1 \\
\vdots \\
H_k
\end{bmatrix}, \quad \beta = \begin{bmatrix}
\beta_1 \\
\vdots \\
\beta_k
\end{bmatrix}, \quad R = \begin{bmatrix}
R_1 \\
\vdots \\
R_k
\end{bmatrix}
\]

where index \( k \) indicates the part corresponding to load case \( k \). Similarly for the yield restrictions

\[
C = \begin{bmatrix}
C_1 \\
\vdots \\
C_k
\end{bmatrix}, \quad C_m = \begin{bmatrix}
C_{m,1} \\
\vdots \\
C_{m,k}
\end{bmatrix}
\]

The problem now consists of \( n = N_e \cdot (N_k + 1) \) optimization variables and \( m = m_H + m_C = N_d \cdot N_k + 2N_e \cdot N_k \) constraint functions.

In the formulation of finite element limit analysis, multiple load cases are in general handled by an expansion of the system of equations, as mentioned in the above. The number of design variables and constraints thus increases linearly with the number of load cases. The size of the constraint matrices increases quadratically; however, as all matrices are defined sparse, the actual problem size increases linearly.

### 2.2 Global stability as semidefinite constraint

To constrain the optimized structures from global stability problems, a semidefinite constraint is introduced on the sum of the elastic and geometric stiffness matrices. In a standard linear elastic finite element analysis, the elastic matrix is introduced to relate displacements and forces in the structure, and the geometric matrix is introduced to handle large deformations. However, in the present formulation, the stiffness matrices are solely introduced to handle the global stability phenomenon.

The local elastic stiffness matrix is given as

\[
k_e = a_e \int_0^{t_e} E \frac{\partial^2 d}{\partial t^2} dx = \frac{E a_e}{t_e} \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix} = a_e k_{0,e}
\]

where \( E \) is the modulus of elasticity. For the optimization problems, the stiffness matrix is defined as \( k_{0,e} \) leaving out the area \( a_e \). The global elastic stiffness matrix \( K(A)^{[N_k \times N_k]} \), defined as a function of the design variables \( A \), is assembled in the usual way

\[
K(A) = \sum_{e=1}^{N_e} k_e(A)
\]

with \( k_e(A)(d_e, d_e) = a_e T_e^T k_{0,e} T_e \)

where \( d_e^{[2d]} \) is an index vector mapping from the local to the global system and \( T_e^{[2d \times 2d]} \) is the transformation matrix from local to global coordinates.

The local geometric stiffness matrix is given as

\[
k_G,e = \beta_e \int_0^{t_e} G^T G dx = \frac{\beta_e}{t_e} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} = \beta_e k_{G,0,e}
\]

where \( G^{[2d \times 2d]} \) is the strain interpolation matrix. For the optimization problems, the stiffness matrix is defined as \( k_{G,0,e} \) without the variable \( \beta_e \). The global stiffness matrix \( K_G(\beta)^{[N_k \times N_k]} \), defined as a function of the design variables \( \beta \), is assembled in the usual way

\[
K_G(\beta) = \sum_{e=1}^{N_e} k_G,e(\beta)
\]

with \( k_G,e(\beta)(d_e, d_e) = \beta_e T_e^T k_{G,0,e} T_e \)

The linear eigenvalue problem defining the global stability limit, see e.g. Cook et al. (1989), is given as

\[
(K + \lambda K_G) v = 0
\]

with eigenvalue \( \lambda \) and corresponding eigenvector \( v \). Here, a priori, the load parameter/eigenvalue is \( \lambda = 1 \), and the buckling condition is governed by the design variables \( \beta \) and \( A \). A sufficient condition for this eigenvalue problem to be fulfilled, and thereby for global stability to be ensured, is to demand the summed stiffness matrix \( K_S \) to be positive definite

\[
K_S = K_G(\beta) + K(A) \succeq 0
\]

Adding the positive semidefinite constraint (15) to the LP (7a–7c) for \( N_k \) load cases, the semidefinite optimization problem (SDP) with linear objective function is given as

\[
\begin{align}
\min_{\beta, A} \quad & A^T L \\
\text{s.t.} \quad & H\beta = R \\
& C \beta - C_m A \leq 0 \\
& K_G(\beta_k) + K(A) \succeq 0 \quad \forall k = 1, \ldots, N_k
\end{align}
\]

The above optimization problem is equivalent to the formulation of the dual problem in the Mosek optimization software (MOSEK 2018a), which is used as the solver. The constraint (16d) is thus, for a single load case, handled as a linear matrix inequality (LMI) in the form

\[
K_S = K_G(\beta_k) + K(A) = S \quad S \succeq 0
\]

where \( K_S \) is converted to a set of linear equality constraints using an implicit semidefinite slack variable \( S \). The semidefinite requirement is thus moved from being explicit on \( K_S \) to being implicit on \( S \). The extra constraints add \( m_K = N_k \times N_d \cdot (N_d + 1)/2 \) linear equality constraint functions to the problem, equivalent to the number of
elements in the lower triangular part of $K_S$ times the number of load cases. The slack variables $S$ add $N_k$ semidefinite constraints.

### 2.3 Local buckling constraint

It is essential to include the local stability of individual members in the optimization when striving towards realistic truss structures. For truss members, only subject to normal forces, local stability can be ensured by imposing that the internal compression force is limited by the critical buckling load. By introducing the critical Euler buckling load for a simply supported member, the local buckling constraint for element $e$ is given as

$$-\beta_e \leq P_{cr,e} = \frac{\pi^2 \cdot EI_e}{L_e^2}$$  \hspace{1cm} (18)

where $L_e$ is the second moment of area. As tension members have positive normal forces $\beta_e \geq 0$, the constraint is only active for compression members where $\beta_e \leq 0$.

Due to the non-linear second moment of area, the constraint is not linear in regard of the design variable $a_e$. To implement the constraint into the optimization problem, a first step is thus to reformulate it into a quadratic function of the cross section area $a$. For a solid rod, with $I_{rod} = \frac{a^4}{4\pi}$, the critical Euler load is simplified to

$$P_{cr,rod} = \frac{\pi^2 \cdot E \cdot a^2}{l^2} = \frac{\pi \cdot E \cdot a^2}{4l^2} = \alpha \cdot a^2$$  \hspace{1cm} (19)

where a constant $\alpha$-factor is introduced, which for the rod is $\alpha_{rod} = \frac{\pi E}{4\pi}$. Similarly, this can be done for thin tubes if the ratio between the tube diameter $D$ and shell thickness $t$ is fixed by a constant $\gamma$ given as $D = \gamma t$. Then, the tube area can be defined as $a = \pi \gamma t^2$ and the second moment of area as $I_{tube} = \frac{\pi}{4\gamma^2} a^2$. The $\alpha$-factor in (19) can thus for a thin tube be replaced by $\alpha_{tube} = \frac{\gamma \pi E}{4\gamma^2}$. In the following, this expression for $\alpha$ is used together with the Eurocode 3 expression for $\gamma$, given for thin tubes with class 1 cross sections (fully plastic) as

$$\gamma = 50 \cdot \frac{235MPa}{f_y}$$  \hspace{1cm} (20)

where the yield stress $f_y$ is given in MPa. Thin tubes are thus assumed as the truss members. However, the expression for $\alpha$ can be modified to many types of cross sections using reasonable assumptions, see e.g. Changizi and Jalalpour (2017).

The simplified local buckling constraint can thus in general be defined as

$$-\beta_e \leq P_{cr,e} \simeq \alpha_e a_e^2$$  \hspace{1cm} (21)

However, the above inequality is concave; hence, it cannot immediately be integrated in the formulation of the convex optimization problem.

#### 2.3.1 Linearization of Euler stability

To include the concave inequality constraint (21) in the convex optimization formulation, the constraint is linearized. Since this is a significant simplification, the linearized equations are updated in an iterative process until convergence. The procedure of linearization is established in the following, whereas the iterative implementation is presented in Section 2.3.2. The idea of handling the Euler stability criterion by an iterative process was also discussed in Parkes (1975) and He and Gilbert (2015).

The general feasible solution domain for element $e$ is shown in Fig. 2. For tension forces ($\beta_e \geq 0$), the domain is constrained by the yield criterion $f_y a_e$. For compression forces ($\beta_e \leq 0$), it is constrained by the quadratic constraint (21) for $0 \leq a_e \leq \frac{f_y}{\alpha_e}$ and by the yield criterion $-f_y a_e$ for $\frac{f_y}{\alpha_e} \leq a_e$.

The quadratic expression for $P_{cr}$ in (21) is linearized in two different ways, tangent and secant, dependent on the member force $\beta_e$. The tangent is used as linearization when $\beta_e < -\epsilon \frac{f_y^2}{\alpha_e}$, where $\epsilon$ is a small number ($\epsilon = 10^{-3}$), hence

![Fig. 2](image-url)  

Feasible solution domain for element $e$ shown in grey. The stress-limit functions are given as functions of $a_e$. Tangent linearization of $P_{cr}$ is shown as a dotted line, with indication of intersection with $P_{cr,e}$.
for most compression members. The tangent $P_{cr,t}$ is shown in Fig. 2 and given as

$$P_{cr,t} = 2\alpha ai_{i-1} - \alpha ai_{i-1}^2$$  \hspace{1cm} (22)$$

where $i$ refers to the current iteration, and $i-1$ refers to the previous iteration in the iterative process. The linearization is thus based upon the solution to a previous solved optimization problem, which is discussed more in detail in Section 2.3.2. To ensure a stable algorithm, $ai_{i-1}$ is not directly used from the previous solution, but calculated based on the truss force from the previous optimization problem as

$$ai_{i-1} = \sqrt{\frac{-\beta_{i-1}}{\alpha}}$$  \hspace{1cm} (23)$$

The tangent is used for most compression members due to it being a conservative simplification. However, the tangent also implicitly imposes a minimum area on the member, given at the intersection with the abscissa (see Fig. 2). To handle this misbehaviour, a secant is used for linearization when $\beta_e \geq -\frac{f_e^2}{2\nu}$, hence for compression members with very small compression forces and tension members in general. The secant $P_{cr,s}$ is shown in Fig. 3 and given as

$$P_{cr,s} = \alpha ai_{i-1} = \sqrt{e f_s a_i}$$  \hspace{1cm} (24)$$

where the term $ai_{i-1}$ has been fixed as

$$ai_{i-1} = \sqrt{e f_s}$$  \hspace{1cm} (25)$$

By using the (non-conservative) secant, member areas are allowed to vanish during the optimization.

From the tangent (22) and secant (24) linearizations, the Euler stability criterion can be formulated generally as a linear inequality constraint in the form

$$-\beta_e - p_e a_e \leq q_e$$  \hspace{1cm} (26)$$

where $p_e$ is the slope of the linearization and $q_e$ the intersection with the ordinate in Figs. 2 and 3. The constraint (26) in matrix form is given as

$$-I\beta - PA \leq q$$  \hspace{1cm} (27)$$

where $I^{[N_e \times N_e \times N_k]}$ is the identity matrix, $P^{[N_e \times N_e \times N_k]}$ is a diagonal matrix containing $p_e$ and $q^{[N_e \times N_k]}$ is a vector containing $q_e$. As (27) is a linear inequality constraint, it can be added to the optimization problem (16a–16d), given as

$$\min_{\beta, A} A^T L$$  \hspace{1cm} (28a)$$

s.t. $H\beta = R$  \hspace{1cm} (28b)$$

$C\beta - C_m A \leq 0$  \hspace{1cm} (28c)$$

$-I\beta - PA \leq q$  \hspace{1cm} (28d)$$

$K_G(\beta_k) + K(A) \succeq 0 \forall k = 1, \ldots, N_k$  \hspace{1cm} (28e)$$

$2.3.2$ Final implementation

As discussed in the previous section, the inclusion of the local stability constraint introduces an iterative solver process, which is shown in Algorithm 1.

**Algorithm 1** Truss optimization including global and local stability.

1: Initialize $L$, $H$, $R$, $C_m$, $K$, and $K_G$
2: Solve (16a–16d) to find $\beta_0$
3: Set $i \leftarrow 1$ and $\Delta V = 1$
4: Initialize $P$ and $q$ in (27) from $\beta_0$
5: while $\Delta V > \varepsilon V$
6: Solve (28a–28e) to find $A_i$ and $\beta_i$
7: Update $P$ and $q$ from $\beta_i$
8: $V_i = A_i^T L$ and $\Delta V = \frac{V_{i-1} - V_i}{V_{i-1}}$
9: $i \leftarrow i + 1$
10: end while

Lines 1–2 initialize and solve the global stability problem (16a–16d) to generate an initial solution $0$ to be used in the local stability constraint (27) initialized in line 4. In lines 5–10, the linearization in (27) is adjusted until convergence of the change in total volume $\Delta V$. The convergence criterion is chosen as a small number $\varepsilon V = 10^{-3}$.

### 3 Numerical applications

The presented method to truss layout optimization with FELA, including global and local stability, is applied to three different two-dimensional structures. Firstly, a simple
two-bar structure subject to both global and local stability issues is studied, and the numerical results are compared with the analytical solution. Secondly, a cantilever beam is studied when subject to a single and multiple point loads. Finally, a column structure subject to a vertical load is studied, and the effects of varying material parameters and slenderness ratios are investigated.

If nothing else is stated, the material parameters are given as $E = 210 \text{ GPa}$ and $f_y = 235 \text{ MPa}$. In the result figures supported nodes are represented by solid circles while loads are applied at the white squares only. For the two latter structures, the meshes are described by the number of nodes in the $x$- and $z$-direction by $n_x$ and $n_z$, respectively, and the connectivity is given by the parameter $r$. If $r = 1$, all nodes are connected with elements to their nearest nodes in all directions, if $r = 2$, all nodes are connected to both the nearest and next-nearest nodes in all directions, and so on.

The code was implemented in Matlab R2016b (Mathworks 2016), and the convex solver was Mosek version 8.1.0.62 (MOSEK 2018b). Mainly results were computed on a desktop with an Intel Core i7-6500U CPU 2.59 GHz and 32 GB RAM. The single case with beyond 8000 elements was computed on a setup with 2x Intel Xeon Processor 2650v4 (12 core, 2.20 GHz) and 256 GB RAM.

### 3.1 Two-bar structure

To validate the implemented methods, a simple two-bar structure is analysed initially. The structure is seen in Fig. 4 and the parameters given as $l_1 = 8 \text{ m}$, $l_2 = 10 \text{ m}$ and $P = 1 \text{ MN}$.

The solution to the optimization problem only containing global stability (16a–16d) is given as $a_1 = 0.038 \text{ cm}^2$ and $a_2 = 42.553 \text{ cm}^2$. Including local stability, the solution from Algorithm 1 is given as $a_1 = 0.038 \text{ cm}^2$ and $a_2 = 49.247 \text{ cm}^2$, hence a slight increase of $a_2$ to prevent column buckling.

The analytical solution to the two-bar structure including global stability is derived in Poulsen and Olesen (2015) and given as

$$a_1 = \frac{P l_1}{E l_2}, \quad a_2 = \frac{P}{f_y}$$

(29)

When local stability is governing, the solution becomes

$$a_1 = \frac{P l_1}{E l_2}, \quad a_2 = \sqrt{\frac{P}{a_2}}$$

(30)

hence, the area of element 2 is now given by the critical Euler load (21). The analytical solution is found to $a_1 = 0.038 \text{ cm}^2$ and $a_2 = 42.553 \text{ cm}^2$ for global stability, and $a_2 = 49.247 \text{ cm}^2$ for local stability, where $a_e$ is given by $a_{tube}$ and (20). The numerical solution thus corresponds to the analytical solution. Furthermore, it is noted that the cross section area of member $a_1$, ensuring global stability, is only about one thousandth of the cross section area of member $a_2$, constituting the main load-bearing structure. The additional volume of material to constrain the structure from global stability issues is thus almost neglectable, but crucial to the structural layout and integrity.

### 3.2 Cantilever beam

The well-known cantilever beam with tip loads is studied. The domain is rectangular with a side length ratio of 2:1, supported along the left edge and loaded with point loads at the upper and lower right corners. The domain, boundary conditions and loads are seen in Fig. 5.

The parameters are given as $L = 5 \text{ m}$ and $P_1 = P_2 = 100 \text{ kN}$.

Four different cases are studied. In cases 1–3, only load $P_1$ is applied, whereas in case 4, loads $P_1$ and $P_2$ are applied as individual load cases. In cases 1–3, the effect of an increasing mesh density is studied. The coarser mesh used in cases 1 and 4 is seen in Fig. 6.

The mesh parameters $n_x$, $n_z$ and $r$ are shown in Table 1 for the four cases, together with the number of nodes $N$.
degrees of freedom $N_d$, number of elements $N_e$ and number of load cases $N_k$. Additionally, the objective values $V$ are shown for the three types of optimization problems ((7a–7c), (16a–16d), (28a–28e)). Finally, the number of elements in the final optimized structures, above a given threshold of $10^{-2} \cdot \max(A)$, is shown for the three optimization problems. The threshold is chosen to remove elements with near zero-area, and given relative to the largest element area.

In Fig. 7, the optimized structures of the cantilever beam case 1 are seen. In Fig. 7a, the solution without any stability constraints is seen, hence a pure material optimization. In Fig. 7b, the global stability constraint is added, and in Fig. 7c, the local stability constraint is added.

From Table 1 and Fig. 7, it is seen how the volume is unaffected when adding the global stability constraint compared with the structure without stability constraints. That being the case, for case 1 with coarse mesh, the final structures from the two optimization problems are almost identical, with only one additional bar in the latter. The structural layout is thus almost unchanged when including global stability in this particular case, which later is seen commonly not to be the case. When adding the local stability constraint, see Fig. 7c, the change is significant in both structural layout, number of elements and volume. The number of elements is reduced substantially as the material is better utilized in fewer but larger elements when constraining local stability. As a result, the design is very different from the previous, and the volume is increased.

From cases 2 and 3 in Table 1, it is seen that when increasing the mesh density, the usual behaviour of decreasing and converging volume is seen in all three optimization problems. Furthermore, it is seen that the number of elements in the structures increases with increasing mesh density. Similar to case 1 and what was seen in Section 3.1, the change in volume from the structures without stability constraints to structures including global stability is insignificant.

In Figs. 8 and 9, the final result (including global and local stability constraints) for cases 2 and 3, with medium and dense meshes, are shown. In Fig. 9, only the main structure is shown with coloured elements, whereas the remaining members are shown in greyscale to simplify the plot.

In Fig. 8, the structure is to a certain extent similar to the one in Fig. 7c with few elements and a clear structural system, though with a smoother curve on the lower compression part. On the contrary, the structure in Fig. 9 contains a large number of elements. The main structure (shown in colour) is though to some degree similar to cases 1 and 2. The compression part is reduced to a few large members, whereas the tension part is distributed over many elements in a fan-like structure. This fan-like structure resembles a thin plate element, which is known to be more optimal than truss-like structures in tension (Sigmund et al. 2016). Furthermore, the entire main structure is supported by several very thin members (shown in greyscale) to ensure global stability. Finally, it is noted how the relative increase in volume when adding local stability is significantly smaller for case 3 compared with cases 1 and 2. However, it is important to recall that when including local stability, the problem becomes concave; hence, no global optima are guaranteed for direct comparison.

As a final remark to the above study, it is noted that the results of all three cases resemble the usually optimized structures of the tip-loaded cantilever beam, see e.g. Changizi and Jalalpour (2017). A quantitative comparison has not been possible, as essential material parameters have not been stated in the paper.

### Table 1  Cantilever beam mesh parameters, number of load cases $N_k$, objective values $V$ and number of elements $N_e$ above given threshold of $10^{-2} \cdot \max(A)$ for the three types of optimization problems

<table>
<thead>
<tr>
<th>Case</th>
<th>Load</th>
<th>$n_x$</th>
<th>$n_z$</th>
<th>$r$</th>
<th>$N$</th>
<th>$N_d$</th>
<th>$N_e$</th>
<th>$N_k$</th>
<th>Objective value $V$ [10^{-4}m^3] / $N_e &gt; 10^{-2} \cdot \max(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$P_1$</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>15</td>
<td>24</td>
<td>78</td>
<td>1</td>
<td>170.2/41</td>
</tr>
<tr>
<td>2</td>
<td>$P_1$</td>
<td>9</td>
<td>5</td>
<td>4</td>
<td>45</td>
<td>80</td>
<td>740</td>
<td>1</td>
<td>158.5/54</td>
</tr>
<tr>
<td>3</td>
<td>$P_1$</td>
<td>17</td>
<td>9</td>
<td>8</td>
<td>153</td>
<td>288</td>
<td>8,712</td>
<td>1</td>
<td>156.7/179</td>
</tr>
<tr>
<td>4</td>
<td>$P_1$ and $P_2$</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>15</td>
<td>24</td>
<td>78</td>
<td>2</td>
<td>180.6/49</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>With global stability: 170.2/42</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>158.5/55</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>156.7/374</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>180.6/49</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>170.2/42</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>158.5/55</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>156.7/374</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>180.6/49</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>203.0/16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>189.3/34</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>160.3/195</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>207.9/16</td>
</tr>
</tbody>
</table>
The last study of the cantilever beam, case 4 with two individual load cases $P_1$ and $P_2$, is seen in Fig. 10.

In Fig. 10a, the structure resembles the one in Fig. 7a, only subject to load $P_1$, but symmetric around the horizontal center axis being able also to support load $P_2$. Hence, including global stability in Fig. 10b does not change the structure for this particular case. Including local stability in Fig. 10c makes a significant change to the layout, where the symmetry in Fig. 10a and b disappears due to the asymmetry in the local stability constraint. The load $P_2$ is thus carried in a tension rod to the top node, utilizing the same ‘expensive’ compression members to carry both loads $P_1$ and $P_2$.

### 3.3 Column structure

Here, a vertically loaded column is studied. The column is confined to a rectangular domain with fixed height $H$ and varying width $W$, hence varying slenderness ratios. The domain is supported along the bottom edge and loaded with a point load centered at the top. The domain, boundary conditions and load are seen in Fig. 11. The fixed parameters are given as $H = 4$ m, $P = 10$ kN and node connectivity $r = 2$.

Four different cases are studied. In the first case, the material parameters are unity, hence $E = 1$ GPa and $f_y = 1$ MPa, and the 235 MPa in (20) is replaced with 1 MPa. For cases 2–4, the material parameters are $E = 210$ GPa and $f_y = 235$ MPa. In cases 2–4, the effect of increasing column slenderness is studied. The mesh used in cases 1 and 2 is seen in Fig. 12. It must be stated, that overlapping elements, e.g. two elements in the same plane extending from the same node to the closest and second closest node, respectively, are plotted in the same plane. In few cases, this is not the best visualization, but in general, this has been considered the most accurate way to visualize the results consistently.

The column width $W$ and mesh parameters are seen in Table 2 for the four cases. Furthermore, the material parameters $E$ and $f_y$ are shown, as well as the objective values $V$ and the number of elements $N_e$ in the final structures for the three types of optimization problems.

In the first case, the material parameters are given as unity; hence, the usual ratio of around 1:1000 between $E$ (GPa) and $f_y$ (MPa) is neglected. This particular case is included, as it is common practice to use unity material parameters in optimization benchmark examples (Sigmund and Bendsøe 2003). This ratio has no effect on the results when optimizing without global stability constraints. However, when including global stability constraints in the optimization problem, this ratio has a considerable impact on the final optimized structures. Despite this, previous works such as (Guo et al. 2005; Descamps and Filomeno Coelho 2014; Tugilimana et al. 2018) use unity material parameters in optimization problems with global stability constraints.
constraint. Case 1 thus demonstrates the importance of using correct material parameters, when working with global stability, opposite to pure material optimization.

In Fig. 13, the results of case 1 are seen. For comparison, the results of case 2 (similar to case 1, but with $E = 210 \text{ GPa}$ and $f_y = 235 \text{ MPa}$) are seen in Fig. 14. In Fig. 13a and 14a, solutions without stability constraints, the structures are seen to be similar, as the load is carried directly from load point to boundary condition. When including the global stability constraint, Figs. 13b and 14b, the differences between the two cases are clear. In case 1, the point load is mainly carried in the outer members with a web of thinner members in the interior. Contrary, in case 2, the load is mainly carried by the center column, which is supported by an outer web of very thin members. When including local stability, Figs. 13c and 14c, the differences in the structural system are similar, though more pronounced. A quantitative comparison is not possible due
Table 2  Column structure mesh parameters, material parameters, objective values $V$ and number of elements $N_e$ above given threshold of $8 \times 10^{-5} \cdot \max(A)$ for the three types of optimization problems

<table>
<thead>
<tr>
<th>Case</th>
<th>$W$</th>
<th>$n_x$</th>
<th>$n_z$</th>
<th>$N_d$</th>
<th>$N_e$</th>
<th>$E$</th>
<th>$f_y$</th>
<th>Objective value $V$ [10^{-6}m^3] / $N_e &gt; 10^{-4} \cdot \max(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[m]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>[GPa]</td>
<td>[MPa]</td>
<td>Without stability</td>
</tr>
<tr>
<td>1</td>
<td>$H/2.5$</td>
<td>5</td>
<td>6</td>
<td>30</td>
<td>50</td>
<td>213</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$H/2.5$</td>
<td>5</td>
<td>6</td>
<td>30</td>
<td>50</td>
<td>213</td>
<td>210</td>
<td>235</td>
</tr>
<tr>
<td>3</td>
<td>$H/10$</td>
<td>3</td>
<td>11</td>
<td>33</td>
<td>60</td>
<td>204</td>
<td>210</td>
<td>235</td>
</tr>
<tr>
<td>4</td>
<td>$H/20$</td>
<td>3</td>
<td>21</td>
<td>63</td>
<td>120</td>
<td>414</td>
<td>210</td>
<td>235</td>
</tr>
</tbody>
</table>

to the different material parameters, but the qualitative comparison above shows the importance of using correct material parameters when working with global stability. As a final comment, it is noted that the structures from case 1 resemble the results in Changizi and Jalalpour (2017), where a similar column structure is optimized. Again, a quantitative comparison has not been possible.

In cases 2–4, the effect on increasing slenderness is studied. The results for the three cases with increasing slenderness are seen in Figs. 14, 15 and 16. In some plots, additional elements, to the number of elements with cross section areas above the threshold shown in Table 2, are plotted in grey-scale. The grey-scale members are plotted to visualize kinematic stable structures, and the number of members in black is used for quantitative comparisons.

For all three cases, the structures, optimized without stability, are identical in layout and volume $V$, see Figs. 14a, 15a and 16a. However, when including global stability, the differences become noticeable. In Fig. 14b, the load is, as mentioned previously, mainly carried by the center column, which is supported by an outer web of very thin members to constrain global stability. In Fig. 15b, the load is still mainly carried by the center column, but towards the bottom, more load is carried by the increasing outer members. In case 4, Fig. 16b, the load is only carried by the center column in the upper part but gradually transferred to the outer members. Thus, in the lower part, the load is mainly carried by the outer members. In both case 3 and 4, the load carrying members are supported by thin members to constraint global stability. A reason for the more
slender columns to transfer the load from the center to the outer is the increased need for bending stiffness near the bottom to resist global stability problems. Another reason for this effect is the decreased distance from the load to the outer members, when slenderness is increasing. The ‘cost’ in material used to transfer the load to the outer members is thus reduced by increasing slenderness. As an additional note, it is remarked that for the more slender columns, the increase in volume, from the structures without stability to the ones with global stability, is more significant. However, it is noted that the increase, in general, is small, similar to what was seen in both Sections 3.1 and 3.2. Here it is thus a general trend, that including global stability does not increase the volume significantly, but has a large impact on the structural layout.

When including the local stability constraint, for all three cases, the effects discussed above are more pronounced, see Figs. 14c, 15c and 16c. Similar to the cantilever structures discussed in Section 3.2, the increase in volume when including local stability is likewise significant for all three column cases. The relative increase is though reduced for the more slender columns, where the material utilization is better.

4 Conclusions

In this paper, a new method to handle global and local stability simultaneously in truss optimization was presented. The method was based on the finite element limit analysis method, allowing very efficient state-of-the-art convex solvers to be used in solving the established optimization problem. The global stability constraint was solved by the introduction of a semidefinite constraint, whereas local stability was handled through the critical Euler buckling load. The overall concave problem was solved in an iterative process with convex sub-problems.

The proposed method was demonstrated through three different applications. The implementation of the method was validated against the analytical solution of a simple two-bar structure. The study of the cantilever beam with tip loads proved the convergence properties of the method by

![Fig. 15](solution_of_column_structure_case_3_with_width_w_4_10_m.png)

(a) Without stability, $V = 170.2 \times 10^{-6} \text{m}^3$, $N_e = 19$ (black)

(b) With global and local stability, $V = 171.3 \times 10^{-6} \text{m}^3$, $N_e = 95$ (77 in black)

(c) With global stability, $V = 191.2 \times 10^{-6} \text{m}^3$, $N_e = 75$ (66 in black)

Fig. 15 Solution of column structure case 3, width $W = 4/10$ m

![Fig. 16](solution_of_column_structure_case_4_with_width_w_4_20_m.png)

(a) Without and local stability, $V = 172.2 \times 10^{-6} \text{m}^3$, $N_e = 192$ (187 in black)

(b) With global stability, $V = 178.5 \times 10^{-6} \text{m}^3$, $N_e = 125$ (123 in black)

Fig. 16 Solution of column structure case 4, width $W = 4/20$ m
increasing mesh densities. The convergence in decreasing material volume was seen for all three optimization problems; pure material optimization, including global stability, and including both global and local stability. With the vertically loaded column structure, the importance of using correct material parameters, when including global stability in the optimization problem, was treated. Hence, the usual practice of using unity material parameters in optimization benchmark examples was challenged by comparison of the structural layout of two identical optimization problems, only differing in applying unity and real material parameters, respectively. The result clearly showed the difference and underlined the recommendation of using correct material parameters in structural optimization including stability. Furthermore, all three studies indicated how the effect of including global stability was small on the volume but made a significant impact on the structural layout. Including local stability led in general to a significant volume increase, and did in several cases also lead to considerable changes in the structural design.

In general, the studies and findings emphasized the eligibility and potential of the method. The main contribution of this paper is thus a newly proposed approach to efficient truss optimization including global and local stability. Hence, the presented basic formulations of the FELA-based optimization problem including global stability, and the proposed algorithm handling the linearized local buckling constraint, form a basis for further studies and method development.

The perspectives of further development are promising, but with several challenges to cope. The large-scale perspectives in the present formulations and in combination with the Mosek solver are limited. Hence, to utilize the method fully in practical applications, where a large amount of elements are desirable to accommodate larger design freedom, some challenges are to be overcome. Another challenge is the local stability constraint leading to a concave optimization problem, hence only local minima as solutions. Possible methods to search the solution space towards more optimal solutions are thus desirable. Finally, methods to reduce the truss complexity, hence number of elements in the final optimized designs, especially desirable for trusses with dense initial meshes, are of interest towards practical applications.

Acknowledgments The presented work is part of an industrial ph.d. project with the title “Innovative design of steel bridge girders in cable supported bridges” and is carried out in cooperation with COWI A/S, DTU Civil Engineering and DTU Mechanical Engineering.

Funding information This study was funded by the COWI Foundation grant C-131.02 and Innovation Fund Denmark grant 5189-00112B.
Paper V

"Large-scale truss optimization including global and local stability"

M. Baandrup, J.F. Olesen & P.N. Poulsen

Submitted to: Structural and Multidisciplinary Optimization, 2019
Large-scale truss optimization including global and local stability

Mads Baandrup · John Forbes Olesen · Peter Noe Poulsen

Received: date / Accepted: date

Abstract Including global and local stability in truss layout optimization is essential in practical applications to avoid unstable structures. However, when doing so, the challenge of solving large-scale optimization problems increases significantly. This challenge has previously been approached in different ways, though without exceeding 10,000 elements in the ground structure. In this paper, a new method for handling global and local stability in large-scale truss optimization is presented. The method is based on finite element limit analysis and is demonstrated to solve problems larger than 50,000 elements. Global stability is introduced with a convex semidefinite constraint. Local stability is constrained by the nonconvex Euler buckling load, solved iteratively as linearized convex sub-problems. The solution strategy is based on convex duality, from where the semidefinite dual variables are interpreted as critical eigenmodes. When solving nonconvex and large-scale optimization problems other challenges arise, such as a solution-space of multiple local minima and increasing truss complexity, both of which are essential to handle in practical applications. Thus, novel methods to search the solution-space for better solutions and to reduce truss complexity are proposed. The proposed methods are demonstrated through three different applications, which illustrate the large-scale capabilities, as well as the nonconvex solution-space search and truss complexity reduction.

Keywords Truss topology optimization · Global and local stability · Limit analysis · Large scale

Acknowledgements The presented work is part of an industrial ph.d. project with the title "Innovative design of steel bridge girders in cable-supported bridges" and is carried out in cooperation with COWI A/S, DTU Civil Engineering and DTU Mechanical Engineering. The project is supported financially by the COWI Foundation grant C-131.02 and Innovation Fund Denmark grant 5189-00112B. The authors are thankful to Erling D. Andersen from Mosek Aps for suggesting solving the dual problem instead of the primal.

1 Introduction

Truss optimization is often based on a ground structure, with the member cross-sectional areas as continuous design variables. In practical application, it is desirable to have a high degree of design freedom in order to obtain good and detailed solutions, hence to allow for large-scale optimization. Here, a high degree of design freedom is obtained by a high number of members in the ground structure. Another essential requirement for practical truss optimization is the inclusion of global and local stability, in order to avoid unstable structural solutions. The classic truss optimization problem without stability constraints can be solved efficiently by state-of-the-art solvers, also for large-scale. However, when including stability requirements in the optimization problem, the challenges of solving large-scale problems are amplified since the numerical problem size increases significantly.

Previously, the problem of including stability requirements in truss optimization has been approached...
in many different ways, see, e.g., Achtziger (1999a,b); Ben-Tal et al. (2000); Kočvara (2002); Madah and Amir (2017). In recent years, the development within software and computational power has made it possible to solve truss structures with beyond 8,000 members including stability constraints. In Descamps and Filomeno Coelho (2014), simultaneous geometry and topology optimization was carried out, implementing global stability through the nominal force method, and local stability through a local buckling criterion. In the paper, a large-scale structure of 984 members was optimized. In Tugilimana et al. (2018), the force equilibrium was formulated elastically, global stability was formulated as a semidefinite constraint on the sum of the global elastic and geometric stiffness matrices, and local stability was handled as a member constraint based on the critical Euler load. In the paper, a large-scale structure of 3,351 members was optimized. Similarly, in Weldeyesus et al. (2019) global stability was defined by a semidefinite constraint. However, local stability was disregarded. In the paper, a large-scale structure imitating 90,100 members was optimized, however, the optimization problem being solved only contained 7,337 members. Recently, an approach to handle global and local stability simultaneously in truss optimization was presented in Poulsen et al. (2019), where the method was based on the finite element limit analysis (FELA) method, which is formulated as a convex linear optimization problem. Global stability was stated as a convex semidefinite constraint, and local stability through the critical Euler load, making the overall problem nonconvex. However, the problem was step-wise linearized, yielding convex sub-problems, forming the steps of an iterative solution strategy. In the paper, a structure of 8,712 members was optimized.

In the present paper, a new formulation of the methods in Poulsen et al. (2019) is presented, which allows for problems beyond 50,000 members to be solved. In the present work, the convex sub-problems solved in Poulsen et al. (2019) are transformed into their dual formulations. Thereby, significantly, reducing the size of each optimization problem, and thus diminishing solver times and increasing the allowable magnitude of models that may be optimized. Furthermore, the duality reveals how the semidefinite dual variables may be interpreted as critical eigenmodes.

The present methods are based on the FELA method, which was introduced with linear optimization in Anderheggen and Knöpfel (1972). The method is based on the finite element method but defines a convex linear optimization problem assuming perfect plastic material behavior. Later the method has been developed in order to handle various kinds of elements, such as beams and slabs, Damkilde and Hayer (1993); Krenk et al. (1994), trusses and plates, Damkilde (1995); Gilbert and Tyas (2003); Gilbert et al. (2014), and in recent years it has been applied in reinforced concrete disks, slabs, and joints, Poulsen and Damkilde (2000); Herfelt et al. (2016, 2017, 2018); Herfelt (2017). A 98 line python script implementing convex truss optimization by limit analysis, without stability constraints though, may be found in He et al. (2019).

The solution of large-scale (and nonconvex) optimization problems made possible by the dual formulations, introduce new challenges to be treated in practical applications. The nonconvex nature of the problem typically encompasses a solution-space with multiple local minima. As the general goal of optimization is to find the best solution, a novel algorithm to search the solution-space for alternative local minima is proposed, in order not to overlook a better solution than the first (possibly local) minimum encountered. This algorithm generates different initial starting points, but due to the applied procedure, they are all near-optimal, and in this way, the number of searches can be limited. With the increasing element numbers, the complexity of the trusses increases, hence the number of joints and construction cost increase. A heuristic method to reduce truss complexity is proposed by imposing penalties on small member areas and short members, as discussed in Parkes (1975); He and Gilbert (2015).

Three cases are studied to demonstrate the proposed methods; a plane cantilever beam, a three-dimensional cantilever beam, and a large-scale roof structure. The study of the plane cantilever beam validates the implementation, as well as demonstrates the algorithm searching the nonconvex solution-space. Furthermore, the plane cantilever beam is used for demonstrating the
performance increase of the dual formulation, in terms of solver time and model size. The interpretation of critical eigenmodes, the method for reducing truss complexity, and the convergence properties of the method, are demonstrated by the three-dimensional cantilever beam. Finally, the roof structure serves as an illustration of the large-scale capabilities, for single and multiple load cases.

2 Truss layout optimization

In Poulsen et al. (2019) the direct and primal formulation of the FELA based truss optimization problem, including global and local stability, was implemented and demonstrated. The efficient convex solver by Mosek MOSEK (2018b) was applied to solve the optimization problem. However, the implementation of the primal formulation in the syntax of the Mosek solver was limited to the model size (number of truss elements and degrees of freedom) due to substantial memory consumption and long computation times. Hence, to solve large-scale models and reduce the solver times significantly, an alternative formulation, in the form of the dual, is applied.

Firstly, the primal truss layout optimization problem, including global and local stability, is briefly revisited; however, the reader is referred to the original paper for full insight. Secondly, the equivalent dual formulation is derived, and thirdly, a heuristic method to reduce truss complexity is proposed. Subsequently, these parts are combined in an algorithm for implementation. Finally, the concavity of the local stability constraint, leading to a solution-space of multiple local minima, is treated. Thus, a method to search this solution-space for better solutions is proposed.

2.1 Primal formulation

The truss optimization problem, including global and local stability constraints, was defined with FELA in Poulsen et al. (2019). FELA is a method similar to the finite element method (FEM), however, with the assumption that materials behave in a perfectly rigid-plastic manner. The load-deflection path is thus not determined, as no deformations occur before yielding. The mathematical problem of FELA is formulated as a linear problem. Thus, a method to search this solution-space leading to a solution-space of multiple local minima, is applied.

The structures studied in this paper are trusses defined by \( N \) nodal points in a \( d \)-dimensional space \( d \in \{2, 3\} \). The number of supports is denoted \( N_s \), and the number of equilibrium equations is \( N_q = d \cdot N - N_s \). The structure is subjected to \( N_k \) load cases given by external load vector \( R_k^{[N_e \times 1]} \) for load case \( k \). The total load vector is denoted as \( R = [R_1^\top \cdots R_k^\top]^\top \). The number of elements in the structure is given by \( N_e \) and the design variable is given as the element areas \( a_e \) collected in \( A^{[N_e \times 1]} \). Similarly, the element lengths are collected in \( L^{[N_e \times 1]} \). The total truss volume is thus given as

\[
V = \sum_{e=1}^{N_e} a_e \ell_e = A^\top L
\]  

The bar force for element \( e \) and load case \( k \) is denoted \( \beta_{e,k} \) and all bar forces are collected in \( \beta^{[N_e \times 1]} \).

The primal formulation of the problem was given as

\[
\min_{\beta, A} A^\top L \tag{2a}
\]

s.t. \( H\beta = R \) \tag{2b}

\[
C\beta - C_m A \leq 0 \tag{2c}
\]

\[
-I\beta - PA \leq q \tag{2d}
\]

\[
K_G(\beta_{e,k}) + K(A) \geq 0 \forall k = 1, \ldots, N_k \tag{2e}
\]

The main design variables are the member areas \( A \) and the secondary design variables are the bar forces \( \beta \). The objective function (2a) being minimized is the total truss volume (1). The first constraint (2b) defines the equilibrium equations, where the equilibrium matrix \( H^{[N_e \times N_e \times N_k]} \) ensures the nodal equilibrium of internal (\( \beta \)) and external (\( R \)) nodal forces for all unsupported nodes. The second constraint (2c) defines the yield condition, or stress constraint by \( C^{[2N_e \times N_e \times N_k]} \) and \( C_m^{[N_m \times N_e \times N_k]} \). The effect of the constraint may be seen as a conversion of the bar forces to their absolute values, which are restricted by the yield stress, \( f_y \), of each bar, respectively. Thus, for each element two inequalities are given for each load case

\[
\begin{bmatrix}
1 & 0 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
\beta_{e,k} - f_y \\
f_y
\end{bmatrix}
\begin{bmatrix}
a_e
\end{bmatrix}
\leq
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\tag{3}
\]

Additionally, constraint (2c) ensures the non-negativity of \( A \), since \( f_y \) is always positive.

The third constraint (2d) defines the restriction on local instability of each element. The local stability constraint is based on the Euler buckling load \( P_{cr} \) for a simply supported member. Hence, the constraint for element \( e \) is given as

\[
-\beta_e \leq P_{cr,e} = \frac{\pi^2 \cdot EI_e}{L^2} \approx \alpha_e a_e^3 \tag{4}
\]

where \( E \) is the modulus of elasticity and \( I_e \) is the second moment of area. Since only one cross-section parameter exists, only one second moment of area exists. This allows for the formulation of the quadratic approximation
term as a function of the design variable $a_e$. The constant $\alpha_e$ depends on the shape of member cross-section, see Achtziger (1999a,b). However, the constraint is non-convex, and it cannot be integrated immediately into the convex optimization problem.

To bypass the nonconvex properties of (4), and thus include it in the convex optimization problem, the quadratic constraint is linearized through its tangent or secant. Hence, the stability criterion can be formulated generally as a linear inequality constraint in the form

$$-\beta_e - p_e a_e \leq q_e$$

where $p_e$ and $q_e$ are linearization parameters defined from $\alpha_e$ and the previous solution to the optimization problem $\beta_e, 1$. Finally, element constraints in the form of (5) are assembled to give the matrix expression in (2d), where $I$ is the identity matrix, $D$ is a matrix containing $p_e$, and $q$ is a vector containing $q_e$.

Since the linearization requires the previous solution, initially problem (2) is solved without constraint (2d), after which the local stability constraint is added. Furthermore, as the linearization is a substantial simplification, the linearized equations are updated in an iterative process until convergence. The iterative update algorithm is similar to the one in line 1-10 in Algorithm 1, presented in Sec. 2.4.

The final constraint (2e) defines the restriction on global instability. The constraint is defined as a semidefinite constraint on the linearized buckling stiffness matrix comprised by the sum of the global geometric stiffness matrix $K_G$, which is a function of the bar forces, and the global elastic stiffness matrix $K_N$, which is a function of the bar areas. The linearized stiffness matrix represents the linear buckling eigenvalue problem. The limit for stable structural solutions is the positive semidefinite limit of this stiffness matrix, see, e.g., Ben-Tal et al. (2000); Kocvara (2002).

To apply both a rigid-plastic force distribution and an elastic linear buckling problem may seem contradictory, however, for a single load case it has been shown that the result of a plastic optimization is also an optimal structure under elastic conditions Dorn et al. (1964). This, and the situation of multiple load cases, were discussed further in Poulsen et al. (2019).

In the Mosek software MOSEK (2018a) the global stability constraint (2e) is handled as a linear matrix inequality (LMI) and formulated as

$$K_S = K_G(\beta) + K(A) = S, \quad S \succeq 0$$

where $K_S$ is converted to a set of linear equality constraints using an implicit semidefinite slack variable $S$. Thus, the semidefinite requirement is shifted from being explicit on $K_S$ to being implicit on $S$. The extra constraints add $m_K = N_k(N_d + 1)/2$ linear equality constraint functions to the problem, equivalent to the number of elements in the lower triangular part of $K_S$ times the number of load cases. The slack variables $S$ add $N_k$ semidefinite constraints to the problem.

The primal problem consist of $n = N_e(N_k + 1)$ design variables, $m = m_H + m_C + m_P + m_K = N_H + N_k + 2N_e \cdot N_c + N_e \cdot N_k + N_k \times N_d \cdot (N_d + 1)/2$ linear constraint functions, and $N_k$ semidefinite constraints.

In the formulation of FELA, multiple load cases are generally handled by an expansion of the system of equations. Hence, the number of design variables and constraints increases (almost) linearly with the number of load cases. Thus, the size of the constraint matrices increases quadratically, but as all matrices are defined sparse, the actual problem size increases linearly.

### 2.2 Dual formulation

By solving the dual formulation of the convex optimization problem (2), with the Mosek solver, the problem size and memory consumption, can be reduced considerably, hence allowing large-scale models to be optimized within a reasonable amount of time. Furthermore, the dual formulation give insight into the physical interpretation of the dual variables.

The main reason why the dual formulation is faster and less memory consuming is found in the global stability constraint (2e). In the primal formulation this constraint adds $m_K = N_k(N_d + 1)/2$ linear equality constraints and $N_k$ semidefinite constraints to the problem, when using the Mosek software due to constraint (6) being formulated explicitly. By solving the dual (equivalent to the formulation of the primal problem in the Mosek software MOSEK (2018a)), the linear equality constraints vanish from the formulation, only leaving the $N_k$ (now explicit) semidefinite constraints, treated as $N_k \times N_d$ conic constraints. Hence, this reformulation reduces the problem size significantly.

Problem (2) is reformulated into its dual form by the general method of establishing the Lagrangian function, see Boyd and Vandenberghe (2004). To handle the semidefinite constraint (2e) in the Lagrangian function, a semidefinite dual variable $Y$ is introduced for each load case. To derive the differential of the Lagrangian function a self-adjoint operator on $Y$ is defined, given
for a single load case as
\[
(K_S, Y)_F = \sum_{ij}^{N_x \times N_d} \left( \beta_i K_{G,0,e} + a_e K_{N,e} \right) Y_{ij} = \left[ \beta^T A^T \right] \left( K_{G,0,1} Y_F \right)_F + \left( K_{G,0,N_e} Y_F \right)_F + \left( K_{0,1} Y_F \right)_F = \left[ \beta^T A^T \right] K_S^F(Y)
\]

where the notation \((\cdot, \cdot)_F\) defines the Frobenius inner product. In the second term \(K_S\) is written as the sum over the \(N_e\) contributions of the global stiffness matrices \(\beta_i K_{G,0,e}\) and \(a_e K_{N,e}\) for element \(e\). Here, subscript 0 indicates the stiffness matrices defined to be factorized with the primal design variables (see Poulsen et al. (2019)). The final term can for multiple load cases be factorized with the dual variables \(y, z, t\), and \(Y\). Hence, the semidefinite part has changed from being defined explicitly on the constraint (2e) to being defined implicitly on the dual matrix variable \(Y_k\) (9e).

When rewriting from the primal to the dual, the number of dual variables is equivalent to the number of primal constraints \(m\). Similarly, the number of dual constraints is equivalent to the number of primal variables \(n\). Hence, in addition to the reduction of linear equality constraints from (2e), as discussed above, a significant reduction in the general linear constraints is seen due to \(m \gg n\) for the given problem. This reduction, despite the equivalent increase in dual variables, gives a further decrease in memory consumption.

The dual variables \(y\) and \(z\) can be interpreted as a representation of the nodal displacements (collapse mechanism) and the plastic strains, respectively, Krenk et al. (1994). The dual variable \(t\) can, similarly to \(y\), be interpreted as nodal displacements, however in connection to local stability. To interpret the semidefinite matrix variable \(Y\), it is first recalled that if \(\varphi^T K_S\varphi = 0\) and \(K_S\) is singular, then \(\varphi\) is an eigenvector to \(K_S\). From the Lagrangian (8a) it holds that \((K_S, Y)_F = 0\) on the limit of global stability, hence \((K_S, Y)_F = \varphi^T K_S \varphi\). From the latter, it is seen that \(\varphi^T = Y \Rightarrow \varphi = \pm \sqrt{\text{diag}(Y)}\), hence the critical eigenmode can be found from \(Y\), where the operator signs are given by the off-diagonal terms. The interpretation of the critical eigenmode from the semidefinite dual matrix variable is demonstrated during the numerical applications.

For later use, the dual version of (2) without the local stability constraint (2d), derived similarly to the above, is given as

\[
\begin{align}
\min_{y, z, t, Y} & -R^T y \\
\text{s.t.} & \begin{bmatrix} H^T & 0 \\ 0 & -C_m^T \end{bmatrix} y - \begin{bmatrix} C^T \\ -C_m^T \end{bmatrix} z + \begin{bmatrix} 1 & P^T \end{bmatrix} t + K_S^F(Y) = \begin{bmatrix} 0 \\ L \end{bmatrix} \\
& z \geq 0 \\
& t \geq 0 \\
& Y_k \geq 0 \quad \forall k = 1, \ldots, N_k
\end{align}
\]

2.3 Reduction of truss complexity

As a consequence of the high number of elements in a large-scale model, leading to costly joints, it is desirable, from an application-oriented perspective, to reduce the complexity of the optimized structure. By reducing the number of members in the final design, both the direct member cost is reduced, as well as the cost of joints.

Two heuristic methods are proposed to penalize short members and members with small cross-sectional areas, respectively. Both penalty methods are introduced
into the primal objective function (1), by adjusting the weighting of the different members. Consequently, the penalty only pushes the design away from using short and thin members, however allowing them if necessary.

The penalty functions are applied and updated in an iterative loop presented in Sec. 2.4.

2.3.1 Penalizing short members

Members are penalized as a function of their length, as shown in Fig. 1. The penalty varies linearly from the longest members not being penalized ($p_l = 0$), to the shortest members being penalized with $p_{l,\text{max}}$. The penalty factor is given as

$$p_{l,e} = -\frac{p_{l,\text{max}}}{\max(L)} \ell_{e,0} + p_{l,\text{max}}$$

where $\ell_{e,0}$ is the true length of member $e$. The contribution to the primal objective function (1) is adjusted accordingly, $\ell_e = \ell_{e,0}(1 + p_{l,e})$. As function (11) is linear w.r.t. $\ell_{e,0}$, $\ell_e$ is quadratic w.r.t. $\ell_{e,0}$, hence $\ell_e$ is quadratically penalized. All members are penalized but due to the quadratic penalty, the effect is only seen on the shortest members. In present studies, $p_{l,\text{max}} = 2$ is found to be an adequate value, as discussed in Section 3.3. The penalty function in Fig. 1 is the same for all members and it is fixed during the update of the local stability constraint, see Sec. 2.4.

The penalty on short members has previously been considered in Parkes (1975); He and Gilbert (2015) with the notional joint length approach, being closely related to the above proposed method. In present studies it was investigated that the two methods yield very similar results.

2.3.2 Penalizing small area members

Members are penalized as a function of their cross-sectional area, as shown in Fig. 2. The penalty is only applied for members with areas $a_e < \frac{f_y}{\alpha_e}$, equivalent to when the local stability constraint is active. The penalty function is given as

$$p_{a,e} = \begin{cases} -\frac{p_{a,\text{max}}}{f_y/\alpha_e} a_{e,i-1} + p_{a,\text{max}} & \text{if } a_{e,i-1} \leq \frac{f_y}{\alpha_e} \\ 0 & \text{if } a_{e,i-1} > \frac{f_y}{\alpha_e} \end{cases}$$

where $a_{e,i-1}$ is the cross-sectional area from the previous iteration and $p_{a,\text{max}}$ is the maximum penalty. In present studies, $p_{a,\text{max}} = 1$ is found to be an adequate value, as discussed in Section 3.3. The contribution to the primal objective function (1) is adjusted accordingly, $\ell_e = \ell_{e,0}(1 + p_{a,e})$. The penalty function in Fig. 2 is unique for each member based on $\alpha_e$ and is updated during the iterative update of the local stability constraint.

2.4 Final implementation

Algorithm 1 provides the implementation of the iterative local stability constraint and the penalty functions.

Algorithm 1 Truss optimization with global and local stability, and penalty functions

1: Initialize $L$, $H$, $R$, $C$, $C_m$, $K$, and $K_G$
2: Solve (10) and find $\beta_0$ from the dual solution
3: Set $i \leftarrow 1$ and $\Delta V = 1$
4: Initialize $P$ and $q$ in (9) from $\beta_0$
5: while $\Delta V > \epsilon_V$ do
6: Solve (9) and find $A_i$ and $\beta_i$ from the dual solution
7: Update $P$ and $q$ from $\beta_i$
8: $V_i = A_i^\top L$ and $\Delta V = \frac{|V_i - V_{i-1}|}{V_{i-1}}$
9: $i \leftarrow i + 1$
10: end while
11: Find $A_{i-1}$ from the dual solution
12: Set $\Delta V = 1$ and $\ell_{e,0} = \ell_e \forall e = 1, \ldots, N_e$
13: while $\Delta V > \epsilon_V$ do
14: Find $p_{l,e}$ from (11) and $p_{a,e}$ from (12) based on $A_{i-1}$
15: Set $\ell_e = \ell_{e,0}(1 + p_{l,e} + p_{a,e}) \forall e = 1, \ldots, N_e$
16: Update $L$ in (9b)
17: Solve (9) and find $A_i$ and $\beta_i$ from the dual solution
18: Update $P$ and $q$ from $\beta_i$
19: $V_i = A_i^\top L$ and $\Delta V = \frac{|V_i - V_{i-1}|}{V_{i-1}}$
20: $i \leftarrow i + 1$
21: end while

Lines 1-2 initialize and solve the global stability problem (10) to generate an initial solution $\beta_0$ to be
used in the local stability constraint (2d) initialized in line 4. In lines 5-10 the linearization in (2d) is adjusted until convergence of the change in volume $\Delta V$. The convergence criterion is chosen as $\varepsilon_V = 10^{-3}$. The problem solved in lines 5-10 are purely based on mechanics, hence after convergence the penalty function(s) $\ell_e = \ell_{e,0}(1 + p_{\ell,e} + p_{a,e})$ is added. The problem including penalty functions is solved iteratively in lines 13-20 until convergence.

The solution to the first step of the iteration will most likely violate the local stability constraint, which, however, indicates the need for the constraint. To ensure a stable algorithm, the local stability constraint is updated based on the stresses of the previous iteration instead of the cross-sectional areas. Thereby, a feasible solution may be retrieved by increasing the cross-sectional area in the present iteration.

### 2.4.1 Searching the nonconvex solution-space

In Algorithm 1 the convex problem (10) becomes nonconvex when the local stability constraint is added (9) and iteratively updated in lines 5-10. Thus, a global optimum cannot be guaranteed, and experience has shown that the final solution satisfying local stability (and penalty functions) in some cases is highly dependent on the initial convex solution $\beta_0$, since $P$ and $q$ are based on this. Therefore, a heuristic strategy to search the nonconvex solution-space of local minima is proposed to find better solutions, see Algorithm 2.

The strategy is to generate $N_j$ "random" initial solutions, designated $\beta_{0,j}$, which will result in $N_j$ different designs with volumes $V_j$. The $N_j$ initial solutions are found by creating new optimization problems based on (10). The primal formulation of (10) is thus modified before the dualization. Firstly, the objective function in the primal formulation of (10) is replaced with the following random function

$$V_{0,j} = \sum_{e=1}^{N_e} a_e R_{e,j}$$

where $R_{e,j}$ is a random number for element $e$ in the interval 0-1. Secondly, to ensure that the solution is close to the initial $V_0$, but allowing it to exceed $V_0$ to have alternative solutions, the following constraint is added to the primal formulation of (10)

$$A^\top L \leq (1 + \delta)V_0$$

where $\delta$ is some perturbation, in the order of 0.01–0.05, of the objective function from the initial solution. From the above described optimization problem a new "random" solution $\beta_{0,j}$ is generated. Due to the nonconvex nature of the optimization problem in lines 5-10 in Algorithm 1, these "random" solutions may lead to final structures with lower volumes than the structures found directly from solution $\beta_0$.

### Algorithm 2 Solution-space search

1: Algorithm 1 line 1-2
2: Find $A_0$ from the dual solution of (10) and set $V_0 = A_0^\top L$
3: for $j$ from 1 to $N_j$ do
4: Modify the primal formulation of (10) by replacing the objective function with (13) and adding the constraint (14). Solve the dual and find $\beta_{0,j}$ from the dual solution
5: Algorithm 1 line 3-20
6: Set $V_j = V_{i,j}$
7: end for
8: Find best solution $V = \min(V_1, \ldots, V_{N_j})$

In lines 1-2, in Algorithm 2, the global stability problem (10) is solved initially and the volume $V_0 = A_0^\top L$ is found from the solution. In lines 3-7 the $N_j$ initial solutions $\beta_{0,j}$, are generated as input for the optimization problem solved in lines 3-20 in Algorithm 1. Finally, in line 8 the best of the $N_j$ solutions is identified. In the following the method in Algorithm 2 is designated solution-space search, in short SSS.

### 3 Numerical applications

The presented methods for large-scale truss layout optimization, including global and local stability, are applied to three different structures. Firstly, the implementation of the dual formulation is validated with a plane cantilever beam, followed by a demonstration of the solution-space search algorithm. Secondly, the performance of the dual formulation is compared to the one of the primal formulation in regard to solver time and model size. Thirdly, a three-dimensional cantilever beam is used to illustrate the penalty functions and the convergence properties of the dual formulation. Here, also a parameter study of the penalty factors is carried out. Furthermore, the critical eigenmode derived from the dual variable $Y$ is visualized. Finally, the large-scale capabilities are illustrated with a cantilevered roof structure.

For all structures, the material parameters are given as $E = 210 \text{ GPa}$ and $f_y = 235 \text{ MPa}$, equivalent, e.g., to steel. In the result figures, for the two first structures, supported nodes are represented by solid circles while loads are applied at the white squares only. For all structures, the meshes are described by the number of nodes in the $x$-, $y$- and $z$-direction by $n_x$, $n_y$, and $n_z$, respectively, and the connectivity is given by the
parameter \( r \). If \( r = 1 \) all nodes are connected with elements to their nearest nodes in all directions, if \( r = 2 \) all nodes are connected to both the nearest and next-to-nearest nodes in all directions, and so on. With \( r > 1 \), the final structure may have overlapping bars, however, in tension this is not an issue, since the combined area is equivalent to a single member. Furthermore, when constraining against local stability, the algorithm will tend to collect material in as few members as possible in order to increase the critical buckling load. The code was implemented in Matlab R2016b MATLAB (2016), and the convex solver was Mosek version 8.1.0.62 MOSEK (2018b). Mainly, results were computed on a desktop with an Intel Core i7-6500U CPU 2.59 GHz and 32 GB RAM. However, if nothing else is stated, the cases with more than 25,000 elements were computed on a setup with 2x Intel Xeon Processor 2650v4 (12 core, 2.20 GHz) and 256 GB RAM.

3.1 Plane cantilever beam

To validate the implementation of the dual formulation a plane cantilever beam is studied initially, and a direct comparison with the results in Poulsen et al. (2019) is made. The domain is rectangular, supported along the left edge, and loaded with a point load at the upper right corner. The domain, boundary conditions, and load are seen in Fig. 3, with the parameters given as \( L = 5 \text{ m} \) and \( P = 100 \text{ kN} \). The mesh (Fig. 4) is defined by the parameters \( n_x = 5 \), \( n_z = 3 \), and \( r = 2 \), and the number of nodes is \( N = 15 \), number of degrees of freedom is \( N_d = 24 \), and number of elements is \( N_e = 78 \).

Three different methods are studied: a) the primal formulation (2) which was introduced in Poulsen et al. (2019), and in which the primal solution is given, b) the dual formulation (9) solved in lines 1-10 in Algorithm 1, and c) the solution-space search shown in Algorithm 2 (SSS algorithm), however leaving out the penalty functions in lines 11-20 in Algorithm 1.

The numerical results for the three methods are shown in Tab. 1, in terms of the objective value \( V \) and the number of elements \( N_e \) for the final optimized structures. However, only principal members with an area above a given threshold, relative to the largest element area, of \( 10^{-4} \cdot \max(A) \) are included in \( N_e \). Furthermore, for clarity, only the principal members are visualized in the plots. The three optimization problems are defined as a) without stability, b) with global stability, and c) with global and local stability. For the latter problem, also the number of iterations of line 5-10 in Algorithm 1 is shown. For the solution-space search, the best of the final solutions is shown. The solution obtained may be checked against the solutions to the convex optimization problems, i.e., the problem without stability and the problem with global stability, respectively. The solutions to the convex problems may be regarded as lower bound solutions for the specific case. The total solver time for each solution in Tab. 1 was below 1 second.

From Tab. 1 the objective values of the two convex optimization problems (without stability and with global stability) are seen to be identical for the primal and dual formulations. The discrepancy in the element numbers after threshold, are due to numerical noise from the Mosek solver on small member areas. The solution from the dual formulation is seen in Fig. 5, where Fig. 5a and 5b are similar to results obtained in Poulsen et al. (2019). For the nonconvex optimization problem, with global and local stability, a different and larger volume is found from the dual formulation compared to the primal. Further, the structural layout seen in Fig. 5c is very different from the one found in Poulsen et al. (2019). The variation is a result of the multiple local minima in the nonconvex optimization problem, with the primal and dual formulations converging towards two different solutions. Indeed, this result justifies the call for the solution-space search algorithm. It is evident from Fig. 5c that hinge cancellation is not considered. However, here hinges only occur between tension members, which does not obscure structural integrity for the specific load case. No unsupported hinges
occurs between compression members as a consequence of the global stability constraint.

The SSS algorithm is applied to the plane cantilever problem, where $N_j = 100$ solutions are generated. The perturbation in (14), of the initial solution $V_0 = 170.2 \times 10^{-4}$ m$^3$, is given as $\delta = 0.01$. In Fig. 6a the distribution of the 100 solutions, with global and local stability, is seen. The majority of the solutions are close to the solution in Fig. 5c found directly from the dual formulation. However, a number of solutions result in lower objective values, where the solution with the lowest final objective value $V = 174.2 \times 10^{-4}$ m$^3$ is seen in Fig. 6c. The “random” initial solution, only with global stability, which made the starting point for the best final solution, is seen in Fig. 6b, considerably different from 5b. From Tab. 1 it is seen that this “random” initial solution has a volume exactly 1.01 times larger than the solution found directly from the dual formulation. However, this small perturbation in the layout results in a 25.2% lower final objective value of the optimization problem with both global and local stability, only slightly larger than the solution without stability.

When comparing the layouts seen in Fig. 5c and 6c the reason for the significant reduction in volume is clear. The two large compression members in Fig. 5c are bisected in Fig. 6c in order to reduce the buckling length, and thus the member area required to ensure local stability. Consequently, an increase in the number of elements from 11 to 23 is seen, since additional supports are required to ensure global stability. However, the minimizing of the volume is significantly improved.

### 3.2 Performance increase by dual formulation

The performance increase by the dual formulation compared to the primal is studied as a function of an increasing number of elements. The basis of the study is the plane cantilever beam studied in the previous section. In the following study, the number of elements $N_e$, and the number of degrees of freedom $N_d$, are increased by an incremental increase in the number of nodes $n_x$ and $n_z$. The connectivity is fixed at $r = 7$, hence until $n_x > 8$ and $n_z > 8$, each node is connected to all other nodes. In this study, only the optimization problem, including global stability ((10) and the equivalent primal formulation) is included, as this is where the performance increase by the dual formulation appears. In the study, solver times and maximum solvable model sizes are measured. All results were computed on the same desktop with an Intel Core i7-6500U CPU 2.59 GHz and 32 GB RAM.

In Figure 7 the solver times as a function of the increasing number of elements are shown for the primal and dual formulations. The maximum model size for the primal formulation is found to be $N_e = 12,684$ and $N_d = 364$ with a solver time of 803 min., whereas the limit of dual formulation is found to be $N_e = 34,496$ and $N_d = 800$ with a solver time of 2,259 min. (not shown in Fig. 7). Thus, the upper limit on the model size is significantly increased by the dual formulation, due to the fewer constraints and thereby less memory demand. When comparing the solver times, the reduction is seen to be significant for the dual formulation, and the difference in solver times is seen to increase exponentially with an increasing number of elements.

Similar trends as discussed above are found when plotting solver times as a function of an increasing number of degrees of freedom, why this plot has been left out. Despite that the results seen in Fig. 7 are highly problem dependent, since the domain, boundary conditions, and loads affect the solver times, the increase in performance due to the dual formulation reflects a general trend.

### 3.3 3D cantilever beam

The plane cantilever beam studied previously is extended to a three-dimensional cantilever beam in order to illustrate the effect of the penalty functions and penalty factors as well as the convergence properties of the dual formulation. The domain is a box supported along the left end surface and loaded with a line load $p$ along the upper free edge. The domain, boundary con-

<table>
<thead>
<tr>
<th>Formulation/method</th>
<th>Objective value $V$ [10$^{-4}$ m$^3$] / $N_e &gt; 10^{-4}$ - max($A$) / Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primal</td>
<td>170.2 / 42 / -</td>
</tr>
<tr>
<td>Dual</td>
<td>170.2 / 37 / -</td>
</tr>
<tr>
<td>Best of SSS</td>
<td>[As dual]</td>
</tr>
</tbody>
</table>

Table 1 Plane cantilever beam objective values $V$, number of elements $N_e$ above given threshold of $10^{-4} \cdot$ max($A$), and number of iterations for three types of optimization problems. Primal results taken from Poulsen et al. (2019)
Table 1 Plane cantilever beam objective values
\[ V = 170.2 \times 10^{-4} \text{ m}^3, N_e = 37 \]
\[ V = 170.2 \times 10^{-4} \text{ m}^3, N_e = 39 \]
\[ V = 232.8 \times 10^{-4} \text{ m}^3, N_e = 11 \]

Fig. 5 Solutions of plane cantilever beam by the dual formulation. Color indication: blue/dark=compression, red/light=tension

Fig. 6 Plane cantilever beam by solution-space search (SSS). Color indication: blue/dark=compression, red/light=tension

Fig. 7 Solver times as function of increasing number of elements, of the optimization problem with global stability, shown for the primal and dual formulations. Markers indicate data points, connecting lines are only shown for clarity.

Two different meshes are studied, with a coarse and fine discretization, with mesh parameters given in Tab. 2. The coarse discretization is similar to Fig. 4, but with two additional planes of nodes in the y-direction. The two meshes are both solved with the dual formulation directly and with the SSS algorithm to identify possibly better solutions. The numerical results, for the in total four cases, are seen in Tab. 3. Here, the objective value \( V \) and number of elements \( N_e \) in the final structures are shown for four optimization problems. The first three optimization problems are similar to Tab. 1, whereas...
the fourth includes the penalty functions in addition to the global and local stability constraints, hence the entire Algorithm 1. For the two latter problems, also the number of iterations of line 5-10 and line 13-21 in Algorithm 1 is shown. The total solver time for each solution of Case 1 and 2 in Tab. 3 was below 15 seconds. The total solver time for the problem with penalty functions for Case 3 and 4 was 13.65 hours and 27.4 hours, respectively. Hence, the computational effort increases significantly with increasing problem size.

For Case 1 in Tab. 3, the volume from the first optimization problem is similar to the one in Tab. 1. This is expected due to the similar initial mesh layout and load, and due to the omission of the stability constraints. When global stability is included, the three-dimensional case is seen to have a slightly higher volume, due to the additional out-of-plane support, as compared to the plane case. The optimized structures for Case 1 are seen in Fig. 9. Here, the additional out-of-plane supports are clearly seen in the comparison between Fig. 9a and 9b. In Fig. 9c the structure optimized with constraints on global and local stability is seen to be similar to Fig. 5c, except for the additional members needed for stability in three dimensions. The effects of the penalty functions are seen in Fig. 9d, where both the amount of tension and compression members are reduced. In total, the number of elements is halved compared to Fig. 9c, however without any change in truss volume.

It is seen that the weight of the structure obtained without global stability constraints is almost the same as that with global stability constraints. However, the two problems yield different solutions. In many cases, the effect of the global stability constraint is insignificant to the objective value. This is due to the stabilizing web evolving from the optimization with the global stability constraint. Hence, the distribution of the material may be very different despite similar objective values.

As discussed in Sec. 2.2, the critical eigenmode $\varphi$ can be derived from the semidefinite dual variable $\mathbf{Y}$. The derived critical eigenmode for the structure in Fig. 9d is seen in Fig. 9e. From the figure, the eigenmode is recognized as out-of-plane instability for all three planes.

For all problems that include the penalty functions, the penalty factors are $p_{t,\text{max}} = 2$ and $p_{a,\text{max}} = 1$. The effect and sensitivity of varying penalty factors are illustrated in Fig. 10 and 11 for $p_{t,\text{max}}$ and $p_{a,\text{max}}$ respectively. The study is shown for Case 1 in Tab. 3. In the figures, the truss volume (solid lines) and number of elements $N_e$ (dashed lines) are shown as function of the penalty values. Furthermore, two situations are shown in each figure: with both penalty functions active (circles markers) and only a single penalty function active (cross markers).

From the figures it is seen that the effect on number of elements is stable with penalty factors beyond 1-2 for both $p_{t,\text{max}}$ and $p_{a,\text{max}}$. For increasing $p_{t,\text{max}}$, the volume increases despite an almost constant number of elements. Hence, in this case, high values of $p_{t,\text{max}}$ are not beneficial. On the contrary, the volume is unaffected of change in $p_{a,\text{max}}$. In general, low penalty factors are found to be adequate.

The SSS algorithm is applied in Case 2, where $N_j = 100$ solutions are generated with a perturbation of $\delta = 0.01$ in (14). In Fig. 12 the distribution of the 100 solutions, with global and local stability, and penalty functions, is seen. The majority of the solutions are close to the Case 1 solution. However, a large number of additional solutions are found with either higher or lower final volumes. The numerical results of the best solution are seen in Tab. 3, row 2. Similarly to the plane cantilever beam, the volume of the best solution of SSS is significantly lower (15.5%) than the solution computed directly by the dual formulation, however, with an increase in the number of elements. The structural explanation for this behavior is thus also similar to the plane case.

The number of elements in the initial ground structure is increased significantly from 747 in the coarse mesh to 25,200 in the fine mesh. The results of the fine mesh solved with the direct dual formulation are seen in Tab. 3 row 3. Compared to Case 1, the volumes of all optimization problems in Case 3 have decreased, as expected, due to the finer mesh discretization. In Case 4 the SSS algorithm is applied, where $N_j = 20$ solutions are generated with a perturbation of $\delta = 0.01$ in (14). In Fig. 13a the distribution of the 20 solutions, with global and local stability, and penalty functions, is seen. In Fig. 13b the best of the 20 solutions is seen. In the structure, the distributed load is mainly carried through the members in planes 2 and 4, whereas the load from planes 1, 3, and 5 is transferred to planes 2 and 4. This way, the number of costly compression members is reduced, hence lowering the total volume by 20.5% from the initial. The asymmetry in the layout is found due to the nonconvex solution-space of the problem, hence an even better symmetric solution is probably achievable. Furthermore, by observing the decreases in volumes from Case 2 to 4 (similar to Case 1 to 3), the convergence properties from applying the finer mesh is apparent.

Finally, for all the four cases, it is noted how significant the reduction in the number of elements is when applying the penalty functions. Especially for cases 3 and 4 with a high number of initial elements, the effect
is noticeable. Because of practical application issues, for instance, constructibility and total construction cost, the significant decrease in the number of elements is considered very beneficial.

### 3.4 Preliminary design of a large-scale roof structure

In the final application example, the design of a large-scale cantilevered roof structure is studied. The goal is to demonstrate the large-scale capabilities of the dual formulation. The domain, boundary conditions, and loads are seen in Fig. 14. The domain is a box supported along two adjacent end surfaces. Two load cases are applied; a

![Image of truss optimization](image.png)

**Table 3** 3D cantilever beam objective values $V$, number of elements $N_e$ above given threshold of $10^{-4} \cdot \max(A)$, and number of iterations for four types of optimization problems

<table>
<thead>
<tr>
<th>Case</th>
<th>Mesh</th>
<th>Formulation/method</th>
<th>Objective value $V$ [$10^{-4}$ m$^3$] / $N_e &gt; 10^{-4} \cdot \max(A)$ / Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Coarse</td>
<td>Dual</td>
<td>170.2 / 123 / -</td>
</tr>
<tr>
<td>2</td>
<td>Coarse</td>
<td>Best of SSS</td>
<td>[As Case 1] / 172.0 / 100 / -</td>
</tr>
<tr>
<td>3</td>
<td>Fine</td>
<td>Dual</td>
<td>158.5 / 290 / -</td>
</tr>
<tr>
<td>4</td>
<td>Fine</td>
<td>Best of SSS</td>
<td>[As Case 3] / 160.2 / 413 / -</td>
</tr>
</tbody>
</table>

(a) Without stability, $V = 170.2 \times 10^{-4}$ m$^3$, $N_e = 123$
(b) With global stability, $V = 170.3 \times 10^{-4}$ m$^3$, $N_e = 155$
(c) With global and local stability, $V = 337.5 \times 10^{-4}$ m$^3$, $N_e = 46$
(d) With global and local stability, and penalty functions, $V = 337.5 \times 10^{-4}$ m$^3$, $N_e = 23$
(e) Critical eigenmode $\phi$ of structure (d). Undeformed structure indicated by black dashed lines

**Fig. 9** Solution of 3D cantilever beam case 1; coarse mesh and dual formulation. Color indication: blue/dark=compression, red/light=tension
downwards uniformly distributed load \( p \) over the entire top surface and a downwards point load \( P \) on the top face at the free corner. The model parameters are given as \( W = 50 \text{ m} \), \( H = 5 \text{ m} \), \( p = 5 \text{ kN/m}^2 \), and \( P = 1 \text{ MN} \).

The large-scale capabilities are demonstrated for a single load case as well as two combined load cases. The maximum model size is only limited by memory and time. As the Mosek solver only supports shared memory, and not distributed memory, the limit is the available memory on a single node. The following problems are solved on a node with 256 GB RAM.

Four different cases are studied. In cases 1-3 a mesh of \( N_e = 32,640 \) elements is used to allow for two simultaneous load cases in the optimization. In Case 4 a mesh of \( N_e = 52,326 \) elements is used for a single load case. In cases 1-3 the mesh parameters are given as \( n_x = n_y = 8 \), \( n_z = 4 \), \( r = 7 \), \( N = 256 \), and \( N_d = 588 \), and for Case 4 the parameters are given as \( n_x = n_y = 9 \), \( n_z = 4 \), \( r = 8 \), \( N = 324 \), \( N_d = 768 \). In cases 1 and 4 only the distributed load \( p \) is applied, in Case 2 only the point load \( P \) is applied, and in Case 3 loads \( p \) and \( P \) are applied. The four cases are only solved with the di-
rect dual formulation. The applied loads, number of elements, number of load cases \( N_k \), and the numerical results for the four optimization problems are seen in Tab. 4. The results of the optimization problem with penalty functions for the four cases are seen in Fig. 15, 16, 17, and 18. In the figures small area members, mainly ensuring global stability, are shown in grey-scale.

Case 1 (Fig. 15) and 2 (Fig. 16) show the optimized structures for the distributed load and point load, respectively. Case 1 and 2 serve as comparison to Case 3, Fig. 17, which shows the optimized structure for the combined load cases. In Case 1 the distributed load is carried perpendicular to the diagonal by a structure with positive moment, and perpendicular to the supports by a structure with negative moment. In Case 2 the point load is partly carried by a long diagonal tension element, and partly by structures with negative moment along the free edges. In Case 3 the main features from cases 1 and 2 are clearly seen to be combined. The gain of optimizing multiple load cases simultaneous is clearly seen when comparing to the volume of a merged structure of cases 1 and 2, as indicated by the volumes shown in the last row of Tab. 4. For all cases the volume of Case 3 is lower than for the merged cases. The total solver time and memory use for Case 1 was 68.8 hours and 45 GB, for Case 2 40.5 hours and 45 GB, and for Case 3 259.4 hours (10.8 days) and 60 GB.

Finally, Case 4 (Fig. 18) optimized for the distributed load, demonstrate the large-scale capabilities of a single load case, with an initial ground structure of 52,326 elements (104,652 design variables and 157,746 constraints). The total solver time and memory use for Case 4 was 323.8 hours (13.5 days) and 117 GB. Other works on large-scale truss optimization including stability constraints have reached 984 elements (8,185 design variables and 6,633 constraint functions) in Descamps and Filomeno Coelho (2014), 3,351 elements (8,416 design variables and 16,832 constraints) in Tugilimana et al. (2018), and 7,337 elements (in the problem actually solved) in Weldeyesus et al. (2019). In the latter case, however, the 7,337 elements imitate, by adaptive ‘member adding’ techniques, the solution of potentially 90,100 elements.

In all cases, the solver times exceed days, which is not practical for application. However, the capabilities of the methods were demonstrated.

In comparison of Case 1 and 4, where the number of elements in the ground structure increases, the objective values decrease in the two first problems. This decrease is guaranteed by the convexity of these optimization problems ensuring a global minimum. However, in the two last problems, which are nonconvex due to the local stability constraint, the objective values increase. Hence, the solution to the two last problems are local minima.
Table 4 Roof structure results. Loads, number of elements $N_e$, number of load cases $N_k$, objective values $V$, number of elements $N_e$ above given threshold of $10^{-2} \cdot \max(A)$, and number of iterations for four types of optimization problems. Last row shows theoretical volumes of case 1 and 2 merged after optimization.

<table>
<thead>
<tr>
<th>Case</th>
<th>Load</th>
<th>$N_e$</th>
<th>$N_k$</th>
<th>Without stability</th>
<th>With global stability</th>
<th>W. global and local stability</th>
<th>With penalty functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Distr. $p$</td>
<td>32,640</td>
<td>1</td>
<td>6.80 / 320</td>
<td>6.80 / 305</td>
<td>6.64 / 279</td>
<td>6.82 / 275</td>
</tr>
<tr>
<td>2</td>
<td>Point $P$</td>
<td>32,640</td>
<td>1</td>
<td>2.42 / 90</td>
<td>2.42 / 80</td>
<td>4.13 / 79</td>
<td>4.17 / 70</td>
</tr>
<tr>
<td>4</td>
<td>Distr. $p$</td>
<td>52,326</td>
<td>1</td>
<td>5.69 / 417</td>
<td>5.69 / 399</td>
<td>6.96 / 365</td>
<td>6.94 / 345</td>
</tr>
<tr>
<td>Merged</td>
<td>max($A_{case 1}$, $A_{case 2}$)</td>
<td>-</td>
<td>-</td>
<td>7.31 / -</td>
<td>7.33 / -</td>
<td>9.63 / -</td>
<td>9.57 / -</td>
</tr>
</tbody>
</table>

Fig. 18 Roof structure case 4, solution with global and local stability, and penalty functions, $V = 6.94 \text{ m}^3$, $N_e = 345$. Color indication: blue/dark = compression, red/light = tension, secondary structure = grey.

4 Conclusions

In this paper, a formulation permitting large-scale truss optimization problems, including global and local stability constraints, was presented. Furthermore, to encourage practical application of the method, two heuristic methods to reduce truss complexity and a method to search the nonconvex solution-space for better solutions were proposed. Finally, the relation between the critical eigenmode and the semidefinite dual matrix variable was identified.

The methods presented were demonstrated through three different applications. The implementation of the dual formulation and the solution-space search algorithm were validated against results given by the primal formulation. Subsequently, the performance of the dual formulation was compared to the one of the primal formulation. The dual formulation was shown to outperform the primal in terms of solver time and model size, hence allowing large-scale models to be solved. Next, the study of a 3D cantilever beam illustrated the effect of applying the penalty functions to reduce truss complexity. Here the number of elements was reduced significantly, however, without any noticeable increase of the volume. Furthermore, additional applications of the solution-space search showed a significant improvement in objective function by finding better local minima. In general, improvements in the range of 15%-25% were found, indicating the benefits of the method. Finally, the large-scale capabilities were demonstrated by the preliminary design of a large-scale roof structure. Here, a model with 52,326 members was optimized for a single load case, and a model with 32,640 members was optimized for two load cases.

In general, the studies and findings indicate the potential of the presented methods to do large-scale truss optimization with stability. Furthermore, the methods proposed to enhance practical application, handle some of the challenges towards constructible and cost-competitive solutions.

Replication of Results The presented work is part of an industrial ph.d. project. Due to property rights from the commercial partner, COWI A/S, it has not been possible to make the code available as supplementary material.

Funding This study was funded by the COWI Foundation grant C-131.02 and Innovation Fund Denmark grant 5189-00112B.

Conflict of Interest The authors declare that they have no conflict of interest.

References


MATLAB (2016) R2016b documentation


"Optimization of truss girders in cable-supported bridges including stability"

M. Baandrup, P.N. Poulsen, J.F. Olesen & H. Polk

Submitted to: Journal of Bridge Engineering, 2019
Optimization of truss girders in cable-supported bridges including stability

Submitted to: Journal of Bridge Engineering, ASCE
November, 2019
Mads Baandrup\textsuperscript{1,2,†}; Peter Noe Poulsten\textsuperscript{1}; John Forbes Olesen\textsuperscript{1}; and Henrik Polk\textsuperscript{2}

\textsuperscript{1}Dept. of Civil Engineering, Technical University of Denmark, Kgs. Lyngby 2800, Denmark
\textsuperscript{2}Dept. of Major Bridges International, COWI A/S, Parallelvej 2, Kgs. Lyngby 2800, Denmark
\textsuperscript{†}Corresponding author: mja@byg.dtu.dk

Abstract

The main design principles for girders in cable-supported bridges have not changed significantly over the past 60 years, and are limited in further development. The design concept suffers from substantial fatigue issues, and will be challenged by self-weight in future very-long bridges with main spans beyond 2 km. In this work, truss topology optimization, including global and local stability, is applied in a conceptual study of new weight-reduced design concepts for girders in cable-supported bridges. The methods are based on finite element limit analysis and convex optimization. A single section of a continuous girder, subject to local and global loads, is optimized to minimize weight while fulfilling constraints on stresses as well as global and local stability. The optimized designs, significantly different in layout from the conventional, show initial weight savings of up to 45% compared to the present principles. Further parameter studies indicate potential weight savings of up to 54%.

Introduction

In the past 60 years, the overall structural system for girders in cable-supported bridges have remained nearly unchanged, Gimsing and Georgakis (2012). The design principles of closed steel box girders with orthotropic decks and transverse diaphragms have been applied in the majority of large suspension bridges since the 1950s, Wolchuk and Harris (1959); Wolchuk (1999). However, the design principles are limited in further development and suffer from substantial fatigue issues, Fisher and Dexter (1997); Kozy and Connor (2010); Zhang (2017). Fatigue issues are inherent in the design, since loads are carried in one direction at a time, leading to stress concentrations at connections, Song and Ding (2014). Furthermore, the design concept will be challenged in the future by its weight, since self-weight is crucial for very-long suspension bridges with main spans beyond 2 km, Gimsing and Georgakis (2012).

Through the years, the conventional design principles have been subject to several studies and improvements, especially with the focus on reducing fatigue issues. Parametric studies have been carried out, both experimentally Aygül et al. (2012); Oh and Bae (2013), analytically Backer et al. (2006), and numerically with focus on cutout details in diaphragms Connor (2004); De Corte (2009); Oh et al. (2011) and the top deck Fettahoglu (2016). Recently, a new approach to improve the design by gradient-based parametric optimization with the focus on weight minimization has been proposed by Baandrup et al. (2019b). In this study, weight savings of 6\% was achieved for current design practice indicating very little room for improvement without altering the concept. Despite many studies the overall structural system has not changed significantly. No major changes have been introduced to overcome fatigue issues or reduce weight, and thereby increase the potential span length. However, in future designs of suspension bridges with main spans beyond 2 km, it is anticipated that greater and more radical design changes will be required. In this regard, the weight-minimization of the girder is a natural starting point, since the reduced girder weight will have knock-on effects on the supporting bridge components such as cables, towers, and anchor blocks.

In the present paper, an optimization method based on truss structures is utilized as the design tool to identify new conceptual design approaches for girders in cable-supported bridges. The structural layout of a simplified bridge girder model is optimized to reduce weight by the use of truss optimization including global and local stability as well as stress constraints.

The girder model subject to optimization is a single section (25 m × 30.1 m × 4.75 m) of the continuous girder in a suspension bridge. Outer geometry,
dimensions, and load cases are based on the recently constructed Osman Gazi Bridge, which opened to traffic in July 2016. The basis of the conceptual study is an assumption of a fixed top plate spanning up to 3 m in two directions and thus distributing the surface loads to an underlying truss structure. The top plate may be constructed as a sandwich element with steel plate skins, see e.g. Battista et al. (2010); Corte (2011); Chu et al. (2018). However, in the present work, the focus is on the optimization of the supporting truss structure. The model is subject to both local and global load cases, identified from a global beam model of the Osman Gazi Bridge.

The applied optimization method is based on finite element limit analysis (FELA) and convex optimization. FELA is based on the finite element concept and defines a convex linear optimization problem assuming perfect plastic material behavior. The basis of the methods used in the present paper, was introduced in Poulsen et al. (2019) and developed to handle large-scale structures in Baandrup et al. (2019a), where methods to reduce truss complexity were formulated. These methods are briefly summarized for completeness and ease of understanding.

Subsequently, the basic methods are extended to handle the bridge girder model with novel constraints imposing the assumed top plate, symmetry conditions, and global section forces. Finally, the full problem formulation is applied to the conceptual bridge girder study. The focus of the study is mainly on potential weight reduction and the identification of the overall structural concept. Hence, construction costs and structural details are not taken into account.

The constraints defining the girder model are validated on a simple model, followed by the optimization of the bridge girder subject to two simple load cases. Subsequently, the complete model is optimized to identify new conceptual designs and possible weight savings. Finally, two parametric studies are carried out to study the effect of varying top plate strength and domain height, respectively.

**Methods**

In the following sections, the bridge girder model to be optimized is presented. Here, the geometry, load cases, and boundary conditions of the FE-model are introduced together with other assumptions. Subsequently, the main principles of FELA-based truss layout optimization are summarized from Poulsen et al. (2019). Finally, the implementation, combining the bridge girder model with the FELA optimization framework, is described.

**Bridge girder model**

The optimization study of the bridge girder model is based on the recently finalized Osman Gazi Bridge, Turkey, which opened to traffic in July 2016 with a main span of 1,550 m. The entire bridge design, made by COWI A/S, was state-of-the-art and is considered a suitable reference for the development of a new bridge girder design. The main dimensions and layout are seen in Fig. 1. The bridge girder design concept is a closed steel box-girder with an orthotropic deck and truss diaphragms every five meters, see Fig. 2. In the remaining, this design principle is designated the *conventional* design concept.

**Figure 2:** Design principle of the closed box girder with orthotropic deck and truss diaphragms in the Osman Gazi Bridge (COWI A/S)

Since the bridge girder is a continuous structure, only one section \((L = 25 \text{ m})\) equivalent to the distance between two sets of hangers is modeled with truss elements and optimized. The width of the girder is \(W = 30.1 \text{ m}\) and the height is \(H = 4.75 \text{ m}\). The domain with outer dimensions is seen in Fig. 3. The outer geometry of the domain is defined by the outer shape of the original design (without walkway), as shown in Fig. 2, thus the reason for the inclined edges. The shape defining the aerodynamic profile is kept, since aerodynamic issues are out of the scope of the present optimization study.

The section is part of a continuous girder and the loads applied are in equilibrium. The kinematic boundary conditions, imposed on the six degrees of freedom shown in Fig. 3, are applied to prevent any rigid body motion.

The section is modeled with a top plate and a supporting structure of truss elements. It is assumed that a top plate, necessary to carry the traffic loads, is at hand (e.g. some sandwich structure with steel skins Battista et al. (2010); Corte (2011); Chu et al. (2018)), spanning up to 3 m in two directions, distributing the surface loads to the nearest nodes in the underlying truss. The specific design of this top plate is not considered and left for future work. However, in the present work, an equivalent steel plate thickness of \(t_{tp} = 20 \text{ mm}\) was selected. In the Osman Gazi Bridge, the equivalent plate thickness was 26 mm for the orthotropic deck (top plate and troughs) spanning 5 m. In the present model, the top
Continuous top plate
Truss top plate

Figure 4: Assumption of a continuous top plate and equivalent modeled truss top plate

Furthermore, since the bridge girder is a symmetric and repetitive structure, symmetry constraints are applied. Symmetry lines are given along the longitudinal center line and in the transverse center line of each section, as shown in Fig. 5. This utilisation of symmetry significantly reduces the number of unknowns allowing for the modelling of larger systems within given computational limits.

The model is subject to 14 load cases, 12 designated global and two designated local. The applied loads are indicated in Fig. 6 and shown in Table 3 in Appendix A.

The 12 global load cases, LC 1-12, consist of section forces from a global beam model of the bridge. The beam model was developed during the design stage, with the COWI A/S in-house finite element software IBDAS (Integrated Bridge Design and Analysis Software). The model, loads (permanent, traffic, wind, seismic action, and temperature), and load combinations were all based on the Eurocode (??) and the UK National Annexes. From the IBDAS beam model, the most critical (static) section forces in typical sections of the girder were identified. In total 12 sets of critical section forces have been identified, equivalent to the maximum and minimum of the six section forces in a beam (\(N_x, M_y, M_z, V_y, V_z, M_1\)). Each set of forces consists of the global section forces acting on each end surface (End a and b) of the local model, as indicated in Fig. 6. The set of section forces is applied together with the distributed load \(p\) on the top surface and hanger forces \(P\) to ensure equilibrium of all loads applied. The section forces in the girder and locally distributed load and hanger forces are given by the beam model.

The two local load cases, LC 13 and 14, consist of a downward uniformly distributed load \(p = 5\) kN/m\(^2\) with equivalent upward hanger forces \(P\) to ensure equilibrium. Additionally, two moments, \(M_{y,a}\) and \(M_{y,b}\), are applied to the end surfaces to imitate the behavior of the continuous girder. In LC 13 the distributed load is applied to the entire top surface, whereas in LC 14 the load is applied only on one side of the longitudinal center line, to give a skew distribution.

Truss layout optimization including global and local stability
The bridge girder model is optimized with FELA-based truss layout optimization. The main principles of the
\begin{align*}
W &= 30:1 \text{ m} \\
H &= 4:75 \text{ m} \\
L &= 25 \text{ m} \\
y & x \\
a_k &= a_l = a_m = a_n \\
\text{Symmetry planes} & \quad \text{Truss members}
\end{align*}

**Figure 5:** Symmetry mapping of the structure indicated by truss members with equal area $a_k = a_l = a_m = a_n$

**Figure 6:** Global section forces applied to end surfaces, local distributed load $p$ (indicated by downward arrows), and hanger forces $P$

applied method were introduced in Poulsen et al. (2019) and are summarized below.

FELA is a numerical method that combines the discretization of a model, known from conventional finite element analysis (FEA), with limit analysis of structures where rigid-plastic material behavior is assumed as well as small displacements. Normally, displacements are the variables in FEA, whereas the variables in the lower-bound method of FELA are the stresses.

The truss optimization takes the ground-structure approach with an initial mesh of nodes connected by members, along with boundary conditions and loads applied at nodes. The objective of the optimization is to minimize the total volume (weight) of the truss structure while fulfilling given constraints. The topological design variables are the cross-sectional areas of all members except the ones in the top plate.

During the optimization, four different constraints must be fulfilled. Firstly, a constraint ensuring equilibrium of nodal forces for all unsupported nodes, thus an admissible stress field is established. Secondly, a constraint ensuring that all member stresses are within the yield stress limits. Finally, two different constraints ensuring restrictions against local and global instability, respectively.

The truss layout optimization problem with the four constraint functions, as formulated in Poulsen et al. (2019), is given as the convex problem

\begin{align*}
\min_{\beta, A} & \quad V = \sum_{e=1}^{N_e} a_e \ell_e = A^\top L \quad \text{(1a)} \\
\text{s.t.} & \quad H\beta = R \quad \text{(1b)} \\
& \quad C\beta - C_mA \leq 0 \quad \text{(1c)} \\
& \quad -I\beta - PA \leq q \quad \text{(1d)} \\
& \quad K_G(\beta_k) + K(A) \succeq 0 \quad \forall k = 1, \ldots, N_k \quad \text{(1e)}
\end{align*}

where $e$ refers to the $N_e$ individual members and $k$ refers to the $N_k$ individual load cases. The main design variables are the member areas collected in $A$ and the secondary design variables are the bar forces collected in $\beta$. The objective function in (1a), defining the total truss volume $V$, is given as the summation of the products of individual member areas $a_e$ and lengths $\ell_e$ (collected in vector $L$). The first constraint (1b) defines the equilibrium equations, where the equilibrium
matrix $H$ ensures the nodal equilibrium of internal ($\beta$) and external ($R$) nodal forces for all unsupported nodes. The second constraint (1c) defines the yield condition, or stress constraint. The effect of the constraint may be seen as a conversion of the bar forces to their absolute values ($C$), which are restricted by the yield stress $f_y$, of each bar ($C_m$). For a single element $e$, the matrices are given as $C_e = [1 - 1]^T$ and $C_{m,e} = [f_y f_y]^T$.

The third constraint (1d) defines the restriction against local instability of each element. The constraint is based on the Euler buckling load $P_{cr}$ for a simply supported member. Hence, the constraint for element $e$ is given as

$$-\beta_e \leq P_{cr,e} = \frac{\pi^2 \cdot EI_e}{l_e^2} \approx \alpha_e a_e^2$$  \hspace{0.5cm} (2)

where $E$ is the modulus of elasticity and $I_e$ the second moment of area. The rightmost quadratic term is exact for a circular cross-section, for other cross-sectional geometries the expression is an approximation, see Poulsen et al. (2019). However, the constraint is concave, and it cannot be integrated immediately into the convex optimization problem.

To eliminate the concave properties of (2), and thus include it in the convex optimization problem, the quadratic constraint is linearized through its tangent or secant. The linearizations of (2) for all elements and load cases are assembled to give the matrix expression (1d), where $I$ is the identity matrix, $P$ is a matrix representing the parts of the linearization which are proportional to $a_e$, and $q$ is a vector containing the constant parts.

The final constraint (1e) defines the restriction against global instability for each of the $N_k$ load cases. The constraint is defined as a semidefinite constraint on the linearized buckling stiffness matrix comprised of the sum of the global geometric stiffness matrix $K_G$, which is a function of the bar forces, and the global elastic stiffness matrix $K$, which is a function of the bar areas.

Multiple load cases are in general handled by an expansion of the system of equations. The vectors $\beta$ and $R$ are expanded to include $N_k$ individual sets of the variables and loads, and similarly the matrices in (1) are expanded to apply the constraints for each load case.

In Baandrup et al. (2019a) the optimization problem (1) was reformulated to handle large-scale models. The primal formulation (1) was rewritten to its dual formulation, resulting in a large reduction in computational time and memory demand. This technique is applied here too, however, for clarity only the primal formulations are shown in the following.

In this conceptual study, emphasis is on practical solutions, and in particular on lowering construction cost, therefore it is desirable to reduce the truss complexity by reducing the number of members and the number of connections (nodes). In Baandrup et al. (2019a) two heuristic methods were formulated to penalize short members and members with small cross-sectional areas. The methods are applied here, too, to reduce structural complexity.

**Arranging the bridge girder model for truss optimization**

The implementation of the fixed top plate, the symmetry constraints, and the global section forces into the basic optimization problem (1) is described below. Finally, the complete optimization problem is stated.

**Fixed top plate**

The assumed fixed top plate with a total steel thickness $t_{tp}$ is modeled as a plane grid with truss members, cf. Fig. 4 and Fig. 7. These truss members are assigned fixed cross-sectional areas, such that the strength of the truss top plate is equivalent to the assumed top plate. Furthermore, the area of the members in the top plate (top layer of the truss mesh) are discarded as design variables.

A part of the plate with dimensions $a \times b$, as indicated in Fig. 7, is studied in order to identify the truss cross-sectional areas $a_x$, $a_y$, and $a_{xy}$ which ensure a strength equivalent to the strength of a solid plate.

The plate and truss structures in Fig. 7a and 7b should have the same uni-axial capacities $\sigma_x$ and $\sigma_y$, and shear capacity $\tau_{xy}$. By establishing force equilibrium equations for the plate and truss structures, the cross-sectional areas are identified as

$$a_x = t_{tp}(b - \beta a)$$ \hspace{0.5cm} (3a)

$$a_y = t_{tp}(a - \beta b)$$ \hspace{0.5cm} (3b)

$$a_{xy} = t_{tp} \frac{\beta}{2} \sqrt{a^2 + b^2}$$ \hspace{0.5cm} (3c)

where a geometric factor is introduced as $\beta = \frac{1}{2\sqrt{3}} \frac{(a+b)^2}{a^2 + b^2}$ in order to fulfill the von Mises yield criterion. The fixed areas are assigned to the top members in the linear equality constraint

$$DA = A_0$$ \hspace{0.5cm} (4)

where $A_0$ contains the fixed areas $a_x$, $a_y$, and $a_{xy}$, and $D$ is an index matrix linking the design variables to the fixed areas. For truss members positioned along the edge of the domain, only half the area is assigned.
Mapping of symmetry constraints
The symmetry mapping (Fig. 5) is implemented with the following linear equality constraint

$$\mathbf{SA} = 0 \quad (5)$$

where $\mathbf{S}$ is an index matrix mapping the symmetry between the regions $k, l, m,$ and $n$. The member areas in each region, represented by $a_k, a_l, a_m,$ and $a_n$ in Fig. 5, are mirrored, hence $a_l = a_k, a_m = a_k,$ and $a_n = a_k,$ which in matrix format can be written as

$$\mathbf{s}_k \mathbf{A} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & a_k \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & a_l \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & a_m \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & a_n \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}^T \begin{bmatrix} \vdots \\ a_k \\ \vdots \\ a_l \\ \vdots \\ a_m \\ \vdots \\ a_n \end{bmatrix} = \mathbf{0} \quad (6)$$

where $\mathbf{s}_k$ is the mapping matrix for a single element in region $k$, which is added to $\mathbf{S}$.

Global section forces
Contrary to the distributed load $p$ and the hanger forces $P$, which are applied through the $\mathbf{R}$-vector, the global section forces are not applied to specific nodes on the end surfaces. This is not possible, since the distribution of material from the optimization is unknown. Instead, section forces are defined as a summation of node forces on the end surfaces, hence the distribution is unrestricted and given as a result of the optimization. The generic summation of each of the six section forces for one end surface with $N_{\text{end}}$ nodes is shown below

$$N_x = \sum_{i=1}^{N_{\text{end}}} F_{x,i} \quad (7a)$$
$$M_y = \sum_{i=1}^{N_{\text{end}}} F_{y,i} \cdot z_i \quad (7b)$$
$$M_z = \sum_{i=1}^{N_{\text{end}}} -F_{x,i} \cdot y_i \quad (7c)$$
$$V_y = \sum_{i=1}^{N_{\text{end}}} F_{y,i} \quad (7d)$$
$$V_z = \sum_{i=1}^{N_{\text{end}}} F_{z,i} \quad (7e)$$
$$M_t = \sum_{i=1}^{N_{\text{end}}} -F_{y,i} \cdot z_i + \sum_{i=1}^{N_{\text{end}}} F_{z,i} \cdot y_i \quad (7f)$$

where $F_{x,i}, F_{y,i},$ and $F_{z,i}$ are the three node forces in node $i$, and $y_i$ and $z_i$ are the distances to the end surface center $(y, z)$ as indicated in Fig. 8. The node forces are found as the sum of components of the bars contributing to node $i$. 

Figure 7: Structural layout and capacities $\sigma_x, \sigma_y,$ and $\tau_{xy}$ on two types of top plates

(a) Plate structure

(b) Truss structure with representative grid elements
The sums in (7) are linear functions of the design variables in \( \beta \) and the equations can be given as the linear equality constraint

\[
G\beta = G_0
\]  

where \( G \) is a section force equilibrium matrix and \( G_0 \) is a vector containing the section force values. The vector for the global load case \( g \) is given as

\[
G_{0,g} = [N_{x,a,g} M_{y,a,g} M_{z,a,g} V_{y,a,g} V_{z,a,g} M_{t,a,g} \nonumber \]  

\[
N_{x,b,g} M_{y,b,g} M_{z,b,g} V_{y,b,g} V_{z,b,g} M_{t,b,g} ]^T
\]  

The elements in the section force equilibrium matrix \( G \) ensures equilibrium with the section forces in \( G_0 \) in all degrees of freedom on the two end surfaces. When global section forces are applied through (8) the equilibrium matrix \( H \) and the load vector \( R \) are reduced, hence the degrees of freedom from the end surfaces are removed from \( H \) and \( R \), such that both constraints (1b) and (8) can be fulfilled. The reduced matrix and the reduced vector are renamed \( H_R \) and \( R_R \), respectively.

The complete optimization problem is constructed by including the three constraints (4), (5), and (8), partly defining the bridge model, into the truss optimization problem (1), and by modifying constraint (1b) according to the above

\[
\min \beta, A \quad A^T L \]  

s.t.

\[
H_R \beta = R_R \]  

\[
C\beta - C_m A \leq 0 \]  

\[
- I\beta - PA \leq q \]  

\[
D A = A_0 \]  

\[
S A = 0 \]  

\[
G\beta = G_0 \]  

\[
K_G(\beta_k) + K(A) \geq 0 \quad \forall k = 1, \ldots, N_k \]

The above optimization problem, shown in its primal formulation, is solved by its dual formulation, similarly to the technique presented in Baandrup et al. (2019a). The derivation of the dual formulation is shown in Appendix B.

### Results

Initially, methods and girder model are validated, followed by an optimization study of the bridge girder. Since the basic optimization problem (1) was validated and demonstrated in Poulsen et al. (2019), focus is on the additional constraints (10e), (10f), and (10g). However, since the constraint on the fixed top plate is evident, and the validity of the symmetry constraints is obvious from the result plots, only the implementation of the global section forces needs to be validated. Subsequently, optimization of the bridge girder model with two individual and simple load cases is demonstrated before the optimization results with all 14 load cases are presented. Finally, two parameter studies of varying top plate thickness and domain height are carried out.

For all structures, the meshes are described by the number of nodes in the \( x \), \( y \), and \( z \) direction by \( n_x \), \( n_y \), and \( n_z \), respectively, and the connectivity is given by the parameter \( r \). If \( r = 1 \) all nodes are connected with elements to their nearest nodes in all directions, if \( r = 2 \) all nodes are connected to the nearest as well as the next-to-nearest nodes in all directions, and so on. In all cases the material parameters are given as \( E = 210 \) GPa and \( f_y = 335 \) MPa, equivalent to the Osman Gazi Bridge. Results with and without the stability constraints are shown. When stability constraints are included, also the heuristic methods to reduce truss complexity are included.

The code was implemented in Matlab R2016b, MATLAB (2016), and the convex solver was Mosek version 8.1.0.62, MOSEK (2018). Mainly results were computed on a setup with 2x Intel Xeon Processor 2660v3 (10 cores, 2.60 GHz), 128 GB RAM. However, the initial validation results were computed on a desktop with an Intel Core i7-6500U CPU 2.59 GHz and 32 GB RAM.

### Validation of global section force implementation

The implementation of the global section forces are validated through a study of the six individual section forces applied to a cubical domain. The cube model with dimension \( L = 5 \) m, boundary conditions, and loads are seen in Fig. 9. The boundary conditions are applied only to avoid a singular system, since all loads are in equilibrium. The loads are defined as \( N_x = 1 \) MN, \( M_y = 2.5 \) MNm, \( M_z = 2.5 \) MNm, \( V_y = 1 \) MN, \( V_z = 1 \) MN, and \( M_t = 5 \) MNm. To apply shear, counteracting bending moments \(-M_z\) and \( M_y\) are applied.
together with \( V_y \) and \( V_z \), respectively.

\[ M_z \]
\[ V_y \]
\[ M_t \]
\[ N_y \]
\[ V_z \]
\[ M_y \]
\[ u_x = 0 \]
\[ u_x = u_y = u_z = 0 \]
\[ L \]
\[ u_x = u_y = u_z = 0 \]
\[ L \]
\[ u_y = 0 \]
\[ V_y \]
\[ M_y \]
\[ M_z \]

\textbf{Figure 9:} Domain, boundary conditions, and loads of cube model for validation

The mesh parameters are given as \( n_x = n_y = n_z = 3 \) and \( r = 2 \) (all nodes inter-connected), generating \( N_e = 335 \) elements in the ground-structure, seen in Fig. 10.

\textbf{Figure 10:} Initial ground-structure of cube model, \( N_e = 335 \)

Initially, the cube model is optimized without the stability constraints, hence problem (10) without (10d), (10e), and (10h). The results for the six global section forces are seen in Fig. 11, and it appears that in all cases the structure is in accordance with the loading.

Next, the cube model is optimized with both stability constraints, hence problem (10) without (10e). The results for the six global section forces are seen in Fig. 12. From the six sub-figures it appears, again, that the structures are in accordance with the loading. As discussed in Poulsen et al. (2019), the inclusion of global stability constraints introduces many thin members, in addition to the main members, to ensure overall global stability. Furthermore, the inclusion of local stability constraints leads to fewer but larger main members, as seen in all cases when Fig. 12 is compared to Fig. 11. The reason for this is found in the benefit of collecting material in fewer and larger members, being less prone to local instability, compared to many slender members. Finally it is emphasized, when including local stability, the optimization problem becomes concave, hence no global optimum is guaranteed, instead many local optima may exist.

\section*{Optimization of bridge girder model}

For all cases of the bridge girder optimization, the same initial ground-structure is applied. The mesh is seen in Fig. 13 and defined by an initial grid of \( n_x = 9, n_y = 11, \) and \( n_z = 3 \) nodes with a connectivity \( r = 2 \). The node grid and mesh are adapted to the shape of the domain and consist in total of \( N = 243 \) nodes and \( N_e = 6,665 \) elements.

When visualizing results a threshold is applied to members with zero or negligible cross-sectional areas for clarity. Two threshold values, relative to the largest cross-sectional area, are applied to identify the main structure \((N_{e,\text{Main}} | a_e > 0.1a_{e,\text{max}})\) and the detailed structure \((N_{e,\text{Det.}} | a_e > 0.01a_{e,\text{max}})\), respectively.

The objective function of volume \( V \) \([\text{m}^3]\), (10a), is transformed to weight per meter of girder, \( W = V \cdot \rho / 25 \text{ m} \), where the density of steel is given as \( \rho = 7.85 \text{ ton/m}^3 \). For comparison, the weight (without walkway) of the Osman Gazi Bridge is 11.13 ton/m, where the top part (top plate and troughs) accounts for 5.25 ton/m (47.2%), and the lower part accounts for 5.88 ton/m. The assumed fixed top plate of \( t_{tp} = 20 \text{ mm} \) accounts for 4.03 ton/m in the truss girder designs.

\section*{Optimization with a single load case}

Firstly, the bridge girder model is subject to a pure bending moment \( M_y = 242 \text{ MNm} \) (equivalent to the largest bending moment in the 12 global load cases) at both end surfaces, similarly to the cube model in Fig. 11b and 12b. The optimization problem is solved with and without the stability constraints (10d) and (10h). The results are seen in Fig. 14. It is clearly seen in Fig. 14a how the structure is optimized to carry the bending moment, with material distributed in top and bottom to maximize the second moment of area. In Fig. 14b the effect of the global stability constraint is seen in the many additional members. Furthermore, the effect of the local stability constraint is seen in the lower part (in compression), where the material is mainly concentrated in three members, as opposed to the five members in Fig. 14a. Despite the changes due to the stability constraints, the structure is clearly still optimized to carry the bending moment.

Secondly, the bridge girder model is optimized for a single local load case with distributed load over the entire top surface and equivalent hanger forces (LC 13 in Table 3 in Appendix A). The complete optimization
Fig. 11: Solutions of cube model for six individual global section forces, with symmetry, without stability constraints. Boundary of domain indicated by dashed lines

Fig. 12: Solutions of cube model for six individual global section forces, with symmetry, and stability constraints. Boundary of domain indicated by dashed lines

Fig. 13: Initial ground-structure of bridge girder section, $N_x = 6,665$

Fig. 14: Bridge girder model optimized for a single load case of pure bending moment $M_y = 242$ MNm applied at both end surfaces. Perspective view with truss top plate. Boundary of domain indicated by thin black lines
problem, (10), including stability and symmetry constraints, and the fixed top plate, is solved. The optimized girder, visualized with the detailed structure of \(N_{e,\text{Det.}} = 382\) elements \(N_{e,\text{Main}} = 286\), is seen in Fig. 15. From the figure it is seen how the material is distributed more densely close to the hanger anchorage, to which all distributed surface load is transferred, and hence the forces are the largest. On the contrary, the amount of material decreases towards the center of the girder and towards the center line between hangers. Thus, the structure is clearly optimized to carry the distributed load to the hangers. The weight is found to 4.27 ton/m, where the top plate of 4.03 ton/m accounts for 94.4% and the lower part thus accounts for 0.24 ton/m. The large ratio between the weight of the upper and lower parts indicates that the top plate is oversized for this comparatively light load case.

**Optimization with 14 load cases**

Next, the bridge girder model is optimized for all 14 load cases (LC 1-14 in Table 3 in Appendix A). A single section of the optimized girder, visualized with the detailed structure of \(N_{e,\text{Det.}} = 472\) elements, is seen in Fig. 16. In Fig. 17 three sections of the continuous girder are shown together to emphasize the design. For clarity, only the main structure with \(N_{e,\text{Main}} = 344\) elements per section is shown. In both Fig. 16 and 17 notable members are highlighted to enhance the design concept.

In Fig. 16 and 17 a significantly different design is seen, compared to the one in Fig. 15. The effects from the global section forces are clearly visible, in particular the large torsion and bending moments are governing the design. Hence, a torsion grid along the circumference of the domain is seen, as well as large longitudinal members in the lower part of the domain. Furthermore, the optimized design is very different from the conventional design, Fig. 2. Not only because truss structures are studied, but also due to the overall structural concept. In general, loads are carried more directly to the hangers, and transferred to the main bottom grid, to be carried by efficient tension members. Thus, the material is utilized more efficiently, compared to the conventional design where loads are carried to the hangers via perpendicular plate structures. In particular, two important design principles are identified. The first principle which is highlighted in red (dark grey in greyscale), includes the largest bottom members spanning across two sections, from the hanger in one side, through the neighboring section, to the hanger in the opposite side. Thus, the interaction between the sections reaches further than to the nearest neighboring sections in order to support bending and torsional moments efficiently. This principle emerges from the optimization, although the mesh (Fig. 13) allows for a span between hangers crossing a single section only. The second principle which is highlighted in blue (black in greyscale), is characterized by the curved narrowing of the girder between hangers, without exploiting the entire domain width.

The weight of the girder is found to 6.09 ton/m, where the weight of the top plate, 4.03 ton/m, accounts for 66.2% and the weight of the lower part is 2.06 ton/m. The total weight is equivalent to 54.7% of the Osman Gazi Bridge. This significant weight reduction indicates the large potential in alternative truss based design concepts. In all weight measures, the contribution from connections (weldings and bolts) has been neglected. Furthermore, construction costs have not been considered. However, concerns about construction cost are subordinate to self-weight issues, which are crucial to the possible achievement of very-long suspension bridges (main span beyond 2 km).

The ratio between the weight of the top plate and the
total weight (66.2%) is higher compared to the Osman Gazi Bridge (47.2%), indicating a conservative choice of the equivalent top plate thickness of 20 mm. Therefore, the effect of reducing the top plate thickness is studied in the following.

**Top plate thickness**
Since the top plate thickness is fixed during optimization, a parameter study of varying thickness, $t_{tp}$, is carried out. The complete girder model with all 14 load cases is studied with fixed thicknesses varying from 10 mm to 25 mm, with 5 mm intervals. The results of the study are summarized in Table 1, where the Osman Gazi Bridge has been included for reference.

In Case 3, the ratio between the top plate and the total weight is higher compared to the conventional design concept in the Osman Gazi Bridge, as noted previously, indicating a further potential saving by reducing the top plate thickness. When this is done, the ratio is reduced, and in Case 1 with $t_{tp} = 10$ mm the ratio of 49.4% is close to the conventional design. With $t_{tp} = 10$ mm the total weight of 4.09 ton/m is equivalent to 36.7% of the Osman Gazi Bridge. However, an equivalent top plate of 10 mm thickness is most likely not capable of spanning 3 m while carrying traffic. With a thicker equivalent top plate of $t_{tp} = 15$ mm, the total weight of 5.11 ton/m is equivalent to 45.9% of the Osman Gazis Bridge.

The truss girder designs for Case 1 and 2 are seen in Fig. 18, visualizing the main structure of a single section. The equivalent visualization of Case 3 is seen in Fig. 17a, while the visualization of Case 4 has been left out. In the designs of both Case 1 and 2, the bottom grid is more uniform, compared to Fig. 17a where the grid consists of distinct primary and secondary members. This effect can be seen as a result of the reduced top plate thickness, hence more material is concentrated in the bottom of the domain. Similarly, in Case 1 pronounced longitudinal members are formed in the bottom to counteract the decreased longitudinal strength in the top part, due to the thinner top plate. Despite the changes to the design, the main principles are similar to Case 3.

**Domain height**
Finally, a study of the effect on the girder weight from varying the domain height $H$ is carried out. The study is carried out by increasing the height $H$ (Fig. 3) while the mesh given in Fig. 13 scales. A summary of the study of domain heights from 4.75 m to 10 m is seen in Table 2.

<table>
<thead>
<tr>
<th>Case:</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height $H$ [m]</td>
<td>4.75</td>
<td>6.0</td>
<td>8.0</td>
<td>10.0</td>
</tr>
<tr>
<td>Weight $W$ [ton/m]</td>
<td>6.09</td>
<td>5.77</td>
<td>5.50</td>
<td>5.42</td>
</tr>
</tbody>
</table>

From the table it is seen that by increasing the domain height, the weight decreases. This effect is expected, due to the benefits from increased height on the torsion and bending moment capacities. However, in general the weight reduction is modest. Further, the effect decreases with increasing height, thus the change from Case 3 to 4 is significantly smaller than from Case 1 to 2.

---

**Figure 16:** Bridge girder result optimized for 14 load cases (LC 1-14), $W = 6.09$ ton/m. Perspective view of detailed structure, $N_{Det.} = 472$. Boundary of domain indicated by thin black lines.
Figure 17: Bridge girder result optimized for 14 load cases (LC 1-14), three spans, $W = 6.09$ ton/m. Main structure without top plate, $N_{e,Main} = 344$ per section.

Table 1: Study of varying top plate thickness $t_{tp}$ for the Osman Gazi bridge and four truss girder cases. Weight savings are given relative to the total weight of the Osman Gazi Bridge. *Equivalent plate thickness of the orthotropic top deck (top plate and troughs)

<table>
<thead>
<tr>
<th>Case</th>
<th>$t_{tp}$</th>
<th>Total weight</th>
<th>Weight saving</th>
<th>Weight of top plate</th>
<th>Weight of lower part</th>
<th>Ratio between top plate and total weight</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[mm]</td>
<td>[ton/m]</td>
<td>[%]</td>
<td>[ton/m]</td>
<td>[ton/m]</td>
<td>[—]</td>
</tr>
<tr>
<td>Osman Gazi</td>
<td>26*</td>
<td>11.13</td>
<td>—</td>
<td>5.25</td>
<td>5.88</td>
<td>47.2%</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>4.09</td>
<td>63.2%</td>
<td>2.02</td>
<td>2.07</td>
<td>49.4%</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>5.11</td>
<td>54.1%</td>
<td>3.03</td>
<td>2.08</td>
<td>59.3%</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>6.09</td>
<td>45.3%</td>
<td>4.03</td>
<td>2.06</td>
<td>66.2%</td>
</tr>
<tr>
<td>4</td>
<td>25</td>
<td>7.09</td>
<td>35.3%</td>
<td>5.04</td>
<td>2.05</td>
<td>71.1%</td>
</tr>
</tbody>
</table>
**Conclusions**

Truss topology optimization, including stress constraints, as well as global and local stability constraints, was applied in the study of new conceptual designs for girders in cable-supported bridges. In addition to the basic optimization problem, novel constraints were added to account for an assumed top plate, symmetry and the application of global section forces, in order to establish the necessary conditions to model a bridge girder section. The girder model was subject to local and global loads, and optimized with the goal of minimizing the total weight. The implementation of the imposed global section forces was validated by simple examples, both without and with stability constraints. Subsequently, the bridge girder model was studied under individual and simple load cases. Finally, optimization including all considered load cases was carried out in order to identify potential new design concepts and weight savings. Due to the truss structure approach, designs showing new and efficient load carrying principles, which are significantly different from the conventional design, emerged from the optimization. Here, the main structure consisted of a torsion grid along the circumference of the domain, as well as large members in the bottom to carry bending moments. Furthermore, load paths were oriented more directly towards the hanger attachments, and running through the bottom part to benefit from tension members. The total girder weight of 6.09 ton/m was equivalent to only 54.7% of the conventionally designed girder of the Osman Gazi Bridge. Hence, the significant weight reduction and new design principles indicate the potential of these alternative design concepts. Moreover, the two parameter studies on the top plate thickness and the domain height indicate possible further weight reduction. The most significant effect was found in reducing the top plate thickness, whereby the ratio between the weight of the top plate and the total weight of the girder approaches the ratio of the conventional design.

In general, the new design principles and possible weight reductions indicate the potential of alternative truss based design concepts. Although the construction cost has been neglected, the potentials are highly relevant to very-long suspension bridges with spans beyond 2 km, for which low self-weight is crucial.

**Data Availability Statement**

Some or all data, models, or code generated or used during the study are available from the corresponding author by request. This includes the Matlab input files to the demonstrated problems.

**Acknowledgements**

The presented work is part of an industrial ph.d. project with the title “Innovative design of steel bridge girders in cable-supported bridges” and is carried out in cooperation with COWI A/S, DTU Civil Engineering and DTU Mechanical Engineering. The project is supported financially by the COWI Foundation grant C-131.02 and Innovation Fund Denmark grant 5189-00112B.

**References**


Large-scale truss optimization including global and local stability. *Structural and Multidisciplinary Optimization* (in review).


### A Table of load cases

The 12 global load cases (LC 1-12) and two local load cases (LC 13-14) are seen in Table 3. Load cases 1-12 are identified from a global beam model of the Osman Gazi Bridge. Each of these is identified based on the maximum and minimum section forces in the girder on end surface $a$, hence LC1 is derived from the maximum $N_x$, LC2 from minimum $N_x$, LC3 from maximum $M_y$, and so on. As seen from the table load case 6 and 8 are identical, hence the minimum bending moment $M_z$ and minimum shear force $V_y$ occur in the same load case and at the same girder location in the global beam model. Consequently, one of these load cases can be neglected in the optimization to reduce the problem size.
Table 3: Bridge girder load cases (LC). LC 1-12: global, LC 13-14: local. *Only distributed load \( p \) on half of top surface and average hanger force \( P \) due to skew load.

<table>
<thead>
<tr>
<th>LC</th>
<th>( N_{x,a} ) [MN]</th>
<th>( M_{y,a} ) [MNm]</th>
<th>( M_{z,a} ) [MNm]</th>
<th>( V_{y,a} ) [MN]</th>
<th>( M_{x,b} ) [MNm]</th>
<th>( M_{z,b} ) [MNm]</th>
<th>( V_{y,b} ) [MN]</th>
<th>( V_{z,b} ) [MN]</th>
<th>( M_{t,b} ) [MN]</th>
<th>Dist. ( p ) [kN/m²]</th>
<th>Hanger ( P ) [kN/m²]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.29</td>
<td>-10.46</td>
<td>18.93</td>
<td>0.16</td>
<td>1.43</td>
<td>-20.12</td>
<td>-9.29</td>
<td>-24.77</td>
<td>-14.91</td>
<td>-0.16</td>
<td>-1.39</td>
</tr>
<tr>
<td>2</td>
<td>-16.58</td>
<td>-75.21</td>
<td>47.72</td>
<td>-0.06</td>
<td>-1.18</td>
<td>47.79</td>
<td>16.58</td>
<td>84.39</td>
<td>-49.34</td>
<td>0.06</td>
<td>-0.45</td>
</tr>
<tr>
<td>3</td>
<td>2.07</td>
<td>236.57</td>
<td>35.42</td>
<td>0.24</td>
<td>1.66</td>
<td>26.45</td>
<td>-2.07</td>
<td>-242.34</td>
<td>-29.53</td>
<td>-0.24</td>
<td>1.20</td>
</tr>
<tr>
<td>4</td>
<td>-4.08</td>
<td>-155.36</td>
<td>-6.27</td>
<td>0.01</td>
<td>0.09</td>
<td>34.30</td>
<td>4.08</td>
<td>119.82</td>
<td>6.65</td>
<td>-0.015</td>
<td>-2.75</td>
</tr>
<tr>
<td>5</td>
<td>-0.08</td>
<td>-5.40</td>
<td>704.06</td>
<td>-4.56</td>
<td>0.07</td>
<td>9.20</td>
<td>0.08</td>
<td>18.82</td>
<td>-818.17</td>
<td>4.56</td>
<td>1.15</td>
</tr>
<tr>
<td>6</td>
<td>1.29</td>
<td>-30.53</td>
<td>-779.78</td>
<td>-5.45</td>
<td>1.64</td>
<td>-24.37</td>
<td>-1.29</td>
<td>8.53</td>
<td>643.56</td>
<td>5.45</td>
<td>-0.12</td>
</tr>
<tr>
<td>7</td>
<td>0.11</td>
<td>-24.28</td>
<td>712.49</td>
<td>5.33</td>
<td>1.25</td>
<td>8.37</td>
<td>0.11</td>
<td>8.57</td>
<td>-579.27</td>
<td>-5.33</td>
<td>0.00</td>
</tr>
<tr>
<td>8</td>
<td>1.29</td>
<td>-30.53</td>
<td>-779.78</td>
<td>-5.45</td>
<td>1.64</td>
<td>-24.37</td>
<td>-1.29</td>
<td>8.53</td>
<td>643.56</td>
<td>5.45</td>
<td>-0.12</td>
</tr>
<tr>
<td>9</td>
<td>6.11</td>
<td>-142.86</td>
<td>-16.49</td>
<td>-0.16</td>
<td>1.54</td>
<td>53.50</td>
<td>-6.11</td>
<td>75.78</td>
<td>12.49</td>
<td>0.16</td>
<td>-3.83</td>
</tr>
<tr>
<td>10</td>
<td>-13.32</td>
<td>42.33</td>
<td>-23.89</td>
<td>0.04</td>
<td>-5.22</td>
<td>-53.72</td>
<td>13.32</td>
<td>46.88</td>
<td>24.89</td>
<td>-0.04</td>
<td>1.92</td>
</tr>
<tr>
<td>11</td>
<td>-0.04</td>
<td>35.64</td>
<td>-230.91</td>
<td>-0.18</td>
<td>1.35</td>
<td>184.41</td>
<td>0.04</td>
<td>-39.93</td>
<td>226.49</td>
<td>0.18</td>
<td>1.01</td>
</tr>
<tr>
<td>12</td>
<td>2.04</td>
<td>41.39</td>
<td>239.93</td>
<td>0.17</td>
<td>1.33</td>
<td>-179.25</td>
<td>-2.04</td>
<td>-45.63</td>
<td>-235.75</td>
<td>-0.17</td>
<td>0.99</td>
</tr>
<tr>
<td>13</td>
<td>-20.10</td>
<td>- -</td>
<td>- -</td>
<td>- -</td>
<td>- -</td>
<td>-20.10</td>
<td>- -</td>
<td>- -</td>
<td>-5.00</td>
<td>- -</td>
<td>- -</td>
</tr>
<tr>
<td>14*</td>
<td>-10.00</td>
<td>- -</td>
<td>- -</td>
<td>- -</td>
<td>- -</td>
<td>-10.00</td>
<td>- -</td>
<td>- -</td>
<td>-5.00</td>
<td>- -</td>
<td>- -</td>
</tr>
</tbody>
</table>
B Dual formulation of complete optimization problem

Similar to the technique in Baandrup et al. (2019a) the dual formulation of problem (10) is derived in the following. To handle the semidefinite constraint (10h) in the formulation of the Lagrangian of the problem, a semidefinite dual variable $Y_k \forall k = 1, \ldots, N_k$ is introduced. To derive the Lagrangian function a self-adjoint operator on $Y_k$ is defined as $\sum_{k=1}^{N_k} (K_{S,k}(\beta_k,A) Y_k) = \beta^T A^T \sum_{k=1}^{N_k} K_{S,k}^*(Y_k)$. The reader is referred to Baandrup et al. (2019a) for details on the adjoint operator. The Lagrangian of problem (10) is given as

$$L(\beta,A,y,z,t,u,v,w,Y_k) = \begin{bmatrix} 0 & L^T \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} - y^T \begin{bmatrix} H_R & 0 \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} - R_R$$

$$+ z^T \begin{bmatrix} [C & -C_m] \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} + t^T \begin{bmatrix} [-I & -P] \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} - \begin{bmatrix} -1 \end{bmatrix} t - \begin{bmatrix} 0 & D \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} - A_0$$

$$- v^T \begin{bmatrix} [0 & S] \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} - w^T \begin{bmatrix} [G & 0] \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} - G_0 - \sum_{k=1}^{N_k} (K_{S,k}(\beta_k,A), Y_k)$$

$$= \begin{bmatrix} \beta^T & L^T \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & L^T \end{bmatrix} - \begin{bmatrix} H_R \\ 0 \end{bmatrix} y - \begin{bmatrix} -I \\ -P^T \end{bmatrix} t - \begin{bmatrix} 0 & D^T \end{bmatrix} u - \begin{bmatrix} 0 & S^T \end{bmatrix} v \\ 0 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} \begin{bmatrix} G^T & 0 \end{bmatrix} w - \sum_{k=1}^{N_k} K_{S,k}^*(Y_k) \end{bmatrix} + R_R y - q^T t + A_0 u + G_0 w$$

(11b)

By differentiating (11b) w.r.t. the primal variables $(\beta,A)$ the dual of (10) can be formed as

$$\min_{y,z,t,u,v,w,Y_k} - R_R y + q^T t - A_0 u - G_0 w$$

$$\text{s.t.} \quad \begin{bmatrix} H_R \\ 0 \end{bmatrix} y - \begin{bmatrix} C^T \\ -C_m^T \end{bmatrix} z + \begin{bmatrix} I \\ P^T \end{bmatrix} t + \begin{bmatrix} 0 \\ D^T \end{bmatrix} u + \begin{bmatrix} 0 \\ S^T \end{bmatrix} v$$

$$+ \begin{bmatrix} G^T \\ 0 \end{bmatrix} w + \sum_{k=1}^{N_k} K_{S,k}^*(Y_k) = \begin{bmatrix} 0 \\ L \end{bmatrix}$$

$$z \geq 0$$

$$t \geq 0$$

$$Y_k \succeq 0 \ \forall k = 1, \ldots, N_k$$

(12b)

(12c)

(12d)

(12e)

with the dual variables $y, z, t, u, v, w,$ and $Y_k$. 

16
Part III

Supplementary Material
Appendix A

Parameter Studies

This appendix contains results in addition to Paper I (Baandrup et al. (2019c)). Hence, the appendix holds parameter studies of ten typical design parameters in the orthotropic deck and diaphragm beam, as shown in Fig. A.1. The results of the studies are shown in Fig. A.2-A.13. The content, besides the FD3 (LC5) function, of Fig. A.2-A.4 and A.13 was already included in both Paper I and in Section 4.2, but included here for entirety. Remark, the vertical axes vary to show all plotted functions clearly.

The output of the studies consist of the four fatigue stress functions (elaborated further in Paper II) normalized with their maximum allowable stress, and the Eurocode stiffness requirement normalized such that unity is the upper allowable limit. Hence, values below unity are acceptable in the design. Furthermore, the girder weight is plotted as a function of the design parameters. A vertical solid line indicates the initial design values.

The parameter $w_{tr,t}$ was studied in two different cases: the width of the trough (with fixed distance between troughs) presented in Fig. A.5 and the distance between troughs (with fixed trough width) illustrated in Fig. A.7. Besides the 11 studies of individual parameters, a study of a trough stiffness scaling is included in Fig. A.13, hence a simultaneous scaling of the parameters $t_{tp}$, $t_{tr}$, $w_{tr,t}$, $w_{tr,b}$, and $h_{tr}$.

![Figure A.1](image)

**Figure A.1**: Indication of ten typical design parameters in an orthotropic deck and diaphragm beam (figure from Baandrup et al. (2019c))
Parameter Studies

Figure A.2: Parameter study of the diaphragm distance $L_d$

Figure A.3: Parameter study of the top plate thickness $t_{tp}$

Figure A.4: Parameter study of the trough plate thickness $t_{tr}$

Figure A.5: Parameter study of the trough top width $w_{tr,t}$ (with fixed distance between troughs)
Figure A.6: Parameter study of the trough bottom width $w_{tr,b}$

Figure A.7: Parameter study of the distance between troughs $w_{tr,t}$ (with fixed trough top width)

Figure A.8: Parameter study of the trough height $h_{tr}$

Figure A.9: Parameter study of the web width $h_w$
Parameter Studies

Figure A.10: Parameter study of the web thickness $t_w$

Figure A.11: Parameter study of the flange width $w_f$

Figure A.12: Parameter study of the flange thickness $t_f$

Figure A.13: Parameter study of the trough stiffness scaling
Appendix B

Parametric Optimization

This appendix contains results in connection to the nine optimization cases presented in Paper II (Baandrup et al. (2019d)). In Table 4.1 in Section 4.3.2, the characteristics of the nine cases and main results in the form of weight and price savings after optimization are summarized. Figures B.1-B.9 provide plots for all cases of the objective and constraint functions, as well as design variables, as functions of the iteration number. Figures B.1-B.3 were already included in Paper II but included here for entirety.
Parametric Optimization

Figure B.1: Optimization results for Case 1 in Table 4.1 (figure from Baandrup et al. (2019d))

Figure B.2: Optimization results for Case 2 in Table 4.1 (figure from Baandrup et al. (2019d))

Figure B.3: Optimization results for Case 3 in Table 4.1 (figure from Baandrup et al. (2019d))
Figure B.4: Optimization results for Case 4 in Table 4.1

Figure B.5: Optimization results for Case 5 in Table 4.1

Figure B.6: Optimization results for Case 6 in Table 4.1
Figure B.7: Optimization results for Case 7 in Table 4.1

Figure B.8: Optimization results for Case 8 in Table 4.1

Figure B.9: Optimization results for Case 9 in Table 4.1
Appendix C

Topology Optimization of Bridge Girders

This appendix contains data and results in addition to Paper III (Baandrup et al. (2020)). In Section C.1 the applied load cases are shown and in Section C.2, additional results from the topology optimization not included in Paper III are presented.

C.1 Load Cases

The 12 global (LC 1-12) and two local load cases (LC 13-14) applied in the topology optimization of the bridge girder are shown in Table C.1. The table was also included in Paper III.

Load cases 1-12 were identified from the global IBDAS beam model of the Osman Gazi Bridge described in Chapter 3. Each of these was identified based on the maximum and minimum section forces on the end surface of the girder. Hence, LC1 was derived from the maximum $N_x$, LC2 from minimum $N_x$, LC3 from maximum $M_y$, and so on.

As shown in Table C.1, load case 6 and 8 are identical, hence the minimum bending moment $M_z$ and minimum shear force $V_y$ occurred in the same load case and at the same girder location in the global beam model. Consequently, one of these load cases can be neglected in the optimization to reduce the problem size.
Table C.1: Loads applied to the FE-model. Load case 1-12: global, load case 13-14: local

<table>
<thead>
<tr>
<th>Load case</th>
<th>( N_x ) [MN]</th>
<th>( M_y ) [MNm]</th>
<th>( M_z ) [MNm]</th>
<th>( V_y ) [MN]</th>
<th>( V_z ) [MN]</th>
<th>( M_t ) [MNm]</th>
<th>Distributed load [kN/m²]</th>
<th>Hanger force [MN]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.3</td>
<td>65.6</td>
<td>8.5</td>
<td>0.2</td>
<td>0.6</td>
<td>-23.1</td>
<td>-6.8</td>
<td>2.3</td>
</tr>
<tr>
<td>2</td>
<td>-16.6</td>
<td>-99.6</td>
<td>52.2</td>
<td>-0.1</td>
<td>0.5</td>
<td>47.8</td>
<td>-12.4</td>
<td>3.7</td>
</tr>
<tr>
<td>3</td>
<td>-2.1</td>
<td>-247.3</td>
<td>-18.7</td>
<td>-0.2</td>
<td>1.4</td>
<td>-26.4</td>
<td>-12.5</td>
<td>3.5</td>
</tr>
<tr>
<td>4</td>
<td>4.7</td>
<td>169.3</td>
<td>6.0</td>
<td>0.0</td>
<td>0.5</td>
<td>-34.3</td>
<td>-7.5</td>
<td>2.3</td>
</tr>
<tr>
<td>5</td>
<td>0.1</td>
<td>28.8</td>
<td>-818.2</td>
<td>4.6</td>
<td>1.1</td>
<td>-10.0</td>
<td>-7.8</td>
<td>2.3</td>
</tr>
<tr>
<td>6</td>
<td>-1.3</td>
<td>-20.8</td>
<td>403.0</td>
<td>5.4</td>
<td>0.3</td>
<td>24.4</td>
<td>-8.8</td>
<td>2.5</td>
</tr>
<tr>
<td>7</td>
<td>0.1</td>
<td>-13.1</td>
<td>-344.2</td>
<td>-5.3</td>
<td>0.3</td>
<td>-7.9</td>
<td>-7.8</td>
<td>2.3</td>
</tr>
<tr>
<td>8</td>
<td>-1.3</td>
<td>-20.8</td>
<td>403.0</td>
<td>5.4</td>
<td>0.3</td>
<td>24.4</td>
<td>-8.8</td>
<td>2.5</td>
</tr>
<tr>
<td>9</td>
<td>6.1</td>
<td>-36.7</td>
<td>-5.9</td>
<td>-0.1</td>
<td>1.3</td>
<td>55.3</td>
<td>-9.2</td>
<td>3.0</td>
</tr>
<tr>
<td>10</td>
<td>-13.4</td>
<td>-125.1</td>
<td>-25.6</td>
<td>0.0</td>
<td>-0.7</td>
<td>-47.4</td>
<td>-11.6</td>
<td>2.5</td>
</tr>
<tr>
<td>11</td>
<td>-0.1</td>
<td>-13.5</td>
<td>226.5</td>
<td>0.2</td>
<td>1.0</td>
<td>-170.8</td>
<td>-11.4</td>
<td>3.4</td>
</tr>
<tr>
<td>12</td>
<td>-2.2</td>
<td>-18.4</td>
<td>-235.8</td>
<td>-0.2</td>
<td>1.0</td>
<td>165.5</td>
<td>-11.0</td>
<td>3.2</td>
</tr>
<tr>
<td>13</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>14*</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

*Only distributed load on half of the top surface and average hanger force due to skew load

C.2 Results

In addition to the main results presented in Paper III (Baandrup et al. (2020)) this section contains additional results of interest, which were not included in the paper.

All the presented optimization studies are summarized in Table C.2. Besides the variations between the studied cases displayed in the table, all aspects of the model and optimization are identical to what is described in Paper III.

In Case 1-13, shown in Fig. C.1-C.13, the girder structures were optimized only for a single load case, hence for each of the 13 individual load cases in Table C.1. In Case 14 (Fig. C.14) the structure was optimized for both load case 13 and 14. In Case 15-18 (Fig. C.15-C.18) the structure was optimized for the five most governing load cases, load case 1, 5, 10, 13, and 14. In Case 15-17 the influence from the local load cases was increased. Finally, in Case 18 the outer skin plates, defining the aerodynamic wind profile, were enforced.
Table C.2: Summary of all studied cases of topology optimization of the bridge girder

<table>
<thead>
<tr>
<th>Case</th>
<th>Load case(s)</th>
<th>Weighting of load case</th>
<th>Enforced outer skin plates</th>
<th>Number of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6/8</td>
<td>—</td>
<td></td>
<td>$50 \times 10^6$</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>13</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>14</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>13, 14</td>
<td>50%, 50%</td>
<td></td>
<td>$180 \times 10^6$</td>
</tr>
<tr>
<td>15</td>
<td>1, 5, 10, 13, 14</td>
<td>20%, 20%, 20%, 20%, 20%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>10%, 10%, 10%, 35%, 35%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>5%, 5%, 42.5%, 42.5%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td></td>
<td></td>
<td>Yes</td>
<td></td>
</tr>
</tbody>
</table>
Topology Optimization of Bridge Girders

Figure C.1: Case 1: Girder structure optimized for load case 1
Topology Optimization of Bridge Girders

Figure C.2: Case 2: Girder structure optimized for load case 2
Figure C.3: Case 3: Girder structure optimized for load case 3
Figure C.4: Case 4: Girder structure optimized for load case 4

(a) Perspective view of a single section with fixed top plate removed to reveal the internal details

(b) Perspective view of five girder sections with fixed top plate

(c) Top view of a single section without fixed top plate

(d) Front view of the girder with fixed top plate
Topography Optimization of Bridge Girders

Figure C.5: Case 5: Girder structure optimized for load case 5

(a) Perspective view of a single section with fixed top plate removed to reveal the internal details

(b) Perspective view of five girder sections with fixed top plate

(c) Top view of a single section without fixed top plate

(d) Front view of the girder with fixed top plate
Figure C.6: Case 6: Girder structure optimized for load case 6 / 8
Figure C.7: Case 7: Girder structure optimized for load case 7
(a) Perspective view of a single section with fixed top plate removed to reveal the internal details

(b) Perspective view of five girder sections with fixed top plate

(c) Top view of a single section without fixed top plate

(d) Front view of the girder with fixed top plate

Figure C.8: Case 8: Girder structure optimized for load case 9
Topology Optimization of Bridge Girders

(a) Perspective view of a single section with fixed top plate removed to reveal the internal details

(b) Perspective view of five girder sections with fixed top plate

(c) Top view of a single section without fixed top plate

(d) Front view of the girder with fixed top plate

Figure C.9: Case 9: Girder structure optimized for load case 10
Figure C.10: Case 10: Girder structure optimized for load case 11
Topography Optimization of Bridge Girders

Figure C.11: Case 11: Girder structure optimized for load case 12
Topology Optimization of Bridge Girders

Figure C.12: Case 12: Girder structure optimized for load case 13
(a) Perspective view of a single section with fixed top plate removed to reveal the internal details

(b) Perspective view of five girder sections with fixed top plate

(c) Top view of a single section without fixed top plate

(d) Front view of the girder with fixed top plate

**Figure C.13**: Case 13: Girder structure optimized for load case 14
(a) Perspective view of a single section with fixed top plate removed to reveal the internal details

(b) Perspective view of five girder sections with fixed top plate

(c) Top view of a single section without fixed top plate

(d) Front view of the girder with fixed top plate

**Figure C.14**: Case 14: Girder structure optimized for load case 13 and 14
Figure C.15: Case 15: Girder structure optimized for load case 1, 5, 10, 13, and 14 with equal weighting of load cases (20% on each)
Figure C.16: Case 16: Girder structure optimized for load case 1, 5, 10, 13, and 14 with increased weighting of local load cases 13 and 14 (from 20% to 35.0%)
Topology Optimization of Bridge Girders

Figure C.17: Case 17: Girder structure optimized for load case 1, 5, 10, 13, and 14 with increased weighting of local load cases 13 and 14 (from 20% to 42.5%)
Figure C.18: Case 18: Girder structure optimized for load case 1, 5, 10, 13, and 14 with increased weighting of local load cases 13 and 14 (from 20% to 42.5%). With enforced outer skin plates (wind profile)
Topology Optimization of Bridge Girders
Appendix D

Truss Optimization of Bridge Girders

This appendix contains data and results in addition to Paper VI (Baandrup et al. (2019b)). In Section D.1 the applied load cases are shown, in Section D.2 the dual formulation of the optimization problem is derived, and in Section D.3 additional results from the truss optimization are presented.

D.1 Load Cases

The 12 global (LC 1-12) and two local load cases (LC 13-14) applied in the truss optimization of the bridge girder are shown in Table D.1. This table was also included in Paper VI.

Load cases 1-12 were identified from the global IBDAS beam model of the Osman Gazi Bridge described in Chapter 3. Each of these was identified based on the maximum and minimum section forces on the end surface, a, of the girder. Hence LC1 was derived from the maximum \( N_x \), LC2 from minimum \( N_x \), LC3 from maximum \( M_y \), and so on.

As shown in the table, load case 6 and 8 are identical, hence the minimum bending moment \( M_z \) and minimum shear force \( V_y \) occurred in the same load case and at the same girder location in the global beam model, similar to what was observed in Section C.1 of Appendix C. Consequently, one of these load cases can be neglected in the optimization to reduce the problem size.

For the two local load cases, LC 13 and 14, in addition to the distributed load \( p \) and hanger forces \( P \), two moments, \( M_{y,a} \) and \( M_{y,b} \), are applied to the end surface to imitate the behavior of the continuous girder. In LC 13, the distributed load is applied to the entire top surface, whereas in LC 14, the load is applied only on one side of the longitudinal center line, to give a skew distribution.
Table D.1: Loads applied to the FE-model. Load case 1-12: global, load case 13-14: local (table from Baandrup et al. (2019b))

<table>
<thead>
<tr>
<th>Load Case</th>
<th>LC</th>
<th>N</th>
<th>x,a</th>
<th>M</th>
<th>y,a</th>
<th>M</th>
<th>z,a</th>
<th>M</th>
<th>t,a</th>
<th>M</th>
<th>x,b</th>
<th>M</th>
<th>y,b</th>
<th>M</th>
<th>z,b</th>
<th>M</th>
<th>t,b</th>
<th>M</th>
<th>Dist.</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.0</td>
<td>-9.29</td>
<td>-10.46</td>
<td>18.93</td>
<td>0.16</td>
<td>1.43</td>
<td>-20.12</td>
<td>-9.29</td>
<td>-24.77</td>
<td>-14.91</td>
<td>-0.16</td>
<td>-1.39</td>
<td>20.12</td>
<td>-7.66</td>
<td>2.45</td>
<td>1.2</td>
<td>19.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>16.58</td>
<td>-75.21</td>
<td>47.72</td>
<td>-0.06</td>
<td>-1.18</td>
<td>47.79</td>
<td>16.58</td>
<td>84.39</td>
<td>-49.34</td>
<td>0.06</td>
<td>-0.45</td>
<td>-47.79</td>
<td>-8.83</td>
<td>3.65</td>
<td>1.2</td>
<td>24.7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2.07</td>
<td>236.57</td>
<td>35.42</td>
<td>0.24</td>
<td>1.66</td>
<td>26.45</td>
<td>-2.07</td>
<td>-242.34</td>
<td>-29.53</td>
<td>-0.24</td>
<td>1.20</td>
<td>-26.45</td>
<td>-15.60</td>
<td>3.58</td>
<td>1.2</td>
<td>29.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>-4.08</td>
<td>-155.36</td>
<td>-6.27</td>
<td>0.01</td>
<td>0.09</td>
<td>34.30</td>
<td>4.08</td>
<td>119.82</td>
<td>6.65</td>
<td>-0.015</td>
<td>-2.75</td>
<td>-34.30</td>
<td>-8.59</td>
<td>4.09</td>
<td>1.2</td>
<td>31.9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>-0.08</td>
<td>-5.40</td>
<td>704.06</td>
<td>-4.56</td>
<td>0.07</td>
<td>9.20</td>
<td>0.08</td>
<td>18.82</td>
<td>-818.17</td>
<td>4.56</td>
<td>1.15</td>
<td>-9.20</td>
<td>-8.77</td>
<td>2.21</td>
<td>1.2</td>
<td>34.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>1.29</td>
<td>-30.53</td>
<td>-779.78</td>
<td>-5.45</td>
<td>1.64</td>
<td>-24.37</td>
<td>-1.29</td>
<td>8.53</td>
<td>643.56</td>
<td>5.45</td>
<td>-0.12</td>
<td>24.37</td>
<td>-9.90</td>
<td>2.42</td>
<td>1.2</td>
<td>36.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>-0.11</td>
<td>-24.28</td>
<td>712.49</td>
<td>5.33</td>
<td>1.25</td>
<td>8.37</td>
<td>0.11</td>
<td>8.57</td>
<td>-579.27</td>
<td>-5.33</td>
<td>0.00</td>
<td>-8.37</td>
<td>-8.77</td>
<td>2.19</td>
<td>1.2</td>
<td>38.8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>1.29</td>
<td>-30.53</td>
<td>-779.78</td>
<td>-5.45</td>
<td>1.64</td>
<td>-24.37</td>
<td>-1.29</td>
<td>8.53</td>
<td>643.56</td>
<td>5.45</td>
<td>-0.12</td>
<td>24.37</td>
<td>-9.90</td>
<td>2.42</td>
<td>1.2</td>
<td>41.1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>6.11</td>
<td>-142.86</td>
<td>-16.49</td>
<td>-0.16</td>
<td>1.54</td>
<td>53.50</td>
<td>-6.11</td>
<td>75.78</td>
<td>12.49</td>
<td>0.16</td>
<td>-3.83</td>
<td>-53.50</td>
<td>-6.46</td>
<td>3.22</td>
<td>1.2</td>
<td>43.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>-13.32</td>
<td>42.33</td>
<td>-23.89</td>
<td>0.04</td>
<td>-5.22</td>
<td>-53.72</td>
<td>13.32</td>
<td>46.88</td>
<td>24.89</td>
<td>-0.04</td>
<td>1.92</td>
<td>-53.72</td>
<td>-2.76</td>
<td>2.54</td>
<td>1.2</td>
<td>45.7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>11</td>
<td>-0.04</td>
<td>35.64</td>
<td>-230.91</td>
<td>-0.18</td>
<td>1.35</td>
<td>284.41</td>
<td>0.04</td>
<td>-39.93</td>
<td>226.49</td>
<td>0.18</td>
<td>-1.01</td>
<td>-184.41</td>
<td>-14.53</td>
<td>3.49</td>
<td>1.2</td>
<td>47.8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>2.04</td>
<td>41.39</td>
<td>239.93</td>
<td>0.17</td>
<td>1.33</td>
<td>-179.25</td>
<td>-2.04</td>
<td>46.63</td>
<td>-235.75</td>
<td>-0.17</td>
<td>0.99</td>
<td>179.25</td>
<td>-14.00</td>
<td>3.34</td>
<td>1.2</td>
<td>50.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>13</td>
<td>-10.10</td>
<td>41.39</td>
<td>239.93</td>
<td>0.17</td>
<td>1.33</td>
<td>-179.25</td>
<td>-2.04</td>
<td>46.63</td>
<td>-235.75</td>
<td>-0.17</td>
<td>0.99</td>
<td>179.25</td>
<td>-14.00</td>
<td>3.34</td>
<td>1.2</td>
<td>52.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>14</td>
<td>-10.10</td>
<td>41.39</td>
<td>239.93</td>
<td>0.17</td>
<td>1.33</td>
<td>-179.25</td>
<td>-2.04</td>
<td>46.63</td>
<td>-235.75</td>
<td>-0.17</td>
<td>0.99</td>
<td>179.25</td>
<td>-14.00</td>
<td>3.34</td>
<td>1.2</td>
<td>54.4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Only distributed load p on one side of the longitudinal center line, and average hanger force P due to skew load
D.2 Dual Formulation of Optimization Problem

Similar to the technique in Baandrup et al. (2019a) the dual formulation of problem (6.7) is derived in the following. To handle the semidefinite constraint (6.7h) in the formulation of the Lagrangian, a semidefinite dual variable $Y_k \forall k = 1, \ldots, N_k$ is introduced. To derive the Lagrangian function a self-adjoint operator on $Y_k$ is defined as

$$\sum_{k=1}^{N_k} \langle K_{S,k}(\beta_k, A), Y_k \rangle.$$  

The reader is referred to Baandrup et al. (2019a) for details on the adjoint operator. The Lagrangian of problem (6.7) is given as

$$L(\beta, A, y, z, t, u, v, w, Y_k) = \begin{bmatrix} 0 & L^\top \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} - y^\top \begin{bmatrix} H_R & 0 \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} - R_R - z^\top \begin{bmatrix} C & -C_m \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} + t^\top \begin{bmatrix} -I & -P \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} - q - u^\top \begin{bmatrix} 0 & D \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} - v^\top \begin{bmatrix} 0 & S \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} - w^\top \begin{bmatrix} G & 0 \end{bmatrix} \begin{bmatrix} \beta \\ A \end{bmatrix} - \sum_{k=1}^{N_k} K_{S,k}^*(Y_k).$$

(D.1a)

By differentiating (D.1b) with respect to the primal variables $(\beta, A)$ the dual of (6.7) can be formed as

$$\begin{align*}
\min & \quad - R_R^\top y + q^\top t - A_0^\top u - G_0^\top w \\
\text{s.t.} & \quad \begin{bmatrix} H_R \\ 0 \end{bmatrix} y - \begin{bmatrix} C^\top \\ -C_m^\top \end{bmatrix} z + \begin{bmatrix} I \\ P^\top \end{bmatrix} t + \begin{bmatrix} 0 \\ D^\top \end{bmatrix} u + \begin{bmatrix} 0 \\ S^\top \end{bmatrix} v \\
& \quad + \begin{bmatrix} G^\top \\ 0 \end{bmatrix} w + \sum_{k=1}^{N_k} K_{S,k}^*(Y_k) = \begin{bmatrix} 0 \\ L \end{bmatrix} \\
& \quad z \geq 0 \\
& \quad t \geq 0 \\
& \quad Y_k \geq 0 \quad \forall k = 1, \ldots, N_k
\end{align*}$$

(D.2a-d)

with the dual variables $y, z, t, u, v, w, \text{and } Y_k$. 

Department of Civil Engineering - Technical University of Denmark
D.3 Results

In addition to the main results presented in Paper VI (Baandrup et al. (2019b)), this section contains additional results of interest, which were not included in the paper. The structures in Case 12, 16, 18, and 19 were illustrated in the paper, however, with a different layout.

All the presented optimization studies and numerical results hereof are summarized in Table D.2. Besides the variations between the studied cases displayed in the table, all aspects of the model and optimization are identical to what is described in Paper VI.

In Case 1-16, the top plate thickness was fixed to $t_{tp} = 20$ mm, whereas in case 17-20 different top plate thicknesses were studied.

In Case 1-13, shown in Fig. D.1-D.13, the girder truss structures were optimized only for a single load case, hence for each of the 13 individual load cases presented in Table D.1. In Case 14 (Fig. D.14) the structure was optimized for both load case 13 and 14, and in Case 15 (Fig. D.15) the structure was optimized for load case 1, 5, 10, 13, and 14, hence the same load cases applied in the topology optimization (see Chapter 5 and Appendix C). Finally, in Case 16-20 (Fig. D.16-D.20) the structure was optimized for all 13 load cases simultaneously.

When visualizing the results, a threshold is applied to members with zero or negligible cross-sectional areas for clarity. Two threshold values, relative to the largest cross-sectional area, are applied to identify the main structure ($N_{e,Main} | a_e > 0.1 \max(\mathbf{A})$) and the detailed structure ($N_{e,Det.} | a_e > 0.01 \max(\mathbf{A})$), respectively.

In all figures, the boundary of the outer domain is indicated by thin black lines.
### Table D.2: Summary of all studied cases of the bridge girder truss optimization.

Weight savings are given as percentage compared to the total weight of 11.13 ton/m of the Osman Gazi Bridge.

<table>
<thead>
<tr>
<th>Case</th>
<th>Load case(s)</th>
<th>Top plate thickness</th>
<th>Total weight [ton/m]</th>
<th>Weight saving</th>
<th>Top plate weight [ton/m]</th>
<th>Weight of lower part [ton/m]</th>
<th>N_e,Main</th>
<th>N_e,Det.</th>
</tr>
</thead>
<tbody>
<tr>
<td>OGB*</td>
<td>—</td>
<td>26†</td>
<td>11.13</td>
<td>—</td>
<td>5.25</td>
<td>5.88</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4.51</td>
<td>59.4%</td>
<td>0.48</td>
<td>310</td>
<td>442</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4.78</td>
<td>57.0%</td>
<td>0.74</td>
<td>322</td>
<td>459</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>5.58</td>
<td>49.9%</td>
<td>1.54</td>
<td>276</td>
<td>348</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>5.35</td>
<td>52.0%</td>
<td>1.31</td>
<td>314</td>
<td>440</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>4.46</td>
<td>59.9%</td>
<td>0.43</td>
<td>294</td>
<td>466</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>6</td>
<td>6 / 8</td>
<td>4.66</td>
<td>58.1%</td>
<td>0.62</td>
<td>340</td>
<td>452</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>4.51</td>
<td>59.5%</td>
<td>0.47</td>
<td>324</td>
<td>484</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>5.31</td>
<td>52.3%</td>
<td>1.27</td>
<td>330</td>
<td>443</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>4.92</td>
<td>55.8%</td>
<td>0.89</td>
<td>364</td>
<td>465</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>5.53</td>
<td>50.3%</td>
<td>1.50</td>
<td>366</td>
<td>494</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>11</td>
<td>12</td>
<td>5.50</td>
<td>50.5%</td>
<td>1.47</td>
<td>366</td>
<td>481</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>12</td>
<td>13</td>
<td>4.27</td>
<td>61.6%</td>
<td>0.24</td>
<td>286</td>
<td>382</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>13</td>
<td>14</td>
<td>4.21</td>
<td>62.2%</td>
<td>0.17</td>
<td>272</td>
<td>374</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>14</td>
<td>13, 14</td>
<td>4.27</td>
<td>61.6%</td>
<td>0.24</td>
<td>284</td>
<td>378</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>15</td>
<td>1, 5, 10, 13, 14</td>
<td>4.98</td>
<td>55.3%</td>
<td>0.94</td>
<td>346</td>
<td>544</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>16 (3*)</td>
<td>1-14</td>
<td>6.09</td>
<td>45.3%</td>
<td>2.06</td>
<td>344</td>
<td>472</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>17</td>
<td>5</td>
<td>3.53</td>
<td>68.2%</td>
<td>1.01</td>
<td>2.52</td>
<td>420</td>
<td>580</td>
<td></td>
</tr>
<tr>
<td>18 (1*)</td>
<td>10</td>
<td>4.09</td>
<td>63.2%</td>
<td>2.02</td>
<td>2.07</td>
<td>342</td>
<td>498</td>
<td></td>
</tr>
<tr>
<td>19 (2*)</td>
<td>15</td>
<td>5.11</td>
<td>54.1%</td>
<td>3.03</td>
<td>2.08</td>
<td>332</td>
<td>464</td>
<td></td>
</tr>
<tr>
<td>20 (4*)</td>
<td>25</td>
<td>7.09</td>
<td>35.3%</td>
<td>5.04</td>
<td>2.05</td>
<td>348</td>
<td>472</td>
<td></td>
</tr>
</tbody>
</table>

*Osman Gazi Bridge
†Equivalent plate thickness of the orthotropic top deck (top plate and troughs)
‡Without walkways
⋆Equivalent Case number in Table 6.2 in Section 6.2.2 and in Table 1 in Paper VI
**Truss Optimization of Bridge Girders**

Figure D.1: Case 1: Truss girder optimized for load case 1, $W = 4.51$ ton/m. Detailed structure of three girder sections, $N_{e,Det.} = 442$ per section.
Figure D.2: Case 2: Truss girder optimized for load case 2, $W = 5.78$ ton/m. Detailed structure of three girder sections, $N_{e,\text{Det.}} = 459$ per section
Truss Optimization of Bridge Girders

(a) Perspective view with solid top plate

(b) Perspective and section view without solid top plate

(c) Top view without solid top plate

**Figure D.3:** Case 3: Truss girder optimized for load case 3, $W = 5.58 \text{ ton/m}$. Detailed structure of three girder sections, $N_{e,Det.} = 348$ per section
Figure D.4: Case 4: Truss girder optimized for load case 4, $W = 5.35$ ton/m. Detailed structure of three girder sections, $N_{e,Det.} = 440$ per section.
Truss Optimization of Bridge Girders

Figure D.5: Case 5: Truss girder optimized for load case 5, \( W = 4.46 \text{ ton/m} \).
Detailed structure of three girder sections, \( N_{e,Det.} = 466 \) per section
Figure D.6: Case 6: Truss girder optimized for load case 6/8, $W = 4.66$ ton/m. Detailed structure of three girder sections, $N_{c,Det.} = 452$ per section.
Figure D.7: Case 7: Truss girder optimized for load case 7, $W = 4.51$ ton/m. Detailed structure of three girder sections, $N_{e,\text{Det.}} = 484$ per section.
Figure D.8: Case 8: Truss girder optimized for load case 9, $W = 5.31$ ton/m. Detailed structure of three girder sections, $N_{e,Det.} = 443$ per section
Truss Optimization of Bridge Girders

Figure D.9: Case 9: Truss girder optimized for load case 10, $W = 4.92 \text{ ton/m}$. Detailed structure of three girder sections, $N_{e,Det.} = 465$ per section.
Figure D.10: Case 10: Truss girder optimized for load case 11, \( W = 5.53 \text{ ton/m} \). Detailed structure of three girder sections, \( N_{e,Det.} = 494 \) per section.
Truss Optimization of Bridge Girders

Figure D.11: Case 11: Truss girder optimized for load case 12, $W = 5.50$ ton/m. Detailed structure of three girder sections, $N_{c,Det.} = 481$ per section
Figure D.12: Case 12: Truss girder optimized for load case 13, $W = 4.27$ ton/m. Detailed structure of three girder sections, $N_{e,Det.} = 382$ per section.
Truss Optimization of Bridge Girders

Figure D.13: Case 13: Truss girder optimized for load case 14, $W = 4.21$ ton/m. Detailed structure of three girder sections, $N_{e,Det.} = 374$ per section.
Figure D.14: Case 14: Truss girder optimized for load cases 13 and 14, $W = 4.27$ ton/m. Detailed structure of three girder sections, $N_{e,Det.} = 378$ per section.
Figure D.15: Case 15: Truss girder optimized for load cases 1, 5, 10, 13, and 14, $W = 4.98$ ton/m. Three girder sections
Truss Optimization of Bridge Girders

(a) Perspective view with solid top plate. Detailed structure

(b) Perspective and section view without solid top plate. Main structure, $N_{e,\text{Main}} = 344$ per section

(c) Top view without solid top plate. Detailed structure, $N_{e,\text{Det.}} = 472$ per section

**Figure D.16**: Case 16: Truss girder optimized for load cases 1-14, with 20 mm top plate thickness, $W = 6.09$ ton/m. Three girder sections
Truss Optimization of Bridge Girders

(a) Perspective view with solid top plate. Detailed structure

(b) Perspective and section view without solid top plate. Main structure, $N_{e,\text{Main}} = 420$ per section

(c) Top view without solid top plate. Detailed structure, $N_{e,\text{Det.}} = 580$ per section

**Figure D.17**: Case 17: Truss girder optimized for load cases 1-14, with 5 mm top plate thickness, $W = 3.53$ ton/m. Three girder sections
Truss Optimization of Bridge Girders

Figure D.18: Case 18: Truss girder optimized for load cases 1-14, with 10 mm top plate thickness, $W = 4.09$ ton/m. Three girder sections.

(a) Perspective view with solid top plate. Detailed structure

(b) Perspective and section view without solid top plate. Main structure, $N_{e,\text{Main}} = 342$ per section

(c) Top view without solid top plate. Detailed structure, $N_{e,\text{Det.}} = 498$ per section
Figure D.19: Case 19: Truss girder optimized for load cases 1-14, with 15 mm top plate thickness, \( W = 5.11 \text{ ton/m} \). Three girder sections
Truss Optimization of Bridge Girders

Figure D.20: Case 20: Truss girder optimized for load cases 1-14, with 25 mm top plate thickness, $W = 7.09$ ton/m. Three girder sections
Truss Optimization of Bridge Girders
Notes
Note I

Critical Fatigue Details

The appended note is a supplement to Paper I (Baandrup et al. (2019c)) and Paper II (Baandrup et al. (2019d)).

The note contains a brief document prepared by COWI A/S with the title "Critical fatigue details to be used for deck optimization". The document was prepared in relation to the specific parameter studies and parametric optimization of the conventional design concept, and should not be used in any other context.

It should be noted that the thesis author has not contributed to the creation of the note. Hence, only the results stated have been used in relation to the work otherwise presented in the thesis.
Critical Fatigue Details
Critical fatigue details to be used for deck optimization

CONTENTS

1  Introduction  1
2  Locations  1
3  Fatigue load model and vehicles  2
4  Fatigue assessment  3

1  Introduction

The following presents the most critical details in a fatigue verification based on the experience gained from the Ozman Gazi Bridge and the 1915 Canakkale Bridge (ongoing design).

2  Locations

Four locations in the orthotropic deck have shown to be most critical in terms of fatigue. The four locations are listed and shown below.

- Location 1: Diaphragm cut-out
- Location 2: Diaphragm to trough weld, direct stress in the diaphragm
- Location 3: Trough to top plate weld, direct stress in the underside of the top plate
- Location 4: Top plate to diaphragm weld, direct stress in the underside of the top plate
Table 1  
Detail categories for critical fatigue details

<table>
<thead>
<tr>
<th>Location</th>
<th>SN curve type</th>
<th>Detail category</th>
<th>Detail in EN 1993-1-9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Direct stress range</td>
<td>125</td>
<td>Table 8.1 detail 5</td>
</tr>
<tr>
<td>2</td>
<td>Direct stress range</td>
<td>80</td>
<td>Table 8.5 detail 2</td>
</tr>
<tr>
<td>3</td>
<td>Direct stress range</td>
<td>71 (100 used for Canakkale, since EN detail not valid for the considered effect)</td>
<td>(Table 8.8 detail 7)</td>
</tr>
<tr>
<td>4</td>
<td>Direct stress range</td>
<td>71</td>
<td>Table 8.4 detail 8</td>
</tr>
</tbody>
</table>

Figure 1  Locations of critical fatigue details

3  Fatigue load model and vehicles

The Eurocode fatigue load model 4 is the most detailed fatigue load model in the Eurocode. However, in reason projects the fatigue load model of the UK Annex has been used, since this supplies a larger variation of vehicles in comparison to the Eurocode and is less conservative. Hereby 23 different vehicles are accounted for compared to 5 vehicles in the Eurocode. The FLM4 UK Annex axle and vehicle types are shown below.

Among the 23 vehicles of the UK annex, one vehicle have shown to contribute far more to the total damage than any other vehicles, the 5A-H vehicle. This is due to a combination of a high load on a single axle and many repetitions. It is assessed that in a deck optimization task only vehicle 5A-H needs to be considered.
Fatigue assessment

In order to simplify the fatigue verification in an optimization process, it is suggested to limit the maximum stress from vehicle 5A-H in the critical details.
based on the experience from the Ozman Gazi Bridge and 1915 Canakkale Bridge.

Both bridges are designed to have 2 million vehicles/year in the heavy lane and have a 100 year design life. This results in 18.1 million passing vehicle 5A-H in the heavy lane.

The size of influence on each detail differs, thus for some details only one stress range contributes to the fatigue damage (e.g. the three rear axles of vehicle 5A-H located above the cut out) while for other several axles contributes with individual stress ranges (e.g. the two first axles of vehicle 5A-H on detail location 3).

Based on time history analysis it has been found that vehicle 5A-H only gives a positive or a negative stress in most details, hereby the stress range can be estimated based on the maximum/minimum stress. However, for a few details both a positive and negative stress occur in the same stress range as seen in the stress history below for detail location 3. Here it is important to allow for a larger range than the maximum/minimum stress indicates.

Figure 4  SYY stress history for detail location 3, vehicle 5A-H in the centre of lane 1, 1915 Canakkale Bridge.

Contributing axles, the number of damage contributing ranges per vehicle and the target stress range and direction is shown below.
Table 2  

<table>
<thead>
<tr>
<th>Location</th>
<th>Detail category</th>
<th>Contributing vehicle axles</th>
<th>Damage contributing ranges per vehicles</th>
<th>Stress direction</th>
<th>Target stress range</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>125</td>
<td>Three rear axles</td>
<td>1</td>
<td>Min. SSS</td>
<td>60-70 MPa</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>First and second axles (centred above) or three rear axles (almost centred above)</td>
<td>1</td>
<td>Max. principal stress</td>
<td>30-40 MPa</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>First and second axle, individually, approx. 0.6 from the diaphragm</td>
<td>2</td>
<td>Min. SYY</td>
<td>35-45 MPa¹</td>
</tr>
<tr>
<td>4</td>
<td>71</td>
<td>First axle, right next to the diaphragm</td>
<td>1</td>
<td>Min. SSS</td>
<td>35-40 MPa</td>
</tr>
</tbody>
</table>

¹ Range from approx. +5 to -40 MPa.

Initially, the details of interest can be reduced to location 1 (diaphragm cut-out) and 3 (top plate bottom stress at trough).
Critical Fatigue Details
The main design principles for girders in cable-supported bridges have remained largely unchanged for the past 50 years and have reached a point where the potential for further development is limited. To accommodate the main challenges of decreasing self-weight and reducing fatigue issues, it is anticipated that radical design changes will be required.

In this thesis, to identify innovative girder design concepts, three different methods of structural optimization are applied: parametric optimization, topology optimization, and truss-layout optimization. From the optimization studies, significantly different design concepts are identified with savings in self-weight in the range of 13%-45%. Additionally, the material-efficient structures reveal new and improved load-carrying principles.