On the Pitchfork Bifurcation of the Folded Node and Other Unbounded Time-Reversible Connection Problems in $\mathbb{R}^3$

Kristiansen, Kristian Uldall

Published in:
SIAM Journal on Applied Dynamical Systems

Link to article, DOI:
10.1137/20M1326180

Publication date:
2020

Document Version
Peer reviewed version

Citation (APA):
ON THE PITCHFORK BIFURCATION OF THE FOLDED NODE
AND OTHER UNBOUNDED TIME-REVERSIBLE CONNECTION
PROBLEMS IN $\mathbb{R}^3$

K. ULDALL KRISTIANSEN

Abstract. In this paper, we revisit the folded node and the bifurcations of secondary canards
at resonances $\mu \in \mathbb{N}$. In particular, we prove for the first time that pitchfork bifurcations occur at
all even values of $\mu$. Our approach relies on a time-reversible version of the Melnikov approach in
[29], used in [30] to prove the transcritical bifurcations for all odd values of $\mu$. It is known that
the secondary canards produced by the transcritical and the pitchfork bifurcations only reach the
Fenichel slow manifolds on one side of each transcritical bifurcation for all $0 < \epsilon \ll 1$. In this paper,
we provide a new geometric explanation for this fact, relying on the symmetry of the normal form
and a separate blowup of the fold lines. We also show that our approach for evaluating the Melnikov
integrals of the folded node – based upon local characterization of the invariant manifolds by higher
order variational equations and reducing these to an inhomogeneous Weber equation – applies to
general, quadratic, time-reversible, unbounded connection problems in $\mathbb{R}^3$. We conclude the paper
by using our approach to present a new proof of the bifurcation of periodic orbits from infinity in
the Falkner-Skan equation.

Key words. Folded node, bifurcations, canards, secondary canards, global bifurcations, geo-
metric singular perturbation theory, blowup, Falkner-Skan equation, Nosé equations, two-fold.

AMS subject classifications. 37C27, 37C29, 37G25, 34A26, 34E15.

1. Introduction. In slow-fast systems with one fast variable and two slow ones,
the folded node $p$ is a singularity of the slow flow on the fold line of a critical manifold $C$.
See an illustration in Fig. 1.1. Upon desingularization $p$ corresponds to a stable
node with eigenvalues $\lambda_s < \lambda_w < 0$, and its strong stable manifold $v$, tangent to
the eigenvector associated with $\lambda_s$, produces a funnel region on the critical manifold,
where orbits approach the singularity tangent to the weak eigendirection (associated
with the eigenvalue $\lambda_w$). Due to the contraction within the funnel region, the folded
node – upon composition with a global return mapping – provides a mechanism for
producing attracting limit cycles $\Gamma_\epsilon$, see [2]. In fact, a blowup of the folded node
reveals one orbit $\gamma$, along which extended versions $W^{cu}$ and $W^{cs}$ of the attracting
and repelling critical manifolds, respectively, twist or rotate; the number of rotations
being described by $\mu := \lambda_s/\lambda_w > 1$. The twisting is such that these manifolds
intersect transversally whenever $\mu \notin \mathbb{N}$. As a consequence, for these values of $\mu$,
there exists a ‘weak canard’ connecting extended versions of the Fenichel slow manifolds.
This orbit acts, due to the twisting of $W^{cu}$ along $\gamma$, as a ‘center of rotation’ and
trajectories on either side will therefore experience small oscillations before they leave
a neighborhood of $p$ by following its unstable set; in Fig. 1.1 this unstable set coincides
with the positive $z$-axis. Consequently, the limit cycles $\Gamma_\epsilon$ will be of mixed-mode type
where small oscillations are followed by larger ones. Such oscillations appear in many
applications, perhaps most notably in chemical reaction dynamics, and the folded
node has therefore gained glory as a (relatively) simple mathematical model of this
phenomenon, see e.g. the review article [4].

Bifurcations of the weak canard occurs whenever $\mu \in \mathbb{N}$; in this case, for $\epsilon = 0$,
the twisting of $W^{cu}$ and $W^{cs}$ is such that these manifolds intersect tangentially along
$\gamma$. These bifurcations were described for $\epsilon = 0$ by the reference [30], working on the
normal form
\[ \dot{x} = \frac{1}{2} \mu y - (\mu + 1) z, \]
\[ \dot{y} = 1, \]
\[ \dot{z} = x + z^2, \]
and using the Melnikov approach developed in [29], following [28]. The system (1.1) is related to the blowup of the folded node \( p \) for \( \epsilon = 0 \). In particular for (1.1), \( \gamma \) takes the following form
\[ \gamma : (x, y, z) = \left( -\frac{1}{4} t^2 + \frac{1}{2} t, \frac{1}{2} t \right), \]
(1.2)

For each odd \( n \), it was shown that there is a transcritical bifurcation of ‘canards’ connecting \( W^{cu}(\mu) \) and \( W^{cs}(\mu) \) for (1.1). For all \( 0 < \epsilon \ll 1 \), this bifurcation produces additional transversal intersections of the slow manifold. The resulting new canards – the ‘secondary canards’ – produce bands on the attracting slow manifold where different number of small oscillations occur, see [2, 3, 30]. For any even \( n \), it was conjectured that a pitchfork bifurcation occurs. This was supported by numerical computations. Furthermore, in [19, App. A] a way was found to compute a ‘third order’ Melnikov integral using Mathematica for all even \( n \) and explicit computations demonstrated that the integral was nonzero for even values of \( n \) up to 20. Following the work of [30] this also shows that a pitchfork bifurcation occurs, at least for these values.

In this paper, we prove the pitchfork bifurcation for every even \( n \) by evaluating the third order Melnikov integral analytically. Our approach is based upon a time-reversible version of the Melnikov theory of [29]. However, the most important insight of this paper is to characterize the manifolds \( W^{cs}(\mu) \) and \( W^{cu}(\mu) \) locally by solutions to higher order variational equations; this is in contrast to [30] which uses an integral representation of these manifolds. We show that this approach, relying on reducing these variational equations to an inhomogeneous Weber equation, extends to a very general class of time-reversible, quadratic, unbounded connection problems in \( \mathbb{R}^3 \). Regardless, for the folded node the ‘time-reversible approach’ also allows us to provide a more detailed blowup picture of the folded node, including a rigorous description of the additional transverse intersections of \( W^{cs} \) and \( W^{cu} \) that arise due to the pitchfork bifurcation.

The bifurcations of canards for (1.1), is closely related to bifurcations of periodic orbits from heteroclinic cycles at infinity. The Falkner-Skan equation
\[ x''' + x''x + \mu(1 - x^2) = 0, \]
(1.3)
and the Nosé equation:
\[ \dot{x} = -y - xz, \]
\[ \dot{y} = x, \]
\[ \dot{z} = \mu(1 - x^2), \]
(1.4)
are well-known examples of systems (without equilibria) possessing such bifurcations, see e.g. [24], and [18] for other examples. The Falkner-Skan equation (1.3) initially appeared in the study of boundary layers in fluid dynamics, see [5]. In this context,
the physical relevant parameter regime is $\mu \in (0, 2)$. However, the equation has subsequently been studied by other authors [23, 22, 24] for all $\mu > 0$ on the basis of the rich dynamics it possesses (including chaotic dynamics and a novel bifurcation of periodic orbits from infinity). On the other hand, the Nosé equations (1.4) model the interaction of a particle with a heat bath [20]. The system also has interesting dynamics without any equilibrium and it possesses many similar properties to (1.1) and (1.3). Nevertheless, the existing description of the bifurcating periodic orbits in the literature [23, 22, 24], for both of these systems, is – as noted by [24] – long and cumbersome, and to a large extend, independent of standard methods of dynamical systems theory. Therefore, although the folded node will be our primary focus, a subsequent aim of the paper, is to apply the Melnikov theory, and our classification of $W^{cs}$ and $W^{cu}$ through solutions of an inhomogeneous Weber equation, to such bifurcations and present a simpler description of the emergence of periodic orbits, based on normal form theory and invariant manifolds and therefore more in tune with dynamical systems theory. For simplicity, we will focus our attention on the Falkner-Skan equation, delaying the details for Nosé to a separate manuscript [15].

1.1. The folded node: Further background. Following [30, Proposition 2.1], see also [27], any folded node can be brought into the ‘normal form’:

$$
\dot{x} = \epsilon \left( \frac{1}{2} \mu y - (\mu + 1)z + O(x, \epsilon, (y + z)^2) \right),
$$

$$
\dot{y} = \epsilon,
$$

$$
\dot{z} = x + z^2 + O(xz^2, z^3, xyz) + \epsilon O(x, y, z, \epsilon),
$$

(1.5)

by only using scalings, translations and a regular time transformation. Here $\mu := \lambda_s/\lambda_w > 1$. The system (1.5) can be studied using Geometric Singular Perturbation Theory (GSPT), [6, 7, 8, 12]. In particular, for $\epsilon = 0$ we find that the critical manifold $C$ is approximately given by the parabolic cylinder $x = -z^2, z < 0$ ($C_a$) being stable and $z > 0$ ($C_r$) being unstable. See Fig. 1.1. Here $C_a$ is in blue, $C_r$ is in red, whereas the degenerate line $F: x = z = 0$, being the fold line, is in green. For (1.5), the folded node $p$ (pink), on $F$, is at the origin. Furthermore, if we for simplicity ignore the $O$-terms in (1.5), then the reduced problem on $C$ is given by

$$
\dot{y}' = 1,
$$

$$
2z'z' = -\frac{1}{2} \mu y + (\mu + 1)z.
$$

Consider $C_a$ where $z < 0$. Then multiplication of the right hand side by $-2z$ gives the topologically equivalent system

$$
\dot{y}' = -2z,
$$

$$
\dot{z}' = \frac{1}{2} \mu y - (\mu + 1)z,
$$

(1.6)

on $C_a$, see [30]. The point $(y, z) = (0, 0)$ is then a stable node of these equations with eigenvalues $-1$ and $-\mu$ and associated eigenvectors:

$$
(2, 1)^T,
$$

and $(2, \mu)^T$, respectively. See illustration of the reduced flow in Fig. 1.2. Notice that the orbits on $C_r$, where $z > 0$, are also orbits of (1.6), but their directions have to be reversed.

3
The folded node singularity $p$. Upon desingularization of the reduced problem, the folded node singularity becomes a stable node. The strong eigenvector associated with the node gives rise to a strong stable manifold $\nu$ (orange) that forms a boundary of a funnel region [27] (shaded), bounded on the other side by $F$, where trajectories approach the folded node $p$ (in finite time before desingularization), tangent to a weak eigendirection. For the system (1.5) without the $O$-terms, the weak eigenvector also produces an invariant space and an orbit $\gamma$, which we show in purple.

**GSPT and blowup analysis.** Compact submanifolds (with boundaries) $S_a$ and $S_r$ of $C_a$ and $C_r$, respectively, bounded away from the fold line, perturb by Fenichel’s theory to attracting and repelling slow manifolds $S_{a,e}$ and $S_{r,e}$ for all $0 < \epsilon \ll 1$, see [6, 7, 8, 12]. We will refer to these manifolds as ‘Fenichel’s (slow) manifolds’. They are nonunique but $O(e^{-c/\epsilon})$-close. Extended versions of these invariant manifolds up close to the folded node $p$ is obtained in [25] by blowing up the point $(x, y, z) = 0$ for $\epsilon = 0$. In further details, the authors apply the following blowup transformation $B : [0, r_0) \times S^3 \to \mathbb{R}^4$, given by

$$
\begin{align*}
B : [0, r_0) \times S^3 &\to \mathbb{R}^4 \\
[r, (\bar{x}, \bar{y}, \bar{z}, \bar{\epsilon})] &\mapsto (x, y, z, \epsilon) = (r^2 \bar{x}, r \bar{y}, r \bar{z}, r^2 \bar{\epsilon}),
\end{align*}
$$

(1.8)

to the extended system ((1.5), $\dot{\epsilon} = 0$). For this extended system, $x = y = z = \epsilon = 0$ is fully nonhyperbolic – its linearization having only zero eigenvalues – but upon the blowup (1.8), we gain hyperbolicity of $r = 0$ after desingularization through division of the resulting right hand sides by $r$. In particular, setting $\bar{x} = -1$ in (1.8) produces
The reduced flow on $C_r$, recall Fig. 1.1, projected onto the $(y,z)$-plane. The strong canard $\nu$ is shown in orange, whereas the weak canard $\gamma$, obtained from (1.5) upon ignoring the $O$-terms, is shown in purple.

The local coordinates $(r_1, y_1, z_1, \epsilon_1)$ provide a coordinate chart ‘$\bar{x} = -1$’, covering $[0, r_0) \times S^3_{\bar{x} < 0}$ where $S^3_{\bar{x} < 0} := S^3 \cap \{\bar{x} < 0\}$. Here $r_1 = 0$ corresponds to $r = 0$. In this chart, one gains hyperbolicity of $C_a$ and $C_r$ for $r = 0$ upon division of the right hand side by $r_1$. By center manifold theory, this then enables an extension of the Fenichel slow manifolds $S_{a,\epsilon}$ and $S_{r,\epsilon}$ as the $B$-image of $\epsilon =$-const. sections of three-dimensional invariant manifolds $M_a$ and $M_r$, respectively, for all $0 < \epsilon \ll 1$. Following [17], we shall abbreviate these extended manifolds in the $(x,y,z)$-space by $S_{a,\sqrt{\epsilon}}$ and $S_{r,\sqrt{\epsilon}}$, respectively; see [25] for further details.

**Remark 1.1.** It is important to highlight that, due to the contraction towards the weak canard, the forward (backward) flow of the Fenichel manifold $S_{a,\epsilon}$ ($S_{r,\epsilon}$) is only a subset of $S_{a,\sqrt{\epsilon}}$ ($S_{r,\sqrt{\epsilon}}$, respectively). Therefore when we intersect $S_{a,\sqrt{\epsilon}}$ and $S_{r,\sqrt{\epsilon}}$, extended by the forward and backward flow, it does not follow directly that the Fenichel manifolds $S_{a,\epsilon}$ and $S_{r,\epsilon}$ also intersect.

Notice that the blowup approach, following (1.9) and the conservation of $\epsilon$, provides control of $S_{a,\sqrt{\epsilon}}$ and $S_{r,\sqrt{\epsilon}}$ up to $O(\sqrt{\epsilon})$-close to the folded node $p$, justifying the use of the subscripts. To describe these manifolds beyond this, we have to look at the

\[
\begin{align*}
(x, y, z, \epsilon) \mapsto \begin{cases} 
 x &= -r_1^2, \\
 y &= r_1 y_1, \\
 z &= r_1 z_1, \\
 \epsilon &= r_1^2 \epsilon_1.
\end{cases}
\]

(1.9)
(scaling) chart obtained by setting \( \bar{\epsilon} = 1 \). This produces the following local blowup

transformation

\[
(r_2, x_2, y_2, z_2) \mapsto \begin{cases} 
  x = r_2^2 x_2, \\
  y = r_2 y_2, \\
  z = r_2 z_2, \\
  \epsilon = r_2^2.
\end{cases}
\]

(1.10)

using the chart-specified coordinates \((x_2, y_2, z_2, r_2)\). The corresponding coordinate chart \( \bar{\epsilon} = 1 \) covers \([0, r_0) \times \mathbb{S}_0^3 \) where \( \mathbb{S}_0^3 := \mathbb{S}^3 \cap \{ r > 0 \} \). By inserting (1.10) into (1.5), dividing the right hand side by \( r_2 \) and subsequently setting \( r_2 = \sqrt{\epsilon} = 0 \), we obtain (1.1), repeated here for convenience:

\[
\begin{align*}
  \dot{x} &= \frac{1}{2} \mu y - (\mu + 1) z, \\
  \dot{y} &= 1, \\
  \dot{z} &= x + z^2.
\end{align*}
\]

(1.11)

In (1.11), we have also dropped the subscripts on \((x_2, y_2, z_2)\). Two explicit algebraic solutions are known for this unperturbed system, one:

\[
v : (x, y, z) = \left(-\frac{\mu^2}{4} t^2 + \frac{\mu}{2} t, \frac{\mu}{2} t\right)
\]

corresponding to the ‘strong canard’, while \(\gamma\) in (1.2), repeated here for convenience:

\[
\gamma : (x, y, z) = \left(-\frac{1}{4} t^2 + \frac{1}{2} t, \frac{1}{2} t\right),
\]

(1.12)

corresponds to the ‘weak canard’, which we will focus on in this paper. The system (1.1) is time-reversible with respect the following symmetry

\[
\sigma = \text{diag}(1, -1, -1) : \quad \text{If } (x, y, z)(t) \text{ is a solution of (1.11) then so is } \sigma(x, y, z)(-t).
\]

(1.13)

The orbit \(\gamma\) is a fix-point of this symmetry; we say that such orbits are symmetric.

**Remark 1.2.** Notice that the projection of (1.12) onto the \((y, z)\)-plane coincides with the span of the weak eigenvector (1.7), explaining the use of ‘weak’ in ‘weak canard’. Also, the orbit (1.12) is unique as an orbit on the blowup sphere with these properties. This is obviously in contrast with reduced flow on \(C_a\) where all trajectories within the funnel is asymptotic to the weak canard.

On a related issue, notice we abuse notation slightly: Most often, \(\gamma\) will refer to (1.12) as an orbit of (1.11). But by the coordinate chart \(\bar{\epsilon} = 1\), this orbit also becomes a heteroclinic connection on \( r = 0 \), \( \mathbb{S}_0^3 \), connecting partially hyperbolic points \(\sigma_p\) and \(p_w\) on the equator \(\bar{\epsilon} = 0\) for the blowup system. We will use the same symbol for this orbit. At the same time, in Fig. 1.2 we also use the symbol \(\gamma\) to highlight the weak eigendirection of the folded node as an attracting node of the desingularized reduced problem on \(C_a\). A similar misuse of notation occurs for \(v\).

Restricting the center manifolds \(M_a\) and \(M_r\), obtained in the chart \(\bar{x} = -1\), to \(r = 0\) we obtain, when writing the result in the chart \(\bar{\epsilon} = 1\), center-stable \(W^{cs}(\mu)\) and center-unstable manifolds \(W^{cu}(\mu)\) of (1.11) and \(z \to \pm \infty\), respectively, consisting of
solutions that grow algebraically as \( t \to \pm \infty \), respectively. Following [25], a simple calculation shows that \( W^{cs}(\mu) \) takes the local form:

\[
W^{cs}_{loc}(\mu) : x = -z^2 + \frac{1}{2}(\mu + 1) - \frac{1}{4}\mu y z^{-1} + z^{-2}m(y z^{-1}, z^{-2}), \tag{1.14}
\]

for all \( z \) sufficiently large and some smooth \( m : I \times [0, \delta] \to \mathbb{R} \) for an appropriate interval \( I \subset \mathbb{R} \) and \( \delta > 0 \) sufficiently small. Due to the invariance of \( \upsilon \) and \( \gamma \), \( m \) also satisfies \( m(2,z^{-2}) = m(2/\mu, z^{-2}) = 0 \). By using the time-reversible symmetry \((x,y,z,t) \mapsto (x,-y,-z,-t)\) of (1.11), a simple calculation shows that the manifold \( W^{cs}_{loc}(\mu) \) takes an identical form, with the expression in (1.14) now valid for all \( z \) sufficiently negative.

By applying the flow of (1.11), we extend these manifolds \( W^{cs}_{loc}(\mu) \) and \( W^{ca}_{loc}(\mu) \) to global manifolds \( W^{cs}(\mu) \) and \( W^{ca}(\mu) \) that intersect along \( \upsilon \) and \( \gamma \). In particular, by considering the variational equations of (1.11) along \( \gamma \) the following was shown in [25].

**Lemma 1.3.** \( W^{cs}(\mu) \) and \( W^{ca}(\mu) \) intersect transversally along \( \gamma \) if and only if \( \mu \not\in \mathbb{N} \).

We illustrate the results of the blowup analysis in Fig. 1.3. See figure caption for further description. Furthermore, we emphasize that on the hemisphere \( S^3_{\delta > 0} \), \( \upsilon \) and \( \gamma \) become heteroclinic connections of points on the equator sphere at \( \bar{e} = 0 \). In particular, \( \upsilon \) connects the partially hyperbolic points \( \sigma p_a \) and \( p_a \), whereas \( \gamma \) connects \( \sigma p_c \) with \( p_w \), as illustrated in Fig. 1.3. There are other special points on the equator: \( q_{in}, q_{out} \) and \( q_{\pm} \). We shall see that these points are important for the description of secondary canards. Here \( q_{in} \) and \( q_{out} = \sigma q_{in} \) correspond to the intersections of the nonhyperbolic critical fiber of the folded node with the blowup sphere. These two points are fully hyperbolic. In fact, as points on the invariant sphere, \( q_{in} \) is an unstable node whereas \( q_{out} \) is a stable node. \( q_{+} \) and \( q_{-} = \sigma q_{+} \), on the other hand, correspond to the intersection of the fold line \( F \) with the blowup sphere and are fully nonhyperbolic points.

By regular perturbation theory, the extension of the slow manifolds \( S_{a,\sqrt{\epsilon}} \) and \( S_{r,\sqrt{\epsilon}} \) by the flow are smoothly \( \mathcal{O}(r_2 = \sqrt{\epsilon}) \)-close to \( W^{ca}(\mu) \) and \( W^{cs}(\mu) \), respectively, in compact subsets of the chart \( \bar{e} = 1 \). Hence, as a consequence of Lemma 1.3, for every \( \mu \not\in \mathbb{N} \) there exist a transverse intersection of \( S_{a,\sqrt{\epsilon}} \) and \( S_{r,\sqrt{\epsilon}} \) which is \( \mathcal{O}(\sqrt{\epsilon}) \)-close to \( \gamma \) in fixed compact subsets of the scaling chart. In general, recall Remark 1.1, it seems that there do not exist any results on how far this perturbed ‘weak canard’ extends and whether ‘it’ (being nonunique) actually reaches the true Fenichel slow manifolds \( S_{a,\epsilon} \) and \( S_{r,\epsilon} \). The situation is different for \( \upsilon \). First and foremost, \( W^{cs}(\mu) \) and \( W^{cs}(\mu) \) always intersect transversally along this orbit. Consequently, \( \upsilon \) always perturbs as a ‘strong maximal canard’ for all \( 0 < \epsilon \ll 1 \), and this perturbed version always reaches the Fenichel manifolds. This latter property is a consequence of the repelling nature of \( \upsilon \), ‘forcing’ \( S_{a,\sqrt{\epsilon}} (S_{r,\sqrt{\epsilon}}) \) and the forward (backward) flow \( S_{a,\epsilon} \) \( (S_{r,\epsilon}, \) respectively) to coincide near this object.

**1.2. Main result.** Using a Melnikov approach, it was shown in [30, Theorem 3.1] that a transcritical bifurcation of the intersection of \( W^{cs}(\mu) \) and \( W^{ca}(\mu) \) occurs along \( \gamma \) for any odd \( \mu = 2k - 1, k \in \mathbb{N} \). As a result, additional (secondary) ‘canards’, connecting \( S_{a,\sqrt{\epsilon}} \) with \( S_{r,\sqrt{\epsilon}} \), exist near \( \mu = 2k - 1 \), for all \( 0 < \epsilon \ll 1 \) by regular perturbation theory. In this paper, we prove the existence of a pitchfork bifurcation for \( \mu = 2k \). We then have the following complete result regarding the bifurcations of ‘canards’ for (1.11):

7
In this figure, we represent \( r = 0 \), \( S_r \bar{\epsilon} \geq 0 \) – by projection – as a solid ‘ball’ in the \((\bar{x}, \bar{y}, \bar{z})\)-space, emphasizing those objects that are inside by using dotted lines. The \( S^2 \) sphere, being the boundary of the ball, corresponds to \( r = \bar{\epsilon} = 0 \), whereas everything inside of the ball corresponds to \( r = 0, \bar{\epsilon} > 0 \). Outside of the ball, we represent \( r > 0, \bar{\epsilon} = 0 \), highlighting, in particular, the critical manifolds \( C_a \) and \( C_r \) and their reduced flow. Through the blowup we gain hyperbolicity of \( C_a \) and \( C_r \) for \( r = 0 \) (indicated by triple-headed arrows) along the lines (in blue and red, respectively) of partially hyperbolic equilibria. By center manifold theory, these lines produce two three-dimensional manifolds, \( M_a \) and \( M_r \) (not shown), having submanifolds within \( r = 0 \), denoted by \( W^c_u \) and \( W^c_s \). These local two-dimensional manifolds are shown in lighter blue and red, respectively, since they extend inside the sphere. Also, within the sphere \( r = 0, \bar{\epsilon} > 0 \) we illustrate the orbits \( \nu \) (orange) and \( \gamma \) (purple), the ‘singular canards’, connecting partially hyperbolic points within \( r = \bar{\epsilon} = 0 \) on \( W^c_u \) and \( W^c_s \), respectively. The transversality of \( W^c_u \) and \( W^c_s \) along \( \nu \), and along \( \gamma \) for any \( \mu \notin \mathbb{N} \), produce transverse intersections of \( S_{\bar{\epsilon}} \sqrt{\epsilon} \) and \( S \gamma \bar{\epsilon} \), since these objects, obtained as \( \epsilon = \text{const.} \), sections of \( M_a \) and \( M_r \), respectively, are smoothly \( \mathcal{O}(r^2 = \sqrt{\epsilon}) \)-close on \( \bar{\epsilon} > 0 \) to \( W^c_u \) and \( W^c_s \), respectively.

**Theorem 1.4.** Consider any \( n \in \mathbb{N} \) and let \( k \in \mathbb{N} \) be so that

\[
n = \begin{cases} 
2k - 1 & n = \text{odd} \\
2k & n = \text{even}
\end{cases}
\]

Set \( \mu = n + \alpha \) and let

\[
D(v, \alpha) = 0,
\]

be the bifurcation equation (to be defined formally below in (2.17) locally near \((v, \alpha) = (0, 0)\)) where each solution \((v, \alpha)\) corresponds to an intersection of \( W^c_s(\mu) \) and \( W^c_u(\mu) \). In particular, \( D(0, \alpha) = 0 \) for all \( \alpha \) due to the existence of the connection \( \gamma \). Then
1. For \( n = \text{odd} \), (1.15) is locally equivalent with the transcritical bifurcation:

\[
\tilde{v}(\tilde{\alpha} + (-1)^k \tilde{v}) = 0.
\]  

(1.16)

2. For \( n = \text{even} \), (1.15) is locally equivalent with the pitchfork bifurcation:

\[
\tilde{v}(\tilde{\alpha} + \tilde{v}^2) = 0.
\]

In each case, the local conjugacy \( \phi : (v, \alpha) \mapsto (\tilde{v}, \tilde{\alpha}) \) satisfies 

\[
D\phi(0, 0) = \text{diag}(d_1(n), d_2(n)) \quad \text{with} \quad d_i(n) > 0 \quad \text{for every} \quad k.
\]  

(1.17)

Theorem 1.4 item (1) is covered by [30]. In particular, it is shown (see [30, Propositions 3.2 & 3.3]) that \( \text{sign} \frac{\partial^2 D}{\partial v^2}(0, 0) = (-1)^k, \frac{\partial^2 D}{\partial v \partial \alpha}(0, 0) > 0 \), which produces (1.17) by singularity theory [10]. We will therefore only prove Theorem 1.4 item (2) in the following. Notice, however, that in [30], the Melnikov function is defined for all \( r_2 = \sqrt{\epsilon} \) sufficiently small, measuring the intersection of \( S_{\alpha}, \sqrt{\epsilon} \) and \( S_{r, \sqrt{\epsilon}} \) directly (rather than measuring the \( \epsilon = 0 \) objects \( W_{cu}(\mu) \) and \( W_{cs}(\mu) \)). Nevertheless, seeing that the bifurcations in Theorem 1.4 are for the \( \epsilon = 0 \) system, we will in this paper just focus on \( \epsilon = 0 \) (and will only describe the perturbation of transverse intersection points into \( 0 < \epsilon \ll 1 \), see e.g. Remark 4.3 and Remark 4.5).

### 1.3. Overview

The remainder of the paper is organized as follows: In the next Section 2, we review the Melnikov theory in [30] in further details and extend this approach to time-reversible systems, see also [13]. The result is collected in Theorem 2.8. This reduces the proof of our main result, Theorem 1.4 item (2) on the pitchfork bifurcation, to evaluating two integrals; one of which is already covered by [30], while the other one is the ‘third order’ Melnikov integral mentioned above. We evaluate these integrals by characterizing the manifolds \( W_{cu}(\mu) \) and \( W_{cs}(\mu) \) locally through solutions of ‘higher order variational equations’ rather than, how it is done in [19], their (implicit) formulation through integral equations. We describe our approach further in Section 2.2 and how these variational equations can be solved upon reduction to an inhomogeneous Weber equation. In this section, we also present a general class of systems, that include the folded node normal form, the Falkner-Skan equation and the Nosé equations, for which we can show, see Theorem 2.10, that our method produces closed form expressions of the appropriate Melnikov integrals. These results rely on properties of Hermite polynomials \( H_n, n \in \mathbb{N}_0 \). All information on these orthogonal polynomials that is relevant to the present manuscript is available in Appendix A.

The proof of Theorem 1.4 is presented in Section 3 below, see Lemma 3.4 proven towards the end of the section. We show that our expressions agree with the computations in [19, App. A] in Appendix B.

Next, in Section 4 we will use the time-reversible setting and the blowup approach to show that the additional intersections produced by the pitchfork bifurcation do not reach the actual Fenichel slow manifolds for any \( 0 < \epsilon \ll 1 \), see Proposition 4.4 and Remark 4.5. They do therefore not produce actual canards. In fact, the statement in Proposition 4.4 about the \( \epsilon = 0 \) system is as follows: The additional intersections produced by the pitchfork, come in pairs: \( \gamma_{sc}(\mu) \) and \( \sigma\gamma_{sc}(\mu) \) that are related by the time-reversible symmetry \( \sigma \), recall (1.13); on the sphere \( S^3_{\epsilon \geq 0} \), see Fig. 1.3, one, say \( \gamma_{sc}(\mu) \), connects the point \( \sigma p_s \), with stable node \( q_{\text{out}} \) while the other one \( \sigma\gamma_{sc}(\mu) \) connects the unstable node \( q_{\text{in}} \) with \( p_s \). Also due to the fact that \( q_{\text{out}} \) and \( q_{\text{in}} \) are
hyperbolic nodes, these new heteroclinic connections are in fact nonunique. This situation is in contrast to the transcritical case $\mu = 2k - 1$, where it is known that true ‘secondary’ canards are produced for every $\mu > 2k - 1$ and $0 < \epsilon \ll 1$. We also provide a new geometric explanation for this property in Section 4. In particular, in Proposition 4.2 we show that for $\epsilon = 0$ the additional intersection $\gamma^{sc}(\mu)$ produced in this case is symmetric with respect to $\sigma$; on the sphere $S_{\epsilon}^3 \geq 0$, see Fig. 1.3, $\gamma^{sc}(\mu)$ connects $q_{in}$ with $q_{out}$ for $\mu < 2k - 1$ whereas for $\mu > 2k - 1$ it connects $\sigma p_s$ with $p_s$. In this latter case, where such a connection is unique, $\gamma^{sc}(\mu)$ therefore has the same asymptotic properties as $\nu$. This also explains why such $\gamma^{sc}(\mu)$ perturbs to a true canard connecting the Fenichel slow manifolds for all $0 < \epsilon \ll 1$ whenever $\mu > 2k - 1$.

Although these statements about canards are probably known to most experts in the field, we believe that we present the first rigorous proofs of these facts.

In our final Section 5, we consider the Falkner-Skan equation (1.3), for which our time-reversible Melnikov approach in Section 2 is also applicable. In particular, we provide a new geometric proof of the emergence of symmetric periodic orbits from infinity in this systems. The approach also applies to the Nosé equations, see the forthcoming manuscript [15]. We conclude the paper in Section 6.

2. A Melnikov theory for time-reversible systems. The reference [29] describes a Melnikov theory for connection problems of nonhyperbolic points at infinity. In this section, we will review this approach in the context of time-reversible systems. For simplicity, we restrict to $\mathbb{R}^3$ and consider a general smooth ODE

$$\dot{x} = f(x, \alpha),$$

for $x = x(t) \in \mathbb{R}^3$, depending on a parameter $\alpha \in \mathbb{R}$, and assume the following:

(H1) There exists a time-reversible symmetry

$$(x, t) \mapsto (\sigma x, -t),$$

with $\sigma \in \mathbb{R}^{3 \times 3}$ being an involution: $\sigma^2 = \text{id}$, $\text{id} = \text{diag}(1, 1, 1)$ being the identity matrix in $\mathbb{R}^{3 \times 3}$, such that

$$f(\sigma x, \alpha) = -\sigma f(x, \alpha),$$

for all $x$ and all $\alpha$.

Therefore:

If $x(t)$ is solution of (2.1) then so is $\sigma x(-t)$.

As is standard, we say that an orbit $x$ with parametrization $x(t)$, which is a fix-point of the symmetry: $x(t) = \sigma x(-t)$ for all $t$, is ‘symmetric’. On the other hand, in general two orbits $x_1 \neq x_2$, for which $x_2(t) = \sigma x_1(-t)$, is said to be ‘symmetrically related’.

Furthermore, we say that a solution $x(t)$ of (2.1) has algebraic growth for $t \to \infty$ if there exists a $\nu > 0$ such that $\sup_{t \geq 0} |x(t)|(|t| + 1)^{-\nu} < \infty$ for $\nu > 0$ large enough. Specifically, we define the Banach space

$$C_{b,+}(\nu) := \{ x \in C([0, \infty), \mathbb{R}^3) | \sup_{t \geq 0} |x(t)|(|t| + 1)^{-\nu} < \infty \},$$

for $\nu > 0$ fixed, see [29]. (Here $b$ is for (algebraic growth) ‘bound’ whereas ‘+’ indicates that the functions are defined for $t \geq 0$.) Similarly, a solution $x(t)$ of (2.1)
has algebraic growth for $t \to -\infty$ if $\sup_{t \leq 0} |x(t)| (|t| + 1)^{-\nu} < \infty$ for $\nu > 0$ large enough. Accordingly, we define

$$C_{b,-}(\nu) := \{ x \in C((-\infty, 0], \mathbb{R}^3) | \sup_{t \leq 0} |x(t)| (|t| + 1)^{-\nu} < \infty \},$$

with ‘−’ indicating that the functions are defined for $t \leq 0$. We will suppress $\nu$ in $C_{b,+}(\nu)$ and $C_{b,-}(\nu)$ whenever it is convenient to do so.

Next, we assume

(H2) For $\alpha = 0$ there exists a symmetric solution $\gamma$ with parametrization $\gamma(t)$, $t \in \mathbb{R}$, of at most algebraic growth for $t \to \pm \infty$, i.e. $\gamma(t) \in C_{b,+}(\nu)$ with $\nu > 0$ large enough. Without loss of generality we suppose that

$$\gamma(0) = 0.$$

(H3) There exists a three-dimensional smooth invariant manifold $W^{cs}$ in the extended system

$$\dot{x} = f(x, \alpha),$$

$$\dot{\alpha} = 0. \tag{2.3}$$

Here $W^{cs}$ denotes the center-stable manifold consisting of all solution curves $(x(t), \alpha)$ of (2.1) near (and including) $(x, \alpha) = (\gamma(t), 0)$ (in a sense specified below) for which $x(t) \in C_{b,+}(\nu)$, for $\nu > 0$ large enough. $W^{cs}$ is foliated by two-dimensional invariant manifolds $W^{cs}(\alpha)$ of (2.1) for fixed values of $\alpha$, sufficiently small.

By (H1) and (H3), there exists a center-unstable manifold

$$W^{cu}(\alpha) := \sigma W^{cs}(\alpha) := \{ \sigma q | q \in W^{cs}(\alpha) \}, \tag{2.4}$$

consisting of all solution curves $(z(t), \alpha)$ of (2.1) near (and including) $(x, \alpha) = (\gamma(t), 0)$ for which $x(t) \in C_{b,-}(\nu)$.

(H4) Let $U := \text{span}(\dot{\gamma}(0))$. Then for $\alpha = 0$ there exists a one-dimensional linear space $V$ such that

$$T_{\gamma(0)} W^{cs}(0) \cap T_{\gamma(0)} W^{cu}(0) = U \oplus V,$$

is a two-dimensional subspace. (H4) implies that the manifolds $W^{cs}(0)$ and $W^{cu}(0)$ intersect tangentially along $\gamma$ for $\alpha = 0$. In fact, seeing that $W^{cu} = \sigma W^{cs}$ we have

**Lemma 2.1.** The following statements are equivalent:

1. (H4) holds.
2. $T_{\gamma(0)} W^{cs}(0) = T_{\gamma(0)} W^{cu}(0)$ and the intersection of $W^{cs}(0)$ and $W^{cu}$ along $\gamma$ is tangential.
3. $T_{\gamma(0)} W^{cs}(0)$ is an invariant subspace for $\sigma$: $x \in T_{\gamma(0)} W^{cs}(0) \implies \sigma x \in T_{\gamma(0)} W^{cs}(0)$.
4. The variational equation along $\gamma$ for $\alpha = 0$:

$$\dot{z} = A(t) z, \tag{2.5}$$

where $A(t) = D_{x}f(\gamma(t), 0)$, has two linearly independent solutions $z_1(t) = \dot{\gamma}(t)$ and $z_2(t)$ for which $z_i \in C_{b,+} \cap C_{b,-}$.
Proof. (1) ⇔ (2) is trivial, seeing that $W^{cs}(0)$ and $W^{cu}(0)$ are two-dimensional manifolds. (3) ⇔ (2) follows from the following computation: $T_{\gamma(0)}W^{cs}(0) = T_{\gamma(0)}W^{cu}(0)$ by (H1), recall (2.4). Finally, (1) ⇔ (4) is standard, see [25, Proposition 4.4]. Indeed, variations along the two-dimensional space $T_{\gamma(0)}W^{cs}(0) \cap T_{\gamma(0)}W^{cu}(0)$ correspond to algebraic growth as $t \to \pm \infty$. [bbox]

Next, following [30] let

$$W = T_{\gamma(0)}W^{cs}(0)^\perp. \quad (2.6)$$

Then $\mathbb{R}^3 = U \oplus V \oplus W$. Let $e_u$, $e_v$ and $e_w$ be unit vectors spanning $U$, $V$ and $W$, respectively, and denote the coordinates of any $x \in \mathbb{R}^3$ with respect to this basis $\{e_u, e_v, e_w\}$ by $(u, v, w)$. Fix $r > 0$ small and let $B_r$ be the ball of radius $r$ centered at $\gamma(0)$. We then define a local section $\Sigma$ transverse to $\gamma$ at $\gamma(0) = 0$ by

$$\Sigma = \{V \oplus W\} \cap B_r.$$  

Notice that $\Sigma$ – in the $(u, v, w)$-coordinates – is contained within the $(v, w)$-plane.

Next, we write $x = z + \gamma(t)$ following [29] such that

$$\dot{z} = A(t)z + g(t, z, \alpha), \quad (2.7)$$

where $g(t, z, \alpha) = f(\gamma(t)+z, \alpha) - f(\gamma(t), 0) - A(t)z$. Also $g(t, 0, 0) = 0$, $D_zg(t, 0, 0) = 0$ and notice that (2.5) is the variational equation along $\gamma(t)$. Furthermore:

**Lemma 2.2.**

$$\sigma A(-t) = -A(t)\sigma, \quad \sigma g(-t, z, \alpha) = -g(t, \sigma z, \alpha),$$

for all $t, z, \alpha$.

**Proof.** Follows directly from the time-reversible symmetry of $f$, recall (H1). [bbox]

Let $\Phi(t, s)$ be the state-transition matrix of (2.5). Then by (H3) and (H4) there exists a continuous projection $P : [0, \infty) \to \mathbb{R}^3$ such that

$$\text{range } P(0) = U \oplus V, \quad \ker P(0) = W,$$

and

$$P(t)\Phi(t, s) = \Phi(t, s)P(s),$$

for all $t, s \geq 0$. Furthermore, if $Q(s) = I - P(s)$ then

$$\ker Q(0) = U \oplus V, \quad \text{range } Q(0) = W. \quad (2.8)$$

and it follows that

$$\|\Phi(t, s)P(s)\| \leq K(t - s + 1)^\theta, \quad (2.9)$$

$$\|\Phi(s, t)Q(t)\| \leq K e^{-\eta(t-s)},$$

for some $K \geq 1$, $\theta, \eta \geq 0$ and all $0 \leq s \leq t$, see e.g. [29, 28]. By assumption (H1), Lemma 2.2 and (2.9) we also have:

**Lemma 2.3.** $\Phi$ is symmetric in the following sense:

$$\sigma \Phi(t, s) = \Phi(-t, -s)\sigma.$$
Also, \( t \mapsto \sigma P(-t)\sigma^{-1} \) and \( t \mapsto \sigma Q(-t)\sigma^{-1} \) are continuous projection operators such that

\[
\| \Phi(t, s)\sigma P(-s)\sigma^{-1} \| = \| \sigma \Phi(-t, -s)P(-s)\sigma^{-1} \| \leq K(s - t + 1)^	heta,
\]

\[
\| \Phi(s, t)\sigma Q(-t)\sigma^{-1} \| = \| \sigma \Phi(-s, -t)Q(-t)\sigma^{-1} \| \leq Ke^{-\eta(s - t)}.
\]

for all \( t \leq s \leq 0 \).

**Proof.** Straightforward calculation. \( \square \)

Consider the adjoint equation of (2.5):

\[
\dot{\psi} + A(t)T \psi = 0,
\]

and notice that \((\psi, t) \mapsto (\sigma T \psi, -t)\) is a time-reversible symmetry for (2.10) by (H1). Then

**Lemma 2.4.** Let \( \psi_*(t) \) be a solution of (2.10). Then \( \psi_*(t) \) decays exponentially for \( t \to \pm \infty \) if and only if \( \phi_*(0) \in W \).

**Proof.** Standard, see [29].

In the following, we fix a specific \( \psi_*(t) \) by setting \( \psi_*(0) = e_w \). Since \( \Phi^T(t, s) = \Phi^{-T}(t, s) \) is a state-transition matrix of (2.10), we can then write \( \psi_*(t) \) as

\[
\psi_*(t) = \Phi^T(0, t)e_w.
\]

(2.12)

We now have the following important result.

**Lemma 2.5.** \( V \) and \( W \) are one-dimensional invariant subspaces of \( \sigma \) and \( \sigma^T \), respectively. Hence, there exists \( \sigma_i \in \{ \pm 1 \} \), \( i = v, w \), such that

\[
\sigma|_V = \sigma_v \text{id}, \quad \sigma^T|_W = \sigma_w \text{id},
\]

where \( \sigma_i = \pm 1 \) for \( i = v, w \).

**Proof.** First, regarding the \( \sigma^T \)-invariance of \( W \): The solution \( \psi_*(t) \) of (2.11) is exponentially decaying for \( t \to \pm \infty \). Clearly, the symmetrically related solution \( \sigma^T \psi_*(-t) \) satisfies the same properties, and hence \( \sigma^T \psi_*(0) \in W \) by Lemma 2.4; the \( \sigma^T \)-invariance of \( W \) therefore follows. Next, since \( \gamma(t) \) is symmetric it follows by differentiation with respect to \( t = 0 \) that \( \sigma|_U = -\text{id} \). But then since \( U \oplus V \) is invariant with respect to \( \sigma \), recall Lemma 2.1 item (3), and \( \sigma^2 = \text{id} \), the statement about \( \sigma|_V \) also follows from a straightforward calculation. \( \square \)

In fact, in the \((u, v, w)\)-coordinates

\[
\sigma = \text{diag}(-1, \sigma_v, \sigma_w).
\]

(2.13)

Next, for (2.7), it can by variation of constants – following [29] – be shown that \( z(t) \in C_{b,+} \), with \( z(0) \in \Sigma \), if and only if there exists a \( v \in V \) such that

\[
z(t) = \Phi(t, 0)v + \int_0^t P(t)\Phi(t, s)g(s, z(s), \alpha)ds + \int_{-\infty}^t Q(t)\Phi(t, s)g(s, z(s), \alpha)ds.
\]

(2.14)

This enables an analytic characterization of the (nonunique) invariant manifold \( W^{cs}(\mu) \), which is essential for the Melnikov approach, as follows. Let the mapping \( z \mapsto T(z) \)
be defined on $C_{b,+}$ so that $T(z)(t)$ is the right hand side of (2.14) and consider a sufficiently small neighborhood $N$ of $(v,\alpha) = (0,0)$. Then, upon possible modification (or cut-off) of $f$ (and therefore of $g$ in (2.7)), as in center manifold theory [3], we obtain for each $(v,\alpha) \in N$, a unique fix-point $z_*(v,\alpha)$ of $T$: $T(z_*) = z_*$, see [29]. Henceforth we will assume that such a modification of $f$ (and therefore of $g$) has been made. It is also standard, see also [29], to show that $z_*$ is smooth, with each partial derivative belonging to $C_{b,+}(\nu)$ for $\nu$ large enough. In this way,

$$W^{cs}(\alpha) = \{z_*(v,\alpha)(t) | (v,\alpha) \in N, t \in \mathbb{R}\}.$$  

**Remark 2.6.** In our examples, including (1.11), the invariant manifolds $W^{cs}(\mu)$ will be obtained, not as fix-points of (2.14), but as center manifolds upon appropriate Poincaré compactification. For the analysis of the implications of the bifurcations of $\gamma$, it will be important to study the reduced problem on such center manifolds. For the Falkner-Skan equation and the Nosé equations the ‘selection’ of these nonunique manifolds is crucial to the analysis.

**2.1. The Melnikov function.** For $\alpha$ sufficiently small, we write $W^{cs}_0(\alpha) = W^{cs}(\alpha) \cap \Sigma$ and $W^{cu}_0(\alpha) = W^{cu}(\alpha) \cap \Sigma$ locally within the $(v,w)$-plane as smooth graphs

$$w = h_{cs}(v,\alpha),$$  

and

$$w = h_{cu}(v,\alpha),$$

respectively, over $v$. Following [29], we then define the Melnikov function as

$$D(v,\alpha) = h_{cu}(v,\alpha) - h_{cs}(v,\alpha).$$

Clearly, a root of $D$ corresponds to an intersection of $W^{cs}_0(\alpha)$ and $W^{cu}_0(\alpha)$ and, hence, to an intersection of $W^{cs}(\alpha)$ and $W^{cu}(\alpha)$. Furthermore, the intersection is transverse if and only if the root is simple. We now prove the following.

**Lemma 2.7.**

$$D(v,\alpha) = \sigma_w h_{cs}(\sigma_v v,\alpha) - h_{cs}(v,\alpha).$$

**Proof.** Following (2.17), we simply have to show that

$$h_{cu}(v,\alpha) = \sigma_w h_{cs}(\sigma_v v,\alpha),$$

for all $v$ and $\alpha$. We will show this using the integral representation (2.14) as follows. Let $z_*(v,\alpha) \in C_{b,+}$ be the fix-point of the mapping $T$, with $T(z)(t)$ being defined as the right hand side of (2.14), so that $z_*(v,\alpha)(t) \in W^{cs}(\alpha)$ for all $t$ and $z_*(v,\alpha)(0) = (v, h_{cs}(v,\alpha))$ within $\Sigma$ in the $(v,w)$-coordinates. Then $\sigma z_*(v,\alpha)(-t) \in C_{b,-}$ so that $\sigma z_*(v,\alpha)(-t) \in W^{cu}$ with

$$\sigma z_*(v,\alpha)(0) = (\sigma_v v, \sigma_w h_{cs}(v,\alpha)),$$

writing the right hand side in the $(v,w)$-coordinates, recall (2.13). Since $W^{cu} = \sigma W^{cs}$, we conclude from (2.16) that

$$h_{cu}(\sigma_v v,\alpha) = \sigma_w h_{cs}(v,\alpha).$$
This shows (2.19), seeing that $\sigma^2_v = 1$. □

Now, we are ready to present the following, final result on the Melnikov integral, which is a translation of [29, Theorem 1] to the time-reversible setting in $\mathbb{R}^3$.

**Theorem 2.8.** Let $t \mapsto z_*(v, \alpha)(t) \in C_{b,+}$ be the solution of (2.7) with initial conditions

$$z_*(v, \alpha)(0) = (v, h_{cs}(v, \alpha)),$$

(2.20)

with respect to the $(v, w)$-coordinates, on $W^{cs}_0(\alpha)$ within $\Sigma \subset \{u = 0\}$. Then

$$D(v, \alpha) = \int_0^\infty \langle \psi_*(s), g(s, z_*(v, \alpha)(s), \alpha) - \sigma wg(s, z_*(\sigma v, \alpha)(s), \alpha) \rangle ds.$$  

(2.21)

**Proof.** The result follows from [29, Theorem 1], upon setting $h_{cs}e_w = h_+$ and $\sigma^T h_{cs}e_w = h_-$. In further details, we simply set $t = 0$ in (2.14):  

$$z_*(v, \alpha)(0) = v - Q(0) \int_0^\infty \Phi(t, s) g(s, z_*(v, \alpha)(s), \alpha) ds.$$  

(2.22)

The last term on the right hand side – by (2.8) – belongs to $W$ for all $|v|, |\alpha| \leq \delta$; whence,

$$h_{cs}(v, \alpha) = (e_w, -Q(0) \int_0^\infty \Phi(0, s) g(s, z_*(v, \alpha)(s), \alpha) ds).$$  

(2.23)

Then upon using (2.12), we obtain the desired form

$$h_{cs}(v, \alpha) = -\int_0^\infty \langle \psi_*(s), g(s, z_*(v, \alpha)(s), \alpha) \rangle ds.$$

The result then follows from (2.18). □

**2.2. A recipe for computing the appropriate Melnikov integrals.** We can use (2.21) to describe the bifurcations of heteroclinic connections of $W^{cs}_0(\alpha)$ and $W^{cs}_0(\alpha)$, provided that we can determine the partial derivatives of $D$. We now describe a set of assumptions, covering all of the cases we study below, where we can describe a specific procedure for doing this. We start with the following.

(H5) Suppose that $\gamma = (0, 0, t)$ for all $t \in \mathbb{R}$ and all $\alpha$.

Upon a linear change of coordinates, we may also suppose without loss of generality that $\sigma$ is diagonal, recall (2.13).

(H6) Suppose that $\sigma = \text{diag}(1, -1, -1)$.

Notice that $\gamma$ is symmetric with respect to this $\sigma$.

(H7) Suppose that $f(\cdot, \alpha)$ is quadratic.

Then in combination with the following lemma, we can show that all relevant Melnikov integrals can be evaluated in closed form.

**Lemma 2.9.** For any $\beta \in \mathbb{R}$, define the second order operator $L_\beta$ by the general expression:

$$L_\beta q := \ddot{q} - t\dot{q} + \beta q,$$

(2.24)

for any $q \in C^2$, so that $L_\beta q = 0$ is the (homogeneous) Weber equation. Finally, for any $n \in \mathbb{N}_0$, let $H_n$ denote the $n$th degree Hermite polynomial.
Then for all non-negative integers \( m \) and \( l \), the following holds
\[
L_m H_l (t/\sqrt{2}) = (m-l) H_l (t/\sqrt{2}).
\] (2.25)

In particular,
\[
H_m (t/\sqrt{2}) \in \ker L_m.
\] (2.26)

**Proof.** Follows from (A.1) and (A.2) in Appendix A. \( \square \)

The next theorem presents the main result of this section.

**Theorem 2.10.** Suppose (H3) and (H5)-(H7).

1. Then \( f \), upon scaling \( x_3 \) and \( t \) if necessary, takes the following form
\[
f(x, \alpha) = \begin{pmatrix}
x_2 a_2 + x_1 x_2 a_{12} + x_1 x_3 \\
x_1 b_1 + x_1^2 b_{11} + x_2 b_{22} \\
1 + x_1 c_1 + x_1^2 c_{11} + x_2 c_{22} + x_2 x_3 c_{23}
\end{pmatrix}.
\] (2.27)

with each coefficient \( a_2, a_{12}, b_1, b_{11}, b_{22}, c_1, c_{11}, c_{22}, c_{23} \) depending smoothly upon the parameter \( \alpha \).

2. Furthermore, let
\[
\beta := -(a_2 b_1 + 1).
\]

Then (H4) is satisfied if and only if \( \beta \in \mathbb{N}_0 \).

3. Next, let \( \alpha = 0 \) be so that \( \beta \in \mathbb{N}_0 \) and consider \( D(v, \alpha) \) as in (2.21). Then \( D(0, \alpha) = 0 \) for all \( \alpha \) and we have the following:

   (a) Suppose \( \beta \) is odd. Then \( \sigma_v = -1, \sigma_w = 1 \) and \( v \mapsto D(v, \alpha), \) see (2.21), is an even function, so that \( \frac{\partial^2 D}{\partial v^2} (0, 0) = 0 \) for all \( j \in \mathbb{N}_0 \). Furthermore, the following partial derivatives of \( D \) can be evaluated in closed form:
\[
\frac{\partial^2 D}{\partial v \partial \alpha} (0, 0), \quad \frac{\partial^3 D}{\partial v^3} (0, 0).
\]

If these quantities are nonzero, then the bifurcation equation \( D(v, \alpha) = 0 \) is locally equivalent with the pitchfork normal form.

   (b) Suppose \( \beta \) is even. Then \( \sigma_v = 1, \sigma_w = -1 \). In this case, the following partial derivatives can be evaluated in closed form
\[
\frac{\partial^2 D}{\partial v \partial \alpha} (0, 0), \quad \frac{\partial^2 D}{\partial v^2} (0, 0).
\]

If these quantities are nonzero, then the bifurcation equation \( D(v, \alpha) = 0 \) is locally equivalent with the transcritical normal form.

**Proof.** Regarding (1): The general form in (2.27) is a simple consequence of \( f(0, 0, t, \alpha) = (0, 0, 1)^T \) by (H5), \( \sigma f(x, \alpha) + f(\sigma x, \alpha) = 0 \) for all \( x \) and all \( \alpha \) by (H6). Furthermore, we conclude, using a Poincaré compactification and center manifold theory, that (H3) implies that the term \( x_1 x_3 a_{13} \) in the first component of \( f \) satisfies \( a_{13} > 0 \) and subsequently that there is no term \( x_2 x_3 b_{23} \) in the second component of \( f \). Seeing that \( a_{13} > 0 \), we finally obtain (2.27) by scaling \( x_3 \) and \( t \) as follows:
\[
\tilde{x}_3 = \sqrt{a_{13}} x_3, \quad \tilde{t} = \sqrt{a_{13}} t.
\]
Next regarding (2): The general form (2.27) produces

\[ A(t) = \begin{pmatrix} t & a_2 & 0 \\ b_1 & 0 & 0 \\ c_1 & t_{23} & 0 \end{pmatrix} . \tag{2.28} \]

But then, upon differentiating the first equation for \( z_1 \) in the variational equation (2.5) one more time with respect to \( t \), we can write the equation for \( z_1 \) as a Weber equation:

\[ L_\beta z_1 = 0, \tag{2.29} \]

recall (2.24). From \( z_1, z_2 \) and \( z_3 \) can be determined by successive integration of

\[ \dot{z}_2 = b_1 z_1 , \tag{2.30} \]

\[ \dot{z}_3 = c_1 z_1 + t_{23} z_2 . \]

By Lemma 2.9, see (2.26), it follows that for any \( \beta \in \mathbb{N}_0 \) there exists an algebraic solution \( x(t) = H_\beta(t/\sqrt{2}) \) of (2.29). Inserting this solution into (2.30) produces, upon using (A.1) and (A.2) in Appendix A, an algebraic solution of (2.5):

\[ z(t) = \begin{pmatrix} H_\beta(t/\sqrt{2}) \\ \frac{b_1}{2(\beta+1)} H_{\beta+1}(t/\sqrt{2}) \\ \frac{b_1 c_2}{2(\beta+1)} \left( \frac{1}{2(\beta+3)} H_{\beta+3}(t/\sqrt{2}) + H_{\beta+1}(t/\sqrt{2}) \right) \end{pmatrix} , \tag{2.31} \]

\( \beta \in \mathbb{N}_0 \Rightarrow (H4) \) is then a consequence of Lemma 2.1, see item 4. On the other hand, if \( \beta \notin \mathbb{N}_0 \) then there are no algebraic solutions of (2.29), see e.g. [1], and therefore (H4) does not hold, see again Lemma 2.1 item 4. This completes the proof of (2).

To finish the proof of Theorem 2.10, we just need to verify the claims about \( \sigma_v, \sigma_w \) and the partial derivatives of \( D \) at \( v = \alpha = 0 \) in item (3). For this we first determine \( \psi_* \). Suppose that \( \beta \in \mathbb{N}_0 \). Then a simple computation shows that the adjoint equation can be written as a second order equation for \( z_1 \):

\[ \ddot{z}_1 = -tz_1 - (\beta + 2) z_1 . \]

Substituting \( z_1 = e^{-t/2} \tilde{z}_1 \) gives

\[ L_{\beta+1} \tilde{z}_1 = 0, \]

recall (2.24), upon dropping the tilde. Using (A.1), we obtain the following expression for \( \psi_* \):

\[ \psi_*(t) = e^{-t^2/2c} \begin{pmatrix} H_{\beta+1}(t/\sqrt{2}) \\ -\frac{a_{12}}{\sqrt{2}} H_\beta(t/\sqrt{2}) \\ 0 \end{pmatrix} , \tag{2.32} \]

for some constant \( c \), ensuring that \( \psi_*(0) = e_w \) has length 1. The statements regarding \( \sigma_v \) and \( \sigma_w \) are then simple consequences of (2.31) and (2.32), recall Lemma 2.5.

Regarding the partial derivatives of \( D \), we focus on \( \beta = \text{odd} \) and the closed form expression for \( \frac{\partial^2 D}{\partial v^2}(0,0) \) in (3a). Both \( \frac{\partial^2 D}{\partial v \partial \alpha}(0,0) \) and \( \beta = \text{even} \) in (3b) are similar, but simpler, and therefore left out. Notice also that the statements about the
local equivalence with the pitchfork and the transcritical normal form follows from singularity theory, see e.g. [10].

Let

\[ z' = (z_1', z_2', z_3')^T := \frac{\partial z}{\partial \nu} (0, 0), \quad z'' = (z_1'', z_2'', z_3'') = \frac{\partial^2 z}{\partial \nu^2} (0, 0). \]  

(2.33)

Notice, by linearization of \((2.14)_{z=\nu, \alpha}\) using \(z_*(0, 0) = 0\), it follows that \(z'\) is determined by an appropriate normalization of \((2.31)\). Then by \((2.32)\) we conclude that \(\frac{\partial^2 P}{\partial \nu^2}(0, 0)\) is a linear combination of terms of the following form

\[ \int_0^\infty e^{-t^2/2} H_\beta(t/\sqrt{2}) z'_m(t) z''_n(t) dt, \]  

(2.34)

for \(n, m = 1, 2, 3\). Obviously, this linear combination can be stated explicitly in terms of \((2.27)\). However, the details are not important here. Next, suppose that \(z''\) is a finite sum of Hermite polynomials:

\[ z'' = \sum_{i \in I} v_i H_i(t/\sqrt{2}), \]  

(2.35)

with \(I \subset \mathbb{N}_0\) being a finite index set and \(v_i \in \mathbb{R}^3\), for any \(i \in I\). Then upon inserting \((2.35)\) into \((2.34)\) we obtain a linear combination of terms of the following form

\[ \int_0^\infty e^{-t^2/2} H_\beta(t/\sqrt{2}) H_i(t/\sqrt{2}) H_j(t/\sqrt{2}) dt, \]  

(2.36)

with \(i, j \in \mathbb{N}_0\) and \(\beta \in \mathbb{N}_0\). Again, the details are not important. However, each term of the form \((2.36)\) can be determined in closed form using \((A.5)\) and the statement of the theorem therefore follows once we have shown \((2.35)\). For this we insert \(z = z_*(v, \alpha)\) into \((2.7)\) and differentiate the resulting equation twice with respect to \(v\). We then evaluate the resulting expression at \((v, \alpha) = (0, 0)\) and again use that \(z_*(0, 0) = 0\). This gives a linear inhomogeneous equation for \(z''\) of the following form

\[ \dot{z}'' = A(t) z'' + \begin{pmatrix} 2(a_{12} z_1' z_2' + a_{13} z_1' z_3') \\ 2(b_{11} z_1'^3 + b_{22} z_2'^2) \\ 2(c_{11} z_1'^2 + c_{22} z_2'^2 + c_{23} z_2' z_3') \end{pmatrix}, \]

By \((2.28)\) we can therefore write the equation for \(z''\) as a second order equation and obtain an inhomogeneous Weber equation:

\[ L_\beta z'' = 2 \frac{d}{dt} \left( a_{12} z_1' z_2' + a_{13} z_1' z_3' \right) + 2 a_2 \left( b_{11} (z_1')^2 + b_{22} (z_2')^2 \right). \]  

(2.37)

Notice that by \((2.31)\) the right hand side of \((2.37)\) is a sum of products of Hermite polynomials. The product rule in \((A.5)\) in Appendix A allows us to write this sum of products as a sum of Hermite polynomials only. A simple calculation, using \((A.2)\), then shows that this sum only consists of Hermite polynomials of even degree. We can then solve \((2.37)\) using Lemma 2.9. In particular, by linearity and the fact that \(m = \beta = \text{odd and } l = \text{even}\), it follows from \((2.25)\) that there exists an algebraic solution of \((2.37)\) of the form

\[ z''_n(t) = \sum_{j \in J} d_j H_{2j}(t/\sqrt{2}), \]

18
for a finite index set $J \subset \mathbb{N}_0$ and $d_j \in \mathbb{R}$, $j \in J$. This is $z''_i$, since (a) it has the desired algebraic growth and (b) $z''(0) = 0$. The latter property (b) is a consequence of $H_{2i}(0) = 0$ for each $i \in \mathbb{N}_0$. Upon integrating the equations for $z''_2$ and $z''_3$ we obtain a solution of the form (2.35). This completes the proof.

**Remark 2.11.** The general procedure for evaluating $\frac{\partial^3 D}{\partial v^3}(0,0)$, described in the proof above, can essentially be summarized as follows:

- **Step (a).** Insert $z_*(v, \alpha)$ into (2.7) and differentiate the resulting equation twice with respect to $v$. This characterizes $z''$ as a solution of a higher order variational equation.
- **Step (b).** This equation can by (H5)-(H7) be reduced to an inhomogeneous Weber equation, see (2.37), with right hand side as a finite sum of products of Hermite polynomials.
- **Step (c).** We can then use (A.5) in Appendix A to reduce these products to sums of Hermite polynomials and solve the resulting equation for $z''$ using Lemma 2.9.
- **Step (d).** Finally, we insert the resulting expression for $z''$ in step (c) into $\frac{\partial^3 D}{\partial v^3}(0,0)$ producing a sum of integrals of the form (2.36). Finally, these integrals can be evaluated using (A.6).

In principle, this procedure can be extended to cases where $f$ is a polynomial of higher degree. It is still possible to reduce the equation for $z''$ to an inhomogeneous Weber equation with a right hand side consisting of a sum of Hermite polynomials. However, I have not managed to find an appropriate general setting for this, where one can show that this right hand side does not involve $H_\beta(t/\sqrt{2})$. These terms belong to the kernel of $L_\beta$, recall (2.26), and we can therefore not apply Lemma 2.9, as described in step (c), to inhomogeneous terms of this form. There are similar issues related to formalising the procedure iteratively to obtain closed form expressions for higher order derivatives of $z_*$ and $D$.

**Remark 2.12.** In [19], in the case of the folded node normal form, a more implicit, integral representation of $z''$ is obtained by differentiating the fix-point equation $z_* = T(z_*)$, with $T$ defined by the right hand side of (2.14). Inserting this expression into the expression for $\frac{\partial^3 D}{\partial v^3}(0,0)$ gives a double integral, which (presumably) can be evaluated in Mathematica. The reference [19] computes all values up to $n = 20$ for the folded node. (We compare these values with our closed form expressions in Appendix B.) However, it is unclear if it is possible to evaluate such double integrals directly.

3. Application of Theorem 2.8 to the folded node: Proof of Theorem 1.4. In this section, we now prove Theorem 1.4. First we follow [30] and rectify $\gamma$, recall (1.12), to the $x_3$-axis by introducing

\[
x_1 := x + z^2 - \frac{1}{2},
\]

\[
x_2 := y - 2z,
\]

\[
x_3 := 2z,
\]

so that

\[
\gamma : x(t) = (0,0,t), \ t \in \mathbb{R},
\]
using – for simplicity – the same symbol for the same object in the new variables. Notice that [19] rectifies $\gamma$ in a slightly different way, see Appendix B. Inserting (3.1) into (1.11) produces

\begin{align*}
\dot{x}_1 &= \frac{1}{2} \mu x_2 + x_1 x_3, \\
\dot{x}_2 &= -2x_1, \\
\dot{x}_3 &= 2x_1 + 1,
\end{align*}

(3.3)

which we, see also [30, Eq. (2.18)], will study in the following. The system (3.3) is time-reversible with respect to the same symmetry as in (1.13). It is easy to see that (3.3) satisfies the assumptions (H5)-(H7) and Theorem 2.10 applies. In particular, we have

$$
\beta = \mu - 1.
$$

We will therefore apply the procedure used in the proof of this result, specifically see Theorem 2.10 item (3a) and Remark 2.11, to (3.3). This will enable a proof of Theorem 1.4.

In the following, let

$$
\mu = n + \alpha.
$$

Then by Lemma 1.3 and the analysis in [30] the assumptions (H1)-(H4) are satisfied. At this stage, we keep $n \in \mathbb{N}$ general. For $n = \text{odd}$ the results in the following are therefore covered by [30]; the only exception is that we exploit the symmetry to simplify some of the expressions.

Writing $x = \gamma(t) + z$ gives

\begin{align*}
\dot{z}_1 &= tz_1 + \frac{n}{2} z_2 + g(z, \alpha), \\
\dot{z}_2 &= -2z_1, \\
\dot{z}_3 &= 2z_1,
\end{align*}

(3.4)

where

$$
g(z, \alpha) = \frac{1}{2} \alpha z_2 + z_1 z_3.
$$

(3.5)

In comparison with (2.7), 'g' for (3.4) is really $(g(z, \alpha), 0, 0)$, but it is useful to allow for a slight abuse of notation and let $g$ here refer to the first nontrivial coordinate function only. Setting $g = 0$ (ignoring the nonlinear terms) in (3.4) produces the variational equations about $\gamma$

$$
\dot{z} = A(t)z \quad \text{with} \quad A(t) = \begin{pmatrix} t & \frac{n}{2} & 0 \\
-2 & 0 & 0 \\
2 & 0 & 0 \end{pmatrix}.
$$

(3.6)

By differentiating the first equation for $z_1$ in (3.6) with respect to $t$, we obtain a Weber equation for $z_1$:

$$
L_{n-1} z_1 = 0,
$$

20
recall (2.24). For \( n \in \mathbb{N} \), it has an algebraic solution:

\[
z_1 = H_{n-1}(t/\sqrt{2}).
\]

(3.7)

Inserting (3.7) into the remaining equations for \( z_2 \) and \( z_3 \), we obtain the following state-transition matrix \( \Phi(t, s) \) of (3.6):

\[
\Phi(t, 0) = \begin{pmatrix}
\frac{1}{nH_{n-1}(0)} H_{n-1}(t/\sqrt{2}) & 0 \\
- \sqrt{2} H_{n}(t/\sqrt{2}) & 0 \\
\end{pmatrix}, \quad n = \text{odd}, \quad (3.8)
\]

\[
\Phi(t, 0) = \begin{pmatrix}
* & - \frac{n}{\sqrt{2}} H_{n-1}(t/\sqrt{2}) \\
* & 1 - \frac{1}{nH_{n}(0)} H_{n}(t/\sqrt{2}) \\
\end{pmatrix}, \quad n = \text{even}, \quad (3.9)
\]

see also [30, Eqns. (3.14)-(3.15)]. Following the notation in [30], the asterisks denote a separate linearly independent solution that we do not specify and which will play no role in the following. Setting \( t = 0 \) in the expressions for \( \Phi \) above, it follows that the \( z_1z_2 \)-plane has the following decomposition:

\[ V \oplus W, \]

where

\[
V = \text{span} \ e_v, \quad \begin{cases} 
\ e_v = (1, 0, 0)^T & n = \text{odd} \\
\ e_v = (0, 1, 0)^T & n = \text{even}
\end{cases}
\]

\[
W = \text{span} \ e_w, \quad \begin{cases} 
\ e_w = (0, 1, 0)^T & n = \text{odd} \\
\ e_w = (1, 0, 0)^T & n = \text{even}
\end{cases}
\]

(3.10)

recall (H4) and (2.6). Also \( U = \text{span}(0, 0, 1)^T \) for all \( n \in \mathbb{N} \). Therefore by (2.18):

**Proposition 3.1.** For (3.3),

\[
\sigma_v = \begin{cases}
1 & n = \text{odd} \\
-1 & n = \text{even}
\end{cases}
\]

\[
\sigma_w = \begin{cases}
-1 & n = \text{odd} \\
1 & n = \text{even}
\end{cases}
\]

and

\[
D(v, \alpha) = \begin{cases}
-2 h_{cs}(v, \alpha) & n = \text{odd} \\
h_{cs}(-v, \alpha) - h_{cs}(v, \alpha) & n = \text{even}
\end{cases}
\]

(3.11)

In particular,

1. \( D(0, \alpha) = 0 \) for all \( \alpha \) and any \( n \).

2. For \( n = \text{even} \), \( v \mapsto D(v, \alpha) \) is an odd function for every \( \alpha \).

**Proof.** Follows from the definition of \( \sigma_i, i = v, w \) in Lemma 2.5 and from (2.18), see also Theorem 2.10. \( \square \)

As a corollary, we have the following.

**Corollary 3.2.** Consider \( n = \text{odd} \). Then solutions of \( D(v, \alpha) = 0 \) bifurcating from \( v = \alpha = 0 \) correspond to symmetric solutions of (3.4), i.e. they are fix-points of the time-reversible symmetry \( \sigma \).
Proof. Follows from (3.11)_{n=\text{odd}} and the fact that any solution of $D(v, \alpha)$ in this case lies within $w = 0$, corresponding to $x_2 = x_3 = 0$, being the fix-point set of the symmetry $\sigma$. \(\Box\)

In contrary, when $n = \text{even}$ bifurcating solutions come in pairs (as a pitchfork bifurcation) that are related by the symmetry. 

Now, finally by Theorem 2.8 we have.

**Lemma 3.3.** Let $z_*(v, \alpha)(\cdot)$ be the solution with $z_*(v, \alpha)(0) = (v, h_{cs}(v, \alpha)) \in W^s_0(\alpha) \subset \Sigma$ in the $(v,w)$-coordinates. Then $z_*(v, \alpha) \in C_{b,+}$ and

$$D(v, \alpha) = \int_0^\infty e^{-t^2/2} \times$$

$$\begin{cases}
\frac{2\sqrt{2}}{\pi H_{n-1}(0)} H_n(t/\sqrt{2}) & n = \text{odd} \\
\frac{1}{H_n(0)} H_n(t/\sqrt{2}) & n = \text{even}
\end{cases}
\begin{cases}
g(z_*(v, \alpha)(t), \alpha) \\
g(z_*(v, \alpha)(t, \alpha) - g(z_*(-v, \alpha)(t, \alpha))
\end{cases}
t \, dt. \tag{3.12}
$$

Proof. We have in these expressions used that $\psi_*(t) = e^{-t^2/2}\left(\frac{2\sqrt{2}}{\pi H_{n-1}(0)} H_n(t/\sqrt{2}) \begin{cases} H_{n-1}(t/\sqrt{2}) & n = \text{odd} \\
0 & n = \text{even} \end{cases}\right)$, for $n$ odd,

$$\psi_*(t) = e^{-t^2/2}\left(\frac{1}{\sqrt{2}H_n(0)} \begin{cases} H_n(t/\sqrt{2}) & n = \text{even} \\
0 & n = \text{odd} \end{cases}\right),$$

for $n$ even, see [30, Eq. (3.12)], is the solution of the adjoint equation (2.10) with $\psi_*(0) = e_w$ which decays exponentially for $t \to \pm \infty$, recall Lemma 2.4. \(\Box\)

We are now ready to prove Theorem 1.4 item (2). 

**Proof of Theorem 1.4 item (2).** Let $n = 2k$, $k \in \mathbb{N}$. By Proposition 3.1, items (1) and (2), it follows that

$$D(0, \alpha) = \frac{\partial^2 D}{\partial v^2}(0, \alpha) = 0, \tag{3.13}$$

for all $\alpha$ and all $i \in \mathbb{N}$. Next, we have the following lemma.

**Lemma 3.4.** For $n = 2k$ with $k \in \mathbb{N}$ the following expressions hold:

$$\frac{\partial^2 D}{\partial v \partial \alpha}(0, 0) = \sqrt{\pi}(2k)!! \frac{\sqrt{2}(2k-1)!!}{\sqrt{2}(2k-1)!!}, \tag{3.14}$$

$$\frac{\partial^3 D}{\partial v^3}(0, 0) = 3\sqrt{2}\pi(2k+1)(2k)!!^4$$

$$\times \sum_{j=0}^{2k-1} \frac{(4k-1-2j)!}{(2k-1-2j)!j!(j+1)!((2k-1-j)!!)^2((2k-j)!!)^2}. \tag{3.15}$$

Let $c_{jk}$ be the elements of the sum in (3.15):

$$c_{jk} := \frac{(4k-1-2j)!}{(2k-1-2j)!j!(j+1)!((2k-1-j)!!)^2((2k-j)!!)^2}.$$
for \( j = 0, \ldots, 2k - 1 \). Then for every \( k \in \mathbb{N} \)
\[
\begin{cases}
  c_{kj} > 0 & \text{for } j = 0, \ldots, k - 1, \\
  c_{kj} < 0 & \text{for } j = k, \ldots, 2k - 1,
\end{cases}
\]
and
\[
\left| \frac{c_{k(k-l)}}{c_{k(k+l-1)}} \right| > 2^{2l-1} \geq 2,
\]
(3.16)
for all \( l = 1, \ldots, k \).

We turn to the proof of Lemma 3.4 once we have shown that Lemma 3.4 implies Theorem 1.4. For this, we first estimate the negative terms (where \( j = k, \ldots, 2k - 1 \)) of the sum in (3.15) using (3.16) to obtain the following positive lower bound,
\[
\frac{\partial^3 D}{\partial v^3}(0, 0) > 3 \sqrt{2 \pi (2k + 1)/(2k)!} \sum_{j=0}^{k-1} \frac{1}{2} c_{kj},
\]
(3.17)
of \( \frac{\partial^3 D}{\partial v^3}(0, 0) \), with the right hand side being the sum of only positive terms. Consequently, the expressions (3.13), (3.14), (3.15) – together with singularity theory [10] – proves our main result Theorem 1.4 item (2) on the pitchfork bifurcation.

**Proof of Lemma 3.4.** Let \( z_*(v, \alpha) \) be as described. Recall, that it has algebraic growth as \( t \to \infty \), and that \( z_*(0, \alpha) = 0 \) for all \( \alpha \) since \( \gamma \) is a solution for all \( \alpha \). Furthermore, by differentiating (3.4) with respect to \( v \) and setting \( v = \alpha = 0 \), we obtain the following equation
\[
\dot{z}' = A(t)z',
\]
with \( z' = \frac{\partial z_*(v, \alpha)}{\partial v}(0, 0) \), recall (2.33). Here \( A(t) \) is given in (3.6) with \( n = 2k \). Consequently, by (3.9) we have
\[
z'(t) = \left( -\frac{\sqrt{2k}}{H_{2k}(0)} H_{2k-1}(t/\sqrt{2}) \right)
\left( \begin{array}{c}
  -\frac{\sqrt{2k}}{H_{2k}(0)} H_{2k-1}(t/\sqrt{2}) \\
  1 - \frac{1}{H_{2k}(0)} H_{2k}(t/\sqrt{2})
\end{array} \right),
\]
(3.18)
see also [30]. Let \( z''(t) = \frac{\partial^2 z_*(v, \alpha)}{\partial v^2}(0, 0) \), recall (2.33), denote the second partial derivative of \( z_* \). We now follow the steps in Remark 2.11.

**Step (a).** By differentiating (3.4) once more with respect to \( v \) and setting \( v = \alpha = 0 \) we obtain a ‘higher order variational equation’
\[
\dot{z}'' = A(t)z'' + \begin{pmatrix}
  z'_1 z'_3 \\
  0 \\
  0
\end{pmatrix}.
\]
(3.19)
We have the following.

**Lemma 3.5.**
\[
\frac{\partial^2 D}{\partial v \partial \alpha}(0, 0) = \frac{1}{H_{2k}(0)^2} \int_0^\infty e^{-t^2/2} H_{2k}(t/\sqrt{2})^2 dt
\]
(3.20)
\[
\frac{\partial^3 D}{\partial v^3}(0, 0) = \frac{3}{\sqrt{2} H_{2k}(0)} \int_0^\infty e^{-t^2/2} H_{2k+1}(t/\sqrt{2})z'_3(t)z''_3(t) dt,
\]
(3.21)
where $z_3'$ and $z_3^{''}$ in (3.21) are defined by (2.33), respectively.

Proof. We use (3.12) with $n = 2k$:

$$D(v, \alpha) = \frac{1}{H_{2k}(0)} \int_0^\infty e^{-t^2/2} H_{2k}(t/\sqrt{2}) (g(z_*(v, \alpha)(t), \alpha) - g(z_*(-v, \alpha)(t), \alpha)) dt,$$

recall (3.5). To obtain (3.20) we differentiate this expression partially with respect to $v$ and $\alpha$. This gives

$$\frac{\partial^2 D}{\partial v \partial \alpha}(0, 0) = \frac{1}{H_{2k}(0)} \int_0^\infty e^{-t^2/2} H_{2k}(t/\sqrt{2}) z_2'(t) dt \quad = \frac{1}{H_{2k}(0)^2} \int_0^\infty e^{-t^2/2} H_{2k}(t/\sqrt{2})^2 dt,$$

by (3.18) upon setting $v = \alpha = 0$.

For (3.21), we also perform a direct calculation to obtain

$$\frac{\partial^3 D}{\partial v^3}(0, 0) = \frac{6}{H_{2k}(0)} \int_0^\infty e^{-t^2/2} H_{2k}(t/\sqrt{2}) (z_1'(t) z_3''(t) + z_3''(t) z_3'(t)) dt.$$

Following (3.4),

$$z_1(i) = \frac{1}{2} z_3(i),$$

for $i = 1, 2$, and hence

$$\frac{\partial^3 D}{\partial v^3}(0, 0) = \frac{3}{H_{2k}(0)} \int_0^\infty e^{-t^2/2} H_{2k}(t/\sqrt{2}) \frac{d}{dt} (z_3'(t) z_3''(t)) dt.$$

By integration by parts, using $z_3'(0) = 0$ and (A.1) in Appendix A, we then obtain the result. □

Using that $H_{2k}$ is an even function, the formula in (A.4) in Appendix A then produces the desired expression (3.14) for $\frac{\partial^2 D}{\partial v \partial \alpha}(0, 0)$ in Lemma 3.4.

To prove the remaining expression (3.15) in Lemma 3.4 for $\frac{\partial^3 D}{\partial v^3}(0, 0)$, we determine $z_3''$, which is the only remaining unknown in the expression (3.21).

Step (b). We do so by first writing (3.19) as an inhomogeneous Weber equation for $z_1''$:

$$L_{2k-1} z_1'' = 2 \frac{d}{dt} (z_1' z_3'),$$

(3.22)

recall the definition of second order linear differential operator $L_{2k-1}$ in (2.24).

Step (c). We then use Lemma 2.9 to solve the linear, inhomogeneous equation (3.22) for the algebraic solution $z_1''$ with $z_1''(0) = 0$, once we have written the right hand side of (3.22) as a finite sum of Hermite polynomials. For this we use (A.5):

Lemma 3.6. The following holds true for any $k \in \mathbb{N}$:

$$z_1' z_3' = \frac{\sqrt{2} k}{H_{2k}(0)^2} \sum_{j=0}^{2k-1} \binom{2k-1}{j} \binom{2k}{j} 2^j j! H_{4k-2j}(t/\sqrt{2}) - \frac{\sqrt{2} k}{H_{2k}(0)} H_{2k-1}(t/\sqrt{2}).$$

Proof. Calculation. □
Consequently, we have

**Lemma 3.7.** The following holds true for any $k \in \mathbb{N}$:

\[
\begin{align*}
    z_1''(t) &= -2 \frac{d}{dt} \left( \frac{\sqrt{2k}}{H_{2k}(0)^2} \sum_{j=0}^{2k-1} \frac{1}{2k - 1 - 2j} \binom{2k - 1}{j} \binom{2k}{j} 2^j j! H_{4k-1-2j}(t/\sqrt{2}) \right. \\
    &\quad \left. + \frac{\sqrt{2k}}{H_{2k}(0)} H_{2k-1}(t/\sqrt{2}) \right), \\
    z_3''(t) &= -4 \left( \frac{\sqrt{2k}}{H_{2k}(0)^2} \sum_{j=0}^{2k-1} \frac{1}{2k - 1 - 2j} \binom{2k - 1}{j} \binom{2k}{j} 2^j j! H_{4k-1-2j}(t/\sqrt{2}) \right. \\
    &\quad \left. + \frac{\sqrt{2k}}{H_{2k}(0)} H_{2k-1}(t/\sqrt{2}) \right). 
\end{align*}
\]

**Proof.** The expression in (3.23) follows from a simple calculation using Lemma 2.9, Lemma 3.6 and (A.2). (3.24) is then obtained by integrating $z_3'' = 2z''$, recall (3.19) and using $z_3''(0) = 0$. \(\square\)

**Step (d).** We then have.

**Lemma 3.8.** The following holds for any $k \in \mathbb{N}$:

\[
\frac{\partial^3 D}{\partial v^3}(0,0) = \frac{6k}{H_{2k}(0)^4} \sum_{j=0}^{2k-1} \frac{1}{2k - 1 - 2j} \binom{2k - 1}{j} \binom{2k}{j} 2^j j! \\
\times \int_{-\infty}^{\infty} e^{-t^2/2} H_{2k+1}(t/\sqrt{2}) H_{2k}(t/\sqrt{2}) H_{4k-1-2j}(t/\sqrt{2}) dt.
\]

**Proof.** We simply insert the expressions for $z_1'$ and $z_3''$ in (3.18) and (3.24), respectively, into (3.21). We then use (A.4) and the fact that the integrand is an even function of $t$ to simplify the expression. \(\square\)

The expression (3.15) for $\frac{\partial^3 D}{\partial v^3}(0,0)$ in Lemma 3.4 then follows from (A.6) and (A.3).

To show (3.16) we simply expand the binomial coefficients in the expression for $c_{k,j}$ and obtain

\[
\frac{c_{k(k-1)}}{|c_{k(k+1)}} = \frac{(2k+2l)(2k+2l-1) \cdots (2k+4-2l)(2k+3-2l)}{(k+l)^2(k+l-1)^2 \cdots (k+3-l)^2(k+2-l)^2},
\]

where the numerator and denominator both consist of $2(2l-1)$ factors. We simplify half of these factors by dividing up

\[
\frac{c_{k(k-1)}}{|c_{k(k+1)}} = 2^{2l-1} \frac{(2k+2l-1)(2k+2l-3) \cdots (2k+5-2l)(2k+3-2l)}{(k+l)(k+l-1) \cdots (k+3-l)(k+2-l)}.
\]

We can write the last fraction as a product

\[
(2 - 1/(k+l)) (2 - 3/(k+l-1)) \cdots (2 - 1/(k+2 - l)),
\]

where each factor is $> 1$ for every $l = 1, \ldots, k$. This shows (3.16) and we have therefore completed the proof of Lemma 3.4.
4. Secondary canards: a complete picture. In Fig. 4.1 we present a sketch of the compactified version of (1.11) using the Poincaré compactification induced by (1.8). The diagram is therefore identical to Fig. 1.3, but with \( r = 0 \) (and therefore \( \epsilon = 0 \)). Recall also that the three-dimensional hemisphere \( \mathbb{S}^3_{\epsilon \geq 0}^3 = \{(\bar{x}, \bar{y}, \bar{z}, \bar{\epsilon}) \in \mathbb{S}^3|\bar{\epsilon} \geq 0\} \) is “flattened out” by projection onto the \( (\bar{x}, \bar{y}, \bar{z}) \)-space, so that the sketched two-dimensional-sphere \((\bar{x}, \bar{y}, \bar{z}) \in \mathbb{S}^2 \) corresponds to the “equator” \( \bar{\epsilon} = 0 \) of \( \mathbb{S}^3_{\epsilon > 0} \). On the other hand, everything inside is \( \bar{\epsilon} > 0 \). In the following, we let \( \sigma \) act on \( \mathbb{S}^3_{\epsilon > 0} \) as follows \( \sigma : (\bar{x}, \bar{y}, \bar{z}, \bar{\epsilon}) \mapsto (\bar{x}, -\bar{y}, -\bar{z}, \bar{\epsilon}) \). This action is consistent with (1.13). The red and blue curves on the equator sphere \( \bar{\epsilon} = 0 \) correspond to the intersection with the critical manifold: \( \bar{x} = \bar{z}^2 \), which away from \( \bar{x} = \bar{z} = 0 \) has gained hyperbolicity, recall Fig. 1.3. Applying center manifold theory to these points gives rise to the local center manifolds \( W^{cs}(\mu) \) and \( W^{cu}(\mu) \) also illustrated as shaded surfaces extending into \( \bar{\epsilon} > 0 \). (Recall, that these manifolds are (a) the ones obtained by restricting the 3D manifolds \( M_r \) and \( M_a \) to the sphere \( r = 0 \) and therefore (b) the ‘extensions’ of the critical manifolds \( C_r \) and \( C_a \), respectively, onto the blowup sphere, recall Fig. 1.3.) The manifolds \( W^{cs}(\mu) \) and \( W^{cu}(\mu) \) contain the strong and weak canards (orange and purple dotted lines, respectively), being heteroclinic orbits, within this framework, connecting partially hyperbolic points \( \sigma p_s \) and \( \sigma p_w \), given by

\[
(\bar{x}, \bar{y}, \bar{z}, \bar{\epsilon}) = (-1, -2, -1, 0), \quad (\bar{x}, \bar{y}, \bar{z}, \bar{\epsilon}) = (-1, -2/\mu, -1, 0),
\]

with \( p_s \) and \( p_w \), respectively, on the equator sphere with \( \bar{\epsilon} = 0 \). Another simple calculation in the \( \bar{z} = 1 \) chart shows that the points \( q_{\text{out}} \) and \( q_{\text{in}} = \sigma q_{\text{out}} \) are hyperbolic attracting and repelling nodes, respectively. They correspond to the intersection of the nonhyperbolic critical fiber of the folded node \( p \) (in Fig. 1.1 this fiber coincides with the z-axis) with the blowup sphere. On the other hand, by working in the chart \( \bar{y} = 1 \) it follows that the points \( q_{\pm} : \bar{x} = \bar{z} = \bar{\epsilon} = 0, \bar{y} = \pm 1 \) are fully nonhyperbolic. Notice also \( q_{\pm} = \sigma q_{\pm} \).

In the following, we write \( a < b \) to mean that \( a < b \) while \( b - a \) is ‘sufficiently small’. We define \( a > b \) similarly to mean that \( b < a \). Finally, \( a \sim b \) will mean that \( |b - a| \) is ‘sufficiently small’.

Recall that for (1.11), \( \gamma \) in (1.12) is the ‘weak canard’ written in the \( \bar{\epsilon} = 1 \) chart. This special orbit divides the center manifolds into unique and nonunique subsets. To see this, notice that the local center manifold for \( \bar{z} > 0 \) and \( \bar{\epsilon} \sim 0 \) is unique around \( u \) all up to \( \gamma \), since these points coincide with the stable set of \( p_s \), see Fig. 4.1 and [30, Fig. 9]. We collect this result – using the \( x \)-variables, recall (3.1) – as follows:

**Lemma 4.1.** The local center manifold \( W^{cs}_{\text{loc}}(\mu) \) is unique on the side \( x_2 < 0 \) as the stable set of \( p_s \) but nonunique for \( x_2 > 0 \). Indeed, every point on the nonunique side of \( W^{cs}_{\text{loc}}(\mu) \) with \( \bar{\epsilon} > 0 \) is forward asymptotic to the hyperbolic and attracting node \( q_{\text{out}} \).

**Proof.** Regarding the unique side of \( W^{cs}_{\text{loc}}(\mu) \), we proceed as follows. In terms of the coordinates \((x_1, y_1, \epsilon_1)\) specified by the chart \( \bar{z} = 1 \), the point \( \sigma p_s \) is \((x_1, y_1, \epsilon_1) = (0, 2\mu^{-1}, 0) \) whereas \( \sigma p_w \) is \((x_1, y_1, \epsilon_1) = (0, 2, 0) \). The center manifold \( W^{cs}_{\text{loc}} \) in (1.14) is therefore only unique for \( y_1 \leq 2 \) which upon coordinate transformation becomes \( y_2 \leq 2z_2 \), seeing that \( z_2 \gg 1 \). The result then follows from the definition of \( x_2 \) in (3.1).

On the other hand, to verify the statement about \( x_2 > 0 \), we blowup each \( q_{\pm} \) to a sphere by setting

\[
\bar{x} = \rho^2 \bar{x}, \quad \bar{z} = \rho \bar{z}, \quad \bar{\epsilon} = \rho^3 \bar{\epsilon},
\]

(4.1)
leaving $\bar{y}$ untouched, where $\rho \geq 0$, $(\bar{x}, \bar{\bar{z}}, \bar{\bar{\bar{\bar{\nu}}}}) \in S^2$. Only $S^2_{\bar{\bar{\bar{\bar{\nu}}}} > 0} := S^2 \cap \{\bar{\bar{\bar{\bar{\nu}}}} \geq 0\}$ is relevant. Notice that these weights are the same as those used for blowing up the fold in $R^3$, see [26]. The calculations are also essentially identical to those in [26], so we skip the details and just present the resulting diagram for the blowup of $q_-$, see Fig. 4.2. Notice that the blowup picture for $q_+$ is obtained by the time-reversible symmetry. In Fig. 4.2, the sphere $S^2_{\bar{\bar{\bar{\bar{\nu}}}} > 0}$, obtained from the blowup (4.1), is shown in green. The consequence of these blowups is then that each point on $W^c_{loc}(\mu)$ with $x_2 > 0$, $\bar{\bar{\bar{\bar{\nu}}}} > 0$, is forward asymptotic to $q_{out}$. Seeing that $q_{out}$ is a hyperbolic and attracting node, this means that $W^c_{loc}(\mu)$ is nonunique on this side of $\gamma$. \hfill $\square$

Using the symmetry, we obtain a similar result for $W^{cu}$. In particular, every point on the nonunique side of $W^c_{loc}(\mu)$ with $\bar{\bar{\bar{\bar{\nu}}}} > 0$ is backwards asymptotic to $q_{in}$.

### 4.1. The transcritical bifurcation.

Now, consider the transcritical bifurcation near any odd integer $n = 2k - 1$. Then by Theorem 1.4 item (1) and Corollary 3.2, we have a symmetric secondary canard $\gamma^{sc}(\mu)$ for any $\mu \sim 2k - 1$. For $\mu = 2k - 1$, \(\gamma^{sc}(2k - 1) = \gamma\). Furthermore

**Proposition 4.2.** The following holds for any $k \in \mathbb{N}$:

1. For any $\mu < 2k - 1$, $\gamma^{sc}(\mu)$ is backwards asymptotic to $q_{in} = \sigma q_{out}$ and forward asymptotic to $q_{out}$. In this case, $\gamma^{sc}(\mu)$ is nonunique.
2. For any $\mu > 2k - 1$, $\gamma^{sc}(\mu)$ is backwards asymptotic to $\sigma p_s$ and forward asymptotic to $p_s$. In this case, $\gamma^{sc}(\mu)$ is unique as a heteroclinic connection.

**Proof.** Firstly, the fact that $\gamma^{sc}(\mu)$ is either (1): backwards asymptotic to $q_{in}$ and forward asymptotic to $q_{out} = \sigma q_{in}$ or (2): backwards asymptotic to $\sigma p_s$ and forward asymptotic to $p_s$, is a consequence of $\gamma^{sc}(\mu)$ being symmetric, recall Corollary 3.2. Similarly, the uniqueness of $\gamma^{sc}(\mu)$ is a consequence of the uniqueness of the center manifolds on one side of $\gamma$ only, see discussion above and Lemma 4.1. To complete the proof, suppose that $\mu > 2k - 1$. ($\mu < 2k - 1$ is similar and therefore left out.) Therefore $\alpha > 0$ and by working with the normal form (1.16), recall also (1.17), we realise that $\gamma^{sc} \subset W^{cs} \cap W^{cu}$ intersects $\Sigma$ along $x_2 = 0$. Let $(x_1(\mu), 0, 0)$ denote the intersection point. Then by (1.16)

$$\text{sign } x_1 = \text{sign } (-1)^{k+1}. \quad (4.2)$$

We will now show that the $x_2$-component of $\gamma^{sc}(\mu)$ is negative for all $t$ sufficiently large. For this purpose, consider the first column of the state-transition matrix $\Phi$ in (3.8)$_{n=2k-1}$ and multiply this column by the nonzero number $H_{2k-2}(0)$. This gives the following solution

$$
\begin{pmatrix}
H_{2k-2}(t/\sqrt{2}) \\
-\frac{1}{2k-2} H_{2k-1}(t/\sqrt{2}) \\
\frac{\sqrt{2}}{2k-2} H_{2k-1}(t/\sqrt{2})
\end{pmatrix}, \quad (4.3)
$$

of the variational equations (3.4) with an initial condition

$$\left( H_{2k-2}(0), 0, 0 \right)^T, \quad (4.4)$$

along $V$; recall that $U \oplus V$ is $T_{\gamma(0)} W^{cs}(\mu)$ for $\mu = 2k - 1$. Using (A.3) we realise that the first component of (4.4) has the same sign as (4.2). Fix therefore $T > 0$ large enough so that $H_{2k-1}(t) \geq 1$, say, for all $t \geq T$. Such $T$ exists since $H_{2k-1}(t)$ is polynomial with positive coefficient of the leading order term $t^{2k-1}$. Then specifically, the $x_2$-component of (4.3) is negative for all $t \geq T$, and consequently for $\mu > 2k - 1,$
by regular perturbation theory, the \( x_2 \)-component of the time \( t \geq T \) forward flow of \( \gamma^{sc} \cap \Sigma \) is also negative. This completes the proof since by Lemma 4.1, \( \gamma^{sc}(\mu) \) then belongs to the unique side of \( W^{cs}_{loc}(\mu) \) with \( x_2 < 0 \) being forward asymptotic to \( p_s \). See also Fig. 4.1. \[ \square \]

**Remark 4.3.** Here we recall some basic facts about canards from [2, 25, 30]. Whereas the strong canard always persists as a true (‘maximal’) canard for any \( 0 < \epsilon \ll 1 \), connecting the Fenichel slow manifolds \( S_{a,\epsilon} \) and \( S_{c,\epsilon} \), the perturbation of the weak canard for \( 0 < \epsilon \ll 1 \) to a true (‘maximal’) canard is clearly more invoked. In particular, there is no candidate weak canard on the critical manifold, but rather a funnel of trajectories tangent at \( p \) to the weak eigenvector at the folded node. However, seeing that \( \gamma^{sc}(\mu) \) on the blowup sphere is asymptotic to \( p_s \) and \( \sigma_p \) for fixed \( \mu > 2k-1 \), see Proposition 4.2 (2), this secondary canard has the same asymptotic properties as \( \nu \) and it therefore also perturbs into a true (‘maximal’) canard connecting the Fenichel slow manifolds \( S_{a,c} \) and \( S_{c,\epsilon} \) for \( 0 < \epsilon \ll 1 \), see also [30].

In fact, the secondary canards appearing for \( \mu > 2k-1 \) do not undergo additional bifurcations for \( \mu > 2k-1 \). Therefore if \( \mu \) satisfies \( 2k-1 < \mu < 2k+1 \) for some \( k \), then there exists \( k \) secondary canards for all \( 0 < \epsilon \ll 1 \), see [30, Proposition 4.1]. These canards divide the Fenichel slow manifold into bands \( o(1) \)-close (with respect to \( \epsilon \to 0 \)) to the strong canard with different rotational properties [2]. These bands provide an explanation for mixed-mode oscillations, see also [4].

**4.2. The pitchfork bifurcation.** Next, we consider \( n = 2k \) and the pitchfork bifurcation. Then by Theorem 1.4 item (2) there exists two secondary canards \( \gamma^{sc}(\mu) \) and \( \sigma \gamma^{sc}(\mu) \) for any \( \mu < 2k \) (or \( \alpha < 0 \)). For \( \mu = 2k \), \( \gamma^{sc}(2k) = \gamma \).

**Proposition 4.4.** The secondary canards \( \gamma^{sc}(\mu) \) and \( \sigma \gamma^{sc}(\mu) \) for \( \mu < 2k \) are nonunique heteroclinic connections. One connects \( \sigma p_s \) with \( q_{out} \) while the other one connects \( q_{in} = \sigma q_{out} \) with \( p_s \).

**Proof.** Straightforward working from the diagrams in Fig. 4.1 and Fig. 4.2. These canards are nonunique since they intersect the nonunique parts of the local center manifolds; recall Lemma 4.1 and that \( \gamma^{sc}(\mu) \) is not symmetric in this case. \[ \square \]

Together Proposition 4.2 and Proposition 4.4 provide a rigorous and geometric explanation of [30, Fig. 17]. In this figure, ‘\( \rho = \nu \)’ and the ‘TPB’s are points beyond which \( \gamma^{sc}(\mu) \) does not reach the fixed local version of \( W^{cs}(\mu) \), see (1.14).

**Remark 4.5.** As in Remark 4.3, we will now describe the implications of Proposition 4.4 for \( 0 < \epsilon \ll 1 \). Fix \( \mu < 2k \) and suppose without loss of generality that \( \gamma^{sc}(\mu) \) is the connection from \( \sigma p_s \) to \( q_{out} \). For all \( 0 < \epsilon \ll 1 \), seeing that \( W^{cs}(\mu) \) and \( W^{cw}(\mu) \) are transverse along \( \gamma^{sc}(\mu) \), this secondary canard produces, as for the transcritical bifurcation above, a connection between the extended manifolds \( S_{a,\sqrt{\epsilon}} \) and \( S_{r,\sqrt{\epsilon}} \). But since \( \gamma^{sc}(\mu) \) for \( \epsilon = 0 \) is asymptotic to \( q_{out} \) in forward time, the perturbed ‘canard’ never reaches the Fenichel slow manifold \( S_{c,\epsilon} \). Instead it follows, upon blowing down, the nonhyperbolic critical fiber as \( \epsilon \to 0 \). However, since \( \gamma^{sc}(\mu) \) is close to the strong canard for all \( t \) sufficiently negative, we can flow the perturbed version backwards and conclude that it does in fact originate from the Fenichel slow manifold \( S_{a,\epsilon} \). Here it also divides the subset of \( S_{a,\epsilon} \) between the secondary canard due to the bifurcation at \( 2k-1 \) and the rest of the funnel into regions of separate rotational properties through the folded node, see also [30, Proposition 2.5].

**5. The Falkner-Skan equation: Bifurcation of periodic orbits from infinity.** In [23] it was shown for the Falkner-Skan equation (1.3) that periodic orbits bifurcate from each integer value of \( \mu \in \mathbb{N} \). As noted in [24], the proof is long, complicated and to a large extend – not based upon dynamical systems theory. The
The global dynamics on the sphere $S_3^3 := S_3^3 \cap \{\bar{\epsilon} \geq 0\}$ – by projection – as a solid ‘ball’ in $(\bar{x}, \bar{y}, \bar{z})$-space. Here the unit sphere, being the boundary of the ball, corresponds to $\bar{\epsilon} = 0$, whereas everything inside of the ball corresponds to $\bar{\epsilon} > 0$. Within this framework, the strong and weak canard, $\upsilon$ and $\gamma$, respectively, are symmetric heteroclinic connections of points on the sphere. These orbits belong to $\bar{\epsilon} > 0$, i.e. inside the sphere, and are therefore indicated in orange and purple, recall also Fig. 1.1, using dotted lines. Indicated are also the invariant manifolds $W^{cu}(\mu)$ and $W^{cs}(\mu)$ (suppressing the $\mu$-dependency in the figure), which are locally center manifolds of normally hyperbolic lines of equilibria (blue and red half-circles, respectively). These lines end in nonhyperbolic points, $q_-$ and $q_+$ in green which correspond to the intersection of the fold line $F$, see Fig. 1.1, with the sphere obtained by blowing up the folded node $p$. The manifolds $W^{cs}(\mu)$ and $W^{cu}(\mu)$ intersect along $\gamma$, doing so tangentially for any $\mu \in \mathbb{N}$. This ‘bifurcation’ produces secondary canards through transcritical and pitchfork bifurcations, see Theorem 1.4.

The aim of the following section, is therefore to give a simple proof using the Melnikov approach, in particular Theorem 2.8, and the recipe in Section 2.2, which is based upon – as is more standard in dynamical systems – invariant manifolds. See also [18], for a similar approach in this context. In this reference, however, periodic orbits are constructed through an analysis of a return mapping.

First we write the equation (1.3) as a first order system

$$
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= -xz - \mu(1 - y^2),
\end{align*}
$$

which possesses two special solutions:

$$
\begin{align*}
\gamma &: (x, y, z) = (-t, -1, 0), \\
\upsilon &: (x, y, z) = (t, 1, 0)
\end{align*}
$$
Figure 4.2. Illustration of the blowup of $q_-$, using the same viewpoint as in Fig. 4.1. This blowup allows us to conclude that every point on the local manifold $W^{cu}$, close to the line of equilibria (in blue) and between $p_w$ and $q_-$, will be backwards asymptotic to $q_a$.

and a time-reversible symmetry given by

$$\sigma = \text{diag}(-1, 1, -1).$$

Both $\gamma$ and $\nu$ are symmetric orbits. It is easy to see, upon rectifying $\gamma$ to the $x_3$-axis, setting $(x, y, z) = (-x_3, 1 + x_1, x_2)$, that (5.1) satisfies the assumptions in Section 2.2, recall (H5)-(H7), respectively. In particular,

$$\beta = 2\mu - 1.$$

Theorem 2.10 therefore applies and we can evaluate the relevant integrals at bifurcations in closed form following the recipe outlined in Remark 2.11. To describe the global dynamics relevant for the bifurcations of periodic orbits, and obtain the invariant manifolds $W^{cs}$ and $W^{cu}$, we will compactify the system. For our purposes I find it useful to just compactify $(x, z)$, leaving $y$ untouched, by setting

$$x = \frac{\tilde{x}}{\tilde{w}},$$
$$z = \frac{\tilde{z}}{\tilde{w}}$$

for $(\tilde{x}, \tilde{z}, \tilde{w}) \in S^2$. To describe the dynamics near the equator defined by $\tilde{w} = 0$, we
consider the directional chart \( \bar{x} = -1 \) defined by
\[
\begin{align*}
z_1 &= -\frac{\bar{z}}{\bar{x}}, \\
w_1 &= -\frac{\bar{w}}{\bar{x}}.
\end{align*}
\] (5.3)
The smooth change of coordinates between the \( \bar{w} = 1 \) chart, defined in (5.2), and the \( \bar{x} = -1 \) chart, given by (5.3), is determined by the following equations
\[
\begin{align*}
w_1 &= -x^{-1}, \\
z_1 &= -zx^{-1},
\end{align*}
\] (5.4)
for \( x < 0 \). Using (5.4) we obtain the following equations in the \( \bar{x} = -1 \) chart:
\[
\begin{align*}
\dot{y} &= z_1, \\
\dot{z}_1 &= z_1 + w_1^2(yz_1 - \mu(1 - y^2)), \\
\dot{w}_1 &= w_1^3 y, 
\end{align*}
\] (5.5)
upon multiplication of the right hand sides by \( w_1 \), to ensure that \( w_1 = 0 \) – corresponding to \( \bar{w} = 0 \) under the coordinate map – is invariant. For this system, we notice that any point \((y, 0, 0)\) is a partially hyperbolic equilibrium of (5.5), the linearization having \( \lambda = 1 \) as a single nonzero eigenvalue. Therefore, by center manifold theory there exists a local repelling center manifold \( W^{cs}_{loc}(\mu) \). A simple calculation, using the invariance of \( \gamma \) and \( \upsilon \), shows that it takes the following form
\[
W^{cs}_{loc}(\mu) : z_1 = (1 - y^2)w_1^2(\mu + w_1 m_1(y, w_1)), \quad y \in I, \quad w_1 \in [0, \delta],
\] (5.6)
with \( I \) a fixed sufficiently large interval and where \( \delta > 0 \) is sufficiently small. Also, \( m_1 \) is a smooth function, also depending on \( \mu \). In terms of \((x, y, z)\), \( W^{cs}_{loc}(\mu) \) takes the following form
\[
W^{cs}_{loc}(\mu) : z = -(1 - y^2)x^{-1}(\mu - x^{-1}m_1(y, -x^{-1})), \quad y \in I,
\]
valid for all \( x \) sufficiently negative.
Inserting (5.6) into (5.5) gives the reduced problem on \( W^{cs}_{loc} \):
\[
\begin{align*}
\dot{y} &= (1 - y^2)(\mu + w_1 m_1(y, w_1)), \\
\dot{w}_1 &= w_1^3 y, 
\end{align*}
\] (5.7)
upon desingularization through division of the right hand side by \( w_1^2 \). Notice that \( \dot{y} > 0 \) for \( y \in (-1, 1) \) and all \( w_1 \geq 0 \) sufficiently small. In particular, \((y, w_1) = (-1, 0)\) and \((y, w_1) = (1, 0)\) are saddles, with the orbit \( y \in (-1, 1), \quad w_1 = 0 \) being a heteroclinic connection under the flow of (5.7). For later reference, let \( L \) be the invariant set defined by
\[
z_1 = w_1 = 0, \quad y \in [-1, 1].
\] (5.8)
It becomes \((\bar{x}, \bar{z}, \bar{w}) = (-1, 0, 0), \quad y \in I\) on the cylinder. By applying the symmetry, we obtain a local manifold \( W^{cs}_{loc} \) for all \( x \) sufficiently large. Combining this information we obtain the diagram in Fig. 5.1, see [22, Fig. 1] for a related figure. The global
manifolds $W^{cs}(\mu)$ and $W^{cu}(\mu)$ intersect along $\gamma$ and $\nu$. In particular, along $\gamma$ we have the following

**Lemma 5.1.** The manifolds $W^{cs}(\mu)$ and $W^{cu}(\mu)$ intersect transversally along $\gamma$ if and only if $2\mu \notin \mathbb{N}$.

**Proof.** We use Lemma 2.1 item (4). Consider therefore the variational equations about $\gamma$:

\[ \begin{align*}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= z_3, \\
\dot{z}_3 &= tz_3 - 2\mu z_2,
\end{align*} \tag{5.9} \]

which upon eliminating $z_1$ and $z_2$, can be written as a Weber equation

\[ L_{2\mu-1}z_3 = 0, \tag{5.10} \]

recall (2.24). The result then follows from Lemma 2.9, see also proof of Theorem 2.10. In particular, for $n = 2\mu$, we obtain the following algebraic solution of (5.9):

\[ z = \begin{pmatrix} 
\frac{1}{2n(n+1)} H_{n+1}(t/\sqrt{2}) \\
\frac{1}{\sqrt{2n}} H_n(t/\sqrt{2}) \\
\frac{1}{\sqrt{2n}} H_{n-1}(t/\sqrt{2})
\end{pmatrix}. \tag{5.11} \]

Next, fix any $n \in \mathbb{N}$ and define $\alpha$ by $\mu = n/2 + \alpha$. Then we can define a local Melnikov function $D(v, \alpha)$, the roots of which correspond to intersections of $W^{cs}(\mu)$ and $W^{cu}(\mu)$ near $\gamma$. Using Theorem 2.8, and proceeding as in the proof of Theorem 1.4, we obtain the following.

**Proposition 5.2.** Let $k \in \mathbb{N}$ be so that

\[ n = \begin{cases} 
2k - 1 & n = \text{odd} \\
2k & n = \text{even}.
\end{cases} \]

Then

1. For $n = \text{odd}$, $D(v, \alpha) = 0$ is locally equivalent with a pitchfork bifurcation:

\[ \hat{v}(\hat{\alpha} + \hat{v}^2) = 0. \tag{5.12} \]

2. For $n = \text{even}$, $D(v, \alpha) = 0$ is locally equivalent with the transcritical bifurcation:

\[ \hat{v}(\hat{\alpha} + (-1)^{k+1}\hat{v}) = 0. \tag{5.13} \]

In each case, the local conjugacy $\phi : (v, \alpha) \to (\hat{v}, \hat{\alpha})$ satisfies $\phi(0, 0) = (0, 0)$ and

\[ D\phi(0,0) = \text{diag}(d_1(n), d_2(n)) \quad \text{with } d_i(n) > 0 \text{ for every } n. \]

**Proof.** See Appendix C. □

The bifurcations of $W^{cs}(\mu)$ and $W^{cu}(\mu)$, described in the previous result, produce new transverse intersection for $\mu \sim n/2$ and every $n \in \mathbb{N}$. Notice, however, that they will not always produce periodic orbits. Instead they may simply diverge (by following the invariant green curves in Fig. 5.1 defined by $(\bar{x}, \bar{z}) = 0$). To give rise to
periodic orbits, the intersections have to be symmetric (which rules out the pitchfork bifurcation, whose symmetrically related solutions diverge either as $t \to \pm \infty$ along the green curves in Fig. 5.1) and they have to be on the ‘right’ side so that they follow $L$ and $\nu$. This is qualitatively very similar to the bifurcation of canards, recall Proposition 4.2 and Proposition 4.4. However, whereas for canards, the ‘interesting’ orbits, the true canards, appeared on the unique sides of the invariant manifolds, we will see that for the Falkner-Skan equation, as well as for the Nosé equations – the details of which is delayed to [15] – that the ‘interesting’ periodic orbits now appear on the nonunique sides of the manifolds. To obtain closed orbits we will have to fix copies of the center manifolds. We do so by using the fix-point sets of the symmetries.

In fact, we obtain a new proof of the following result on the bifurcation of periodic orbits from infinity, see [23], now based upon bifurcation theory and invariant manifolds.

**Theorem 5.3.** Let $\Gamma$ be the (singular) heteroclinic cycle obtained from concatenating (a) $\gamma$, (b) the ‘segment’ $L : (x, z, w) = (-1, 0, 0)$, $y \in I$, recall (5.8) in the ‘$x = -1$’ chart, (c) $\upsilon$, and finally (d) $\sigma L$, i.e. the symmetrically related version of the segment $L$ defined in (b). Then symmetric periodic orbits bifurcate from $\Gamma$ for each $\mu \in \mathbb{N}$.

In further details, let $\mu = k$ (so that $n = 2k$). Then symmetric periodic orbits $\Gamma_\mu$ only exist (‘locally’ to $\Gamma$) for $\mu \succ k$ and $\Gamma_\mu \to \Gamma$ as $\mu \to k^+$ for the compactified system.

**Proof.** The manifolds $W^{cs}$ and $W^{cu} = \sigma W^{cs}$ are (again) nonunique. We select unique copies as follows: Consider the strip $I$ defined by $(0, y, 0)$ with $y \sim 1$. Notice that $\sigma I = I$. We then select a unique copy of $W^{cs}(\mu)$ on the $y \geq -1$ side of $\gamma$ by flowing this strip backwards (where $W^{cs}(\mu)$ becomes attracting). Obviously, we let $W^{cu}(\mu)$ be the symmetrically related version of this fixed manifold.

Now, the transcritical bifurcation (5.13) produce a secondary intersection $\gamma^{sc}(\mu)$ of $W^{cs}(\mu)$ and $W^{cu}(\mu) = \sigma W^{cs}(\mu)$ for all $\mu \sim k$ so that $\gamma^{sc}(k) = \gamma$. In particular, we first note – following (C.2) in Appendix C – that $\gamma^{sc}(\mu)$ intersects $\Sigma$ along the $y$-axis. Let $(0, y_0(\mu), 0)$ denote this intersection point where $y_0(\mu) \sim -1$. Consider $\mu \succ k$ so that $\alpha > 0$. Then by (5.13) we have

$$\text{sign}(y_0(\mu) + 1) = \text{sign}(-1)^k. \quad (5.14)$$

Consider now the solution (5.11)$_{n=2k}$ of the variational equations (5.9), repeated here for convenience

$$z = \begin{pmatrix} \frac{1}{2k(2k+1)} H_{2k+1}(t/\sqrt{2}) \\ \frac{1}{2\sqrt{2k}} H_{2k}(t/\sqrt{2}) \\ H_{2k-1}(t/\sqrt{2}) \end{pmatrix}, \quad (5.15)$$

with initial condition

$$\left(0, \frac{1}{2\sqrt{2k}} H_{2k}(0), 0\right)^T. \quad (5.16)$$

By (A.3) in Appendix A, we realise that the sign of the second component of (5.15) coincides with the sign of (5.14). But then, since the second component of (5.15) is positive for all sufficiently large $t$, we conclude that $\gamma^{sc}(\mu)$ for $\mu \succ k$ follows $L$ for $t$ large enough. Subsequently, by following $\upsilon$, see Fig. 5.1, we realise that $\gamma^{sc}(\mu)$ returns to $x = 0$. Since we have fixed the manifolds to intersect $x = 0$ in the strip.
I, this intersection is of the form \((0, y_1(\mu), 0)\) with \(y_1(\mu) \to 1\) as \(\mu \to k^+\). But then

upon applying the time-reversible symmetry, we obtain a closed orbit. The periodic orbit intersects the fix-point set of \(\sigma\), defined by \((0, y, 0)\) twice, once near \(y \sim -1\) at \((0, y_0(\mu), 0)\) and once near \(y \sim 1\) at \((0, y_1(\mu), 0)\).

In the remaining cases (\(\mu < k\) and the \(n = \text{odd}\)), the ‘secondary intersection’ diverge as either \(t \to \pm \infty\) by following the green curves in Fig. 5.1 defined by \((\bar{x}, \bar{z}) = (0, \pm 1)\).

**Remark 5.4.** Notice that the periodic orbits appearing for \(\mu > k\) will rotate (or twist) around \(\gamma\), the number of rotations, as for the folded node, being determined by \(k\). See Fig. 5.2 for examples, and the figure caption for further description. As for the folded node, and the twisting of the secondary canards around the weak one, these rotations around \(\gamma\) can be explained by the solutions (see (C.4)) of variational equations. See also [30] for further details.

**Figure 5.1.** Illustration of the compactification of (5.1). Our viewpoint is from the negative \(y\)-axis, seeing the disc at \(y = -1\), containing \(\gamma\) (purple), from below. Notice that the circles at \(y = \pm 1\) are not invariant; they are just emphasized for illustrative purposes. We find invariant manifolds \(W^{cs}(\mu)\) (red) and \(W^{cu}(\mu) = \sigma W^{cs}(\mu)\) (blue) by application of center manifold theory to the partially hyperbolic lines \(L\) and \(\sigma L\). Together with \(\gamma\) and \(\nu\) (orange and dotted since it is on the disc at \(y = 1\)), these lines produce a (singular) cycle. We obtain a new proof of the bifurcation of periodic orbits by using our time-reversible version of the Melnikov theory to perturb away from this cycle.

**6. Conclusion.** In this paper, we have applied a time-reversible version of the Melnikov theory for nonhyperbolic unbounded connection problems in [29] to the bifurcations of canards in the folded node normal form. In particular, we proved – for the first time – the existence of a pitchfork bifurcation for \(\mu = \text{even}\). Our time-reversible setting also allowed for a new description of the ‘secondary canards’ emerging from the bifurcations at \(\mu \in \mathbb{N}\), see Section 4. The connection to the
Figure 5.2. In (a): Three periodic orbits of the (compactified) Falkner-Skan equation, projected onto the $(\bar{x}, y)$-plane, for three different values of $\mu$ ($\mu = 1.1$ in blue, $\mu = 2.3$ in red, $\mu = 3.4$ in green). These orbits are determined by appropriate backward integration from the set $I$, described in the proof of Theorem 5.3. In (b): The periodic orbits in (a) are now projected onto the $(y, \bar{z})$-plane with a zoom near $\gamma$, appearing as a point $(-1,0)$ in this projection. Notice that the periodic orbits twist (one twist defined as one a $360^\circ$ complete rotation) around $\gamma$. For $\mu = 1.1$ (blue) there is $1/2$ a twist, for $\mu = 2.3$ (red) there are $3/2$ twists, and finally for $\mu = 3.4$ (green) there are $5/2$ twists.

Weber equation as well as properties of the Hermite polynomials were essential to our proof of Theorem 1.4. But the results in Section 2.2, specifically see Theorem 2.10 and Remark 2.11, highlight that the Weber equation is ‘synonymous’ with quadratic, time-reversible systems satisfying (H4) and (H2) with the unbounded symmetric orbit $\gamma$ linear in $t$ and independent of $\alpha$. This is also expected, as noted by [24], since the Weber equation is the ‘simplest’ non-autonomous equation with a non-trivial time-reversible symmetry.

In Section 5, we also applied our approach to the Falkner-Skan equation. In particular, we provided a new proof of the emergence of periodic orbits, bifurcating from a heteroclinic cycle at infinity, using more standard methods of dynamical systems theory. Our approach also applies to the Nosé equations (1.4). In fact, by proceeding as in Section 5, it is possible to show that for $\mu > 1$ periodic orbits only bifurcate from $\mu \in \mathbb{N}$, a result that had escaped [24]. This will be the main result of the forthcoming manuscript [15]. In future work, it would be natural to use the geometric framework provided by this theory to study the emergence of chaos in these two systems.

I believe that it is possible to obtain closed-form expressions for the Melnikov integrals in [19] related to the bifurcations of faux canards for the folded saddle singularity. Although these problems do not fit our general setting in Section 2.2, the Weber equation also appears naturally for these problems.

Finally, the two-fold bifurcation in piecewise smooth systems, see e.g. [9, 11], also possess orbits that are reminiscent of weak and strong canards. Upon viewing the two-fold as a singular limit $\epsilon \to 0$ of a smooth system, see e.g. [16, 14], one can obtain a situation very similar to the folded node by using blowup to desingularize the limit $\epsilon \to 0$. In fact, even though the two-fold does not fit our general framework in Section 2.2, the variational equations along the ‘weak canard’ can also be reduced to the Weber equation, see [14, Eq. (101)]. In particular, for each $\mu \in \mathbb{N}$, this equation has an algebraic solution resulting in a bifurcation scenario similar to folded node, where ‘secondary canards’ (may) emerge. I expect that the details are very similar.
to the folded node above, see also the numerical exploration in [16, Section 8]. I have therefore decided not pursue this problem further in the present manuscript.

Acknowledgement. I would like to thank Martin Wechselberger for his encouragement and for providing valuable feedback on an earlier version of this manuscript.

REFERENCES

Appendix A. Properties on the Hermite polynomials.

The following properties of the “physicist” Hermite polynomials:

\[ H_n(x) = \left( 2x - \frac{d}{dx} \right)^n \cdot 1, \]

is standard, see e.g. [21].

**Lemma A.1.** For every \( n \in \mathbb{N} \)

\[
\begin{align*}
H_{n+1}(t) &= 2sH_n(s) - H_n'(s), \\
H_n'(s) &= 2nH_{n-1}(s), \\
H_n(0) &= \begin{cases} 
0 & n = \text{odd} \\
(-2)^{n/2}(n-1)!! & n = \text{even}
\end{cases}
\end{align*}
\]

and

\[
\int_{-\infty}^{\infty} e^{-t^2/2} H_n(t/\sqrt{2}) H_m(t/\sqrt{2}) = \sqrt{2\pi} 2^n n! \delta_{nm},
\]

where \( \delta_{nm} \) is the Kronecker delta.

Furthermore, for every \( n, m \in \mathbb{N} \):

\[
H_n(s)H_m(s) = \sum_{j=0}^{\min(n,m)} \binom{m}{j} \binom{n}{j} 2^j j! H_{n+m-2j}(s),
\]

Finally, for every \( (n,m,l) \in \mathbb{N}^3 \) that satisfies the triangle property and for which \( s = (n+m+l)/2 \in \mathbb{N} \):

\[
\int_{-\infty}^{\infty} e^{-t^2/2} H_n(t/\sqrt{2}) H_m(t/\sqrt{2}) H_l(t/\sqrt{2}) dt = \sqrt{2\pi} 2^n \frac{n! m! l!}{(s-n)!(s-m)!(s-l)!}.
\]
equivalent and we can map the $\mu \in (0,1)$ system into the $\mu > 1$ system, considered in the present paper, through the following transformation:

$$
(x, y, z, t, \mu) \mapsto \begin{cases} 
    \tilde{x} = \frac{1}{\mu} x, \\
    \tilde{y} = \sqrt{\mu} y, \\
    \tilde{z} = \frac{1}{\sqrt{\mu}} z, \\
    \tilde{t} = \sqrt{\mu} t, \\
    \tilde{\mu} = \mu - 1,
\end{cases}
$$

upon dropping the tildes. Specifically, this transformation maps $\gamma$ and $\upsilon$ for $\mu \in (0,1)$ into $\upsilon$ and $\gamma$ for $\mu^{-1} > 1$, respectively. But [19] also rectifies $\gamma$ (which is $\upsilon$ for $\mu \in (0,1)$) in a slightly different way. A simple computation shows that $(\tilde{x}, \tilde{y}, \tilde{z})$ in [19, Eq. (15)] is related to $(x_1,x_2,x_3)$ in (3.1) as follows:

$$
\begin{align*}
    \tilde{x} &= \frac{1}{\mu} x_1, \\
    \tilde{y} &= \sqrt{\mu} (x_2 + x_3), \\
    \tilde{z} &= -\frac{1}{2\sqrt{\mu}} x_2,
\end{align*}
$$

upon also replacing $\mu$ by $\mu^{-1}$. Let $D_{MW}(v, \mu^{-1})$ be the Melnikov function in [19, Proposition 29] for $\rho = v$ and $r = 0$. Then from (B.1) and [19, Eq. (89)] it follows that:

$$
D(v, \alpha) = (n + \alpha)D_{MW} \left( -\frac{1}{2\sqrt{n + \alpha}} v, \frac{1}{n + \alpha} \right),
$$

where $D$ is the Melnikov function used in the present paper. Hence,

$$
\frac{\partial^3 D}{\partial v^3}(0,0) = -\frac{1}{8\sqrt{n}} \frac{\partial^3 D_{MW}}{\partial v^3}(0,n^{-1}).
$$

In Table B.1, we compare each side of this equation, using our analytical expression (3.15) on the left hand side, whereas on the right hand side we use the numerical values in the table on [19, p. 595]. See further explanation in the table caption. We conclude that the results are in agreement (and attribute the tiny differences, indicated in red, to round off errors).

**Appendix C. The Falkner-Skan equation: Proof of Proposition 5.2.** Let $\tilde{z}$ be defined as $(x, y, z) = \gamma(t) + \tilde{z}$. Then we have

$$
\begin{align*}
    \dot{\tilde{z}}_1 &= \tilde{z}_2, \\
    \dot{\tilde{z}}_2 &= \tilde{z}_3, \\
    \dot{\tilde{z}}_3 &= tz_3 - n\tilde{z}_2 + g(z, \alpha),
\end{align*}
$$

where

$$
g(z, \alpha) := -2\alpha z_2 - z_1 z_3 + \left( \frac{n}{2} + \alpha \right) z_2^2,
$$
We attribute these tiny differences to round off errors.

Comparison of our closed-form Melnikov integral (3.15) with the values in [19]. The first and second column show all of the even values considered in [19]. The third column shows the values of the right hand side of (B.2), using the values in the first two columns, whereas the final column uses the expression in (3.15). In red we indicate the slight deviations between the last two columns. We attribute these tiny differences to round off errors.

\[
\begin{array}{|c|c|c|c|}
\hline
n = 2k & \frac{\partial^3 D_{MW}}{\partial v^3} (0, n-1) & \frac{1}{8\sqrt{\pi}} \frac{\partial^3 D_{MW}}{\partial v^3} (0, n-1) & \frac{\partial^3 D}{\partial v^3} (0, 0) \\
\hline
2 & -4.0837336724863 \times 10^3 & 360.9544714... & 360.9544714... \\
4 & -9.1263550336787 \times 10^5 & 57039.71895... & 57039.71896... \\
6 & -1.2403985652051 \times 10^8 & 6.329882421... \times 10^6 & 6.329882420... \times 10^6 \\
8 & -1.3867566218372 \times 10^{10} & 6.128656321... \times 10^8 & 6.1286563218... \times 10^8 \\
10 & -1.3996176586682 \times 10^{12} & 5.532474570... \times 10^{10} & 5.532474568... \times 10^{10} \\
12 & -1.3282386742790 \times 10^{14} & 4.792686474... \times 10^{12} & 4.792686474... \times 10^{12} \\
14 & -1.2108610331032 \times 10^{16} & 4.045202792... \times 10^{14} & 4.045202792... \times 10^{14} \\
16 & -1.0738223745005 \times 10^{18} & 3.355694922... \times 10^{16} & 3.355694920... \times 10^{16} \\
18 & -9.381989535112 \times 10^{19} & 2.751293251... \times 10^{18} & 2.751293251... \times 10^{18} \\
20 & -8.0059501510523 \times 10^{21} & 2.237731095... \times 10^{20} & 2.237731095... \times 10^{20} \\
\hline
\end{array}
\]

upon dropping the tildes. Then, by (5.11), we obtain the following regarding the state transition matrix:

\[
\Phi(t, 0) = \begin{pmatrix}
1 & \frac{1}{\sqrt{2\pi n(n+1)\eta_{n-1}(0)}} (H_{n+1}(t/\sqrt{2}) - H_{n+1}(0)) \\
0 & \frac{1}{\sqrt{2\pi n(n-1)\eta_{n-1}(0)}} H_n(t/\sqrt{2}) \\
0 & \frac{1}{\pi_{n-1}(0)} H_{n-1}(t/\sqrt{2})
\end{pmatrix}, \quad n = \text{odd},
\]

\[
\Phi(t, 0) = \begin{pmatrix}
1 & \frac{1}{\sqrt{2\pi n(n+1)\eta_{n}(0)}} H_{n+1}(t/\sqrt{2}) \\
0 & \frac{1}{\eta_n(0)} H_n(t/\sqrt{2}) \\
0 & \frac{1}{\sqrt{2\pi n\eta_n(0)}} H_{n-1}(t/\sqrt{2})
\end{pmatrix}, \quad n = \text{even}. \tag{C.1}
\]

Consequently,

\[
V = \text{span } e_v, \quad \begin{cases}
e_v = (0, 0, 1)^T & n = \text{odd} \\
e_v = (0, 1, 0)^T & n = \text{even}
\end{cases}
\]

\[
W = \text{span } e_w, \quad \begin{cases}
e_w = (0, 1, 0)^T & n = \text{odd} \\
e_w = (0, 0, 1)^T & n = \text{even}
\end{cases}
\]

recall (H4) and (2.6). Also \( U = \text{span}(1, 0, 0)^T \) for all \( n \in \mathbb{N} \). Therefore by (2.18):
\[ \sigma_v = \begin{cases} -1 & n = \text{odd} \\ 1 & n = \text{even} \end{cases} \]

\[ \sigma_w = \begin{cases} 1 & n = \text{odd} \\ -1 & n = \text{even} \end{cases} \]

and hence

\[ D(v, \alpha) = \begin{cases} h_{cs}(-v, \alpha) - h_{cs}(v, \alpha) & n = \text{odd} \\ -2h_{cs}(v, \alpha) & n = \text{even} \end{cases} \]

Consequently, for \( n = \text{odd} \), \( v \mapsto D(v, \alpha) \) is an odd function for every \( \alpha \). On the other hand, for \( n = \text{even} \) roots of \( D(\cdot, \alpha) \) correspond to symmetric solutions, being fixed with respect to the symmetry \( \sigma \). Furthermore, using

\[
\psi_*(t) = \begin{pmatrix}
\frac{1}{\sqrt{2}}(n+1)H_{n-1}(t/\sqrt{2}) & 0 \\
\frac{1}{\sqrt{2}H_{n-1}(0)}e^{-t^2/2}H_n(t/\sqrt{2}) & 0
\end{pmatrix}, \quad n = \text{odd}
\]

\[
\psi_*(t) = \begin{pmatrix}
\frac{1}{\sqrt{2}H_{n}(0)}e^{-t^2/2}H_{n-1}(t/\sqrt{2}) & 0 \\
\frac{1}{\sqrt{2}}H_{n-1}(0)e^{-t^2/2}H_n(t/\sqrt{2}) & 0
\end{pmatrix}, \quad n = \text{even}
\]

which follows from a simple calculation, we obtain

\[
D(v, \alpha) = 2\int_0^\infty e^{-t^2/2} \times
\begin{cases}
\frac{1}{2\sqrt{2}nH_{n-1}(0)}H_n(t/\sqrt{2}) (g(z_*(-v, \alpha)(t), \alpha) - g(z_*(v, \alpha)(t), \alpha)) & n = \text{odd} \\
\frac{1}{H_n(0)}H_n(t/\sqrt{2})g(z_*(v, \alpha)(t), \alpha) & n = \text{even}
\end{cases}
\]  

\[ \text{dt}. \]  

(C.3)

We now focus on \( n = \text{even} \), which is easier, and prove the transcritical case. The details of \( n = \text{odd} \) and the pitchfork are lengthier, but similar to the details of the proof of Theorem 1.4 item (2), see also Remark 2.11, and therefore left out.

Let therefore \( n = 2k \), such that \( \mu = k + \alpha \), and write \( z' := \frac{\partial}{\partial v} z_*(0, 0) \). Following (C.1), we have

\[
z' = \begin{pmatrix}
\frac{1}{\sqrt{2}(n+1)H_{n}(0)}H_{n+1}(t/\sqrt{2}) \\
\frac{1}{H_{n}(0)}H_n(t/\sqrt{2}) \\
\frac{\sqrt{2}}{H_{n}(0)}H_{n-1}(t/\sqrt{2})
\end{pmatrix},
\]

(C.4)

Then upon differentiating (C.3)\(_{n=\text{even}}\) with respect to \( \alpha \) and \( v \) we have

\[
\frac{\partial^2 D}{\partial v \partial \alpha}(0, 0) = \frac{2}{H_{2k}(0)} \int_0^\infty e^{-t^2/2}H_{2k}(t/\sqrt{2})(-2z'_2) dt \\
= -\frac{2}{H_{2k}(0)^2} \int_{-\infty}^\infty e^{-t^2/2}H_{2k}(t/\sqrt{2})^2 dt \\
= -\frac{2\sqrt{2\pi(2k)!}}{(2k - 1)!!^2} = -\frac{2\sqrt{2\pi}(2k)!}{(2k - 1)!!^2},
\]

40
using (C.1) as well as (A.3) and (A.4) in Appendix A. Similarly, by differentiating (C.3) \( n = \text{even} \) twice with respect to \( v \) we have

\[
\frac{\partial^2 D}{\partial v^2}(0, 0) = \frac{2}{H_{2k}(0)} \int_0^\infty e^{-t^2/2} H_{2k}(t/\sqrt{2})(-2z_1'z_3' + 2k(z_1')^2)\,dt
\]

:= I_1 + I_2,

where

\[
I_1 = -\frac{4k}{(2k + 1)H_{2k}(0)^3} \int_{-\infty}^\infty e^{-t^2/2} H_{2k}(t/\sqrt{2})H_{2k+1}(t/\sqrt{2})H_{2k-1}(t/\sqrt{2})\,dt,
\]

\[
I_2 = \frac{2k}{H_{2k}(0)^3} \int_{-\infty}^\infty e^{-t^2/2} H_{2k}(t/\sqrt{2})^3\,dt,
\]

using (C.4). To compute these integrals we use (A.6) in Appendix A and obtain the following expression

\[
I_2 = \frac{2k}{H_{2k}(0)^3} \sqrt{2\pi} 2^{3k} (2k)!^3 k!^3 = (-1)^k \frac{2k\sqrt{2\pi}(2k)!^3}{(2k - 1)!^3 k!^3} = (-1)^k \frac{2k\sqrt{2\pi}(2k)!^3}{k!^3},
\]

using (A.3), and, after some simple calculations,

\[
I_1 = -\frac{1}{k + 1} I_2.
\]

Consequently,

\[
\frac{\partial^2 D}{\partial v^2}(0, 0) = \frac{k}{k + 1} I_2 = (-1)^k \frac{2k^2 \sqrt{2\pi}(2k)!^3}{(k + 1)k!^3}.
\]

By singularity theory [10] this proves the transcritical bifurcation and the local equivalence (upon replacing \( D \) by \( -D \)) with the normal form (5.13).