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Topology Optimization of Ultra High Resolution Shell Structures

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Abstract

This work presents a high performance computing framework for ultra large scale, shell-element based topology optimization. The shell elements are formulated using a linear elastic, small strain assumption and are of the solid type, meaning that each quadrilateral shell element is extruded and assigned 24 degrees of freedom. The resulting linear system is solved using a fully parallelized multigrid preconditioned Krylov method, tailored specifically for unstructured quadrilaterial shell meshes. The multigrid approach is shown to have good parallel scaling properties and is able to efficiently handle the ill-conditioning arising from the 'Solid Interpolation of Material Properties' (SIMP) method. For the optimization, the classical minimum compliance design problem with multiple load cases, prescribed minimum length scale and a local volume constraint is investigated. The latter is implemented through efficient PDE-filtering in contrast to usual local image filtering based implementations. Finally, the framework is demonstrated on two idealized examples from civil and aerospace engineering, solving shell optimization problems with up to 11 million shell elements on 800 cores. As an example, this resolution corresponds to a minimum feature size of 1.5 cm on a high-riser of height 80 m.

Keywords: Topology Optimization, Shell Reinforcement, High-performance Computing, Multigrid

1. Introduction

Shell structures can achieve high stiffness-to-weight ratios, and are therefore often found in weight critical applications including aircrafts, high-risers and

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ships. On the other hand, topology optimization is a numerical optimization tool used to create high performance structures, tailored to specific load cases, with little or no prior knowledge of the optimal structure [1]. Thus, topology optimization and shell structures provide a near-perfect combination in the pursuit for stiffness optimal and light-weight constructions.

Performing structural optimization using shell elements presents several possibilities and challenges. For example, the shell thickness can be optimized as a varying field throughout the structure [2]. Alternatively, applying the regular SIMP approach [3, 4] can be used to determine the optimal perforation of a given shell structure. Another large class of shell topology optimization problems is the orientation of anisotropic material features, whether it is stiffening beads [5, 6, 7] or composite laminates [8]. Furthermore, applications of shell element based topology optimization include [9, 10], which present simultaneous topology and shape optimization of curved plates, where the resulting structures carry the loads efficiently through in-plane strains. A discrete material optimization framework is presented in [11], which is specifically developed to choose between several laminate directions in shell structures. This framework is expanded several times to include optimizing the shell thickness [12], and in order to improve the discrete formulation [13]. A case study optimizing the thickness of a reinforcing shell layer of a submarine sail, which forms part of the outer submarine structure, is considered in [14]. Optimization of buckling phenomena of shells is studied in [15], where regularization schemes are included to avoid spurious buckling modes in void regions. A framework for the placement of reinforcing patches is presented in [16], where the patches act as a reinforcement onto an exising shell structure. A case study in which a wind turbine wing is optimized is presented in [17], using a genetic algorithm approach for the outer turbine skin, and topology optimization for the reinforcing spars.

Numerical analysis of shell structures using finite elements results in very high condition numbers of the resulting stiffness matrices compared to those of standard solid elements [18]. The high condition numbers have adverse effect on the numerical accuracy of direct solution methods, and have significant impact on the efficiency of iterative solution methods. The ill-conditioning occurs due to the large difference between high frequency in-plane deformation modes, and low frequency out-of-plane deformation modes, as shown in [18]. Usually, direct solution methods are applied to shell structures, even when tackling large scale problems as e.g. done in [19], which solves a finite element problem using approximately 920,000 quadrilateral shell elements. In [20] the authors present a multigrid method for shell structures, which shows that many iterations are needed for convergence in thin shells, where the conditioning problem is worst. Combining the high condition numbers of shell structures with the highly heterogeneous material parameters encountered in topology optimization, which further increases the condition number, presents a severe challenge that must be overcome in order to allow for ultra high resolution topology optimization of shells structures.

The conjecture that ultra high resolution is a necessity in order to achieve maximum insight into a given design space, has been demonstrated in many recent works including [21, 22, 23, 24, 25, 26]. As an example, [22] discretizes an

entire 26m long aircraft wing, with a maximum element size of 0.8 cm, allowing the optimization to create many local features that would not have been possible using a coarse mesh. Moreover, the infill possibilities provided by additive manufacturing, require a highly resolved design space in order to allow for the formation of an intricate infill layout.

In this work, a framework for performing high resolution topology optimization of shell structures is presented. First, a fully parallelized multigrid preconditioner based on the approach from [22] is developed to facilitate the solving of very large scale shell topology optimization problems. The prolongation operators for the unstructured shell grids are obtained using a variation of the approach presented in [20] for a continuum shell element formulation. The proposed solver setup allows for the solution of systems with up to 11 million quadrilateral shell elements (69 million degrees of freedom) with highly heterogeneous material coefficients. The framework is then employed on selected design problems from civil and aerospace engineering and the findings are summarized and discussed.

2. Shell finite element model

Throughout this work linear elasticity using small strains for shells is considered. We present a methodology for using a continuum shell formulation defined for hexahedral elements, on a mesh of quadrilateral shell elements. This methodology is introduced, since continuum shell formulations have desirable properties with respect to symmetry and multigrid methods (see section 3 for details). Furthermore, as most available meshing software focuses on quadrilateral shell elements, the development of this approach was necessary. The method works by creating an equivalent hexahedral element for each quadrilateral when integrating the element.

The methodology for creating an equivalent hexahedral requires access to the shell normals defined on the element nodes. In order to approximate these vectors the normal vector for each quadrilateral is computed. Then the nodal normal vector is found by averaging the normal vectors of all connected elements, as depicted in fig. 1. When constructing the equivalent hexahedral, all nodes are translated by half the shell thickness h in both directions along the corresponding nodal normal, as also depicted in fig. 1. During the assembly process the nodes in the quadrilateral mesh can be considered as owning the degrees of freedom corresponding to both extruded nodes in the equivalent hexahedral mesh.

A drawback of this extrusion method, is that arbitrary shell thicknesses cannot be considered. The formation of hourglass hexahedral elements is possible if the shell thickness is large relative to the element side length. Therefore careful monitoring of the validity of the extruded mesh is necessary. In this work the scaled Jacobian metric is computed for all elements, and the computations are stopped if minimum value of the metric is unsatisfactory, which for this work is chosen as less than 0.2.

The shell element is a slightly modified version of the continuum shell element presented in [27]. The used formulation employs both the Mixed Interpolation



Figure 1: Illustration of the extrusion process for interpreting quadrilateral shell elements as hexahedrals. The red arrows indicate the normals of the quadrilateral surfaces, while V_3 is used to denote the approximated shell normal at a given node. Each node in the quadrilateral mesh is extruded to two nodes in the corresponding hexahedral mesh, as shown in blue.

of Tensorial Components (MITC) correction when interpolating the out-of-plane shear strains and the Assumed Natural Strains (ANS) correction when computing the normal strain in the out-of-plane normal direction. Unlike the original formulation from [27], this version does not employ the Enhanced Assumed Strains (EAS) correction, as this correction negatively affects the efficiency of iterative solvers. As in the original formulation, it is possible to include multiple layers with varying material properties, which is used to implement passive domains during the optimization when formulating a shell reinforcement problem. Finally, the Scaled Thickness Conditioning (STC) [28] is applied to the formulation, in order to reduce the condition number associated with thin shell elements. For completeness, the full shell element formulation including design dependence is given in appendix Appendix A.

The accuracy of the element implementation was verified using a set of standard test examples. For the centrally loaded circular clamped plate and for the hemispherical problem presented in [29], the element is found to converge to the exact deformation value. For the pinched diaphragm supported cylinder study [30], the element implementation was found to deviate by 5% of the analytical displacement value.

3. Multigrid approach

It is well-known that solving large scale problems using direct methods is not feasible in general. Therefore, iterative methods must be employed. Here we use a Galerkin projection based multigrid preconditioned Krylov subspace solver in order to efficiently solve large scale heterogeneous shell problems [31]. The used multigrid method is based on the scheme presented in [32, 22], which uses a standard V-cycle multigrid method with Galerikin projection as preconditioner for the preconditioned flexible conjugate gradients solver [33, 34]. For topology optimization problems the solver uses the deformation of the previous design as initial guess for the CG iterations, allowing for fast convergence for smaller design changes. Information about the used hierarchy of smoothers is summarized in table 1.

Table 1: Summary of the used smoothers, preconditioners and steps used in the multigrid preconditioner.

level(s)	smoother	preconditioner	steps
Finest	Flexible CG	multigrid	-
Intermediate	Chebyshev [34]	SOR [34]	4
Coarse	GMRES [34]	Algebraic multigrid (PETSc GAMG) [35]	200

The prolongation operator is based on [20] which deals with degenerated, i.e. quadrilateral, shell elements. The prolongation operator maps the displacement fields from a coarse mesh onto a fine mesh. In the presented approach the displacement fields of the interior and exterior nodes of the hexahedrals are treated separately, resulting in a prolongation for quadrilateral shell elements which prolongs six different fields. This treatment of the displacements as two separate fields is performed to avoid smoothing of the field through the shell thickness, as there exists no additional nodes through the thickness in the coarser meshes.

The operator is constructed in parallel using the shape functions of the coarse mesh to interpolate coarse nodal values to a given node in the fine mesh. In practice this requires some additional computations, due to both the lack of structure in the mesh and due to the 2.5D nature of shells. The pseudo-code for computing the prolongation operator between two meshes is shown in algorithm 1. The KD-tree used in algorithm 1 is introduced to avoid performing a search for all potential nodes in the unstructured meshes. Similarly, the second loop, which handles all fine nodes, is introduced to implement a heuristic addition for nodes which were never found to reside within a coarse element. The process of projecting a node onto the element plane, and determining if it is within the said element, is depicted in fig. 2. The potential nodes are identified, and projected onto the plane of the quadrilateral, wherein it is checked whether they are inside the quadrilateral.

3.1. Implementation

The multigrid approach is implemented using the PCMG preconditioner in PETSc [36, 37, 38], which also contains the multigrid cycles and smoothers. The prolongation operator is constructed in parallel as a preprocessing step. The efficiency of the multigrid approach and element is confirmed by the strong scaling results presented in fig. 4, where the fuselage example from fig. 3 (and section section 5.2) is studied with approximately 45 million degrees of freedom. A near linear speedup is observed until 480 processors, or 93000 degrees of freedom pr. core.

Algorithm 1: Pseudo-code to construct prolongation

```
Data: Fine mesh \mathcal{H}^i, coarse mesh \mathcal{H}^{i-1}
Result: prolongation operator P
K := \text{KD} tree of all nodes in \mathcal{H}^i:
n_{\text{touched}} := \text{zero valued array of size nodes in } \mathcal{H}^i;
forall elements e \in \mathcal{H}^{i-1} do
     i_e := global indices associated with e;
     c_e := Center of e;
     l_e := approximate element size of e;
     n_{\text{found}} := \text{nodes in } K \text{ within } 1.5 \times l_e \text{ of } c_e;
     for
all nodes n \in n_{\text{found}} \operatorname{do}
          \tilde{n} := projection of n onto plane of e;
          if \tilde{n} is within e then
               n_{\text{touched}}[n] := n_{\text{touched}}[n] + 1;
               i_n := global indices associated with n;
               N := shape functions in e corresponding to position of \tilde{n};
               Insert N into the submatrix i_n \times i_e of P;
          end
     end
end
Synchronize values of n_{\text{touched}} across processors;
K_c := \text{KD} tree of all nodes in \mathcal{H}^{i-1};
forall nodes n \in \mathcal{H}^i do
     if n_{\text{touched}}[n] is 0 then
          n_{\text{found}} := 3 \text{ nearest nodes to } n \text{ in } K_c;
          w_{\text{found}} := \frac{1}{|n_{\text{found}} - n|^2};
         w_{\text{found}} := \frac{w_{\text{found}}}{\sum w_{\text{found}}};
          Insert w_{\text{found}} into P such that n is coupled to n_{\text{found}};
     end
end
```

The Lotte tower design problem from fig. 5 [39] is used to examine the residual as function of conjugate gradient iterations, see more details in section 5.1. Figure 6 shows the convergence of the multigrid approach for various multigrid levels using homogeneous material parameters in the domain. As can be seen, the required number of iterations to reach a given tolerance increases with the number of multigrid levels. However, the cost of each iteration decreases with more levels, as the coarse problem size decreases.

A study of the residual decrease for the same domain with various shell thicknesses is shown in fig. 7. The number of iterations required to reach convergence increases drastically as the thickness decreases relative to the size of the shell. This is due to the increasing condition number, as the difference between the largest and smallest eigenvalues increases. The increasing condition number drastically reduces the convergence rate of Krylov methods.

Figure 8 shows the required number of CG iterations for convergence during



Figure 2: Illustration of the projection process used to evaluate if a node is within a quadrilateral.



Figure 3: sketch of the used geometry for the fuselage example. The outer shell is shown in yellow, the floor plate in green, and the various reinforcing plates are shown in blue. The three regions with boundary conditions, are shown in red.

design iterations for a typical topology optimization problem. It can be seen that the required number of iterations increases in the beginning, as the heterogeneity of the material parameters increases fast. After some iterations, however, the design changes become smaller and more localized, allowing the CG iterations to benefit from the non-zero initial guess from the previous design iteration.

The presented multigrid approach, and associated implementation, is not limited to shell elements. Similar performance as that presented in fig. 4 has been observed when using the framework to solve topology optimization problems using hexahedral elements, although far fewer iterations are needed to reach convergence in the iterative solving process. The scaling tests, along with the numerical results of section 5.2, were run on the DTU Sophia cluster with two AMD EPYC 7351 16-Core processors and 128 GB memory per node and infiniband interconnect.



Figure 4: Speed-up factor for the first design iteration of fuselage example with approximately 45 million degrees of freedom. The parallel version with 160 cores is used as the base case, due to memory constraints the problem is not solved on fewer compute nodes.

4. Optimization formulation

The considered optimization problem is the well-studied minimum compliance problem for linear elasticity with multiple load cases [1]. Each element in the mesh is assigned a design variable $x_e \in [0, 1]$. The field of design variables is modified through a set of filters in order to obtain the so-called physical density, which is used to interpolate the stiffness and mass of the corresponding element. The robust formulation [40, 41] is applied to ensure both a minimum length scale and a 0-1 final design. The robust formulation removes the need for the usual penalization of intermediate densities, as discussed in [42]. A linear Young's modulus interpolation scheme is used, corresponding to the 'Simplified Isotropic Material with Penalization' (SIMP) method with penalization value p = 1, as shown in eq. (1).

$$E(x_e) = E_{\min} + x_e(E_0 - E_{\min}), \quad E_{\min} = 10^{-6}E_0, \quad 0 \le x_e \le 1$$
 (1)

 E_0 denotes the background stiffness and E_{\min} denotes the stiffness of a weak material used to immitate void.

To prevent numerical artifacts such as checkerboards and mesh dependency we add regularizations in the form of the Helmholtz PDE-filter [43], and the robust formulation [41]. Furthermore, a modified local volume constraint [23] is developed for the problem, which uses the Helmholtz PDE-filter instead of a local average to calculate the local volume fraction.

In order to simplify the notation the PDE filtering operator $F_r : \mathbb{R}^n \to \mathbb{R}^n$ for a given radius r is introduced. Here n denotes the number of elements in the finite element mesh. The operator is defined as $\mathbf{y} = F_r(\mathbf{x})$ where y is the solution to the modified Helmholtz PDE with homogeneous Neumann boundary conditions defined on the same mesh. Two realizations of this operator are used when formulating the optimization problem; the solid filtering, which replaces the density filter F_{solid} with $r = r_{\text{solid}}$, and the local volume filter, which replaces the neighborhood average [23] for the local volume filter F_{LV} with $r = r_{\text{LV}}$. Usually



Figure 5: Geometry of the Lotte tower test example. The square bottom of the tower is clamped. Two load cases, each consisting of two point loads, are depicted with green and blue arrows.

the filter radii are chosen such that $r_{\rm LV} \gg r_{\rm solid}$. The authors note that all of the presented radii are corrected to obtain the scalar coefficient $r^* = \frac{r}{2\sqrt{3}}$ used in the PDE formulation, as discussed in [42].

As with the PDE filter, the heaviside projection is introduced using an operator $H_{\eta} : \mathbb{R}^n \to \mathbb{R}^n$ which depends on two parameters η and β . The operator is defined for a given threshold value η , while the value of β is increased throughout the optimization using a continuation scheme. The operator is defined such that $\mathbf{z} = H_{\eta}(\mathbf{y})$ is applied element-wise in the vectors as

$$z_e = \frac{\tanh \beta \eta + \tanh \beta (y_e - \eta)}{\tanh \beta \eta + \tanh \beta (1 - \eta)}, \quad \forall e \in \{1, .., n_e\}.$$
 (2)

Three realizations of the Heaviside operator are used in the robust formulation; the nominal H_{η^n} , dilated H_{η^d} , and eroded H_{η^e} . In this work, the used threshold values are $\eta^n = 0.5$, $\eta^d = 0.4$ and $\eta^e = 0.6$. The continuation scheme of β begins by setting $\beta := 0.01$, then after every 30 iterations the value is updated $\beta := \beta + 1$. When $\beta = 8$, or at iteration 240, the update scheme changes to $\beta := \frac{6}{5}\beta$ every 30 iterations. This continues until iteration 600, where the value



Figure 6: Residuals for the Lotte tower example (section 5.1) with various multigrid levels. Meshed using 126,024 elements (757,560 dof) and a shell thickness of 0.1.



Figure 7: Residuals for the Lotte tower example (section 5.1) with 4 multigrid levels, and various shell thicknesses. Meshed using 126,024 elements (757,560 dof).

of beta $\beta \approx 59.4$. The optimization is run for a maximum of 650 iterations. This conservative choice of β -continuation is chosen due to the slower increase in value, compared to the usual continuation scheme used in e.g. [41]. The slower increase is not strictly necessary, but results in smaller jumps in β value, which is found to improve convergence stability.

Given a set of n_l load cases with their respective force vectors $\mathbf{f}_i \in \mathbb{R}^n, i \in \{1, ..., n_l\}$, and corresponding weights $w_i \in \mathbb{R}$. The optimization formulation can then be written as:



Figure 8: Used number of CG iterations for all design iterations during the optimization of the Lotte tower example (section 5.1). The model is meshed using approximately 11 million elements, with a shell thickness of 0.015 m.

$$\begin{array}{ll} \underset{\mathbf{x} \in \mathbb{R}^{n_{e}}}{\text{minimize}} & \sum_{i=1}^{n_{l}} w_{i} \mathbf{u}_{i}^{\top} \mathbf{f}_{i} \\ \text{subject to:} \\ \text{state equation} & \mathbf{K}(H_{\eta^{e}}(F_{\text{solid}}(\mathbf{x}))) \mathbf{u}_{i} = \mathbf{f}_{i}, \quad \forall i \in \{1, ..., n_{l}\} \\ \text{global volume} & \frac{1}{\sum_{e=1}^{n_{e}} v_{e}} \sum_{e=1}^{n_{e}} v_{e}(H_{\eta^{d}}(F_{\text{solid}}(\mathbf{x})))_{e} \leq V_{g}^{*} \\ \text{local volume} & ||F_{\text{LV}}(H_{\eta^{d}}(F_{\text{solid}}(\mathbf{x})))||_{p_{\text{IV}}} \leq V_{l}^{*} \\ \end{array} \right. \tag{3}$$

Note that this is the reduced version of the robust formulation, which relies on knowledge about the compliance and both constraints. Namely, it is known that the compliance attains its maximum value for the eroded realization, while both volume constraints attain their maximal values for the dilated realization.

The local volume constraint is aggregated using the p-norm approximation, with a penalty value of $p_{lv} = 16$. All examples presented here are conducted with the global and local volume fractions $V_g^* = V_l^* = 0.5$. It is noted that applying the volume constraint on the dilated field does not directly control the resulting volume in the nominal case. As all examples presented in this paper are purely academic, with arbitrarily chosen volume constraints, a variation in resulting volume fractions is accepted, and the resulting nominal volume fractions are stated. If control of the nominal volume is desired, the volume fraction update scheme presented in [41] may be used. Furthermore, the exact control of the volume fraction is complicated by the approximate nature of the p-norm aggregation used for the local volume constraint. The approximation error in the local volume constraint might prevent the optimizer from using more material, even if the global volume constraint allows for more material.

Finally, the sensitivities are obtained using the discrete adjoint method and the optimization problem is solved using a fully parallelized implementation of the Method of Moving Asymptotes (MMA) [44, 45].

5. Numerical examples

5.1. Lotte Tower - Perforation Design

The first example of high resolution topology optimization using shell elements is the Lotte Tower example, originally suggested in [39]. The tower has a square base of size 10 m by 10 m meters which is clamped, and a circular top with radius 5 m where the loads are applied. The tower is 80 meters high, and is subjected to two sets of point-loads of magnitude 1 N at the circular top, as depicted in fig. 5. The tower itself is modeled as an empty shell where the cross-section linearly varies between the square at the bottom and the circle at the top. The geometry was meshed using Cubit [46] with a mapping algorithm, resulting in 11,637,184 elements on the finest mesh, corresponding to an average element size of 1.6 cm. The coarser meshes were meshed independently using the same approach but using fewer elements. This mesh refinement is so highly resolved that an illustration of the used mesh is difficult to render. Instead, the refinement can be inferred from the close-up of the results presented in fig. 9.

The tower is optimized using a filter radius of $r_{\rm solid} = 8 \,\mathrm{cm}$ or five times average element size. Using the expression presented in [47] it can be found that this corresponds to imposing a length-scale of 3.2 cm or 2 times the average element size. The local volume constraint is computed based on the dilated realization, with a filter radius of $r_{\rm LV} = 96 \,\mathrm{cm}$. The studied problem is that of optimal perforation of a shell, i.e. no passive shell layers have been included.

A close-up of the resulting structure is shown in fig. 9 in order to illustrate the refinement of the structure. It can be seen that the small bone-like features are resolved with multiple elements across their thickness.

An overview of the resulting structure is shown in fig. 10. It can be seen that it is very intricate, consisting of many small bone-like features which are oriented to carry the loads. Arches are formed near the points where the loads are applied and underneath. Near the corners of the square base many vertical substructures can be found, which branch out to the arches throughout the height of the structure. Note that the entire structure is modeled without any symmetry in the design variables. Nevertheless, the resulting designs are near symmetric, although not fully. This is due to numerical noise and non-convexity of the optimization problem.

An alternate version of the tower has been run using the same parameters, with the exception of the filter radii which are changed to $r_{\rm solid} = 22.5 \,\mathrm{cm}$ (15 times the average element size) for the solid filter, and $r_{\rm LV} = 4.5 \,\mathrm{m}$ for the local volume filter. The resulting structure is shown in fig. 11, which reveals many of the same structural features, albeit with a larger minimum member size, than the example using a lower filter radius. The non-symmetry of the solution is much more apparent in this example due to the large structural features. It can be seen that the non-symmetry is particularly concentrated on the middle of the lower flat sections, and between the loaded arches near the top of the tower. This coincides with the areas with lower strain density, where material is applied



Figure 9: Close up of resulting topology for the Lotte Tower example using 11.6 million shell elements and filter radii of $r_{\rm solid} = 8 \,\mathrm{cm}$ and $r_{\rm LV} = 96 \,\mathrm{cm}$.

late in the optimization process, and only has little effect on minimizing the objective function.

The resulting compliance values are 0.334 J for the $r_{\rm solid} = 8 \,\mathrm{cm}, r_{\rm LV} = 96 \,\mathrm{cm}$ filter radii, and 0.316 J for the $r_{\rm solid} = 22.5 \,\mathrm{cm}, r_{\rm LV} = 4.5 \,\mathrm{m}$ filter radii. As the filter radius $r_{\rm solid}$ is increased, the compliance value should increase, as a larger feature size is enforced on the optimization algorithm. Oppositely, when the local volume filter radius $r_{\rm LV}$ is increased, the compliance values should decrease, as less complexity is forced on the resulting structure. Thus when increasing both radii, it is difficult to predict the effect on the compliance value, which in this case is lower.

The final volume fractions for the two designs were 0.46 for the $r_{\rm solid} = 8 \,\mathrm{cm}, r_{\rm LV} = 96 \,\mathrm{cm}$ filter radii, and 0.45 for the $r_{\rm solid} = 22.5 \,\mathrm{cm}, r_{\rm LV} = 4.5 \,\mathrm{m}$ filter radii. These values are lower than the maximally allowed value of 0.5 for the reasons discussed in section 4.

Figure 12 shows the iteration history of the weighted compliance for the Lotte tower example with filter radii $r_{\rm solid} = 8 \,\mathrm{cm}$ and $r_{\rm LV} = 96 \,\mathrm{cm}$ from Figure 9 and 10. It can be seen that the compliance steadily decreases, with the exception of the discontinuities which occur when the β value is increased through the continuation scheme. As expected, the final compliance is higher than the value found for the homogeneous design in the first iteration. This is a consequence of the linear stiffness interpolation, which makes the compliance of the homogeneous design lower than what is usually obtained using standard SIMP with p = 3.



Figure 10: Resulting topology of the Lotte tower example with filter radii of $r_{\rm solid} = 8 \,\mathrm{cm}$ and $r_{\rm LV} = 96 \,\mathrm{cm}$.

Figure 11: Resulting topology of the Lotte tower example with filter radii of $r_{\rm solid} = 22.5 \,\mathrm{cm}$ and $r_{\rm LV} = 4.5 \,\mathrm{m}$.



Figure 12: Weighed compliance over design iterations for the Lotte tower example with filter radius of $r_{\rm solid} = 8 \,\mathrm{cm}$ and local volume filter radius $r_{\rm LV} = 96 \,\mathrm{cm}$. The compliance weights are chosen as $w_1 = w_2 = 1.9898 \times 10^{-2}$, such that $w_1 J_1 = w_2 J_2 = 5$ for the initial conditions.

Both of the tower examples ran for 650 design iterations on 25 compute nodes on the DTU Sophia cluster, i.e. a total of 800 cores. The full optimization procedure takes around 17 hours, or an average of 94 seconds pr. design iteration, and solves the state field a total of 1300 times, due to the two load cases.

5.2. Fuselage - Reinforcement Design

The second example concerns the optimal reinforcement of the fuselage of the NASA common research model [48] depicted in fig. 3. The geometry of the common research model is used to define the outer shell of the fuselage. Additionally, a floor panel and some vertical stiffeners are added. The mesh is generated using a paving algorithm implemented in Cubit [46], and consits of 7,488,576 shell elements, with an average element size of 0.95 cm. The coarse grids for the multigrid prolongation are generated independently in a similar fashion, targeting larger average element sizes. Like the Lotte tower, the mesh refinement is so highly resolved that an illustration is difficult to render. However, the elements in a close-up are shown in fig. 13 to give an impression of the mesh. The filter size for the solid filter for the robust formulation is $r_{\rm solid} = 7.5$ cm or 7.9 times the average element size, corresponding to imposing a length-scale of 3 cm or 3 times the average element size.

The outer shell, shown in yellow in fig. 3 and floor panel shown in green, are both modeled as shells reinforced symmetrically from both sides. In both cases the central 0.13 cm is modeled as a passive domain, while two outer reinforcements, each with thickness 1.2 cm, are modeled using the SIMP approach with a single design variable. This corresponds to the innermost 5% of the shell being passive. The plate structures shown in blue in fig. 3 are modeled without any passive domain and a thickness of 2.54 cm, which corresponds to a shell perforation design problem.

Three variations of the fuselage example are considered. The two first variations vary the local volume filter size, which is used for the local volume constraint. The fuselage is studied using a local volume filter size of $r_{\rm LV} = 127$ cm and



Figure 13: Small local filter size - Overview of the resulting topology of the fuselage example. A white plate has been added to the components which have been modeled using a passive shell. Likewise, a blue background plate is added to the reinforcing plates near the wing, in order to improve the visualization. The example uses a filter radius of $r_{\rm solid} = 7.5 \,\mathrm{cm}$ (7.9 elements) and local volume filter radius $r_{\rm LV} = 127 \,\mathrm{cm}$ (130 elements). The radii of the two enlarged circles are 127 cm and 15 cm, respectively. The elements are shown in the innermost enlargement, in order to illustrate the mesh refinement.



Figure 14: Small local filter size - Close up of the central part of the fuse lage. Computed with a filter radius of $r_{\rm solid}=7.5\,{\rm cm}$ (7.9 elements) and local volume filter radius $r_{\rm LV}=127\,{\rm cm}$ (130 elements).



Figure 15: Small local filter size - Close up of the tail section of the fuselage. The reinforcement structure of the floor can also be seen. Computed with a filter radius of $r_{\rm solid} = 7.5 \,\mathrm{cm}$ (7.9 elements) and local volume filter radius $r_{\rm LV} = 127 \,\mathrm{cm}$ (130 elements).



Figure 16: Large local filter size - Overview of the resulting topology of the fuselage example. It should be noted that a white plate has been added to the components which have been modeled using a passive shell. Likewise, a blue background plate is added to the reinforcing plates near the wing, in order to improve the visualization. Computed with a filter radius of $r_{\rm solid} = 7.5$ cm (7.9 elements) and local volume filter radius $r_{\rm LV} = 300$ cm (260 elements).

 $r_{\rm LV} = 300$ cm, which corresponds to 130 and 260 times the average element size respectively. The final variation studies the effects of including a thicker passive shell in the outer skin and on the floor plate. Here, a passive shell of 40%, instead of the usual 5%, is studied. The variation in passive thickness is performed using a local volume filter radius of $r_{\rm LV} = 127$ cm, corresponding the first variation. The three cases are denoted as 'small local filter size', 'large local filter size', and 'thick passive shell' respectively, to help distinguish the designs.

A symmetry boundary condition is applied on the mid-plane of the fuselage, such that only half the fuselage is modeled. A Dirichlet boundary condition with zero displacement in the 'upwards' direction is applied at two edges at the intersection of the interior vertical stiffeners and the outer shell, and along the



Figure 17: Large local filter size - Close up of the central part of the fuselage. Computed with a filter radius of $r_{\rm solid} = 7.5$ cm (7.9 elements) and local volume filter radius $r_{\rm LV} = 300$ cm (260 elements).



Figure 18: Large local filter size - Close up of the tail section of the fuselage. The reinforcement structure of the floor can also be seen. Computed with a filter radius of $r_{\rm solid} = 7.5 \,\mathrm{cm}$ (7.9 elements) and local volume filter radius $r_{\rm LV} = 300 \,\mathrm{cm}$ (260 elements).



Figure 19: Thick passive shell - Overview of the resulting topology of the fuselage example. It should be noted that a white plate has been added to the components which have been modeled using a passive shell. Likewise, a blue background plate is added to the reinforcing plates near the wing, in order to improve the visualization. Computed with a filter radius of $r_{\rm solid} = 7.5$ cm (7.9 elements) and local volume filter radius $r_{\rm LV} = 127$ cm (160 elements). The outer aircraft skin and floor are modelled with a passive central shell corresponding to 40% of the total shell thickness.

edge which connects a point from each curve. A single point on the top of the outer shell is given a Dirichlet boundary condition in the direction along the length of the fuselage, to avoid rigid body motion.

Two load cases are considered for the fuselage example. The first load case is an internal pressure of magnitude $35 \,\mathrm{N/cm^2}$ on the outer shell. In order to ensure equilibrium in the loads along the length of the fuselage, an additional force is distributed along the intersection of the outer shell and the vertical reinforcement near the tail. The second load case is a simplified gravity load. It is applied as a design independent body load of magnitude $15.6 \,\mathrm{N/cm^3}$ on both the outer shell and the floor panel.

The purpose of this model is to show the efficiency of the proposed method for large unstructured meshes. It should therefore be noted that this geometry does not represent an actual aircraft, due to the lack of windows and several internal reinforcing beams among other things. We remark that the applied load-cases and boundary conditions do not represent the physical loading of a fuselage, as such were not available to the authors. Furthermore, a realistic physical system would need to take additional effects into account, such as buckling, dynamics, thermal and electromagnetic responses. In relation to these considerations it is worth mentioning an additional benefit of using the local volume constraint. This constraint ensures that the final designs are free of long, slender and disconnected structural members, at least for the cases with a thin passive background shell. From the work of [49] it was found that such structures have significantly improved buckling resistance compared to designs obtained using the classical minimum compliance formulation. However, since this has not been proven or demonstrated by buckling analysis of the obtained



Figure 20: Thick passive shell - Close up of the central part of the fuselage. Computed with a filter radius of $r_{\rm solid} = 7.5$ cm (7.9 elements) and local volume filter radius $r_{\rm LV} = 127$ cm (160 elements). The outer aircraft skin and floor are modelled with a passive central shell corresponding to 40% of the total shell thickness.



Figure 21: Thick passive shell - Close up of the tail section of the fuse lage. The reinforcement structure of the floor can also be seen. Computed with a filter radius of $r_{\rm solid}=7.5\,{\rm cm}$ (7.9 elements) and local volume filter radius $r_{\rm LV}=127\,{\rm cm}$ (160 elements). The outer aircraft skin and floor are modelled with a passive central shell corresponding to 40% of the total shell thickness.



Figure 22: Comparison of the various fuselage examples. From top to bottom: small local filter size, large local filter size, and thick passive shell.

shell structures, this interesting research question together with inclusion of global buckling constraints, is left for future investigations.

Figures 13, 16 and 19 shows an overview of the resulting structures of the fuselage for all considered cases. For all three realizations it can be seen that rings are formed along the radial direction of the cylinder as a reinforcement against the internal pressure. Near the center of the fuselage, the rings are also connected by axial reinforcements, which carry the simplified gravity loading. These axial reinforcements slowly curve into the radial reinforcements in a smooth transition.

In the realizations with a thin passive support small load carrying arms appear in the unreinforced patches of the outer skin, as can be seen in figs. 13 and 16. These arms appear to prevent the unreinforced patch to locally have a large deformation by adding some additional reinforcement. The arms are not observed in the case with a thick background plate, where the background bending stiffness is higher. This phenomenon is studied further in Appendix B. These two designs have compliance values of 818 kJ and 766 kJ, for the small and large local filter size respectively. It can be seen that the compliance value of the structure with larger allowed feature size is lower as expected, due to the increased design freedom.

In the case with a thicker passive domain, many of the supporting structures are not connected, as seen in fig. 19. This effect is due to the high background stiffness of the passive shell, which will sufficiently carry the loads in areas with low strain energy density without the need for reinforcement. This design has a compliance value of 571 kJ, which is higher than both realizations with a thinner passive plate. This is expected, as more material is used compared to the two other designs. The full designs can be compared in fig. 22, where the full fuselages are shown side by side.

It can be seen that all three realizations contain the same basic features. The smaller feature size results in more substructures, which form web like reinforcements. As the filter size is increased the optimization algorithm generates larger substructures. In fig. 13 the first spyglass is a circle with radius corresponding to the used local volume filter. It can be seen that even the smallest features of the resulting structure are resolved by a large number of finite elements. It can also be observed that even the smallest used local volume filter is quite large compared to the overall structure. Due to the weighting factor which occurs in the PDE filtering, the distance between elements has a large effect on the resulting local volume fraction. Therefore, a large filter must be employed in order to achieve the desired local volume constraint.

The internal reinforcements near the wing connection are shown in figs. 14, 17 and 20. They form a webbing support, which helps carrying the loads across the various sections. In the intersection of the support with the other shells it can be seen that the supporting material is placed as an extension of the reinforcements.

The floor panels are stiffened with cross beams, as shown in figs. 14, 17 and 20 for the small local filter size. A line runs along the center of the plate with no reinforcing material, due to low bending moments at this point in the structure.

We remark that the 'small local filter size' case has a global volume fraction of 0.4524, which is considerably lower than the constraint. Likewise, the 'large local filter size' case has a volume fraction of 0.4517 and the 'thick passive shell' case has a volume fraction of 0.4316. This is due to the application of the volume constraint on the dilated field, and the local volume filter, which overestimates the volume in the p-norm aggregation used for the local volume constraint, as discussed in [23]. This could in principle be circumvented by an adaptive constraint technique, but has not been further pursued here.

6. Conclusion

This paper presents a design approach for generating optimized reinforcement or perforation of shell structures. The approach is based on a solid-shell element formulation and a multigrid preconditioned Krylov iterative method, which allows to efficiently solve the series of state equations associated with the optimization process. The multigrid preconditioner employs geometric multigrid restriction for the fine levels, and an algebraic multigrid method to obtain the solution of the coarse space problem. The approach overcomes the ill-conditioning problems arising partly due to shell formulations and partly due to high stiffness contrast in the element-based design parameterization. Using the proposed design method a series of academic optimization problems are solved, each repeatedly solving finite element problems using up to 11.6 million shell elements with +69.8 million degrees of freedom. This paves the way for solving large unstructured systems with shell elements for real life applications in future work.

The approach makes use of local-density control which ensures distributed material and hence a certain robustness towards unpredicted loads. For reinforcements problems, like the airplane fuselage problem considered, skin stiffness itself as well as applied minimum length-scale eliminate the need for locally connected reinforcements. This may seem counterintuitive but satisfies the applied pressure loading and optimization setting. Future studies includes buckling constraints, dynamics, thermal effects, electromagnetic effects, local load fluctuations or even finer design resolutions (and correspondingly smaller length scales imposed) which should help eliminate the aforementioned artifacts.

7. Reproducibility

The used meshes, geometry files, and final designs can be made available upon reasonable request.

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Appendix A. Shell element formulation

This appendix provides a detailed summary of the used shell element formulation in order to facilitate reproduction of the proposed framework. For the original formulation, the reader is referred to [27].

Appendix A.1. Transformation matrix

In order to perform the numerical integration a transformation matrix is constructed. With onset in the normal vector to the shell surface at node I, here denoted \mathbf{V}_3^I , two additional vectors are computed using the method proposed by [30], yielding

$$\mathbf{V}_1^I = \mathbf{e_2} \times \mathbf{V}_3^I, \quad \mathbf{V}_2^I = \mathbf{V}_3^I \times \mathbf{V}_1^I.$$
(A.1)

or as

$$\mathbf{V}_2^I = \mathbf{V}_3^I \times \mathbf{e_1}, \quad \mathbf{V}_1^I = \mathbf{V}_2^I \times \mathbf{V}_3^I.$$
(A.2)

for the case where \mathbf{V}_3 and \mathbf{e}_2 are parallel. The node director basis inside a given element \mathbf{V}_i is found by interpolating the basis vectors defined at each node \mathbf{V}_i^I using the quadrilatereal bilinear shape functions.

The transformation matrix from coordinate system A to coordinate system B is constructed using direction cosines as

$$t_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j, \quad \mathbf{v}_B = \begin{cases} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{cases} \mathbf{v}_A.$$
(A.3)

which allows for the construction of the transformation matrix

$$\mathbf{T} = \begin{bmatrix} (t_{11})^2 & (t_{12})^2 & (t_{13})^2 & t_{11}t_{12} & t_{11}t_{13} & t_{12}t_{13} \\ (t_{21})^2 & (t_{22})^2 & (t_{23})^2 & t_{21}t_{22} & t_{21}t_{23} & t_{22}t_{23} \\ (t_{31})^2 & (t_{32})^2 & (t_{33})^2 & t_{31}t_{32} & t_{31}t_{33} & t_{32}t_{33} \\ 2t_{11}t_{21} & 2t_{12}t_{22} & 2t_{13}t_{23} & t_{11}t_{22} + t_{12}t_{21} & t_{11}t_{23} + t_{13}t_{21} & t_{12}t_{23} + t_{13}t_{22} \\ 2t_{11}t_{31} & 2t_{12}t_{32} & 2t_{13}t_{33} & t_{11}t_{32} + t_{12}t_{31} & t_{11}t_{33} + t_{13}t_{31} & t_{12}t_{33} + t_{13}t_{32} \\ 2t_{21}t_{31} & 2t_{22}t_{32} & 2t_{23}t_{33} & t_{21}t_{32} + t_{22}t_{31} & t_{21}t_{33} + t_{23}t_{31} & t_{22}t_{33} + t_{23}t_{32} \\ \end{array}$$

which allows the strain and constitutive laws to be transformed as follows

$$\sigma_B = \mathbf{T}\sigma_A, \quad \mathbf{C}_B = \mathbf{T}^\mathsf{T}\mathbf{C}_A\mathbf{T}. \tag{A.5}$$

Appendix A.2. Isoparametric formulation

In the continuum shell element formulation the displacement field and physical coordinates are interpolated using standard trilinear hexahedral shapefunctions [30].

The Jacobian matrix used to transform from the isoparametric reference space to the usual orthonormal basis [30] is defined as follows.

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \sum_{I=1}^{8} \begin{bmatrix} N_{I,\xi}x_I & N_{I,\eta}x_I & N_{I,\zeta}x_I \\ N_{I,\xi}y_I & N_{I,\eta}y_I & N_{I,\zeta}y_I \\ N_{I,\xi}z_I & N_{I,\eta}z_I & N_{I,\zeta}z_I \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 & \mathbf{G}_3 \end{bmatrix},$$
(A.6)

$$\mathbf{J}^{-1} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{bmatrix} = \begin{bmatrix} \mathbf{G}^{\mathbf{1}^{\mathsf{T}}} \\ \mathbf{G}^{\mathbf{2}^{\mathsf{T}}} \\ \mathbf{G}^{\mathbf{3}^{\mathsf{T}}} \end{bmatrix}.$$
(A.7)

The covariant, \mathbf{G}_{I} , and contravariant, $\mathbf{G}^{I^{\mathsf{T}}}$, bases are found directly from the Jacobian and its inverse [27, 30].

The design dependent constitutive matrix for an isotropic material using the SIMP model is expressed as follows

$$\mathbf{C} = \frac{\rho^{p} E_{0}}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0\\ & 1-\nu & \nu & 0 & 0 & 0\\ & & 1-\nu & 0 & 0 & 0\\ & & & \frac{1-2\nu}{2} & 0 & 0\\ & & & & \frac{1-2\nu}{2} & 0\\ & & & & & \frac{1-2\nu}{2} \end{bmatrix}$$
(A.8)

where ρ is the design variable and p = 1 is the penalty parameter. The constitutive matrix is formulated in the contravariant basism, i.e. Equation (A.4) is applied with $t_{ij} = \mathbf{V}_i \cdot {\{\mathbf{G}^j\}}$ to obtain $\mathbf{T}^{\mathbf{V}}$.

$$\bar{\mathbf{C}} = \mathbf{T}^{\mathbf{V}^{\mathsf{T}}} \mathbf{C} \mathbf{T}^{\mathbf{V}} \tag{A.9}$$

Appendix A.3. Strain displacement matrix

The bi-linear strain interpolation matrix **B** interpolates deformations at element nodes to strains in the covariant basis at an internal point defined by some given $\xi, \eta, \zeta \in [-1, 1]$. The matrix uses the regular interpolations for the in plane components $\tilde{\epsilon}_{11}, \tilde{\epsilon}_{22}$, and $\tilde{\gamma}_{12}$ strains, while alternative interpolations are applied in the remaining three strains to prevent locking.

To prevent out-of-plane shear locking, i.e. in $\tilde{\gamma}_{23}$ and $\tilde{\gamma}_{31}$, the Mixed Integer Tensorial Components (MITC) is employed [50] based on the four tying points A, B, C, and D shown in table A.2.

Additionally, to prohibit locking in the shell normal direction, the Assumed Normal Strain (ANS) interpolation is used for $\tilde{\epsilon}_{33}$ [51, 52], using four new tying points A_1 , A_2 , A_3 , and A_4 as depicted in table A.3.

This leads to following strain-displacement relation for node I

Table A.2: MITC tying points.

	ξ_L	η_L	ζ_L
A	-1	0	0
B	0	-1	0
C	1	0	0
D	0	1	0

Table A.3: ANS tying points.

	ξ	η	ζ
A_1	-1	-1	0
A_2	1	-1	0
A_3	1	1	0
A_4	-1	1	0

$$\mathbf{B}_{I} = \begin{bmatrix} \mathbf{G}_{1}^{\mathsf{T}}N_{I,\xi} \\ \mathbf{G}_{2}^{\mathsf{T}}N_{I,\eta} \\ \sum_{L=1}^{4} \frac{1}{4}(1+\xi_{L}\xi)(1+\eta_{L}\eta)\mathbf{G}_{3}^{\mathsf{L}^{\mathsf{T}}}N_{I,\zeta}^{L} \\ \mathbf{G}_{1}^{\mathsf{T}}N_{I,\eta} + \mathbf{G}_{2}^{\mathsf{T}}N_{I,xi} \\ \frac{1}{2} \begin{bmatrix} (1-\eta)(\mathbf{G}_{3}^{\mathbf{B}^{\mathsf{T}}}N_{I,\xi}^{B} + \mathbf{G}_{1}^{\mathbf{B}^{\mathsf{T}}}N_{I,\zeta}^{B}) + (1+\eta)(\mathbf{G}_{3}^{\mathbf{D}^{\mathsf{T}}}N_{I,1}^{D} + \mathbf{G}_{1}^{\mathbf{D}^{\mathsf{T}}}N_{I,\zeta}^{D}) \\ \frac{1}{2} \begin{bmatrix} (1-\xi)(\mathbf{G}_{3}^{\mathbf{A}^{\mathsf{T}}}N_{I,\eta}^{A} + \mathbf{G}_{2}^{\mathbf{A}^{\mathsf{T}}}N_{I,\zeta}^{A}) + (1+\eta)(\mathbf{G}_{3}^{\mathbf{C}^{\mathsf{T}}}N_{I,2}^{C} + \mathbf{G}_{2}^{\mathbf{C}^{\mathsf{T}}}}N_{I,\zeta}^{C}) \end{bmatrix} \end{bmatrix}$$
(A.10)

Collecting the contribution from each of the nodal points, yields the complete strain-displacement matrix, i.e.

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \dots & \mathbf{B}_8 \end{bmatrix}, \tag{A.11}$$

Appendix A.4. Stiffness matrix integration and loads

The local stiffness matrix is obtained by standard Gaussian quadrature, i.e.

$$\mathbf{K}_{e} = \int_{\Omega_{e}} \mathbf{B}^{\mathsf{T}} \bar{\mathbf{C}} \mathbf{B} \, \mathrm{d}V$$

$$\approx \sum_{GP} |J| \mathbf{B}^{\mathsf{T}} \bar{\mathbf{C}} \mathbf{B}.$$
(A.12)

If multiple material layers are used a new isoparametric space is introduced for each layer, as depicted in fig. A.23. The resulting numerical integration scheme becomes

$$\mathbf{K}_{e} \approx \sum_{L=1}^{nlay} \sum_{GP} \mathbf{B}^{\mathsf{T}} \mathbf{C}_{\mathbf{L}} \mathbf{B} |J^{\xi}| |J^{r}|, \qquad (A.13)$$



Figure A.23: Illustration of the two isoparametric spaces.

The two external force integrals can be approximated by a Gaussian quadrature in the isoparametric space. The body force can integrated by the same quadrature rule s the stiffness matrix

$$\mathbf{f_{vol}} = \int_{\Omega_e} \mathbf{N}_e \hat{\mathbf{b}} \, \mathrm{d}V$$

$$\approx \sum_{GP} |J| \mathbf{N}_e \hat{\mathbf{b}}.$$
(A.14)

The surface forces, i.e. pressure loads and tractions, are obtained by integrating over the corresponding surface, determined by the two isoparametric coordinates $\xi_1, \xi_2 \in \{\xi, \eta, \zeta\}$. Let J_{ξ_1} denote the column of the Jacobian **J** corresponding to ξ_1 . The surface integral is then computed by Gaussian quadrature as

$$\mathbf{f_{surf}} = \int_{\partial \Omega_e} \mathbf{N}_e \hat{\mathbf{t}} \, \mathrm{d}A$$

$$\approx \sum_{GP} ||J_{\xi_1} \times J_{\xi_2}||_2 \mathbf{N}_e \hat{\mathbf{t}}.$$
(A.15)

Now the resulting system of equations can be assembled using the regular

$$\mathbf{Ku} = \mathbf{f_{vol}} + \mathbf{f_{surf}}.$$
 (A.16)

We remark that the resulting system of equations is poorly conditioned and hence, provides a challenge for iterative solvers.

Appendix A.5. Conditioning

To improve the performance of the proposed iterative solver, the Scaled Thickness Conditioning (STC) presented by [28] is included to reduce the condition number of the system matrix for thin continuum shells. The method uses a scaling parameter C, which for thin shells has the following optimal value

$$\mathcal{C}^{\text{opt}} \approx \frac{l_1 + l_2}{2h},\tag{A.17}$$

where l_1 and l_2 denote the element side lengths. The nodal scaling matrices are computed for nodes lying in the element mid-plane corresponding to the ANS nodes shown in table A.3. For consistency, the nodes are denoted $\lambda \in \{A_1, A_2, A_3, A_4\}$ and the nodal scaling matrices are given as

$$\mathbf{s}_{\mathbf{1}}^{\lambda} = \frac{1}{\eta_{\lambda}} \begin{bmatrix} \frac{\mathcal{C}+1}{2\mathcal{C}} & 0 & 0\\ 0 & \frac{\mathcal{C}+1}{2\mathcal{C}} & 0\\ 0 & 0 & \frac{\mathcal{C}+1}{2\mathcal{C}} \end{bmatrix}, \quad \mathbf{s}_{\mathbf{2}}^{\lambda} = \frac{1}{\eta_{\lambda}} \begin{bmatrix} \frac{\mathcal{C}-1}{2\mathcal{C}} & 0 & 0\\ 0 & \frac{\mathcal{C}-1}{2\mathcal{C}} & 0\\ 0 & 0 & \frac{\mathcal{C}-1}{2\mathcal{C}} \end{bmatrix}, \quad (A.18)$$

where η_{λ} denotes the number of elements attached to node λ . Using the nodal scaling matrices, the element scaling matrices are constructed as follows

$$\mathbf{s}_{e} = \begin{bmatrix} \mathbf{s}_{2}^{\mathbf{A}_{1}} & \mathbf{s}_{1}^{\mathbf{A}_{1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{s}_{1}^{\mathbf{A}_{1}} & \mathbf{s}_{2}^{\mathbf{A}_{1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{s}_{1}^{\mathbf{A}_{2}} & \mathbf{s}_{2}^{\mathbf{A}_{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{s}_{2}^{\mathbf{A}_{2}} & \mathbf{s}_{1}^{\mathbf{A}_{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{s}_{1}^{\mathbf{A}_{3}} & \mathbf{s}_{2}^{\mathbf{A}_{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{s}_{2}^{\mathbf{A}_{3}} & \mathbf{s}_{1}^{\mathbf{A}_{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{s}_{1}^{\mathbf{A}_{3}} & \mathbf{s}_{2}^{\mathbf{A}_{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{s}_{2}^{\mathbf{A}_{4}} & \mathbf{s}_{1}^{\mathbf{A}_{4}} \\ \end{bmatrix} .$$
(A.19)

The nodal scaling matrices \mathbf{s}_e are assembled to a symmetric global scaling matrix \mathbf{S} by the regular finite element assembly. The scaling matrix is then applied to obtain the scaled stiffness matrix and force vector, i.e.

$$\mathbf{K}^{\mathcal{C}} = \mathbf{S}\mathbf{K}\mathbf{S}, \quad \mathbf{f}^{\mathcal{C}} = \mathbf{S}\mathbf{f}, \quad \mathbf{u} = \mathbf{S}\mathbf{u}^{\mathcal{C}}.$$
 (A.20)

The resulting scaled linear system of equations now reads

$$\mathbf{K}\mathbf{u} = \mathbf{f} \Leftrightarrow \mathbf{K}^{\mathcal{C}}\mathbf{u}^{\mathcal{C}} = \mathbf{f}^{\mathcal{C}}.$$
 (A.21)

Note, that the matrix \mathbf{S} is never assembled in order to reduce the memory usage. Instead, the scaled stiffness and forces are obtained during assembly by performing the corresponding local products.

Appendix B. A small study on the emergent 'arm' supports

The occurrence of non-connected reinforcements, dubbed 'arms', in the fuselage example merited further study. Intuition states that closed cells provide a better reinforcement, as the reinforcing material supports itself better and thus provides a stiffer reinforcement. In order to study whether the 'arms' provide some benefit two reinforced structures, shown in figs. B.24 and B.25, are studied. Both cases are clamped plates of size 20x20 subjected to a uniform pressure load, where a fourth of the domain is modeled using symmetry conditions.

The reinforcement is placed on both sides of a the base plate. The total thickness with reinforcements is set to a constant of 1, while the fraction of base thickness to reinforcement thickness is swept. Both plates are reinforced with material corresponding to a volume fraction of V = 0.36, such that the compliance values can be compared directly. The plates are resolved with 100x100 elements.





Figure B.24: Plate with overlapping reinforcements shaped as a cross.

Figure B.25: Plate with reinforcement in the form of 'arms', note that the four reinforcements are not connected.

Base plate thickness	Cross	Arms
40%	2.05229	2.442617
20%	3.629288	4.303002
10%	13.88358	12.2136
5%	94.69309	72.91165
1%	5821.474	4370.23
0.1%	11634.7	8735.052
0.01%	11708.35	8818.975

Table B.4: Compliance values for both reinforcement configurations for a series of base plate thicknesses. It can be seen that the cross performs best when the stiffness contrast between the reinforced an non-reinforced areas is low, while the arms perform better when this contrasts is high.

The resulting compliance values are shown in table B.4. From there it can clearly be seen that the cross connected reinforcement performs better when the reinforced plate is thick compared to the reinforcement. As the thickness decreases the 'arm' like structure becomes better performing. Therefore it can be concluded that the 'arm' reinforcements which occur in the fuselage designs are a part of the desired solution, and not some artifact due to enforced lengthscale and volume constraint.

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