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Transverse vibrations of tightened bolts: Simplified modeling of tension-dependent boundary stiffness and damping

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Abstract

An axially pre-stressed Bernoulli-Euler beam with end linear springs and dampers is derived to model tightened bolts. The boundary stiffness and damping are analytically expressed as functions of the tensile axial force. The theoretical frequencies and damping ratios resulting from the beam model compare agreeably with the experimental data obtained from two types of bolts. An attempt to explain the apparently linear behavior of the boundary conditions is given, based on a model comprised of coupled mass-spring chains, which is reduced to a one-degree-freedom mass-spring system whose effective restoring force is sought depending on the number of mass-spring systems. The frequency response of the coupled chains and the shape of the restoring force of the reduced system indicates a nearly linear behavior of the coupled chains as the number of chains increases. The results of this work could potentially be used in a vibration based technique to estimate bolt tension from measured vibration.

1 Introduction

Bolts are widely used as a means to connect two or more parts in mechanical structures. The use of bolts is of great importance in different industries such as building, construction, automotive, etc. When a bolt is tightened, its tension causes complex interactions between asperities at connecting interfaces, which results in a change of the stiffness and damping

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on the interfaces between the bolt head/nut and the structure. The mechanism of change of the local stiffness and damping is yet not well understood [2, 5], thus generating a model for bolts is a challenging task [14]. One approach to model the contact between two surfaces is to use detailed finite element methods [13, 17]. Another approach includes using springs and dashpots [18]. In this paper we adopt the latter approach, where the bolt is modeled as an elastic beam, and the interfaces contact between the bolt head/nut and the structure as linear springs and dampers placed at the beam ends. This work is motivated by the need of better understanding boundary effects in vibration based estimation of bolt tension [4, 19].

The choice of using linear dampers to model damping in the joints is motivated by previous experimental investigations that indicated a nearly linear damping in the case of small excitation amplitudes [3]. The linear behavior of damping was also observed in buildings, where the integrated effect of a lot of different friction damping elements causes the building to respond as if it were equipped with a linear viscous damping system [22]. For the choice of linear springs, we show that when there is a collective of nonlinear springs between two surfaces, the stiffness behavior appears effectively nearly linear as more nonlinear springs are added, and the lowest mode is considered dominant.

The boundary stiffness and damping has an important effect on the dynamics and modal characteristics of jointed structure. Numerous authors have studied the effect of the boundary conditions on beams. Rao and Mirza [16] derived analytical expressions for the mode shapes and natural frequencies for a beam with two transverse and two rotational elastic springs at its end. They showed that both the transverse and rotational spring constants have a significant effect on the first three frequencies. Xing and Wang [24] also considered the same boundary conditions, but with a beam subjected to constant axial load. The natural frequencies and the mode shape functions were derived analytically, and it was shown that the flexibility of the supports had less effect on the mode shape than on the frequency. Sah et al. [19] theoretically reproduced the natural frequencies of a tightened bolt, using a beam model with elastic end supports and a tensile axial force. In the present paper, the work in [19] is extended to theoretically reproduce the measured bolt damping ratio, by adding end dampers to the beam model. For the case of a beam with both elastic and viscous end conditions, Al-Jumaily and Faulkner [1] analytically formulated the frequency equation in the complex domain. Later on Svedholm et al. [21] presented an analytical solution for the case where the beam is under a moving load, by deriving a set of orthogonality conditions.

In this paper the beam model used to model the tightened bolt has the same general boundary conditions as in [1, 19, 21], but the coefficients of both the end springs and dampers are functions of the axial tension in the beam. An axially pre-stressed Bernoulli-Euler beam model with end linear springs and dampers is derived to theoretically reproduce the experimental transverse frequency and damping ratio of two tightened bolts. Unlike previous works, e.g. [21], that considered the end springs and dampers to be independent of the applied tension, in this paper the boundary stiffness and damping are expressed as nonlinear functions of the tensile axial force. This theoretical model helps to understand the measured frequencies and damping ratios of the bolt as it is gradually tightened.

Section 2 introduces the theoretical model, Section 3 and 4 compare the theoretical

frequencies and damping ratios to the experimental data of the two bolts for varying tension, Section 5 presents an attempt to explain the linear behavior of the boundary springs, and Section 6 presents conclusions.

2 Mathematical model

The tightened bolt is modeled as a plane beam having two linear end springs (transverse $\tilde{k}_u$ and rotational $\tilde{k}_\theta$), two linear end viscous dampers (transverse $\tilde{d}_u$ and rotational $\tilde{d}_\theta$), and subjected to a tensile axial force N , see Figure 1. The equation of motion of the beam in-plane transverse deflection $\tilde{u}(\tilde{x}, \tilde{t})$ is obtained using Bernoulli-Euler theory, where $\tilde{x}$ and $\tilde{t}$ denote the longitudinal coordinate and the time, respectively. The Bernoulli-Euler equation for the pre-stressed beam has the nondimensional form [10, 23]:

$$\ddot{u} + u'''' - p u'' + d_e \dot{u} + d_i (\dot{u}'''' - p \dot{u}'') = 0, \quad (1)$$

where $u = u(x, t)$ is the nondimensional transverse deflection at time t and $x \in [0, 1]$ is the axial coordinate. The parameter p denotes the nondimensional tensile axial force, and d_e and d_i denote the nondimensional external and internal damping coefficients, respectively, the latter assuming material damping proportional to the time rate of change of the transverse restoring force $u'''' - p u''$. The boundary conditions are:

$$-u'''(0, t) + p u'(0, t) = k_u u(0, t) + d_u \dot{u}(0, t), \quad (2)$$

$$u''(0, t) = k_\theta u'(0, t) + d_\theta \dot{u}'(0, t), \quad (3)$$

$$-u'''(1, t) + p u'(1, t) = -k_u u(1, t) - d_u \dot{u}(1, t), \quad (4)$$

$$u''(1, t) = -k_\theta u'(1, t) - d_\theta \dot{u}'(1, t), \quad (5)$$

and all parameters and variables are nondimensional:

$$\begin{aligned} x &= \frac{\tilde{x}}{l}, & t &= w_0 \tilde{t}, & u &= \frac{\tilde{u}}{l}, & p &= \frac{N l^2}{EI} \\ k_u &= \frac{l^3 \tilde{k}_u}{EI}, & k_\theta &= \frac{l \tilde{k}_\theta}{EI}, & d_e &= \frac{w_0 l^4 \tilde{d}_e}{EI} \\ d_b &= w_0 \tilde{d}_e, & d_u &= \frac{w_0 l^3 \tilde{d}_u}{EI}, & d_\theta &= \frac{w_0 l \tilde{d}_\theta}{EI}, \end{aligned} \quad (6)$$

where EI and ρA are the bending stiffness and mass per unit length, respectively. In Eqs. (1)-(5) dots and primes denote differentiation w.r.t. to t and x , respectively, subscripts u and θ denote “transverse” and “rotational”, respectively. The axial coordinate x and transverse deflection u are nondimensionalized by the beam length l , which models the effective length of the bolt. Time t is nondimensionalized by the characteristic frequency $w_0 = \sqrt{EI/\rho A l^4}$. Eqs. (1)-(5) are the same as in [19], with added damping terms, and the same as in [9], with added beam damping terms.

A solution form $u(x, t) = \phi(x) e^{\lambda t}$ inserted in Eqs. (1)-(5) gives

Figure 1: Simple bolt model - a tensioned slender beam with finite boundary stiffness.

$$\phi''''(x) - p\phi''(x) + \lambda^2\phi(x) + d_e\lambda\phi(x) + d_i\lambda\left(\phi''''(x) - p\phi(x)''\right) = 0, \quad (7)$$

$$-\phi'''(0) + p\phi'(0, t) = k_u\phi(0) + d_\theta\lambda\phi(0), \quad (8)$$

$$\phi''(0) = k_\theta\phi'(0) + d_u\lambda\phi'(0), \quad (9)$$

$$-\phi'''(1) + p\phi'(1) = -k_u\phi(1) - d_u\lambda\phi(1), \quad (10)$$

$$\phi''(1) = -k_\theta\phi'(1) - d_\theta\lambda\phi'(1). \quad (11)$$

The undamped natural frequency w and the damping ratio ζ are related to λ such that:

$$\lambda = -\zeta w + i w \sqrt{1 - \zeta^2} \quad (12)$$

where the damping ratio is considered small, $\zeta \ll 1$, in this study. The undamped natural frequency can be solved for by setting the external damping, beam damping and boundary damping equal to zero. The corresponding undamped mode shape is written as the general solution to Eq. (7) with $d_e = d_i = 0$, i.e.:

$$\phi(x) = C_1 \sin(\alpha x) + C_2 \cos(\alpha x) + C_3 \sinh(\beta x) + C_4 \cosh(\beta x), \quad (13)$$

where α and β are given by [11]:

$$\alpha = \sqrt{\sqrt{\left(\frac{p}{2}\right)^2 + w^2} - \frac{p}{2}} \quad \text{and} \quad \beta = \sqrt{\sqrt{\left(\frac{p}{2}\right)^2 + w^2} + \frac{p}{2}}, \quad (14)$$

and where C_1 , C_2 , C_3 and C_4 are constants to be determined from the boundary conditions Eqs. (8)-(11), with damping coefficients set equal to zero ($d_e = d_i = d_u = d_\theta = 0$). Inserting

Eq. (13) into Eqs. (8)-(11) and solving for non-zero $\mathbf{C} = \{C_1 \ C_2 \ C_3 \ C_4\}^T$, by requiring the determinant to vanish, a frequency equation is obtained in the form:

$$g(\alpha, p, k_\theta, k_u) = 0, \quad (15)$$

where $\beta = \sqrt{\alpha^2 + p}$ (from (14)) has been substituted. Eq. (15) is then solved for each root α to produce the corresponding undamped natural frequency:

$$w = \alpha^2 \sqrt{1 + p/\alpha^2}, \quad (16)$$

from Eq. (14), as in [19].

Since the damping is small, its effect on the mode shapes is negligible. Therefore, the damping can be expressed in terms of the undamped mode shapes and boundary parameters [9]: For a specific mode ϕ , multiplying Eq. (7) by ϕ , integrating over the beam length [0,1] and then solving for λ^2 gives:

$$\lambda^2 = -\frac{\int_0^1 \phi(\phi'''' - p\phi'' + \lambda D\phi)dx}{\int_0^1 \phi^2 dx}, \quad D = d_e + d_b \left(\frac{\partial^4}{\partial x^4} - p \frac{\partial^2}{\partial x^2} \right) \quad (17)$$

The first two integral terms in Eq. (17) is integrated by parts, with boundary terms replaceable by Eqs. (8)-(11), then multiplied by the complex conjugate $\bar{\lambda}$. Inserting Eq. (12) and separating real from imaginary parts then gives:

$$w^2 = \frac{\int_0^1 (\phi''^2 + p\phi'^2) dx + k_u (\phi(0)^2 + \phi(1)^2) + k_\theta (\phi'(0)^2 + \phi'(1)^2)}{\int_0^1 \phi^2 dx} \quad (18)$$

and

$$\zeta = \frac{d_e}{2w} + \frac{w d_i}{2} + \frac{d_u (\phi(0)^2 + \phi(1)^2) + d_\theta (\phi'(0)^2 + \phi'(1)^2)}{2w \int_0^1 \phi^2 dx} \quad (19)$$

Figure 2 shows the frequency w calculated from Eq. (16), obtained directly from the root α in Eq. (15), compared to the frequency obtained using the undamped mode shape, Eq. (18). The good agreement between the two frequencies indicates the validity of the approximate expressions (18)-(19) for small damping.

The damping ratio, given by Eq. (19), could alternatively be obtained by solving the frequency equation (15) for the complex-valued roots α and β , however given that the parameters ($k_u, k_\theta, d_u, d_\theta$) are all nonlinear functions of the tension makes it a difficult task.

3 Boundary stiffness vs. tension

When the bolt is tightened the boundary stiffness and damping change with the applied pre-load. For completeness this section summarizes the main results regarding boundary stiffness given in [19] and in addition shows the undamped mode shapes.

Figure 2: The lowest beam natural frequency w as given by Eq. (16) (solid line) obtained directly from solving Eq. (15) for the root α , compared to the frequency (circles) obtained using Eq. (18). Parameters as given in the caption of Figure 3a.

The relationship between the boundary stiffness and tension is complicated, as it depends on the distribution and shape of asperities on mating surfaces. Here introduced in [19], which relates the normal (i.e. axial) stiffness k_n to the axial tension p such that:

$$k_n(p) = \frac{c_1 \tanh(c_2 p)}{1 + c_3 e^{-c_4 p}} \quad (20)$$

where c_1 , c_2 , c_3 and c_4 are positive constants. The term $\tanh(c_2 p)$ tracks the nonlinear behavior of the stiffness as the tension p is changing at low tension values, and also ensures the condition of zero axial stiffness at zero tension, while the constant c_1 controls the limit behavior for high tension. The term $c_3 \exp(-c_4 p)$ controls the transition of the stiffness between low and high tension. The rotational and transverse boundary stiffnesses are then expressed in terms of the normal stiffness as [19]:

$$k_\theta = c_\theta k_n \quad k_u = c_u k_n. \quad (21)$$

Eq. (15) then takes the form:

$$g(\alpha, p, k_u, k_\theta) = \tilde{g}(\alpha, p, c_u, c_\theta, c_1, c_2, c_3, c_4) = 0, \quad (22)$$

where $k_u = k_u(c_u, c_1, c_2, c_3, c_4)$ and $k_\theta = k_\theta(c_\theta, c_1, c_2, c_3, c_4)$, and where the parameters c_1, c_2, c_3 and c_4 are to be obtained by fitting to experimental frequency data. The α 's are obtained by numerically solving Eq. (22) for a set of values of (c_1, c_2, c_3, c_4) , and the corresponding frequency solutions, $f = w/2\pi$, are deduced using Eq. (16), and then compared to the measured frequencies f_j for every value of measured bolt tension p_j , where subscript j denotes a specific tension level. The parameters (c_1, c_2, c_3, c_4) are obtained so as to minimize, in the least square sense, the difference between the solution f and the measured frequencies f_j . The parameter c_1 controls the limit at high tension, c_2 models the behavior at low tension, and c_3 and c_4 control the transition from low to high tension.

Figure 3 shows a comparison of the dimensional frequency $\tilde{f}_1 = w_0 f_1 / 2\pi$ of the first mode obtained by solving Eq. (22) for α and using the relation (16) against the experimental frequency of two bolts M12×260 and M12×140 as a function of applied tension N . The experimental data was taken from [4]. The frequency solution with fitted parameters (blue solid line) for the first mode of both bolts agrees excellently with the measured frequencies (markers) at all tensions. The markers (circle) refer to two tests of measurements, where in each test the bolt tension is gradually increased to 70% of its yield force.

Figure 4a shows the undamped mode shapes for three tension levels in the case of the long bolt. As appears the mode shape for $N = 1.6$ kN (dotted line) resembles the case of free-free boundary conditions (i.e. zero curvature), while as the tension increases the mode shape starts corresponding that of clamped-clamped boundary conditions ($\phi = \phi' = 0$ at the boundaries), e.g the case $N = 42$ kN (solid line). Figure 4b is a zoom near the right beam end $x = 1$ of Figure 4a. The zoom reveals that the displacement and the slope of the mode shape approach zero as the tension is increased (solid line).

4 Boundary damping vs. tension

The overall damping in jointed systems mainly originates from micro-slipping interfaces at the component boundaries [15]. However, the effect of external and internal damping on the bolt shaft is expected to become dominant for higher tensions, when the relative motion between the bolt-head/nut and the structure reduces due to increased boundary stiffness. To explain the experimental data for the damping ratio [4], the boundary damping d_u and d_θ in Eq. (19) are here assumed to be proportional to the transverse and rotational boundary stiffnesses k_u and k_θ . The proportional dependence of the damping on the stiffness turns out to work for the mathematical model to reproduce the experimental results, so we let:

$$d_u = \bar{c}_u k_u, \quad d_\theta = \bar{c}_\theta k_\theta. \quad (23)$$

Figure 3: First mode frequency vs. applied tension for the two bolts, M12×260 (a) and M12×140 (b). Theoretical results (solid line), and experimental data (circles) adapted from [4]. Parameters: $E = 207$ GPa, $\rho = 7850$ kg/m³; (a) $c_1 = 2 \cdot 10^5$, $c_2 = 0.35$, $c_3 = 20$, $c_4 = 0.8$, $\bar{c}_u = \bar{c}_\theta = 1.2 \cdot 10^{-3}$, $l = 253.5$ mm ; (b) $c_1 = 2 \cdot 10^5$, $c_2 = 0.3$, $c_3 = 20$, $c_4 = 0.6$, $\bar{c}_u = 0.6 \cdot 10^{-3}$; $\bar{c}_\theta = 3.6 \cdot 10^{-3}$, $l = 127$ mm.

The parameters $\bar{c}_u$ and $\bar{c}_\theta$ together with d_e and d_b are obtained so as to minimize, in least square sense, the difference between the damping ratio ζ calculated from Eq. (19) and the experimental damping ratio given in [4] for the two bolts M12×260 and M12×140. Figure 5 shows the damping ratio (19) (blue solid line) against experimental data (markers). As appears from the analytical predictions (19) agrees excellently with the experimental data. For small tension N , the damping is dominated by boundary damping, caused by the relative motion between bolt-head/nut and the structure. As the tension increases this motion is reduced and the damping becomes dominated by internal and external damping in the bolt shaft. Based on the damping ratio expression (19), the contribution of the internal damping increases with frequency and therefore with tension, however the experimental damping ratio was shown to decrease as the tension increases. Therefore the analytical results shown in Figure 5 were produced by only taking into account the boundary damping (d_u, d_θ) and the external damping d_e in Eq. (19), while setting the internal damping d_i to zero.

Figure 4: (a) Undamped lowest mode shape, Eq. (13), of the long bolt (M12×260) for three different values of the tension p . (b) Zoom of (a) near the boundary at $x = 1$. Parameters as given in the caption for Figure 3a

5 Effectively linear boundaries

Figures 3 and 5 shows that the model presented in this paper successfully can explain experimental data. In this model the boundary conditions were assumed to be linear. However, in real applications the boundaries are more likely to behave nonlinearly, at the least at the microscale. In this section we suggest how the collective behavior of the “local” nonlinear stiffnesses can result in a linear behavior at the macroscale **by introducing a model, whose purpose is only to explain and justify the use of a linear boundary stiffness, and not to reproduce the experimental data in Figures 3 and 5.**

Figure 5: Damping ratio vs. applied tension for the two bolts M12x260 (a) and M12x140 (b). Theoretical results (solid line) and experimental data (circles) adapted from [4]. Parameters as given in the caption of Figure 3, and for (a) $\bar{c}_u = \bar{c}_\theta = 1.2 \cdot 10^{-3}$, $d_e = 5 \cdot 10^{-3}$, $d_i = 0.0$; (b) $\bar{c}_u = 0.6 \cdot 10^{-3}$, $\bar{c}_\theta = 3.6 \cdot 10^{-3}$, $d_e = 8 \cdot 10^{-10}$, $d_i = 0.0$.

Under low-amplitude excitation, experiments on jointed structures indicated nearly linear *damping* [3]. The linear behavior of damping was also observed in buildings, where the resulting effect of different friction damping elements behaves as a linear viscous damping [22]. For the case of *stiffness*, an attempt is made here to explain the linear behavior of the locally (micro-scale level) nonlinear stiffness. A system of multiple chains is considered, each connected to a common base by a nonlinear spring $\bar{k}$, while the other end is preloaded P , with one preloaded end subjected to an impact $f(t) = f_0 \delta(t)$, where δ denotes Dirac's delta function (Figure 6a). Each nonlinear spring models the nonlinear stiffness for a single material asperity, while the preload models the different asperities heights. The equation of motion of the system, is:

$$\mathbf{M} \ddot{\mathbf{x}} + \mathbf{C} \dot{\mathbf{x}} + \mathbf{K}_c \mathbf{x} + \mathbf{K} \mathbf{x} + \bar{\mathbf{K}} \mathbf{x}^3 = \bar{\mathbf{F}}, \quad (24)$$

where $\mathbf{x} = \{x_1 \ x_2 \ \dots \ x_n\}^T$ with x_i corresponding to the displacement of a mass m_i (Figure 6a), and $\mathbf{x}^3 \equiv \{x_1^3 \ x_2^3 \ \dots \ x_n^3\}^T$, and where $\mathbf{K}$, $\mathbf{C}$, $\mathbf{K}_c$, $\mathbf{M}$, $\bar{\mathbf{K}}$, $\bar{\mathbf{F}}$ are given respectively as follows:

$$\mathbf{K} = \begin{bmatrix} 2k & -k & 0 & -k & 0 & 0 & \dots \\ -k & 3k & -k & 0 & -k & 0 & \dots \\ 0 & -k & 2k & 0 & 0 & -k & \dots \\ -k & 0 & 0 & 3k & -k & 0 & \dots \\ 0 & -k & 0 & -k & 4k & -k & \dots \\ 0 & 0 & -k & 0 & -k & 3k & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (25)$$

and

$$\begin{aligned} \mathbf{C} &= \text{diag}(c \ 0 \ 0 \ c \ 0 \ 0 \dots), \quad \mathbf{K}_c = \text{diag}(k_c \ 0 \ 0 \ k_c \ 0 \ 0 \dots), \\ \mathbf{M} &= m \mathbf{I}, \quad \bar{\mathbf{K}} = \text{diag}(\bar{k} \ 0 \ 0 \ \bar{k} \ 0 \ 0 \dots), \\ \bar{\mathbf{F}} &= \{f(t) \ 0 \ P_1 \ 0 \ 0 \ P_2 \ \dots \ 0 \ 0 \ P_{n/3}\}^T. \end{aligned} \quad (26)$$

The restoring force $\mathbf{K} \mathbf{x} + \bar{\mathbf{K}} \mathbf{x}^3$ is chosen to be of a hardening nature since an asperity is most likely to undergo, under a continuous load, a deformation against a flat surface without experiencing a buckling-like phenomenon (multiple equilibrium points). It should also be noted that although the restoring force only contains the first two symmetric terms of the actual nonlinear force, the quadratic term is indeed implicitly included in Eq. (24) due to the presence of the static preloads P 's.

Figure 7a shows the Fourier transform of a numerically obtained solution to Eq. (24) for one of the masses in Figure 6a, for the cases of 1, 2, 3, and 4 chains. As appears the frequency response starts taking the shape as the frequency response of a linear system as the number of the chains increases. For example, when there is only one chain (blue), the response appears to contain multiple frequency components, while in the case of four chains (green) the frequency components in the response decreases to apparently having a single component.

To further investigate this linear tendency as the number of chains increases, the multi-degree-freedom system (Figure 6a) is reduced to a first-mode equivalent single-degree freedom spring-mass system (Figure 6b), where the restoring force of the spring represents an effective stiffnesses in the original multi-degree of freedom system. The equation of motion for the reduced system in Figure 6b is:

$$m \ddot{x} + c \dot{x} + k_c x + F_{\text{cor}}(x) = 0, \quad (27)$$

where m is the mass, c the damping, k_c the connecting stiffness and $F_{\text{cor}}(x)$ the force to be found. The force $F_{\text{cor}}(x)$ can be thought of as a correction to the restoring force $k_c x + F_{\text{cor}}(x)$ of the reduced model (27); it is obtained by numerically simulating the response of any mass of a chain in the frequency domain, and then solving for the correction force such that:

$$\bar{F}_{\text{cor}}(\omega) = -(-m\omega^2 + k_c + i\omega) \bar{x}(\omega), \quad (28)$$

where $\bar{x}(\omega)$ is the Fourier transform of the displacement $x_i(t)$ of any mass m_i of the original system Eq. (24). In this analysis all the masses have the same mass m . Jalali et al [12] used force rate mapping from time-domain acceleration records to identify a nonlinear joint model. Here, we consider the Fourier transform data $\bar{x}(\omega)$ around the first mode, which is obtained by numerically integrating Eq. (24), collecting one of the masses response, and Fourier transform it. Then $\bar{F}_{\text{cor}}(\omega)$ is obtained using Eq. (28), with the parameters m , c , k_c are the ones used in the original systems Eq. (24). An inverse Fourier transform is performed on $\bar{F}_{\text{cor}}(\omega)$ to get $F_{\text{cor}}(t)$, and a parametric plot is produced to display $F_{\text{cor}}(t)$ vs. $x(t)$; then a least square method is used to fit that data such that the coefficients k_1 and k_3 in

$$F_{\text{cor}}(x) = k_1 x + k_3 x^3, \quad (29)$$

are estimated for each original system of one, two or any number of chains.

Table 1 gives the values of the estimated coefficients k_1 and k_3 for different number of chains in Eq. (24) (Figure 6a). As appears when number of chains increases, the magnitude of the nonlinear stiffness k_3 of $F_{\text{cor}}(x)$ in the reduced model Figure 6b decreases; this means the original system tends to behaving more linearly.

Figure 7b shows the correction function $F_{\text{cor}}(x)$ as function of x . It appears the number of chains increases the correction force characteristic approaches a linear shape. The negative slope of $F_{\text{cor}}(x)$ (i.e., $k_1 < 0$) shows that any chain in Figure 6a as a whole becomes softer than the contact stiffness k_c . Furthermore, if the correction function $F_{\text{cor}}(x)$ also contains a quadratic term, such that:

$$F_{\text{cor}}(x) = k_1 x + k_2 x^2 + k_3 x^3, \quad (30)$$

Figure 8 shows that also here, as the number of chains increases, the correction force characteristic approaches a linear shape.

In this analysis, Fourier transform of the response is performed to single out the low frequency components corresponding to the first mode of the response. Alternatively, low pass filtering was also used to confirm the obtained results.

The above results suggest that, for small amplitude vibrations, and for an increased number of asperities in contact, a linear dependency of the contact force on the average asperities separation distance can be adopted. On the other hand, the assumption of an exponential increase of the number of asperities in contact in the classical statistical Greenwood and Williamson model (GW) [7] gives an exponential dependency of the contact force on separation distance. However, this is not directly contradicting our results, since the experimental restoring characteristic can be locally linearized for the case relevant here of small amplitude vibrations, and second, the GW model assumes either a single or a very large number of asperities, neither of which applies to our case of limited number of asperities.

Table 1: Linear and nonlinear correction coefficients k_1 and k_3 for different number of chains.

Number of chains	$ k_1 $	$ k_3 $
1	0.0649	0.0158
2	0.0541	0.0034
3	0.0508	0.0010
4	0.0460	0.0005

Figure 6: Coupled mass-spring chains modeling the contact between the structure and the bolt head/nut (a); reduced one-degree-of-freedom mass-spring system (b).

6 Conclusion

In this work we showed how a tightened bolt can be successfully modeled by Bernoulli-Euler beam with linear transverse and rotational springs and dampers at its end. The boundary stiffness and damping were expressed analytically as function of bolt tension. In particular, we showed that the model reproduces excellently the experimentally observed lowest natural frequency and damping ratio versus tension for two tests, when gradually tightened. A tentative explanation was suggested on how the collective of nonlinear stiffness behaviors of local asperities may result in an effectively linear boundary stiffness behavior.

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Figure 7: (a) Frequency response for the system (24) (Figure 6a) subject to impact at m_3 ; (b) Force correction $F_{cor}(x) = k_1 x + k_3 x^3$, Eq. (29). Parameters **used in Eq. (24) (Figure 6a)**: $m_1 = m_2 = \dots = m_3 = 1$, $P_1 = 100$, $P_2 = 400$, $P_3 = 750$, $P_4 = 950$, $c = 0.5$, $k_c = 1000$, $\bar{k} = 1000$, and $f_0 = 50$, (N=1), 100 (N=2), 140 (N=3), and 200 (N=4).

Figure 8: Force correction $F_{cor}(x) = k_1 x + k_2 x^2 + k_3 x^3$, Eq. (30). Parameters used in Eq. (24) (Figure 6a): $m_1 = m_2 = \dots = m_3 = 1$, $P_1 = 50$, $P_2 = 400$, $P_3 = 700$, $P_4 = 900$, $c = 0.5$, $k_c = 1000$, $\bar{k} = 1000$, and $f_0 = 50$ (N=1), 100 (N=2), 140 (N=3), and 200 (N=4).

References

- [1] Al-Jumaily, A., Faulkner, L., 1977. Vibration of continuous systems with compliant boundaries. J. Sound Vib. 54 (2), 203–213.
- [2] Brake, M.R.W., 2018. The Mechanics of Jointed structures, Springer.
- [3] Bograd, S., Reuss, P., Schmidt A., Gaul L., Mayer M., 2011. Modeling the dynamics of mechanical joints. Mech. Syst. Signa Pr. 25, 2801-2826.
- [4] Brøns, M., Thomsen, J.J., Sah, S.M., Tcherniak D., Fidlin A., Estimating bolt tension from vibrations: transient features, nonlinearity, and signal processing (Submitted for journal publication).
- [5] Dohner, J.L., Gregory, D.L., Segalman, D., Martinez, D.R., 2000. White paper: On the Development of Methodologies for Constructing Predictive Models of Structures with Joints and Interfaces, Structural Dynamics Department, Sandia National Laboratories, Albuquerque, NM.
- [6] Gaul, L., Bohlen, S., 1987. Identification of nonlinear structural joint models and implementation in discretized structure models, in: Proceedings of the 11th ASME Confer-

- ence on Mechanical Vibration and Noise: The Role of Damping in Vibration and Noise, Boston(USA), pp. 213-219.
- [7] Greenwood, J.A., Williamson, J.B.P., 1966. Contact of nominally flat surfaces. *Proc. R. Soc. Lond.* 295, 300-319.
 - [8] Hashimoto, M., Marui, E., Kato, S., 1994. Estimation of contact stiffness at interfaces in machine structures by a beam model on an elastic foundation. *Tribol Int.* 27 (6), 423-431.
 - [9] Hermansen, M.B., Thomsen J.J., 2018. Vibration-based estimation of beam boundary parameters. *J. Sound Vib.* 429, 287-304.
 - [10] Inman, D.J., 1996. *Engineering Vibration*, Prentice Hall, NJ.
 - [11] Rao S.S., 1995 *Mechanical vibrations*, third edition. Reading, MA, Addison-Wesley Publishing Company.
 - [12] Jalali, H., Ahmadian, H., Mottershead, J.E., 2007 Identification of nonlinear bolted lap-joint parameters by force-state mapping. *Int. J. Solids Struct.*, 44 (25–26), 8087-8105.
 - [13] Jeong Kim, B.S.K., Yoon J.C., 2007. Finite element analysis and modeling of structure with bolted joints, *Appl. Math. Model.* 31 (5), 895–911.
 - [14] Lacayo, R., Pesaresi, L., Gross, J., Fochler, D., Armand, J., Salles, L., Schwingshackl, C., Allen, M., Brake, M. 2019 Nonlinear modeling of structures with bolted joints: a comparison of two approaches based on a time-domain and frequency-domain solver, *Mech. Syst. Signal Process.* 114, 413-438.
 - [15] Padmanabhan, K., 1992. Prediction of damping in machined joints, *Int. J. Mach. Tool Manu.* 32, 305-312.
 - [16] Rao, C.K., Mirza, S. 1989. A note on vibration of generally restrained beams. *J. Sound Vib.* 130, 453-465.
 - [17] Reid, J.D., Hiser, N.R., 2005. Detailed modeling of bolted joints with slippage. *Finite Elem. Anal. Des.* 41 (6), 547-562.
 - [18] Sanliturk, K.Y., Imregun, M., Ewins, D.J., 1997. Harmonic balance vibration analysis of turbine blades with friction dampers. *J. Vib. Acoust.* 119 (1), 96-103.
 - [19] Sah, S.M., Thomsen, J.J., Brøns, M., Fidlin, A., Tcherniak, D., 2018. Estimating bolt tightness using transverse natural frequencies. *J. Sound Vib.* 431, 137–149.
 - [20] Shi, X., Polycarpou, A.A., 2005. Measurement and modeling of normal contact stiffness and contact damping at the meso scale. *J. Vib. Acoust.* 127, 52-60.

- [21] Svedholm, C., Zangeneh, A., Pacoste, C., François, S., Karoumi, R., 2016. Vibration of damped uniform beams with general end conditions under moving loads. *Eng. Struct.* 126, 40–52.
- [22] Tamura, Y., 2012. Amplitude dependency of damping in buildings and critical tip drift ratio, *Int. J. High-Rise Build.* 1 (1), 1-13.
- [23] Thomsen J.J., 2003. *Vibrations and Stability: Advanced Theory, Analysis, and Tools*, Springer-Verlag Berlin Heidelberg.
- [24] Xing, J.Z, Wang, Y.G., 2013. Free vibrations of a beam with elastic end restraints subject to a constant axial load. *Arch. Appl. Mech.* 83, 241–252.