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Peter Beelen and Mrinmoy Datta

ABSTRACT. In 1991 Sørensen proposed a conjecture for the maximum number of points on the intersection of a surface of degree d and a non-degenerate Hermitian surface in $\mathbb{P}^3(\mathbb{F}_{q^2})$. The conjecture was proven to be true by Edoukou in the case when $d = 2$. In this paper, we prove that the conjecture is true for $d = 3$. For $q \geq 4$, we also determine the second highest number of rational points on the intersection of a cubic surface and a non-degenerate Hermitian surface. Finally, we classify all the cubic surfaces that admit the highest and, for $q \geq 4$, the second highest number of points in common with a non-degenerate Hermitian surface. This classification disproves a conjecture proposed by Edoukou, Ling and Xing.

1. Introduction

Hermitian varieties defined over a finite field have received a lot of attention in the literature since they were introduced [1] in 1966 by Bose and Chakravarti. The geometry of the Hermitian varieties has been studied extensively in [1] and further in [2]. In particular, the line-plane incidence with respect to the non-degenerate Hermitian surfaces gives rise to beautiful combinatorial structures. Various combinatorial studies related to Hermitian surfaces include [4, 11, 14] among others. Further, Hermitian varieties have turned out to be very efficient from the perspective of error correcting codes since they have a large number of rational points. For discussions on codes defined by homogeneous polynomials of a fixed degree on Hermitian surfaces, we refer to [15, Example 6.6] and more generally on Hermitian varieties of arbitrary dimension in [8, Section 3]. In order to determine the minimum distance of the codes on Hermitian surfaces mentioned above, as well as from independent interests one may ask the following question:

QUESTION 1.1. Let $F \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ be a homogeneous polynomial of degree d and V_2 denote a non-degenerate Hermitian surface in $\mathbb{P}^3(\mathbb{F}_{q^2})$. What is the maximum number of $\mathbb{F}_{q^2}$-rational points in $V(F) \cap V_2$?

Note that answering this question will lead to determining the maximum number of rational points on a hyperplane section of the d-uple embedding of the Hermitian surface. It appears that this question was first addressed by Sørensen in his Ph.D. thesis [20] in order to generalize the work of Chakravarti [3] towards understanding the 2-uple embedding of the cubic surface defined by the equation $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$ in $\mathbb{P}^3(\mathbb{F}_4)$. Sørensen made the following conjecture:

CONJECTURE 1. [**20**, Page 9] *Let F and V_2 be as in Question 1.1. If $d \leq q$, then*

$$|V(F)(\mathbb{F}_{q^2}) \cap V_2| \leq d(q^3 + q^2 - q) + q + 1.$$

Further, the surfaces given by a homogeneous polynomial $F \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ attaining the above upper bound are given by a union of d planes in $\mathbb{P}^3(\mathbb{F}_{q^2})$ that are tangent to V_2, each containing a common line ℓ intersecting V_2 at $q + 1$ points.

In [**1**, Sections 7 and 8] Bose and Chakravarti have analysed the linear sections of general Hermitian varieties and determined the number of $\mathbb{F}_{q^2}$-rational points they contain. In particular, this answers the Question 1.1 for $d = 1$. Moreover, it can easily be seen that their answer validates Conjecture 1 in this case. Sørensen further observed that the combinatorial complexity of finding the maximum number of points of intersection of the Hermitian surface and a surface of degree d increases significantly with d. Conjecture 1 stands open till date.

The first breakthrough towards proving the conjecture was made by Edoukou in 2007. In [**8**], he proved that the conjecture is true for $d = 2$. Subsequently, in [**9**] Edoukou, Ling and Xing determined the first five highest number of points that an intersection of a quadric surface and a non-degenerate Hermitian surface can have in $\mathbb{P}^3(\mathbb{F}_{q^2})$. In the same article, the authors made several conjectures related to the configuration of hypersurfaces which may admit several highest numbers of points of intersection with non-degenerate Hermitian surfaces.

In the current paper, we answer Question 1.1 for $d = 3$. Certainly, cubic surfaces have been one among the most fascinating and studied objects in algebraic geometry and in particular, some of the results in Chapter 7 of [**17**] have turned out be extremely useful for us. In Theorem 5.3 and Corollary 5.4 we give a proof of Conjecture 1 for $d = 3$. To prove Theorem 5.3 we make extensive use of the underlying combinatorial structure of line-plane incidence with respect to Hermitian surfaces. Further, for $q \geq 4$ we classify all the cubic surfaces that attain the second highest number of points of intersection of a cubic surface and a non-degenerate Hermitian surface. This, in particular, disproves one of the conjectures in [**9**].

This paper is organized as follows. In Section 2, we recall various well-known properties of non-degenerate Hermitian surfaces, revisit some preliminary results from algebraic geometry and some basic bounds on the number of rational points on varieties defined over a finite field. Section 3 deals with results on cubic surfaces. In particular, we analyze cubic surfaces in $\mathbb{P}^3(\mathbb{F}_{q^2})$ containing of a pair of skew lines. In Section 4 we derive various results that are helpful in partially answering Question 1.1 for general values of d. Finally, in Section 5 we present our main results for $d = 3$.

2. Preliminaries

Fix a prime power q for the remainder of this paper. As usual, $\mathbb{F}_q$ and $\mathbb{F}_{q^2}$ denote the finite fields with q and q^2 elements respectively. For $m \geq 0$, we denote by $\mathbb{P}^m$, the projective space of dimension m over the algebraic closure $\overline{\mathbb{F}}_q$, while $\mathbb{P}^m(\mathbb{F}_{q^2})$ will denote the set of all $\mathbb{F}_{q^2}$-rational points of $\mathbb{P}^m$. Similarly, $\mathbb{A}^m$ and $\mathbb{A}^m(\mathbb{F}_{q^2})$ will denote the affine space of dimension m over $\overline{\mathbb{F}}_q$ and $\mathbb{F}_{q^2}$ respectively. Further, for a homogeneous polynomial $F \in \mathbb{F}_{q^2}[x_0, \ldots, x_m]$, we denote by $V(F)$, the set of zeroes of F in $\mathbb{P}^m$ and by $\mathsf{V}(F) = V(F)(\mathbb{F}_{q^2})$ the set of all $\mathbb{F}_{q^2}$-rational points of $V(F)$. By an algebraic variety we will mean a set of zeroes of a certain set of polynomials in the affine space or projective space, depending on the context. In particular, an algebraic variety need not be irreducible. We remark that, whenever we say that a variety is irreducible or nonsingular, we will mean that the variety is irreducible or nonsingular over $\overline{\mathbb{F}}_q$. This section is divided into three subsections: in the first subsection, we recall several known facts about Hermitian varieties over finite fields, the second subsection is dedicated to some useful results from basic algebraic geometry, while the third subsection concerns some known upper bounds

on the number of rational points on varieties defined over a finite field. The proofs of the results in this section can be found in the indicated references.

2.1. Hermitian varieties over finite fields. In this subsection, we recall the definition of Hermitian varieties and various of their well-known properties (cf. [**1, 2**]) that will be used in the latter part of this paper. At the end of this subsection, we recall the result proved by Edoukou [**8**] where he proves Conjecture 1 for the case $d = 2$. We begin with the following.

DEFINITION 2.1. For an $(m + 1) \times (m + 1)$ matrix $A = (a_{ij})$ $(0 \le i, j \le m)$ with entries in $\mathbb{F}_{q^2}$, we denote by $A^{(q)}$, the matrix whose (i, j)-th entry is given by $a_{ij}{}^q$. The matrix A is said to be a *Hermitian matrix* if $A \ne 0$ and $A^T = A^{(q)}$.

A *Hermitian variety* of dimension $m - 1$, denoted by V_{m-1}, is the set of zeroes of the polynomial $x^T A x^{(q)}$ inside $\mathbb{P}^m$, where A is an $(m + 1) \times (m + 1)$ Hermitian matrix and $x = (x_0, \ldots, x_m)^T$. The Hermitian variety is said to be *non-degenerate* if rank $A = m + 1$ and *degenerate* otherwise.

It is a well-known fact that if the rank of a Hermitian matrix is r, then by a suitable change of coordinates, we can describe the corresponding Hermitian variety by the zero set of the polynomial

$$(1) \qquad x_0^{q+1} + x_1^{q+1} + \cdots + x_{r-1}^{q+1} = 0.$$

For a proof of the above fact the reader is referred to [**1**, Equation (5.6)]. We note that the polynomial $x_0^{q+1} + \cdots + x_{r-1}^{q+1}$ is irreducible over the algebraic closure of $\mathbb{F}_q$ whenever $r \ge 3$. This shows that Hermitian varieties corresponding to Hermitian matrices of rank at least 3 are irreducible. For the purpose of this paper we will, from now on, restrict our attention to Hermitian curves and Hermitian surfaces, i.e. Hermitian varieties of dimensions 1 and 2 respectively. The linear sections of Hermitian surfaces are extremely well understood. Indeed, the hyperplane section of a Hermitian variety is also a Hermitian variety. We recall the following two results of Bose and Chakravarti [**1**] concerning the intersections of lines and planes with Hermitian surfaces in $\mathbb{P}^3(\mathbb{F}_{q^2})$.

LEMMA 2.2. [**1**, Section 7] *Any line in $\mathbb{P}^3(\mathbb{F}_{q^2})$ satisfies precisely one of the following.*
  (i) *The line intersects V_2 at exactly 1 point.*
  (ii) *The line intersects V_2 at exactly $q + 1$ points.*
  (iii) *The line is contained in V_2.*

Reflecting these three possibilities, we give the following definition.

DEFINITION 2.3. Let ℓ be a line in $\mathbb{P}^3(\mathbb{F}_{q^2})$. The line ℓ is called
  (a) a *tangent line* if it intersects V_2 at exactly 1 point.
  (b) a *secant line* if it intersects V_2 at exactly $q + 1$ points.
  (c) a *generator* if it is contained in V_2.

THEOREM 2.4. [**1**, Section 10] *Let V_2 denote a non-degenerate Hermitian surface in $\mathbb{P}^3$. Let Π be any hyperplane in $\mathbb{P}^3(\mathbb{F}_{q^2})$. If Π is a tangent to V_2 at some point $P \in V_2$, then Π intersects V_2 at exactly $q + 1$ generators, all passing through P. Otherwise, Π intersects V_2 at a non-degenerate Hermitian curve V_1. In particular,*

$$|V_2 \cap \Pi| = \begin{cases} q^3 + q^2 + 1 & \text{if } \Pi \text{ is a tangent plane,} \\ q^3 + 1 & \text{if } \Pi \text{ is not a tangent plane.} \end{cases}$$

REMARK 2.5. Let Π be tangent to V_2 at a point P. Theorem 2.4 shows that there are $q + 1$ lines passing through P contained in Π that are generators. The remaining $q^2 - q$ lines contained

in Π that pass through P are tangent lines. Further, any line that passes through P but is not contained in Π is a secant line. We refer to [**1**, Section 10] for the proof of these results.

In the course of proving our main results in Section 5, we shall make frequent use of the set of all planes containing a given line. Let ℓ be any line in $\mathbb{P}^3(\mathbb{F}_{q^2})$. By *the book of planes around* ℓ, denoted by $\mathcal{B}(\ell)$, we mean the set of all planes in $\mathbb{P}^3(\mathbb{F}_{q^2})$ that contain ℓ. This set of planes is also called the pencil or sheaf of planes with axis ℓ. We note that, for any line ℓ in $\mathbb{P}^3(\mathbb{F}_{q^2})$, the corresponding book has cardinality $q^2 + 1$. The following proposition and its corollary will prove to be instrumental in the latter part of this article.

PROPOSITION 2.6. *Let ℓ be a line in $\mathbb{P}^3(\mathbb{F}_{q^2})$ and $\mathcal{B}(\ell)$ be the book of planes around ℓ.*

(a) [**2**, Lemma 5.2.3] *If ℓ is a generator, then every plane in $\mathcal{B}(\ell)$ is tangent to V_2 at some point of ℓ.*

(b) [**2**, Lemma 5.2.6] *If ℓ is a tangent line, then exactly one plane in $\mathcal{B}(\ell)$ is tangent to V_2 at the point where ℓ meets V_2.*

(c) [**2**, Lemma 5.2.5] *If ℓ is a secant line, intersecting V_2 at $q + 1$ points $P_0, \ldots, P_q$, then out of the $q^2 + 1$ planes in $\mathcal{B}(\ell)$, exactly $q + 1$ are tangent to V_2 at some point distinct from $P_0, \ldots, P_q$.*

COROLLARY 2.7. *Let Π_1 and Π_2 be two planes in $\mathbb{P}^3(\mathbb{F}_{q^2})$ meeting at a line ℓ. Then*

(a) *If Π_1, Π_2 are both tangent to V_2, then ℓ is either a generator or a secant line.*

(b) *If one of Π_1, Π_2 is not a tangent plane, then ℓ is either a secant or a tangent line.*

PROOF. Part (a) follows trivially by noticing that the book of a tangent line contains exactly one tangent plane (see Prop. 2.6 (b)), while part (b) is an easy consequence of the fact that all the planes in the book of a generator are tangent planes (see Prop. 2.6 (a)). □

Based on the above combinatorial structure, Sørensen considered [**20**] the following arrangement of planes that attains the conjectured upper bound. We include a proof for the convenience of the reader.

PROPOSITION 2.8. *Let $\Pi_1, \ldots, \Pi_d$ be d distinct planes that are tangent to V_2. Further assume that they contain a common line which is a secant. Then,*

$$|(\Pi_1 \cup \cdots \cup \Pi_d) \cap V_2| = d(q^3 + q^2 - q) + q + 1.$$

Moreover, there exists a homogeneous polynomial $F \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ of degree d such that $V(F) = \Pi_1 \cup \cdots \cup \Pi_d$, where the Π_i-s are defined as above.

PROOF. Let $\ell = \Pi_1 \cap \cdots \cap \Pi_d$. By hypothesis the line ℓ is a secant to V_2. Now for $k = 1, \ldots, d$ we have $|(\Pi_k \setminus \ell) \cap V_2| = q^3 + q^2 + 1 - (q + 1) = q^3 + q^2 - q$. Consequently,

$$|(\Pi_1 \cup \cdots \cup \Pi_d) \cap V_2| = \sum_{k=1}^{d} |(\Pi_k \setminus \ell) \cap V_2| + |\ell \cap V_2| = d(q^3 + q^2 - q) + q + 1.$$

Since $d \leq q$, the existence of distinct planes $\Pi_1, \ldots, \Pi_d$ tangent to V_2 each containing a common secant line is guaranteed by Proposition 2.6 (c). Existence of $F \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ homogeneous of degree d such that $V(F) = \Pi_1 \cup \cdots \cup \Pi_d$ follows trivially. □

Finally, to conclude this subsection, we recall the following results of Edoukou.

THEOREM 2.9. [**8**, Thm. 5.11 and 6.2] *Let $F \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ be a homogeneous quadratic polynomial. Then,*

$$|V(F) \cap V_2| = 2q^3 + 2q^2 - q + 1 \qquad \text{or} \qquad |V(F) \cap V_2| \leq 2q^3 + q^2 + 1.$$

Also, $|V(F) \cap V_2| = 2q^3 + 2q^2 - q + 1$ if and only if $V(F)$ is the union of two tangent planes intersecting at a secant line.

2.2. Preliminaries from algebraic geometry. In the present subsection we recall various basic results from algebraic geometry that will be needed for proving our main theorem. We will make use of the notions of dimension, degree and singularity of a variety, as can be found in standard textbooks of Algebraic Geometry, for example, the book of Harris [**12**]. We begin with the following definitions. A variety is said to be of *pure dimension* or *equidimensional* if all the irreducible components of the variety have equal dimension. Further, two equidimensional varieties $X, Y \subset \mathbb{P}^m$ are said to *intersect properly* if $\operatorname{codim}(X \cap Y) = \operatorname{codim} X + \operatorname{codim} Y$. It turns out that good upper bounds for the number of rational points on varieties defined over a finite field, depend on the degree and dimension of the variety. Because of this reason, the following proposition from [**12**] will be indispensable for us.

PROPOSITION 2.10. [**12**, Cor. 18.5 and 18.6] *Let X and Y be equidimensional varieties that intersect properly in $\mathbb{P}^m$. Then $\deg(X \cap Y) \leq \deg X \deg Y$ and equality holds only if both X and Y are smooth at a general point of any component of $X \cap Y$.*

As can be seen in [**12**, p. 54], the phrase "X is smooth at a general point of any component of $X \cap Y$" means the following: for any irreducible component C of $X \cap Y$, the set of points on C where X is nonsingular, contains an open dense subset of C. In particular, we have the following proposition concerning the intersection of two hypersurfaces.

PROPOSITION 2.11. *Let $F, G \in \mathbb{F}_{q^2}[x_0, x_1, \ldots, x_m]$ be nonconstant homogeneous polynomials having no common factors. Then*

(a) *$V(F)$ and $V(G)$ intersect properly.*
(b) *$V(F) \cap V(G)$ is equidimensional of dimension $m - 2$.*
(c) *$\deg(V(F) \cap V(G)) \leq \deg F \deg G$. Moreover, if there exists an irreducible component C of $V(F) \cap V(G)$ such that $V(G)$ is singular at every point of C, then*

$$\deg(V(F) \cap V(G)) \leq \deg F \deg G - 1.$$

PROOF. Part (a) can be found in [**5**, Lemma 2.2]. Part (b) is proved using Macaulay's unmixedness theorem (See [**21**, Chapter 7, Theorem 26]). Part (c) follows straightaway from Proposition 2.10. □

REMARK 2.12. As an immediate consequence of the irreducibility of non-degenerate Hermitian varieties of dimension at least 1 and Proposition 2.11, we see that a surface given by a nonconstant homogeneous polynomial of degree $d \leq q$ in $\mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ intersects the Hermitian surface at an equidimensional variety of dimension 1 and degree at most $d(q+1)$. A similar consequence for non-degenerate Hermitian curves can be derived.

2.3. Basic upper bounds. In this subsection, we recall some well-known upper bounds on the number of rational points on varieties defined over a finite field $\mathbb{F}_q$ with given degree and dimension. We start with a result from [**16**].

PROPOSITION 2.13. [**16**, Prop. 2.3] *Let X be an equidimensional projective (resp. affine) variety defined over a finite field $\mathbb{F}_q$. Further assume that $\dim X = \delta$ and $\deg X = d$. Then*

$$|X(\mathbb{F}_q)| \leq d p_\delta \qquad (\text{resp. } |X(\mathbb{F}_q)| \leq d q^\delta),$$

where $p_\delta = 1 + q + \cdots + q^\delta$.

The projective part of the above proposition appears in [**15**, Proposition 2.3] and in [**10**, Proposition 12.1] in somewhat incorrect form. In [**16**], the authors observed that the condition of equidimensionality has to be added in the hypothesis to make it correct; in Proposition 2.3 of the same article, the upper bound is proved for irreducible varieties and the statement above follows immediately. An alternative proof can be found in [**6**, Proposition 2.3].

The following theorem was proved by Serre [**18**] and independently by Sørensen [**19**]. It concerns the maximum number of zeroes a homogeneous polynomial of degree d in $m+1$ variables can have in $\mathbb{P}^m(\mathbb{F}_q)$. The upper bound is often referred to as *Serre's inequality* in the literature.

THEOREM 2.14 (Serre's inequality). *Let $F \in \mathbb{F}_q[x_0, x_1, \ldots, x_m]$ be a nonconstant homogeneous polynomial of degree $d \le q$. Then*

$$|\mathsf{V}(F)| \le dq^{m-1} + p_{m-2},$$

where $p_{m-2} = 1 + q + \cdots + q^{m-2}$. Moreover, equality holds if and only if $V(F)$ is a union of d hyperplanes defined over $\mathbb{F}_q$ all containing a common linear subspace of codimension 2.

3. Cubic surfaces

For future use, we derive some results on cubic surfaces $X \subset \mathbb{P}^3$ defined over a field $\mathbb{F}$ and containing a skew pair of lines ℓ and m. In later sections, we will apply these results for the field $\mathbb{F} = \mathbb{F}_{q^2}$. We may assume after a linear change of coordinates, if necessary, that $\ell = V(x_2, x_3)$ and $m = V(x_0, x_1)$. Such a cubic surface is the zero set of a homogeneous polynomial of the form

$$(2) \qquad F = Ax_0^2 + Bx_0x_1 + Cx_1^2 + Dx_0 + Ex_1,$$

where $A, B, C, D, E \in \mathbb{F}[x_2, x_3]$, with $\deg A = \deg B = \deg C = 1$ and $\deg D = \deg E = 2$. In this section, we are interested in estimating the cardinality of the set of planes

$$\mathcal{T}_\ell := \{\Pi \in \mathcal{B}(\ell) \mid V(F) \cap \Pi \text{ is a union of lines}\}.$$

The reason for studying such sets of planes will become clear in later sections.

To study the behavior of the cubic surface $X = V(F)$ restricted to a plane $\Pi \in \mathcal{B}(\ell)$, we start with the following lemma, which essentially is a direct consequence of the proof of Proposition 7.3 from [**17**]. However, since in [**17**] it was assumed that the cubic is smooth and the proof was given for the case that $\text{char}(\mathbb{F})$ is odd, we provide a proof for the convenience of the reader.

LEMMA 3.1. *Let $\mathbb{F}$ be an algebraically closed field and $a, b, c, d, e \in \mathbb{F}$ not all zero. Further, let $g := ax_0^2 + bx_0x_1 + cx_1^2 + dx_0t + ex_1t \in \mathbb{F}[x_0, x_1, t]$. Then g is reducible if and only if $-ae^2 + bde - cd^2 = 0$.*

PROOF. Since the polynomial g has degree two, it is reducible if and only if there exists a projective point which is a common zero of g and its partial derivatives. Considering the partial derivatives of g, we see that the coordinates of such a point $P = [x_0 : x_1 : t]$ need to satisfy

$$\begin{pmatrix} 2a & b & d \\ b & 2c & e \\ d & e & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let us denote the 3×3 matrix occurring above by M. Observe that $\det M = 2(-ae^2 + bde - cd^2)$.

Now assume that g and its partial derivatives have some projective point as common zero. Then necessarily $\det M = 0$. If $\text{char}(\mathbb{F}) \ne 2$, then $-ae^2 + bde - cd^2 = 0$, as desired. If $\text{char}(\mathbb{F}) = 2$, it is easy to see that the rank of M equals two unless $(e, d, b) = (0, 0, 0)$. If $(e, d, b) = (0, 0, 0)$, clearly $-ae^2 + bde - cd^2 = 0$, while otherwise the only common zero of the partial derivatives is the projective point $[e : d : b]$. By assumption, this point also needs to be a zero of g, which implies that $-ae^2 + bde - cd^2 = 0$.

Now conversely assume that $-ae^2 + bde - cd^2 = 0$. As in the proof of Proposition 7.3 in [**17**], if $\text{char}(\mathbb{F}) \ne 2$, then g defines a singular conic and is hence reducible. If $\text{char}(\mathbb{F}) = 2$ and $(e, d, b) = (0, 0, 0)$, then g is a square. This leaves the case that $\text{char}(\mathbb{F}) = 2$ and $(e, d, b) \ne (0, 0, 0)$. In this case, the projective point $[e : d : b]$ is a singularity on $V(g)$, again implying that g is reducible, since $\deg g = 2$. □

We now obtain the following proposition on the cardinality of $\mathcal{T}_\ell$. If X is smooth, this proposition is a direct consequence of Proposition 7.3 in [**17**], but the proof given there applies for any cubic surface. For the convenience of the reader, we include the proof.

PROPOSITION 3.2. *Let $X = V(F)$ be a cubic surface, with F as in equation (2). Define $\ell = V(x_2, x_3) \subset X$ and $\mathcal{T}_\ell = \{\Pi \in \mathcal{B}(\ell) \mid V(F) \cap \Pi$ is a union of lines$\}$. Then $\mathcal{T}_\ell = \mathcal{B}(\ell)$ or $|\mathcal{T}_\ell| \leq 5$.*

PROOF. It is enough to show the lemma in case $\mathbb{F}$ is algebraically closed. Indeed, if the proposition is true under this assumption, it will be true for any field. Any $\Pi \in \mathcal{B}(\ell)$ is of the form $\Pi = V(\lambda_1 x_2 - \lambda_2 x_3)$, where $[\lambda_1 : \lambda_2] \in \mathbb{P}^1(\mathbb{F})$. Now, the last two coordinates of any point on Π are given by $[\lambda_2 : \lambda_1]$. Here, without loss of generality we may assume that the first nonzero coordinate of $[\lambda_2 : \lambda_1]$ is 1. We have

$$F|_{(\Pi \setminus \ell)} = ax_0^2 + bx_0x_1 + cx_1^2 + dx_0t + ex_1t,$$

where $a = A(\lambda_2, \lambda_1), b = B(\lambda_2, \lambda_1), c = C(\lambda_2, \lambda_1), d = D(\lambda_2, \lambda_1), e = E(\lambda_2, \lambda_1)$, and t is x_2 if $\lambda_1 \neq 0$ and x_3 otherwise. Now apart from ℓ, $V(F|_\Pi)$ contains additional lines if and only if $g(x_0, x_1, t) = ax_0^2 + bx_0x_1 + cx_1^2 + dx_0t + ex_1t$ is reducible. However, Lemma 3.1 implies that g is reducible if and only if $-ae^2 + bde - cd^2 = 0$.

This implies that, for any $\Pi \in \mathcal{B}(\ell)$, the cubic curve $V(F) \cap \Pi$ is a union of lines if and only if $-ae^2 + bde - cd^2 = 0$. Note that, either $-AE^2 + BDE - CD^2$ is the zero polynomial or it is a nonzero quintic. In the first case, we have $\mathcal{T}_\ell = \mathcal{B}(\ell)$, while in the second case $|\mathcal{T}_\ell| \leq 5$. □

Note that if $|\mathbb{F}| \geq 5$, then the cases $\mathcal{T}_\ell = \mathcal{B}(\ell)$ and $|\mathcal{T}_\ell| \leq 5$ are disjoint. Our next result concerns a description of cubic surfaces defined by a polynomial as in equation (2) satisfying $-AE^2 + BDE - CD^2 = 0$. We start with a definition and a lemma.

DEFINITION 3.3. Let $X \subset \mathbb{P}^3$ be a cubic surface defined over a field $\mathbb{F}$. A point P on X is called a weak Eckardt point, if it is a point of intersection of three lines on X. If the point P is a smooth point on X, it is called an Eckardt point.

Note that since an Eckardt point P is assumed to be a smooth point on X, there are exactly three lines contained in X through P, all contained in the tangent plane of X at P. By definition, a singular point P of X cannot be an Eckardt point, though it may be a weak Eckardt point. In this case more than three lines contained in X may pass through P. One possibility is that X is a cone with center P, but also if X is not a cone, this may happen. Consider for example $X = V(F)$, with $F = x_0x_1(x_2 - ax_3) - (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$ and where $a \in \mathbb{F} \setminus \{0, 1\}$. Then $P = [1 : 0 : 0 : 0]$ is a weak Eckardt point lying on exactly six distinct lines each contained in X.

We now show that if a line on X contains an Eckardt point, then any other weak Eckardt point on that line is smooth and hence an Eckardt point.

LEMMA 3.4. *Let $X \subset \mathbb{P}^3$ be an irreducible cubic surface defined over a field $\mathbb{F}$ and ℓ, m two skew lines on X. Assume that ℓ is not a double line. If ℓ contains an Eckardt point, then any point on ℓ is smooth. In particular, any other weak Eckardt point of X on ℓ is actually an Eckardt point.*

PROOF. Let $P_1, P_2 \in \ell$ and P_1 be an Eckardt point. Choosing a suitable coordinate system x_0, x_1, x_2, x_3 for $\mathbb{P}^3(\mathbb{F})$, we may assume that $\ell = V(x_0, x_1)$, $m = V(x_2, x_3)$, $P_1 = [0 : 0 : 1 : 0]$ and $P_2 = [0 : 0 : 0 : 1]$. This means that $X = V(F)$, with F of the form

$$F = a_2x_0^2x_2 + a_3x_0^2x_3 + b_2x_0x_1x_2 + b_3x_0x_1x_3 + c_2x_1^2x_2 + c_3x_1^2x_3$$
$$+ d_{22}x_0x_2^2 + d_{23}x_0x_2x_3 + d_{33}x_0x_3^2 + e_{22}x_1x_2^2 + e_{23}x_1x_2x_3 + e_{33}x_1x_3^2.$$

Since P_1 is a smooth point of X, we have $(d_{22}, e_{22}) \neq (0,0)$ and after a further change of coordinates, we may assume that the tangent plane of X at P_1 is $V(x_1)$; i.e. $d_{22} = 0$ and $e_{22} = 1$.

Since P_1 is an Eckardt point, three lines contained in $V(x_1)$ (the tangent plane of X at P_1) need to pass through it. The first line is ℓ, while the two other lines need to be of the form $\ell_\alpha := P_1 \cup \{[1 : 0 : t : \alpha] \mid t \in \mathbb{F}\}$. It is not hard to see that $\ell_\alpha \subset X$ if and only if $a_2 + d_{23}\alpha = 0$ and $a_3\alpha + d_{33}\alpha^2 = 0$. Since we need two possibilities for α, we see that necessarily $a_2 = d_{23} = 0$ and $d_{33} \neq 0$. Further replacing the coordinate x_0 by a suitable scalar multiple, we may assume that $d_{33} = 1$. Replacing x_0 by $x_0 + e_{33}x_1$, we may assume that $e_{33} = 0$. Hence we may assume that F is of the form

$$(3) \qquad F = a_3 x_0^2 x_3 + b_2 x_0 x_1 x_2 + b_3 x_0 x_1 x_3 + c_2 x_1^2 x_2 + c_3 x_1^2 x_3 + x_0 x_3^2 + x_1 x_2^2 + e_{23} x_1 x_2 x_3.$$

But this implies that the point $P_2 = [0 : 0 : 0 : 1]$ is a smooth point of X, since the partial derivative of F w.r.t. x_0 does not vanish in P_2. Hence if P_2 was a weak Eckardt point, we conclude that P_2 actually is an Eckardt point of X. $\qquad \square$

It is clear that if X is a smooth cubic surface there is no distinction between weak Eckardt points and Eckardt points. The following lemma states how many weak Eckardt points a line on a cubic surface can contain. If X is a smooth cubic surface, this result is already stated in Section 1.2 of [7].

LEMMA 3.5. *Let $X \subset \mathbb{P}^3$ be an irreducible cubic surface over a field $\mathbb{F}$ and ℓ, m be two skew lines on X. Assume that ℓ is not a double line. Then the number of weak Eckardt points on ℓ is at most 5 if $\mathrm{char}(\mathbb{F}) = 2$ and at most 2 otherwise.*

PROOF. Let $X = V(F)$ for a suitably chosen cubic polynomial F. Since ℓ is not a double line, at least one of the partial derivatives of F remains nonzero when restricted to ℓ. This implies that ℓ can contain at most two singular points, since the partial derivatives of F are homogeneous polynomials of degree two. In view of Lemma 3.4, we conclude that either ℓ contains up to two weak Eckardt points, or that any weak Eckardt point on ℓ actually is an Eckardt point. In the first case we are done. Therefore, from now on we assume that ℓ contains two Eckardt points P_1 and P_2 and that any additional weak Eckardt point on ℓ actually is an Eckardt point.

Reasoning as in the proof of Lemma 3.4, we may assume without loss of generality that $\ell = V(x_0, x_1)$, $m = V(x_2, x_3)$, $P_1 = [0 : 0 : 1 : 0]$, $P_2 = [0 : 0 : 0 : 1]$ and $X = V(F)$, with F of the form as in equation (3). However, since we now assume that P_2 is an Eckardt point as well, one can reason as for the point P_1 in the proof of Lemma 3.4 and obtain that $c_3 = e_{23} = 0$. Hence F is of the form

$$F = a_3 x_0^2 x_3 + b_2 x_0 x_1 x_2 + b_3 x_0 x_1 x_3 + c_2 x_1^2 x_2 + x_0 x_3^2 + x_1 x_2^2.$$

If ℓ contains a third Eckardt point $P = [0 : 0 : 1 : k]$ for some $k \in \mathbb{F} \backslash \{0\}$, then ℓ is contained in the tangent plane Π of X at P, which implies that $\Pi = V(x_1 - \lambda x_0)$ for some $\lambda \in \mathbb{F} \backslash \{0\}$. Restricting F to Π by eliminating x_1 we obtain that $F|_\Pi = x_0 Q(x_0, x_2, x_3)$, with

$$Q(x_0, x_2, x_3) := (a_3 + b_3\lambda)x_0 x_3 + (b_2\lambda + c_2\lambda^2)x_0 x_2 + x_3^2 + \lambda x_2^2.$$

Since P is an Eckardt point, $Q(x_0, x_2, x_3)$ is the product of two degree one polynomials, each having P as zero. Hence P is a singular point of $V(Q)$, implying that $k^2 + \lambda = 0$ (P a zero of Q), $(a_3 + b_3\lambda)k + (b_2\lambda + c_2\lambda^2) = 0$ (P a zero of $\frac{\partial Q}{\partial x_0}$) and $2\lambda = 0$ (P a zero of $\frac{\partial Q}{\partial x_2}$). If $\mathrm{char}(\mathbb{F}) \neq 2$, then such a P does not exist, since $\lambda \neq 0$. Otherwise, we have $(a_3 + b_3 k^2)k + (b_2 k^2 + c_2 k^4) = 0$, which can have at most three non-zero solutions for k, implying that ℓ contains at most five Eckardt points. This completes the proof. $\qquad \square$

We now state the main result of this section.

THEOREM 3.6. *Let $X \subset \mathbb{P}^3$ be a cubic surface defined over a field $\mathbb{F}$ containing two skew lines ℓ_1 and ℓ_2. Then one of the following holds:*

    (a) *X is reducible, or*

    (b) *X contains a double line, or*

    (c) *$|\mathcal{T}_{\ell_1}| \leq 5$ or $|\mathcal{T}_{\ell_2}| \leq 5$.*

PROOF. Let X be an irreducible cubic surface not containing a double line. We need to show that $|\mathcal{T}_{\ell_1}| \leq 5$ or $|\mathcal{T}_{\ell_2}| \leq 5$. If $|\mathbb{F}| \leq 4$, this is trivial. Therefore we will assume from now on that $|\mathbb{F}| \geq 5$. In view of Proposition 3.2 it is enough to show that $\mathcal{T}_{\ell_1} \neq \mathcal{B}(\ell_1)$ or $\mathcal{T}_{\ell_2} \neq \mathcal{B}(\ell_2)$. Suppose on the contrary that $\mathcal{T}_{\ell_1} = \mathcal{B}(\ell_1)$ and $\mathcal{T}_{\ell_2} = \mathcal{B}(\ell_2)$.

Any plane $\Pi \in \mathcal{B}(\ell_1)$ intersects the line ℓ_2 in exactly one point, which we will denote by P_Π. Since $\mathcal{T}_{\ell_1} = \mathcal{B}(\ell_1)$, for each $\Pi \in \mathcal{B}(\ell_1)$, there exists a line $m_\Pi \subset \Pi$ such that $P_\Pi \in m_\Pi \subset X$. Since ℓ_1 and ℓ_2 are skew, the lines m_Π intersect the line ℓ_1 in exactly one point. For a given point P on ℓ_1, at most two of the lines m_Π may contain P. Indeed, such lines are contained in the intersection of X and the plane containing P and ℓ_2, which contains two lines apart from ℓ_2. Since by assumption $|\mathcal{B}(\ell_1)| = |\mathbb{F}| + 1 \geq 6$, this implies that we can find three mutually skew lines m_1, m_2, and m_3 contained in X meeting each of the lines ℓ_1 and ℓ_2 in exactly one point.

It is well known that there exists a unique, smooth quadric Q containing the lines m_1, m_2, and m_3. Moreover, by construction of the lines m_1, m_2, m_3 we have $|Q \cap \ell_i| \geq 3$ for $i = 1, 2$, implying that both ℓ_1 and ℓ_2 are contained in Q. Since X does not contain a double line, Proposition 2.10 (i.e., [**12**, Cor.18.5 and 18.6]) implies that $Q \cap X$ contains a sixth line ℓ_3. We claim that the lines ℓ_1, ℓ_2, and ℓ_3 are mutually skew. Indeed, if ℓ_1 and ℓ_3 would intersect, there would exist a plane $\Pi \in \mathcal{B}(\ell_2)$ containing $\ell_1 \cap \ell_3$. Then $\Pi \cap Q$ consists of two lines: ℓ_2 and a line n containing the point $\ell_1 \cap \ell_3$. Then there would exist three distinct lines contained in Q, each passing through the point $\ell_1 \cap \ell_3$. However, since Q is smooth, this implies that the tangent plane of Q at $\ell_1 \cap \ell_3$ intersects Q in at least the lines ℓ_1, ℓ_3, and n. This in turn shows that Q contains the tangent plane entirely, which gives a contradiction. Hence ℓ_1 and ℓ_3 are skew lines. A similar argument shows that ℓ_2 and ℓ_3 are skew.

Now consider a point P on ℓ_3. The plane passing through P and ℓ_i $(i = 1, 2)$, contains a line $n_{i,P} \subset X$ passing through P. Clearly $n_{i,P}$ is not one of the lines ℓ_1, ℓ_2, or ℓ_3, since these three lines are mutually skew, while $n_{i,P}$ intersects ℓ_3 at P. If $n_{1,P} = n_{2,P}$, then this line intersects all of the lines ℓ_1, ℓ_2, and ℓ_3. But then $n_{1,P}$ intersects Q in at least three points and therefore is contained in Q. Since $n_{1,P}$ is contained in X as well and $X \cap Q = \ell_1 \cup \ell_2 \cup \ell_3 \cup m_1 \cup m_2 \cup m_3$, we conclude that if $n_{1,P} = n_{2,P}$, then $n_{1,P} \in \{m_1, m_2, m_3\}$. In particular $n_{1,P} = n_{2,P}$ can occur for at most three points P on ℓ_3. For the remaining points P on ℓ_3, we find at least three lines contained in X passing through P, namely $n_{1,P}, n_{2,P}$, and ℓ_3. Hence ℓ_3 contains at least $|\ell_3| - 3 = |\mathbb{F}| - 2$ weak Eckardt points. Since $|\mathbb{F}| \geq 5$, we directly find a contradiction in odd characteristic using Lemma 3.5. If $\mathrm{char}(\mathbb{F}) = 2$, the assumption that $|\mathbb{F}| \geq 5$, implies that $|\mathbb{F}| \geq 8$. Hence ℓ_3 contains at least 6 Eckardt points in this case. Again Lemma 3.5 gives a contradiction. $\square$

One can in fact show that in case (c) $|\mathcal{T}_{\ell_1}| \leq 5$ and $|\mathcal{T}_{\ell_2}| \leq 5$, but the current formulation is strong enough for our purposes.

4. General results towards a proof of Sørensen's conjecture

The results in this section are oriented towards an attempt to resolve Sørensen's conjecture in the general case. We show that by studying the incidence structures of lines and planes in $\mathbb{P}^3(\mathbb{F}_{q^2})$ with respect to the surface of degree d in question, can lead to significant progress towards proving Conjecture 1. First, we mention that throughout this section F will denote a nonzero homogeneous polynomial of degree d, where $2 \leq d \leq q$ in $\mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$. For a plane Π, which we always assume to be defined over $\mathbb{F}_{q^2}$, we derive various upper bounds on

$|\mathsf{V}(F) \cap V_2 \cap \Pi|$ depending on the line arrangements on $V(F)$ in Π. Also when considering a line contained in Π, we will always mean a line defined over $\mathbb{F}_{q^2}$.

LEMMA 4.1. *Let Π (resp. $\ell \subset \Pi$) be tangent to (resp. a generator of) V_2 and suppose that Π is not contained in $V(F)$. Then we have the following.*

(a) *If $\ell \subset V(F)$, then*

$$|\mathsf{V}(F) \cap V_2 \cap \Pi| \leq dq^2 + 1 \quad \text{and} \quad |\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| \leq (d-1)q^2.$$

(b) *If $\ell \subset V(F)$ and ℓ is the only generator of V_2 contained in $V(F) \cap \Pi$, then*

$$|\mathsf{V}(F) \cap V_2 \cap \Pi| \leq q^2 + (d-1)q + 1 \quad \text{and} \quad |\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| \leq (d-1)q.$$

(c) *If $V(F) \cap \Pi$ does not contain any generators of V_2, then*

$$|\mathsf{V}(F) \cap V_2 \cap \Pi| \leq d(q+1) \quad \text{and} \quad |\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| \leq dq.$$

PROOF. We begin by noting that $F|_\Pi \neq 0$.

(a) Suppose that Π is a tangent to V_2 at a point P. By Theorem 2.4, we know that Π contains exactly $q+1$ generators each passing through P. Since ℓ is one of the $q+1$ generators mentioned, clearly $P \in \ell$. Theorem 2.14 implies that $|\mathsf{V}(F) \cap V_2 \cap \Pi| \leq dq^2 + 1$ and moreover this upper bound is attained if and only if $P \in \mathsf{V}(F)$ and $V(F)$ contains d of the $q+1$ generators passing through P. Further $|\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| \leq (d-1)q^2$.

(b) If moreover, ℓ is the only generator in the plane Π that is contained in $V(F)$, then for any generator $\ell' \subset \Pi$ with $\ell \neq \ell'$, we have $|\mathsf{V}(F) \cap \ell'| \leq d$. This implies that $|\mathsf{V}(F) \cap V_2 \cap (\ell' \setminus \{P\})| \leq d-1$. We have thus proved that $|\mathsf{V}(F) \cap V_2 \cap \Pi| \leq q^2 + (d-1)q + 1$. We also deduce that, in the case when $V(F)$ contains only one generator ℓ in Π, then $|\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| \leq (d-1)q$.

(c) Since, for any generator $\ell \subset \Pi$, we have $|\mathsf{V}(F) \cap \ell| \leq d$, we obtain $|\mathsf{V}(F) \cap V_2 \cap \Pi| \leq d(q+1)$. We further note that, if ℓ is a generator of V_2 contained in Π, then $V_2 \cap (\Pi \setminus \ell)$ is an affine curve of degree q, whereas $V(F) \cap (\Pi \setminus \ell)$ is an affine curve of degree d. Since they have no common components, we deduce from Bézout's theorem (or Proposition 2.13) that $|\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| \leq dq$.

$\square$

LEMMA 4.2. *Let Π be a plane that is not a tangent to V_2 and ℓ be any line contained in Π. If $\Pi \not\subset V(F)$ then $|\mathsf{V}(F) \cap V_2 \cap \Pi| \leq d(q+1)$. Further, if $\ell \subseteq V(F)$ then $|\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| \leq (d-1)(q+1)$.*

PROOF. First, we note that $F|_\Pi \neq 0$. Theorem 2.4 implies that $V_2 \cap \Pi$ is a non-degenerate Hermitian curve and hence is irreducible, as noted in Section 2. Again, by applying Proposition 2.13 we see that, $|\mathsf{V}(F) \cap V_2 \cap \Pi| \leq d(q+1)$. Now suppose that $V(F)$ contains a line $\ell \subseteq \Pi$. In this case $V(F) \cap (\Pi \setminus \ell)$ is an affine curve of degree $d-1$ and from Proposition 2.13 we deduce that $|\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| \leq (d-1)(q+1)$. $\square$

We now make use of Lemma 4.1 and 4.2 to derive various upper bounds for $|\mathsf{V}(F) \cap V_2|$.

LEMMA 4.3. *Suppose that $q > 2$ and that $V(F)$ contains no generators of V_2. Then*

$$|\mathsf{V}(F) \cap V_2| \leq d(q^3 + q + 1) < dq^3 + (d-1)q^2 + 1.$$

PROOF. Evidently, if ℓ is a generator of V_2, then $|\mathsf{V}(F) \cap \ell| \leq d$. For any $\Pi \in \mathcal{B}(\ell)$, as noted in Lemma 4.1(c), we have $|\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| \leq dq$. Hence,

$$|\mathsf{V}(F) \cap V_2| \leq \sum_{\Pi \in \mathcal{B}(\ell)} |\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| + |\mathsf{V}(F) \cap V_2 \cap \ell|$$
$$\leq dq(q^2 + 1) + d$$
$$= dq^3 + dq + d < dq^3 + (d-1)q^2 + 1.$$

This completes the proof. □

REMARK 4.4. Note that in the situation of Lemma 4.3, the set $V(F) \cap V_2$ is an algebraic curve of degree at most $d(q+1)$ not containing any lines. For any q (including $q = 2$), the bound $|\mathsf{V}(F) \cap V_2| \leq dq^3 + (d-1)q^2 + 1$ then follows directly by applying a bound on the number of rational points on a curve containing no lines due to Homma [**13**, Theorem 1.1]. Lemma 4.3 shows that, in this case, apparently a better bound on $|\mathsf{V}(F) \cap V_2|$ is possible for $q > 2$.

LEMMA 4.5. *Suppose that $V(F)$ contains a generator ℓ of V_2, but contains no two skew generators. Further assume that $V(F)$ does not contain any plane. Then*

$$|\mathsf{V}(F) \cap V_2| \leq (d-1)q^3 + dq^2 + 1 < dq^3 + (d-1)q^2 + 1.$$

PROOF. Since $V(F)$ does not contain any plane, it is evident that $F|_\Pi$ is a nonzero polynomial for any plane in $\mathbb{P}^3(\mathbb{F}_{q^2})$. The proof is divided into two cases.

Case 1: Suppose that $V(F)$ contains another generator ℓ'. By hypothesis, the generators ℓ and ℓ' are contained in a plane Π. Let $\Pi' \in \mathcal{B}(\ell)$ with $\Pi' \neq \Pi$ and suppose that ℓ_1 is a generator of V_2 contained in Π'. If $\ell_1 \neq \ell$, then it follows from Theorem 2.4 and Proposition 2.6 (a) that ℓ' and ℓ_1 are skew lines. By the hypothesis on $V(F)$, we have $\ell_1 \not\subset V(F)$. Thus, using Lemma 4.1 (a) and (b), we have $|\mathsf{V}(F) \cap V_2 \cap \Pi| \leq dq^2 + 1$ and $|\mathsf{V}(F) \cap V_2 \cap (\Pi' \setminus \ell)| \leq (d-1)q$. This proves that,

$$|\mathsf{V}(F) \cap V_2| = |\mathsf{V}(F) \cap V_2 \cap \Pi| + \sum_{\Pi' \neq \Pi} |\mathsf{V}(F) \cap V_2 \cap (\Pi' \setminus \ell)|$$
$$\leq dq^2 + 1 + q^2(d-1)q$$
$$= (d-1)q^3 + dq^2 + 1 < dq^3 + (d-1)q^2 + 1.$$

Case 2: Suppose that $V(F)$ contains no other generators. Using Lemma 4.1 (b), we deduce that $|\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| \leq (d-1)q$ for any $\Pi \in \mathcal{B}(\ell)$. Hence,

$$|\mathsf{V}(F) \cap V_2| \leq \sum_{\Pi \in \mathcal{B}(\ell)} |\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| + |\ell|$$
$$\leq (q^2 + 1)(d-1)q + q^2 + 1$$
$$= (d-1)q^3 + q^2 + (d-1)q + 1$$
$$= (dq^3 + (d-1)q^2 + 1) - (q^3 + (d-2)q^2 - (d-1)q) < dq^3 + (d-1)q^2 + 1.$$

This completes the proof. □

Having investigated cases where $V(F)$ does not contain a plane, in the remainder of this section we move our attention to cases where $V(F)$ contains at least one plane in $\mathbb{P}^3(\mathbb{F}_{q^2})$.

LEMMA 4.6. *Suppose that $V(F)$ contains a plane which is not tangent to V_2. If Conjecture 1 is true for polynomials of degree at most $d - 1$, then $|\mathsf{V}(F) \cap V_2| \leq dq^3 + (d-1)q^2 + 1$. Further, for $d > 2$ we have $|\mathsf{V}(F) \cap V_2| \leq dq^3 + (d-1)q^2 - (d-2)q + 2 < dq^3 + (d-1)q^2 + 1$.*

PROOF. **Case 1:** Let $d = 2$. Then $V(F)$ is a union of two planes, say Π_1 and Π_2. We assume without loss of generality that, Π_1 is a plane not tangent to V_2. From Theorem 2.4, we see that $|V_2 \cap \Pi_1| = q^3 + 1$, while $|V_2 \cap \Pi_2| \leq q^3 + q^2 + 1$. Further, from Lemma 2.3 we have, $|V_2 \cap \Pi_1 \cap \Pi_2| \geq 1$. This shows that, $|\mathsf{V}(F) \cap V_2| \leq 2q^3 + q^2 + 1$.

Case 2: Assume that $d \geq 3$. We may write $F = HG$, where $H, G \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$, with $\deg H = 1$ and $\deg G \geq 2$. Further, $V(H)$ is a not a tangent to V_2. Theorem 2.4 shows that $|\mathsf{V}(H) \cap V_2| = q^3 + 1$. Further, since Conjecture 1 is assumed to be true for polynomials of degree at most $d - 1$, we see that $|\mathsf{V}(G) \cap V_2| \leq (d-1)(q^3 + q^2 - q) + q + 1$. Consequently,

$$\begin{aligned}
|\mathsf{V}(F) \cap V_2| &\leq |\mathsf{V}(G) \cap V_2| + |\mathsf{V}(H) \cap V_2| \\
&\leq (d-1)(q^3 + q^2 - q) + q + 1 + q^3 + 1 \\
&= dq^3 + (d-1)q^2 - (d-2)q + 2 \\
&= dq^3 + (d-1)q^2 + 1 - ((d-2)q - 1) < dq^3 + (d-1)q^2 + 1.
\end{aligned}$$

This completes the proof. $\qquad \square$

LEMMA 4.7. *Suppose that, $V(F)$ contains a generator ℓ and at least two planes in $\mathcal{B}(\ell)$. Further assume that the conjecture is true for polynomials of degree at most $d - 1$. Then*

$$|\mathsf{V}(F) \cap V_2| \leq dq^3 + (d-1)q^2 + 1.$$

PROOF. Let $\Pi_1, \ldots, \Pi_s \in \mathcal{B}(\ell)$ such that $\Pi_k \subseteq V(F)$ for all $k = 1, \ldots, s$. Since, by assumption, $V(F)$ contains at least two planes from $\mathcal{B}(\ell)$, we have $2 \leq s \leq d$. First assume that $s = d$. Then $V(F) = \Pi_1 \cup \cdots \cup \Pi_d$ and consequently, $|\mathsf{V}(F) \cap V_2| = dq^3 + q^2 + 1$. So from now on we may assume that $s < d$.

Case 1: Let $(d, s) = (3, 2)$. We may write $V(F) = \Pi_1 \cup \Pi_2 \cup \Pi_3$, where $\Pi_1 \cap \Pi_2 = \ell$ and $\ell \not\subset \Pi_3$. We note that $|(\Pi_1 \cup \Pi_2) \cap V_2| = 2q^3 + q^2 + 1$, whereas from Theorem 2.4 we see that $|\Pi_3 \cap V_2| \leq q^3 + q^2 + 1$. Since, any line intersects V_2 in at least one point (see Lemma 2.2), we see that $|(\Pi_1 \cup \Pi_2) \cap \Pi_3 \cap V_2| \geq 1$. This implies that,

$$\begin{aligned}
|\mathsf{V}(F) \cap V_2| &= |(\Pi_1 \cup \Pi_2) \cap V_2| + |\Pi_3 \cap V_2| - |(\Pi_1 \cup \Pi_2) \cap \Pi_3 \cap V_2| \\
&\leq 2q^3 + q^2 + 1 + q^3 + q^2 + 1 - 1 \\
&= 3q^3 + 2q^2 + 1.
\end{aligned}$$

Case 2: Let $(d, s) \neq (3, 2)$. We may write $F = H_1 \ldots H_s G$, where $H_1, \ldots, H_s, G \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ with $\deg H_1 = \cdots = \deg H_s = 1$ and $\deg G = d - s$. Further, we write that $\Pi_1 = V(H_1), \ldots, \Pi_s = V(H_s)$ and note that $\ell = \Pi_1 \cap \cdots \cap \Pi_s$. We have,

$$|(\Pi_1 \cup \cdots \cup \Pi_s) \cap V_2| = sq^3 + q^2 + 1 \quad \text{and} \quad |\mathsf{V}(G) \cap V_2| \leq (d-s)(q^3 + q^2 - q) + q + 1.$$

While the first assertion above follows trivially, the second one is a direct consequence of the hypothesis that the conjecture is true for polynomials of degree at most $d - 1$. This implies that

$$\begin{aligned}
|\mathsf{V}(F) \cap V_2| &\leq |(\Pi_1 \cup \cdots \cup \Pi_s) \cap V_2| + |\mathsf{V}(G) \cap V_2| \\
&\leq sq^3 + q^2 + 1 + (d-s)(q^3 + q^2 - q) + q + 1 \\
&= dq^3 + (d-s+1)q^2 - (d-s-1)q + 2 \\
&= dq^3 + (d-1)q^2 + 1 + \left(-(s-2)q^2 - (d-s-1)q + 1\right) \leq dq^3 + (d-1)q^2 + 1.
\end{aligned}$$

Note that for $2 \leq s \leq d - 1$ the quantity $(s-2)q^2 + (d-s-1)q$ is nonnegative and zero if and only if $d = 3$ and $s = 2$. The last inequality now follows since $(d, s) \neq (3, 2)$. $\qquad \square$

REMARK 4.8. To prove Sørensen's conjecture for $d = 2$, Edoukou has made use of the classification of quadric surfaces and proved the conjecture for each class of quadrics. We remark that, if we use the results in this section then the classification of quadric surfaces will not be needed anymore. Indeed, if a quadric surface is reducible over $\mathbb{F}_{q^2}$, then it is given by union of two planes. In the case when one of the planes is not tangent to V_2, we can apply Lemma 4.6 with the fact that the conjecture is true for $d = 1$, to get the desired inequality. The other case of a reducible quadric surface occurs if the surface is union of two tangent planes. As already noted in Corollary 2.7, the two tangent planes can intersect at a secant or at a generator. In the case when they intersect at a secant the upper bound in Sørensen's conjecture is attained (see Proposition 2.8) while in the latter case, the desired inequality is easily derived from Lemma 4.7. This leads us to the case when the quadric surface is irreducible over $\mathbb{F}_{q^2}$. For irreducible quadrics containing no generators or containing a generator but no two skew generators, we could obtain the desired inequalities by using Lemma 4.3 and 4.5 respectively. Finally, after a linear change of variables, any quadric containing two skew lines is given by an equation of the form $x_0 x_1 + x_2 x_3 = 0$, which is a hyperbolic quadric. This case was proved in §5.2.1 in [**8**]. Although this proof does not necessarily shorten or essentially simplify the proof of Edoukou's theorem (since, after all, the most nontrivial part of his proof lies in §5.2.1 of [**8**]), it may give an idea of generalizing the proof for larger values of d.

5. Proof of Sørensen's conjecture for cubic surfaces

In this section, we will make use of the results that we have derived in Sections 3 and 4 to show that Conjecture 1 is true for $d = 3$ whenever $q \geq 3$. We begin with the following:

PROPOSITION 5.1. *Let* $F \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ *be a reducible cubic. Further assume that any linear factor of* F *corresponds to a plane tangent to* V_2. *Then either*

$$|\mathsf{V}(F) \cap V_2| = 3(q^3 + q^2 - q) + q + 1$$

or

$$|\mathsf{V}(F) \cap V_2| \leq \max\{3q^3 + 2q^2 + 2, 3(q^3 + q^2 - q) + 1\} < 3(q^3 + q^2 - q) + q + 1.$$

In particular, if $q \geq 4$ *and* $|\mathsf{V}(F) \cap V_2| < 3(q^3 + q^2 - q) + q + 1$, *then* $|\mathsf{V}(F) \cap V_2| \leq 3(q^3 + q^2 - q) + 1$.

PROOF. Since F is reducible, we may write $F = HQ$, where $H, Q \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ with $\deg H = 1$ and $\deg Q = 2$. Further, by assumption, $V(H)$ is a plane that is tangent to V_2. This immediately proves that $|\mathsf{V}(H) \cap V_2| = q^3 + q^2 + 1$.

Case 1: Let Q be a homogeneous quadratic polynomial such that $|\mathsf{V}(Q) \cap V_2| < 2(q^3 + q^2 - q) + q + 1$. In this case, Theorem 2.9 implies that $|\mathsf{V}(Q) \cap V_2| \leq 2q^3 + q^2 + 1$. This shows that,

$$|\mathsf{V}(F) \cap V_2| \leq |\mathsf{V}(H) \cap V_2| + |\mathsf{V}(Q) \cap V_2| \leq 3q^3 + 2q^2 + 2 < 3(q^3 + q^2 - q) + q + 1,$$

the last inequality follows since $q \geq 3$. We remark that, if $q > 3$, then $|\mathsf{V}(F) \cap V_2| \leq 3(q^3 + q^2 - q)$.

Case 2: Let Q be a homogeneous quadratic polynomial such that $|\mathsf{V}(Q) \cap V_2| = 2(q^3 + q^2 - q) + q + 1$. Then $V(Q)$ is a union of two planes Π_1, Π_2, both tangent to V_2, such that the line $\ell = \Pi_1 \cap \Pi_2$ intersects V_2 at $q + 1$ points. We write $\Pi_0 = V(H)$. Note that, if $\ell \subseteq \Pi_0$ then Proposition 2.8 implies that $|\mathsf{V}(F) \cap V_2| = 3(q^3 + q^2 - q) + q + 1$. We may thus assume that $\ell \nsubseteq \Pi_0$. If Π_0 intersects Π_1 or Π_2 at a generator, then Lemma 4.7 applies (since Conjecture 1 is true for $d = 2$) and the proposition follows. Thus, in view of Corollary 2.7, it is enough to prove the proposition in the case when

$$(4) \qquad |\Pi_0 \cap \Pi_1 \cap V_2| = q + 1 \quad \text{and} \quad |\Pi_0 \cap \Pi_2 \cap V_2| = q + 1.$$

Applying the above conditions and using inclusion-exclusion principle, we have

$$|\mathsf{V}(F) \cap V_2| = \sum_{i=0}^{2} |\Pi_i \cap V_2| - \sum_{i<j} |\Pi_i \cap \Pi_j \cap V_2| + |\Pi_0 \cap \Pi_1 \cap \Pi_2 \cap V_2|$$
$$\leq 3(q^3 + q^2 + 1) - 3(q+1) + 1$$
$$= 3(q^3 + q^2 - q) + 1.$$

This completes the proof. $\qquad\square$

In Section 3 we came across a case where the cubic surface in question may contain a double line. The following Lemma shows that the upper bound of Conjecture 1 holds in this case.

LEMMA 5.2. *Let $F \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ be an irreducible cubic. Suppose that $V(F)$ contains a double line ℓ. Then $|\mathsf{V}(F) \cap V_2| \leq 3q^3 + 2q^2 + 1$.*

PROOF. Since F is irreducible in $\mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$, for every plane $\Pi \in \mathbb{P}^3(\mathbb{F}_{q^2})$, the polynomial $F|_\Pi$ is a nonzero cubic. Since ℓ is a double line, for any $\Pi \in \mathcal{B}(\ell)$, we have $V(F) \cap \Pi = \ell \cup \ell_\Pi$ for some line ℓ_Π contained in Π.

Case 1: The line ℓ is a tangent line. Let Π denote the unique plane containing ℓ which is tangent to V_2. Then $|\mathsf{V}(F) \cap V_2 \cap \Pi| \leq q^2 + 1$ and equality holds if and only if ℓ_Π is a generator. Since $|\mathsf{V}(F) \cap V_2 \cap \ell| = 1$, we conclude that $|\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| \leq q^2$. Let $\Pi_1, \ldots, \Pi_{q^2}$ denote the remaining planes containing ℓ. It follows from Proposition 2.6 (b) that the Π_j-s are not tangent to V_2. Consequently, for each $j = 1, \ldots, q^2$, we see that $V_2 \cap \Pi_j$ is a non-degenerate Hermitian curve which intersects the line ℓ_{Π_j} at most $q+1$ points outside ℓ. In other words, $|\mathsf{V}(F) \cap V_2 \cap (\Pi_j \setminus \ell)| \leq q+1$ for each $j = 1, \ldots, q^2$. Hence,

$$|\mathsf{V}(F) \cap V_2| = |\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| + \sum_{j=1}^{q^2} |\mathsf{V}(F) \cap V_2 \cap (\Pi_j \setminus \ell)| + |\mathsf{V}(F) \cap V_2 \cap \ell|$$
$$\leq q^2 + q^2(q+1) + 1$$
$$= q^3 + 2q^2 + 1 < 3q^3 + 2q^2 + 1.$$

Case 2: The line ℓ is a secant line. Let $\Pi_1, \ldots, \Pi_{q+1}$ denote the $q+1$ planes containing ℓ that are tangent to V_2 and $\Pi_{q+2}, \ldots, \Pi_{q^2+1}$ denote the remaining planes containing ℓ that are not tangent to V_2. For $j = 1, \ldots, q+1$, we claim that $|\mathsf{V}(F) \cap V_2 \cap (\Pi_j \setminus \ell)| \leq q^2$. The assertion follows trivially if ℓ_{Π_j} is not a generator, since ℓ_{Π_j} intersects each of the $q+1$ generators in Π_j at a single point. In the case when ℓ_{Π_j} is a generator, we have $\mathsf{V}(F) \cap V_2 \cap \ell_{\Pi_j} = \ell_{\Pi_j}$ and the point of intersection of ℓ and ℓ_{Π_j} belongs to V_2. This proves the claim. Moreover, using a similar argument as in Case 1, we see that $|\mathsf{V}(F) \cap V_2 \cap (\Pi_j \setminus \ell)| \leq q+1$ for $j = q+2, \ldots, q^2+1$. Thus,

$$|\mathsf{V}(F) \cap V_2| = \sum_{j=1}^{q+1} |\mathsf{V}(F) \cap V_2 \cap (\Pi_j \setminus \ell)| + \sum_{j=q+2}^{q^2+1} |\mathsf{V}(F) \cap V_2 \cap (\Pi_j \setminus \ell)| + |\mathsf{V}(F) \cap V_2 \cap \ell|$$
$$\leq (q+1)q^2 + (q^2 - q)(q+1) + q + 1$$
$$= (q+1)(2q^2 - q + 1) = 2q^3 + q^2 + 1 < 3q^3 + 2q^2 + 1.$$

Case 3: The line ℓ is a generator. First assume that there exists $\Pi \in \mathcal{B}(\ell)$ such that ℓ_Π is also a generator. Then

$$|\mathsf{V}(F) \cap V_2 \cap \Pi| = \begin{cases} 2q^2 + 1, & \text{if } \ell \neq \ell_\Pi \\ q^2 + 1, & \text{if } \ell = \ell_\Pi \end{cases}$$

Since ℓ is an irreducible component of $V(F) \cap V_2$ and $V(F)$ is singular at every point on ℓ, Proposition 2.11 implies that $V(F) \cap V_2$ is an equidimensional variety of dimension 1 and degree at most $3(q+1) - 1$. Since Π contains two components, namely ℓ and ℓ_Π, of $V(F) \cap V_2$ of degree 1 each, we see that $V(F) \cap V_2 \cap \Pi^C$ is an affine variety of degree at most $3q$ if $\ell \neq \ell_\Pi$ and at most $3q + 1$ if $\ell = \ell_\Pi$. Further, the equidimensionality of $V(F) \cap V_2$ readily implies the equidimensionality of $V(F) \cap V_2 \cap \Pi^C$ as an affine variety. Applying Proposition 2.13, we obtain

$$|\mathsf{V}(F) \cap V_2 \cap \Pi^C| \leq \begin{cases} 3q^3, & \text{if } \ell \neq \ell_\Pi \\ 3q^3 + q^2 & \text{if } \ell = \ell_\Pi \end{cases}$$

Thus,

$$|\mathsf{V}(F) \cap V_2| = |\mathsf{V}(F) \cap V_2 \cap \Pi| + |\mathsf{V}(F) \cap V_2 \cap \Pi^C| \leq 3q^3 + 2q^2 + 1.$$

Now suppose that for each $\Pi \in \mathcal{B}(\ell)$, the line ℓ_Π is either a tangent or a secant to V_2. Then

$$|\mathsf{V}(F) \cap V_2| = |\ell| + \sum_{\Pi \in \mathcal{B}(\ell)} |V_2 \cap \ell_\Pi| \leq q^2 + 1 + (q^2 + 1)(q+1) = q^3 + 2q^2 + q + 2 < 3q^3 + 2q^2 + 1.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

We are now ready to state and prove the main theorem of this paper.

THEOREM 5.3. *Let $q \geq 3$. For any homogeneous cubic $F \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ we have* $|\mathsf{V}(F) \cap V_2| \leq 3(q^3 + q^2 - q) + q + 1$. *Moreover, if $q \geq 4$, then*

$$|\mathsf{V}(F) \cap V_2| = 3(q^3 + q^2 - q) + q + 1 \quad \text{or} \quad |\mathsf{V}(F) \cap V_2| \leq 3(q^3 + q^2 - q) + 1.$$

PROOF. **Case 1:** F is reducible. If $V(F)$ contains a plane not tangent to V_2, Lemma 4.6 applies and shows that $|\mathsf{V}(F) \cap V_2| < 3q^3 + 2q^2 + 1$. Otherwise, Proposition 5.1 applies.

Case 2: F is irreducible. If $V(F)$ contains no generator, then Lemma 4.3 shows that $|\mathsf{V}(F) \cap V_2| < 3q^3 + 2q^2 + 1$. So we may assume that $V(F)$ contains a generator. If $V(F)$ contains no two skew generators then from Lemma 4.5 we see that $|\mathsf{V}(F) \cap V_2| < 3q^3 + 2q^2 + 1$. This leads us to investigate the case when $V(F)$ contains two skew generators. From the results of Section 3, we see that such cubics satisfy one of the following two properties:

(a) $V(F)$ contains a double line.
(b) There exists a generator ℓ contained in $V(F)$, such that $|\mathcal{T}_\ell| \leq 5$.

For (a), Lemma 5.2 applies and shows that $|\mathsf{V}(F) \cap V_2| \leq 3q^3 + 2q^2 + 1$. Now suppose that (b) holds. Let $\Pi_1, \ldots, \Pi_s$ be s planes containing ℓ such that $V(F) \cap \Pi_j$ is a union of lines. Further let $0 \leq t \leq s$ be such that $V(F) \cap \Pi_j$ is a union of lines forming a triangle for $1 \leq j \leq t$, that is to say that the three lines do not share a common point. Then for $t+1 \leq j \leq s$, the intersection $V(F) \cap \Pi_j$ is a union of three lines passing through a common point P_j on ℓ. This implies that P_j is a weak Eckardt point for $t+1 \leq j \leq s$. We have

$$|\mathsf{V}(F) \cap V_2 \cap (\Pi_j \setminus \ell)| \leq q^2 + q - 1 \text{ for } 1 \leq j \leq t$$

and

$$(5) \qquad\qquad |\mathsf{V}(F) \cap V_2 \cap (\Pi_j \setminus \ell)| \leq 2q^2 \text{ for } t+1 \leq j \leq s.$$

Finally using Lemma 4.1(b) we obtain, $|\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)| \leq 2q$ for the remaining planes $\Pi \in \mathcal{B}(\ell)$. In this case $V(F) \cap \Pi$ is a union of ℓ and an irreducible conic.

This shows that,

$$|\mathsf{V}(F) \cap V_2| = |\ell| + \sum_{i=1}^{s} |\mathsf{V}(F) \cap V_2 \cap (\Pi_j \setminus \ell)| + \sum_{\substack{\Pi \in \mathcal{B}(\ell) \\ \Pi \neq \Pi_j}} |\mathsf{V}(F) \cap V_2 \cap (\Pi \setminus \ell)|$$

$$\leq q^2 + 1 + t(q^2 + q - 1) + 2(s - t)q^2 + (q^2 + 1 - s)2q$$

(6) $$= 2q^3 + (2s - t + 1)q^2 - (2s - t - 2)q + 1 - t.$$

Since $0 \leq t \leq s \leq 5$, a simple calculation shows that if $q \geq 8$, then $2q^3 + (2s - t + 1)q^2 - (2s - t - 2)q + 1 - t < 3(q^3 + q^2 - q)$. Note that that $s - t \leq 2$ in odd characteristic by Lemma 3.5. Therefore, we obtain that for $q \geq 5$ and odd: $2q^3 + (2s - t + 1)q^2 - (2s - t - 2)q + 1 - t < 3(q^3 + q^2 - q)$.

Now we look at the remaining cases $q = 3$ and $q = 4$. Note that equality holds in equation (5) if and only if all lines in $V(F) \cap \Pi_j$ are generators of V_2. If this is not the case, then the estimate can be improved to $|\mathsf{V}(F) \cap V_2 \cap (\Pi_j \setminus \ell)| \leq q^2 + q$. Hence, if none of the planes $\Pi_{t+1}, \ldots, \Pi_s$ intersect X in three generators, then the estimate in (6) can be improved to

$$|\mathsf{V}(F) \cap V_2| \leq 2q^3 + (s + 1)q^2 - (s - 2)q + 1 - t.$$

Since $s \leq 5$ and $s - t \leq 2$ if q is odd, we find for $q = 3$ and $q = 4$ that $|\mathsf{V}(F) \cap V_2| < 3(q^3 + q^2 - q)$. Therefore we may now assume that there exists a plane $\Pi \in \mathcal{B}(\ell)$ such that $V(F) \cap \Pi$ contains three generators. Let $m \neq \ell$ be such a generator. Define

$$g = |\{m \subset V(F) \text{ a generator} \mid m \neq \ell \text{ and } m \subset \Pi \text{ for some } \Pi \in \mathcal{B}(\ell)\}|.$$

It is easy to see that

$$|\mathsf{V}(F) \cap V_2| \leq q^2 + 1 + gq^2 + (2(q^2 + 1) - g)q = (q^2 + 1)(2q + 1) + g(q^2 - q).$$

This implies that $|\mathsf{V}(F) \cap V_2| \leq 3(q^3 + q^2 - q) + 1$ if $g \leq q + 2$. For $q = 4$, the inequality is strict. Therefore, we may now assume that $V(F) \cap \Pi$ contains three generators $\ell_1 := \ell$, ℓ_2 and ℓ_3 such that for all i, there exist $q + 1$ generators distinct from ℓ_1, ℓ_2 and ℓ_3, that intersect ℓ_i. These $3(q + 1)$ generators are all mutually distinct, since otherwise there would exist a triple of generators forming a triangle. However, this is not possible, since co-planar generators intersect in a common point. Therefore, counting ℓ_1, ℓ_2, ℓ_3 as well, we have found in total $3(q + 2)$ distinct generators contained in $V(F) \cap V_2$. However, this is not possible, since the degree of $V(F) \cap V_2$ is at most $3(q + 1)$. This completes the proof. $\qquad \square$

COROLLARY 5.4. *Let $q \geq 3$. The cubic surfaces defined over $\mathbb{F}_{q^2}$ that attain the upper bound in Conjecture 1 are given by a union of three planes tangent to V_2 containing a common line which intersects V_2 at precisely $q + 1$ points.*

PROOF. Follows immediately from the proofs of Proposition 5.1 and Theorem 5.3. $\qquad \square$

Note that, Theorem 5.3 and Corollary 5.4 prove Conjecture 1 for $d = 3$. For $q \geq 4$ analyzing the proofs of Proposition 5.1 and Theorem 5.3 actually yields a classification of all the homogeneous cubics $F \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ such that $|\mathsf{V}(F) \cap V_2| = 3(q^3 + q^2 - q) + 1$.

COROLLARY 5.5. *Let $q \geq 4$. Let $F \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ be a homogeneous cubic such that $|\mathsf{V}(F) \cap V_2| = 3(q^3 + q^2 - q) + 1$. Then $V(F)$ is a union of three distinct planes Π_1, Π_2, Π_3 tangent to V_2 satisfying*

    (a) *For $i \neq j$, the line $\ell_{ij} = \Pi_i \cap \Pi_j$ is a secant.*
    (b) *$|\Pi_1 \cap \Pi_2 \cap \Pi_3 \cap V_2| = 1$.*

The following theorem guarantees the existence of a homogeneous cubic in $\mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ satisfying the assertions of Corollary 5.5.

THEOREM 5.6. *There exist three tangent planes Π_1, Π_2, Π_3 to V_2 such that*

(a) *For $i \neq j$, the line $\ell_{ij} = \Pi_i \cap \Pi_j$ is a secant.*
(b) $|\Pi_1 \cap \Pi_2 \cap \Pi_3 \cap V_2| = 1.$

In particular, there exists a homogeneous cubic $F \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ such that $|\mathsf{V}(F) \cap V_2| = 3(q^3 + q^2 - q) + 1$. Consequently, the second highest number of points of intersection of cubic surfaces defined over $\mathbb{F}_{q^2}$ and V_2 is given by $3(q^3 + q^2 - q) + 1$.

PROOF. Let $P \in V_2$ and Π be tangent to V_2 at P. As noted in Theorem 2.4, the plane Π intersects V_2 at $q + 1$ generators passing through P. We fix three such generators ℓ_1, ℓ_2, ℓ_3 and a line ℓ passing through P that is not contained in Π. Remark 2.5 shows that the line ℓ is a secant. Choose Π_1 to be the unique plane containing ℓ and ℓ_1 and Π_2 the unique plane containing ℓ and ℓ_2. Clearly, $\Pi_1 \cap \Pi_2 = \ell$ and we define $\ell_{12} := \ell$. Since Π_1, Π_2 contain the generators ℓ_1, ℓ_2 respectively, they are tangent to V_2 at some point other than P. Fix a line ℓ_{13} contained in Π_1 passing through P with the property that $\ell_{13} \neq \ell_1$. Since ℓ_{13} does not lie completely in Π, it is a secant line (see Remark 2.5). Define Π_3 to be the unique plane that contains ℓ_{13} and ℓ_3. Clearly $\ell_{13} = \Pi_1 \cap \Pi_3$. Since Π_3 contains a generator ℓ_3 it is tangent to V_2 but since $\ell_{13} \not\subset \Pi$ we see that $\Pi_3 \neq \Pi$. Further, it is clear that Π_1, Π_2, Π_3 are distinct. We define $\ell_{23} = \Pi_2 \cap \Pi_3$. Note that $P \in \ell_{23}$ since $P \in \Pi_2$ and $P \in \Pi_3$. Since Π_2, Π_3 are tangent planes, it follows from Corollary 2.7 (a) that ℓ_{23} could either be a generator or a secant. If ℓ_{23} would be a generator, then Π_3 would contain two generators, namely ℓ_3 and ℓ_{23}, both passing through P, which would imply that $\Pi_3 = \Pi$, leading to a contradiction. Thus ℓ_{23} is a secant line. Hence Π_1, Π_2, Π_3 satisfy assertion (a) and since $\{P\} = \Pi_1 \cap \Pi_2 \cap \Pi_3 \cap V_2$ the second assertion follows as well.

Clearly, the planes Π_1, Π_2, Π_3 are defined over $\mathbb{F}_{q^2}$ and therefore are given by the zero set of linear polynomials $H_1, H_2, H_3 \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$. Take $F = H_1 H_2 H_3$. Note that,

$$
\begin{aligned}
|\mathsf{V}(F) \cap V_2| &= \sum_{i=1}^{3} |\Pi_i \cap V_2| - \sum_{i<j} |\Pi_i \cap \Pi_j \cap V_2| + |\Pi_1 \cap \Pi_2 \cap \Pi_3 \cap V_2| \\
&= 3(q^3 + q^2 + 1) - 3(q + 1) + 1 = 3(q^3 + q^2 - q) + 1.
\end{aligned}
$$

This completes the proof. $\qquad\square$

In [**9**] several conjectures were made related to the intersection of a hypersurface of degree d and a Hermitian variety. Specialized to the case of non-degenerate Hermitian surfaces, Conjecture 2(i) in [**9**] can be rephrased as follows.

CONJECTURE 2. [**9**, Conjecture 2. (i)] *Let w_i ($1 \leq i \leq 2d+1$) denote the $2d+1$ highest possible values of $|\mathsf{V}(F) \cap V_2|$, if $F \in \mathbb{F}_{q^2}[x_0, x_1, x_2, x_3]$ is a homogeneous polynomial of degree d. Then for each i there exist linear homogeneous polynomials $H_1, \ldots, H_d$ such that $V(H_1), \ldots, V(H_d)$ contain a common line and $|\mathsf{V}(H_1 \cdots H_d) \cap V_2| = w_i$.*

PROPOSITION 5.7. *If $d = 3$ and $q \geq 4$ then Conjecture 2 is false.*

PROOF. Let $q \geq 4$. By Theorem 5.6 the second highest number of points that a cubic surface and V_2 have in common is given by $3(q^3 + q^2 - q) + 1$. Now, let Π_1, Π_2, Π_3 be three planes in $\mathbb{P}^3(\mathbb{F}_{q^2})$ containing a common line ℓ. Take $X = \Pi_1 \cup \Pi_2 \cup \Pi_3$. If one of the Π_j-s is not tangent to V_2, then from Lemma 4.6 we see that $|X \cap V_2| < 3q^2 + 2q^2 + 1$. Now suppose that Π_1, Π_2, Π_3 are all tangent to V_2. Then ℓ is either a secant line or a generator. If ℓ is a secant line, then by Proposition 2.8 we have $|X \cap V_2| = 3(q^3 + q^2 - q) + q + 1$. On the other hand, if ℓ is a generator, then $|X \cap V_2| = 3q^3 + q^2 + 1$. Thus such a configuration does not give rise to the desired number of points of intersections with V_2. $\qquad\square$

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