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Damping of Coupled Bending-Torsion Beam Vibrations by Spatially Filtered Warping Position Feedback

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Abstract

Damping of coupled bending-torsion beam vibrations is realized by local actuators, operating axially on an end cross-section of a fully three-dimensional structure by a collective positive position feedback control with a spatial filter that specifically targets the out-of-plane warping component of the coupled response. To reduce the computational effort, a beam element model is introduced, in which a single control bimoment is supplemented by a fictitious stiffness that accounts for the inability of the local actuators to fully restrain warping. This fictitious stiffness is conveniently calibrated by frequency matching in the limit of infinite control gain and it enables accurate estimation of both complex roots and the stability limit from beam element analysis. It is demonstrated by a numerical example that substantial modal damping is attained by positive position feedback of the torsional warping component from the fully coupled response.

Keywords:
Damping, coupled vibrations, positive position feedback, active control, thin-walled beams, finite element method

1. Introduction

Many beam structures exhibit properties which inherently couple basic deformation modes. Varying cross-sectional properties, pretwist, inhomogeneous material distribution and lack of cross-sectional symmetry may all lead to coupling between axial, bending and/or torsional deformation modes of beam structures. Areas of application in large-scale structures may include wind turbine blades, bridges and tall buildings [1]. In all of these areas, structural vibrations may be an issue and supplemental damping by discrete devices may be required to increase the life-time of structures prone to fatigue damage, reduce the risk of aeroelastic instabilities like flutter and secure accelerations below comfort levels.

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Early descriptions of coupled beam vibrations because of non-coinciding elastic and shear centres were given by Gere [2]. A thorough derivation of the coupled differential equations with contributions from rotary and warping inertia can be found in [3, 4] while [5] provides solutions to the governing coupled equations with forced motion. Special emphasis on the effect of warping was given in [6], concluding that significant errors occurred when not taking this supplemental deformation into account. Coupled vibrations in slender beam structures are effectively analysed by beam elements [7, 8], as also shown in the present paper. Furthermore, external dampers and discrete masses directly influence the dynamic behaviour of beams with bending and torsion coupling, as investigated in [9, 10, 11] by both a finite element approach with beam elements as well as exact solutions.

Wind turbine blades are large-scale structures with pronounced coupled bending-torsion vibrations because of asymmetric cross-section geometry and inhomogeneous material distribution, making them prone to aeroelastic flutter instability with apparent negative damping [12]. Because of the continuously increasing size of wind turbines and their blades, it is of interest to alleviate excessive dynamic vibrations, as previously proposed with tuned mass dampers [13, 14].

The coupled vibrations imposed by aeroelastic interaction is also present in streamlined bridge decks, as reported for long-span suspension bridges in [15, 16, 17]. As it may be a limiting factor in design of future multi-deck bridges, efforts are put into increasing the flutter wind speed limit by adding damping, using for example tuned mass dampers [18, 19]. Other structures with coupled bending-torsion vibrations are composite beams or plates, used in many areas of application where limited weight combined with high strength is of importance [20, 21, 22].

This paper specifically considers damping of free coupled bending-torsion vibrations by partially restraining the out-of-plane, axial warping displacements that occur due to the torsional part of the fully coupled response. Figure 1a shows a viscous bimoment, restraining the cross-section warping of a 1D beam model. It has been demonstrated in [23] that the restraint of warping by a special boundary condition for the bimoment may realize a significant increase in frequency and thereby a great damping potential. This effect may also
be observed in the case of forced harmonic vibrations, for which the steady-state amplitude at resonance is directly inverse proportional to the modal damping ratio. The promising results for the ideal beam model have recently been confirmed by a full three-dimensional finite element (3D FE) analysis [24], with several discrete viscous dampers placed in configurations constituting the desired bimoment on a beam cross-section, as shown in Fig. 1b for four corner forces acting oppositely on a single-symmetric cross section. However, the discrete actuators in Fig. 1b only resist warping of the cross-section locally where they are placed, thereby leaving non-vanishing warping displacements to occur in the remainder of the cross-section. This partial restrainment of warping is associated with an excess flexibility in the system, not accounted for by the beam element model in Fig. 1a. Therefore, an equivalent control law for the beam element in Fig. 1a is supplemented by a fictitious spring stiffness that represents this additional flexibility experienced by the actual model in Fig. 1b. The inclusion of the fictitious spring reduces the frequency of the system, by an elastic boundary condition, as described for torsional damping in [25] or for structures with flexible or non-ideal supports in [26, 27, 28].

For large and densely meshed structures it may be computationally ineffective to perform a detailed damping analysis with the full 3D FE model in Fig. 1b. Therefore, this paper instead solves the complex eigenvalue problem for the efficient beam element model in Fig. 1a, augmented with the fictitious spring to account for the excess warping associated with the local actuator control in Fig. 1b.
As torsional warping displacements may be quite small, an active control with positive position feedback (PPF) [29, 30, 31, 32, 33] of the warping displacement is proposed to achieve a substantial increase in attainable damping. The displacements are effectively measured by e.g. strain sensors at each actuator location and processed by a spatial filter that combines the individual signals into a resulting out-of-plane warping deflection of the cross section, without affecting deformations from the extension and bending content of the coupled response. A positive position feedback control law provides the desired damping performance and the individual target force processed to the individual actuator on the cross section is then distributed by the same spatial filter used to modulate the sensor signal. Thus, the control scheme is collectively collocated [34], although the feedback for each individual actuator is not entirely decentralized.

A proposed first-order linear filter with assumed ideal actuator dynamics has previously been proposed for vibration damping in [35, 36], and has furthermore been considered for damping of pure torsional vibrations in [37]. It is demonstrated how the excess flexibility from the partial warping restrainment, included in the beam model by the fictitious spring stiffness, affects and changes the stability limit [38, 39] of the combined system with PPF. It is finally demonstrated that substantial damping ratios are obtained by only restraining the torsional part of the coupled bending-torsion response of a beam structure.

2. Coupled vibrations with active control

A main goal of the present paper is the formulation of a simple beam model, that accurately captures the detailed dynamics of an underlying 3D beam structure with local positive position feedback (PPF) control of the out-of-plane warping displacement from torsion. This section therefore derives the governing equations for coupled bending-torsion vibrations of a pure beam model with an equivalent bimoment representing the active boundary control.

2.1. Governing differential equations

Consider a beam with length ℓ, longitudinal axis z and transverse axes \{x_1, x_2\}. For a cross-section without double symmetry the elastic centre \( C = (c_1, c_2) \) - which for isotropic and homogeneous cross-sections coincide with the centre of mass - and shear centre \( A = (0, 0) \)
do not coincide, as shown in Fig. 2a. The \( \{x_1, x_2\} \)-directions are in this case not principal axes and define the translational displacements \( \xi_1(z) \) and \( \xi_2(z) \) with respect to the shear centre. As bending-torsion-coupling is investigated, the extension deformation is excluded in the present formulation. The angle of twist with respect to the shear centre around the \( z \)-axis is \( \theta(z) \). The coupling between bending and torsion is explicitly given by the inertial forces acting in \( C \), generating a torsional moment through the distances \( c_1 \) and \( c_2 \) to the shear centre \( A \). To keep a compact notation the arguments \( (x_1, x_2, z) \) are excluded in the following.

The equations governing coupled bending and torsional vibrations of a general thin-walled beam may be derived as shown in e.g. [3, 4],

\[
\begin{align*}
& EI_{11} \dddot{\xi}_1 + EI_{12} \dddot{\xi}_2 + \rho A \ddot{\xi}_1 - \rho A c_2 \ddot{\theta} = 0 \\
& EI_{21} \dddot{\xi}_1 + EI_{22} \dddot{\xi}_2 + \rho A \ddot{\xi}_2 + \rho A c_1 \ddot{\theta} = 0 \\
& EI_{\psi} \dddot{\theta} - GK \dddot{\theta} + \rho A c_1 \ddot{\xi}_2 - \rho A c_2 \ddot{\xi}_1 + \rho J \ddot{\theta} = 0
\end{align*}
\] (1)

The cross-sectional parameters consist of the bending stiffnesses \( EI_{\alpha\beta} \), the torsional stiffness \( GK \) associated with homogeneous torsion (St. Venant), the warping stiffness \( EI_{\psi} \) associated with inhomogeneous torsion (Vlasov), the mass per unit length \( \rho A \) and the torsional inertia per unit length \( \rho J \). Differentiation with respect to \( z \) is indicated by \( (\ )' = \partial(\ )/\partial z \), while \( (\ )' = \partial(\ )/\partial t \) describes differentiation with respect to time \( t \). Furthermore \( \psi \) is the warping function or sector-coordinate associated with free homogeneous torsion. The coupling specifically depends on the distances \( c_\alpha \) between the elastic centre and shear centre, shown in Fig. 2a.

2.2. Discretization by beam elements

A main objective of the paper is to demonstrate the accuracy of the beam element model relative to a full 3D FE model of the beam structure. For beam elements the classic cubic Hermitian shape functions are often sufficient to interpolate between nodes, as they linearly resolve the curvature for \( \xi_\beta \) and \( \theta \) in (1). Without the extensional deformation of the element, each node has six degrees-of-freedom, as indicated in Fig. 2b. The discretization of the beam model then yields a global stiffness matrix \( \mathbf{K} \) and mass matrix \( \mathbf{M} \), with further details on the specific finite element implementation provided in the Appendix A, summarizing explicit
expressions for the system matrices by appropriate sub-matrices, that may include higher-order effects, such as rotary and warping inertia, by supplemental terms.

The equations of motion for the structure modelled by beam elements are represented in terms of the $n$ degrees-of-freedom in the displacement vector $\mathbf{q}(t)$ of the linear system,

$$M\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}$$

(2)

when neglecting any inherent structural damping. Free vibrations are throughout the paper examined by assuming harmonic solutions of the form: $\mathbf{q}(t) = \tilde{\mathbf{q}}e^{i\omega t}$, where $\tilde{()}$ denotes the vibration amplitude and $\omega$ is the angular frequency. By substitution into (2), the undamped eigenvalue problem takes the form

$$\left( -\omega_0^2 \mathbf{M} + \mathbf{K} \right) \tilde{\mathbf{q}}_{0,j} = \mathbf{0}$$

(3)

where $\omega_{0,j}$ are the undamped natural frequencies and $\tilde{\mathbf{q}}_{0,j}$ are the corresponding mode shape vectors, with subscript 0 denoting the undamped structure and $j$ referring to the specific mode number.

2.3 Damping by warping position feedback

Efficient damping of the beam element model is introduced by a PPF control, where the local actuator forces on the 3D structure in Fig. 1b act on the local axial displacements from torsional warping extracted from the coupled bending-torsion response. For the present beam model in Fig. 1a, the combined actuator forces constitute a resulting bimoment $B_d$.
that is energy conjugated to the gradient of the angle of twist \( \theta'_d \), representing the warping intensity. For the beam element model the set of equations of motion is therefore given as

\[
\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = -\mathbf{w}\mathbf{B}_d(t)
\] (4)

The effect of the resulting bimoment is here added to the beam model by a connectivity vector \( \mathbf{w} \), which identifies the specific actuator configuration and location. For the beam model, the connectivity vector has the simple structure \( \mathbf{w} = [\ldots, 1, \ldots]^T \), where the single unit value is located at the degree-of-freedom representing the warping intensity of the controlled node. It furthermore defines the representative displacement

\[
\theta'_d(t) = \mathbf{w}^T\mathbf{q}(t)
\] (5)

that produces work with the equivalent bimoment \( \mathbf{B}_d \) in (4).

For the proposed PPF control, the bimoment is assumed linear and collocated, with the feedback force formulated as a linear filter [35, 36],

\[
\mathbf{B}_d(t) + \tau\dot{\mathbf{B}}_d(t) = g\theta'_d(t)
\] (6)

where \( \tau \) is a filter constant defining the cut-off frequency at \( 1/\tau \). Thus, the gain \( g \) here governs the magnitude of the collective feedback control for the equivalent bimoment on the simple beam model, whereas Section 5 introduces the actual control for the full 3D FE structure with a gain for the axial force actuators.

2.4. Frequency relations for PPF

Based on the measured displacement \( \theta'_d(t) \), the equivalent bimoment \( \mathbf{B}_d(t) \) is fed back to the system by the control law in (6) for the representative beam model. The properties of this filter may be examined in the frequency domain by assuming the harmonic solutions \( \mathbf{B}_d(t) = \tilde{\mathbf{B}}_d e^{i\omega t} \) and \( \mathbf{\theta}_d(t) = \tilde{\mathbf{\theta}}_d e^{i\omega t} \), leading to the frequency control relation

\[
\tilde{\mathbf{B}}_d = H\tilde{\mathbf{\theta}}'_d
\] (7)

which introduces the temporal dependence by the frequency function

\[
H = \frac{g}{1 + i\omega\tau}
\] (8)
Thus, the effect and tuning of the actuator is governed by the gain $g$, while the frequency characteristics are represented by the denominator of $H$. The complex $H = \left| H \right| e^{i\varphi}$ is conveniently decomposed into its phase angle $\varphi$ and the magnitude $\left| H/g \right|$ of the frequency function normalized with the gain $g$,

$$\tan \varphi = -\omega \tau, \quad \left| H/g \right| = \frac{1}{\sqrt{1 + (\omega \tau)^2}}$$

The phase angle and normalized magnitude are shown in Fig. 3a. It is seen by the solid line that the PPF imposes a control component that operates ahead of velocity by a phase angle $\varphi > \pi/2$, which approaches the viscous limit $\varphi \simeq \pi/2$ in the high-frequency domain $\omega \tau \to \infty$. This phase lead $\varphi > \pi/2$ corresponds to apparent negative stiffness [36], whereby the PPF locally increases actuator deflection and improves the overall energy dissipation. Therefore, PPF is advantageous for small deformation problems, such as the warping feedback control considered in the present case.

The magnitude of the transfer function, represented by the dashed line in Fig. 3a, avoids undesirable influence from high-frequency vibrations, as $\left| H \right| \to 0$ above the cut-off frequency, whereas the filter ensures large damping in the low frequency domain below the cut-off frequency. In order for the filter in (6) to dissipate energy, the imaginary part $\text{Im}[H] > 0$. Figure 3b shows both the real and imaginary part of the frequency function $H$, where the imaginary part of the frequency function (solid curve) is seen to be negative, whereby the gain $g < 0$. Conversely, the real part $\text{Re}[H] < 0$ imposes the previously mentioned negative stiffness that locally increases the actuator stroke and improves performance over classic
viscous damping or direct velocity feedback [40].

3. Beam model with warping-restrained flexibility

Figure 4a shows a flange of the thin-walled cross-section in Fig. 4b and Fig. 1b. The 3D FE structural model is discretized by finite elements and two axial actuator forces act locally at the free end. During the PPF control the actuators will restrain the warping displacements at their specific location, even completely preventing the axial motion in the limit $g \to -\infty$. However, the cross-section may still be able to warp in between the actuator locations, as indicated in Fig. 4b. This residual warping displacement is associated with an excess flexibility, which is not included in the beam model. Thus, the PPF control law in (6) for the effective beam model must augmented to capture the correct dynamics and this additional flexibility of the underlying 3D FE model [24].

3.1. Augmented control with fictitious stiffness

The additional flexibility to be used when controlling the equivalent beam model may be modelled as a spring with stiffness $k$ in series with the active controller in (6) with gain $g$, as schematically indicated in Fig. 5. The fictitious spring stiffness is introduced by the linear relation

$$B_d(t) = k\theta'_k(t)$$

(10)

in which the excess component $\theta'_k$ is associated with the cross-section warping in Fig. 4b for locking of the actuators at $g \to -\infty$. It follows from Fig. 5 that the total gradient of
the angle of twist $\theta_{d*}'$ is given as the sum of the contributions from the bimoment and from the fictitious spring,

$$\theta_{d*}'(t) = \theta_k'(t) + \theta_d'(t)$$ \hspace{1cm} (11)

Introducing (6) and (10) into (11) then yields the augmented control equation

$$\left(\frac{1}{g} + \frac{1}{k}\right)B_d(t) + \frac{\tau}{g}B_d(t) = \theta_{d*}'(t)$$ \hspace{1cm} (12)

to be used in the following for calibration of the gain $g$ when controlling the beam model. The fictitious stiffness $k$ is in the following determined such that the frequency of the equivalent beam at infinite gain ($g \to -\infty$) recovers the correct frequency when locking the discrete actuators on the actual cross-section in the full 3D FE model in Fig. 4.

The gain-dependent frequency function $H_*$ of the augmented control equation (12) can now be written as

$$H_* = \left[\left(\frac{1}{g} + \frac{1}{k}\right) + i\omega\frac{\tau}{g}\right]^{-1}$$ \hspace{1cm} (13)

whereby $H_* \to H$ in (8) as $k \to \infty$. When introducing the actuator force (12) in the global system (4), the equations of motion may be written in the frequency domain as

$$\left(-\omega_j^2M + K + H_*ww^T\right)\bar{q}_j = 0$$ \hspace{1cm} (14)

with the effect of the augmented controller included at a specific single degree-of-freedom by the matrix $ww^T$. The combined controller, governing the actuator bimoment $B_d$, is shown in Fig. 5, where the physical controller with gain $g$ is placed in series with the fictitious spring $k$. Thus, when $k \to \infty$ the fictitious spring will not be deformed, whereby the boundary condition recovers the non-modified controller in (6) with gain $g$. However, for a finite value of $k$, the additional flexibility implies an increase in displacement ($\theta_{d*}' > \theta_d'$) at the beam boundary, thus modifying both the performance, calibration and stability limit of the proposed PPF control.

3.2. Determination of residual flexibility

The accurate determination of the fictitious spring stiffness $k$ is important for the dynamic damping analysis using the simple beam model. The residual warping-restrained
stiffness $k$ in (12) is obtained from the dynamics of the structure in the limit $g \to -\infty$, in which the controller equation (13) reduces to additional stiffness term $k\mathbf{w}\mathbf{w}^T$ in

$$
\left(-\omega_{\infty,j}^2\mathbf{M} + \mathbf{K} + k\mathbf{w}\mathbf{w}^T\right)\tilde{\mathbf{q}}_{\infty,j} = 0
$$

(15)

For this limiting eigenvalue problem $\omega_{\infty,j}$ is the frequency of the beam model associated with infinite gain, while $\tilde{\mathbf{q}}_{\infty,j}$ is the corresponding mode shape vector. In the following, $\omega_{\infty,j}$ is therefore referred to as the infinitely damped (natural) frequency. In the present case the stiffness $k$ is determined so that the infinitely damped frequency $\omega_{\infty,j}$ from (15) matches the corresponding natural frequency $\omega_{3D}$ from the full 3D FE model with infinite gain. Thus, the stiffness $k$ is calibrated by the frequency matching $\omega_{\infty,j} = \omega_{3D}$, which only requires the solution of a real-valued eigenvalue problem for the large 3D FE model.

The fictitious stiffness $k$ can be isolated in (15) by applying the matrix determinant lemma of the Sherman-Morrison formula [41]. Substituting the calibration frequency $\omega_{\infty,j} = \omega_{3D}$ into the beam model in (15), this lemma directly gives

$$
\det\left(-\omega_{3D}^2\mathbf{M} + \mathbf{K} + k\mathbf{w}\mathbf{w}^T\right) = \det\left[-\omega_{3D}^2\mathbf{M} + \mathbf{K}\right]\left(1 + k\mathbf{w}^T\left[-\omega_{3D}^2\mathbf{M} + \mathbf{K}\right]^{-1}\mathbf{w}\right) = 0
$$

(16)

where non-trivial solutions require the determinant to be zero. This explicitly gives the residual stiffness as

$$
k = -\frac{1}{\mathbf{w}^T\left[-\omega_{3D}^2\mathbf{M} + \mathbf{K}\right]^{-1}\mathbf{w}}
$$

(17)

based on the beam model components $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{w}$, and the frequency $\omega_{3D}$ from the full 3D FE model with infinite actuator gain. As the beam model will have a limited number of degrees-of-freedom, the inversion of the matrix in (17) is computationally inexpensive.
4. Stability condition

With the warping-restrained stiffness $k$ introduced in the modified active filter in (12) and (13), the combined system for the beam model is now described and the effect of the flexibility on the stability of the system is investigated. Fig. 6 shows a root locus tracing a damped frequency $\omega$ in the complex plane from $\omega_0$ (×) for $g = 0$ to $\omega_\infty$ (○) for $g \to -\infty$. An unbounded response occurs beyond the stability limit (□), that depends on the control law (12) with the excess flexibility comprised by $k$.

4.1. Combined closed-loop system

When solving the combined closed-loop system, comprised by the beam structure model in (4) and the equivalent control equation (12), the damped frequencies $\omega_j$ trace a family of root loci in the complex $\omega$-plane, as indicated in Fig. 6 for gain $g = 0$ ($\omega_0$) to $g \to -\infty$ ($\omega_\infty$). These root loci can be obtained by solving the eigenvalue problem associated with the coupled equations (4) and (12), which can be written as

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}(t) \\ \dot{B}_d(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \nu^{-1}T \end{bmatrix} \begin{bmatrix} \ddot{q}(t) \\ \ddot{B}_d(t) \end{bmatrix} + \begin{bmatrix} K & w \\ -gw^{-1}w^T & 1 \end{bmatrix} \begin{bmatrix} q(t) \\ B_d(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(18)

where $\nu = (1 + \frac{g}{k})$ contains the correction from the fictitious stiffness $k$. From this block matrix formulation, the combined system is solved in the corresponding state-space format,
for which the complex roots $\omega_n$ are expressed as

$$\omega_j = |\omega_j| \left( \sqrt{1 - \zeta_j^2} + i\zeta_j \right)$$  \hspace{1cm} (19)

introducing the apparent damping ratio as the relative imaginary part

$$\zeta_j = \frac{\text{Im}[\omega_j]}{|\omega_j|}$$  \hspace{1cm} (20)

The damping ratio is therefore a rational measure for the modal damping attainable by the active PPF control.

4.2. Stability limit

It follows from Fig. 3b that $\text{Re}[H] < 0$, whereby the active warping control with positive position feedback and $g < 0$ imposes apparent negative stiffness on the structure. Thus, for a limiting value of the gain $g = g_{\text{stab}}$ the negative control stiffness makes the combined system non-positive definite and thus unstable with unbounded response. It is crucial to avoid instabilities when applying active control, and the determination of the precise gain limit $g = g_{\text{stab}}$ is therefore a necessity.

The equivalent first-order control equation (6) implies that a pure imaginary root is added to the family of complex conjugated natural frequencies obtained by the eigenvalue problem associated with (18). Stability requires all eigenvalues to have a positive imaginary part, whereby instability occurs when a single of these pure imaginary roots becomes negative, as indicated by the trajectory ($\times$ to $\circ$) along the imaginary axis in Fig. 6. A stability criterion is constructed by inspection of (18), in which the augmented mass and damping matrices will be guaranteed positive semi-definite for $\nu > 0$ and thus $g > -k$. Although this inequality indicates a stability gain from modal spill-over, the condition is often not limiting because of the large values of $k$ obtained by the small to moderate reductions in the infinitely damped frequency $\omega_{\infty,j}$ found e.g. in the numerical example in Section 7.

The augmented stiffness matrix must as well be positive definite. Because of the PPF control, this introduces a limit on the negative gain $g = g_{\text{stab}} < 0$. Both the structural and controller equations are now pre-multiplied with $-g\nu^{-1}$, whereby the original structural coordinate $\mathbf{q}(t)$ is normalized as $\mathbf{q}^*(t) = -g\nu^{-1}\mathbf{q}(t)$. Hereby, the elastic energy from the
The augmented stiffness matrix in (18) can be expressed as

\[
V_e = \begin{bmatrix}
q^*(t)^T & B_d(t)
\end{bmatrix}
\begin{bmatrix}
\mathbf{K} & -g\nu^{-1}\mathbf{w} \\
-g\nu^{-1}\mathbf{w}^T & -g\nu^{-1}
\end{bmatrix}
\begin{bmatrix}
q^*(t) \\
B_d(t)
\end{bmatrix}
\]  

(21)

which can be re-written into

\[
V_e = \frac{1}{2}q^*(t)^T [\mathbf{K} + g\nu^{-1}\mathbf{w}\mathbf{w}^T] q^*(t) + \frac{1}{2} \left( B_d(t)(g\nu^{-1})^{1/2} - (g\nu^{-1})^{1/2}\mathbf{w}^T q^*(t) \right)^2 \\
- B_d(t)g\nu^{-1}B_d(t) - q^*(t)^T g\nu^{-1}q^*(t)
\]  

(22)

Stability requires the energy in (22) to be positive, i.e. \( V_e > 0 \), which directly identifies a limit of stability from the first term in (22), as the remaining terms are defined positive for \( g < 0 \). Therefore the matrix inside the square brackets of the first term (22) must be positive definite. By the matrix determinant lemma in (16) this corresponds to the determinant requirement

\[
det (\mathbf{K} + g\nu^{-1}\mathbf{w}\mathbf{w}^T) = det (\mathbf{K}) \left( 1 + g\nu^{-1}\mathbf{w}^T \mathbf{K}^{-1}\mathbf{w} \right) > 0
\]  

(23)

which leads to an explicit solution for the stability gain \( g_{stab} \),

\[
\frac{1}{g_{stab}} = -\mathbf{w}^T \mathbf{K}^{-1}\mathbf{w} - \frac{1}{k}
\]  

(24)

The gain limit \( g = g_{stab} \) is thus comprised of the structural flexibility at the location of the control bimoment and the added flexibility from the partial warping restraint. This added flexibility \( 1/k \) is seen to reduce the stability gain, which is otherwise ignored by the original PPF control equation in (6) for the beam model.

5. Spatial filtering and warping control

The accuracy of the simple beam element model with a PPF control for the equivalent bimoment relies on a precise calibration of the fictitious stiffness \( k \) in the augmented control equation (12). As mentioned previously, the stiffness \( k \) is calibrated by frequency matching \( \omega_{\infty,j} = \omega_{3D} \), where \( \omega_{3D} \) denotes the infinitely damped frequency from the full 3D FE model with infinite actuator gains. Initially, the collective warping control for the full 3D FE model is obtained by constructing a connectivity vector \( \mathbf{w}_{3D} \) with a spatial filter that only restrains warping, while ignoring axial deformations from extension and bending.
5.1. Control equations for 3D FE model

The actual 3D FE model with discrete axial actuators placed as in Figs. 1b and 4a is inherently different in size and element type compared to the simple beam element model in Fig. 1a with the actuator bimoment in Fig. 5. Thus, the system matrices \( \mathbf{M}, \mathbf{K} \) and displacement vector \( \mathbf{q}(t) \) will be larger in size for the full 3D FE model. The governing system of equations for the full FE model may be written in standard form as

\[
\mathbf{M} \ddot{\mathbf{q}}(t) + \mathbf{K} \mathbf{q}(t) = -\mathbf{f}_d(t)
\]  

(25)

in which the individual discrete actuator forces in Fig. 1b are contained in the force vector \( \mathbf{f}_d(t) \). The present PPF on the full 3D structure is constructed in a collective format, with a single displacement

\[
q_d(t) = \mathbf{w}_{3D}^T \mathbf{q}(t)
\]  

(26)

introducing a connectivity vector \( \mathbf{w}_{3D} \), which describes the spatial filtering of the individual sensor signals. The collective displacement \( q_d \) for the full 3D FE model is then processed by a PPF control equation similar to that in (6) for the beam element bimoment \( B_d \),

\[
f_d(t) + \tau \dot{f}_d(t) = g_{3D} q_d(t)
\]  

(27)

determining a corresponding collective control force \( f_d \), which scales with the common control gain \( g_{3D} \). The actual control force provided by the individual actuator in the 3D FE model is then finally obtained by multiplication with the connectivity vector,

\[
f_d(t) = \mathbf{w}_{3D} f_d(t)
\]  

(28)

Next, the connectivity vector \( \mathbf{w}_{3D} \) is constructed by a spatial filter that only targets the warping displacement of the full 3D FE model, while the relation between the actual gain \( g_{3D} \) and the equivalent bimoment gain \( g_{\text{beam}} \) for the beam model is established.

5.2. Connectivity and balancing

For a vibration mode with inherent coupling between torsion and bending, the axial displacements of the end cross-section may derive from a mix of inclinations associated with bending and out-of-plane warping from torsion. In the current strategy, only the axial displacements from warping are to be restrained by the PPF control, thus leaving the
displacements from bending unaffected. Figure 7a,b shows the bending inclination and the warping displacement for the I-profile with a single line of symmetry, previously considered in Figs. 1 and 4. The connectivity vector $w_{3D}$ must now be constructed such that only warping is affected by the PPF control system for the full 3D FE model.

Multiple actuators act on the cross-section of the 3D model to realize the equivalent bimoment $B_d$, used in the equivalent beam model. Thus, the individual non-vanishing components in $w_{3D}$ must be relatively balanced to work optimally together in extracting the warping displacement in (26) and conversely imposing the ideal bimoment by the force vector $f_d(t)$ in (28).

For warping different parts of the cross-section deform in opposite directions, as shown in Fig. 7b, whereby the entries in $w_{3D}$ must have the $\pm$-signs according to the warping function in Fig. 7c. The connectivity vector is therefore with zero entries, except at the degree-of-freedom with an actuator, for which a balancing factor $\alpha_i$ is introduced for actuator $i$,

$$w_{3D} = [..., \alpha_i, ...]^T$$ (29)

A proper balance of the individual actuators $i = 1, 2, \ldots$ is secured by balancing factors proportional to the square-root of the sector-coordinate at the actuator location

$$\alpha_i = \text{sign}(\psi_i) \sqrt{|\psi_i|}$$ (30)

in which case the sign-function secures the desired spatial filtering that extracts the sector-coordinate in e.g. Fig. 7c, without affecting axial displacements from extension and bending.
Consider the four-actuator configuration in Fig. 1b with the analytical sector-coordinate shown in Fig. 7c. In this case, the four non-zero entries in the connectivity vector will be $\pm a/\sqrt{18}$ for the two actuators placed at the free ends of the top flange and $\mp 2a/\sqrt{18}$ for the corresponding two actuators acting at the free ends of the bottom flange. When the top-flange warping function is considered as a common factor between the four entries, the connectivity vector is conveniently scaled as

$$w_{3D} = [\ldots, 1, -1, -2, 2, \ldots]^T$$

(31)

whereby the common scaling factor $a/\sqrt{18}$ is simply absorbed by the corresponding control gain $g_{3D}$, as demonstrated in Section 6. Thus, for the normalized connectivity vector, the balancing factors are then $\alpha_{1,2} = \pm 1$ and $\alpha_{3,4} = \mp 2$ for the four actuator forces in Fig. 1b.

For more complicated cross-section geometries and/or actuator configurations, a specific value of the sector-coordinate $\psi_i$ at actuator location $i$ may conveniently be obtained by available numerical cross-section analysis tools, such as [42] used in the numerical example of Section 7.

5.3. Locking cross-sectional warping

The targeted natural frequency $\omega_{3D}$, used for calibration of the fictitious spring stiffness by (17), is obtained from the full 3D FE model in the gain limit $g_{3D} \to -\infty$. This requires solving a large eigenvalue problem in state-space form, which may be computationally inefficient. Instead the particular locking of warping, without affecting the cross-section deformations from extension and bending, may be realized by enforcing a constraint equation based on the connectivity vector composed in the Section 5.2. This constraint condition is effectively imposed by use of Lagrange multipliers [43], with the constraint for warping simply given as $w_{3D}^T \bar{q} = 0$. For free vibrations this leads to solving the following eigenvalue problem,

$$\begin{pmatrix} K & w_{3D} \\ w_{3D}^T & 0 \end{pmatrix} - \omega^2 \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{q} \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(32)

where $\lambda$ is the Lagrange multiplier and $w_{3D}$ specifies the constraint structure using the connectivity vector $w_{3D}$ established by the procedure indicated in (29). The eigenvalue prob-
lem in (32) determines the real-valued natural \( \omega_{3D} \) that is used by the frequency matching \( \omega_{\infty,j} = \omega_{3D} \) to determined the fictitious spring stiffness \( k \) by the (17).

6. Control gain equivalence

The gain \( g_{\text{beam}} \) in (12) for the beam model and \( g_{3D} \) for the 3D FE model in (27) are inherently different due to the different nature of the models and scaling of the respective connectivity vectors \( w \) and \( w_{3D} \). The equivalence between the two models therefore requires a correct conversion between the two gains. This is achieved by equivalence of the virtual work produced by the bimoment in the beam model and the actuators in the 3D FE model. The control bimoment in the beam model produces virtual work by multiplication with the virtual rate of the angle of twist. In the frequency domain this virtual work relation can be written as

\[
\delta V_{\text{beam}} = -\delta \tilde{\theta}_d \tilde{B}_d = -H \delta \tilde{\theta}_d \tilde{\theta}_d'
\]

introducing the frequency function \( H \) in (8). As this relation ignores the excess flexibility by \( k \to \infty \), the virtual work may be considered as a perturbation from the undamped state.

The virtual work in the 3D FE model consists of the virtual work produced by each of the discrete actuators,

\[
\delta V_{3D} = -\delta \tilde{q}_d \tilde{f}_d = -H \delta \tilde{q}_d^T w_{3D} \tilde{q}_{3D}^T \tilde{q}
\]

identifying \( f_d \) and \( q_{d} \) as resulting scalar variables for the collective control effort. In the 3D FE model the collective displacement \( \tilde{q}_{d} = w_{3D}^T \tilde{q} = \sum_i \alpha_i \psi_i \tilde{\theta}_d' \) represents the sum of the magnitude of axial displacements from warping at the individual actuator locations. Thereby, the virtual work in the 3D FE model can be written as

\[
\delta V_{3D} = H \left( \sum_i \alpha_i \psi_i \right)^2 \tilde{\theta}_d' \tilde{\theta}_d'
\]

By \( \delta V_{\text{beam}} = \delta V_{3D} \) and substitution of \( H \) from (8) with different gains in the two models, the equivalence between these gains can be obtained as

\[
g_{3D} = \frac{g_{\text{beam}} \left( \sum_i \alpha_i \psi_i \right)^2}{\left( \sum_i \alpha_i \psi_i \right)^2}
\]

Even though (36) describes an approximate relation, the result appears to be fairly accurate, as demonstrated in the subsequent numerical example of Section 7.
7. Numerical example

In this section an example is given to demonstrate the method of using beam elements with the warping-restrained flexibility added to analyse the damping potential of warping control applied to a beam with bending-torsion coupling. In the following the damping efficiency will be independently demonstrated for the two lowest coupled modes of a beam.

The present beam is simply supported (pinned) in both ends with respect to bending in the \( x_1 \)-direction, as seen in Fig. 8a, and 'simply' supported with respect to torsion, thus restraining the rotation at both beam ends while allowing warping, as indicated in Fig. 8b.

The beam has length \( \ell \) and the cross-section is seen in Fig. 8c with overall thickness \( a/40 \) and dimensions as depicted in the figure. The material is elastic with properties \( E = 210 \) GPa, \( \nu = 0.3 \) and \( \rho = 7850 \) kg/m\(^3\). For the 3D FE model, the beam structure is discretized with isoparametric elements consisting of a bi-cubic-linear element for flange parts and a cubic-bi-linear element for corners and junctions, as indicated in Fig. 4. The beam is discretized with 30 length-wise elements, where half of them are distributed over \( \ell/25 \) at the beam end where the actuators are placed, while a single element per flange part is sufficient according to [24] for the in-plane mesh. The corresponding boundary conditions in the 3D FE model restrain all nodes at both beam end cross-sections in the vertical \( x_2 \)-direction, while restraining the nodes along the \( x_2 \)-axis in all three directions. Thereby, the beam cross-section may rotate around the \( x_2 \)-axis and displace in the axial \( z \)-direction in the flanges.

Figure 9a shows the vibration shape of the lowest mode obtained by the 3D FE model. It is
seen that the torsion couples with the lateral displacement component in the $x_1$-direction. Figure 9b-d shows the displacement components of the coupled mode from the equivalent beam element model, with the discrete (red) circles representing average values from the full 3D FE model in Fig. 9a. It is seen that good agreement is obtained between the undamped modes from the beam element model and the full 3D FE model.

7.1. Actuator configuration

The bottom flange in Fig. 8c has half the width of the top flange, whereby the elastic and shear centres do not coincide along the $x_2$-axis. For the beam model, an actuator bimoment $B_d$ is applied at the right end, as shown in Fig. 8b, which in the full 3D FE model is represented by a set of eight discrete actuators acting in the axial direction. The discrete actuators are thus structured in the connectivity vector $w_{3D}$ in (29) such that they collectively produce a bimoment that only operates on the warping displacement shown in Fig. 7c.

Two actuators are placed across the flange thickness in each corner of the four corners (free ends) of the cross-section in the 3D FE model. Because of the cross-section’s symmetry with respect to the $x_2$-axis, the four actuators placed at the left ends of the flanges are scaled by balancing factors of same magnitude and opposite sign as their right flange counterparts.
In Fig. 8d,e the circles indicate the specific location of the four left side actuators, denoted as a to d from top to bottom, respectively. The connectivity vector for the left half of the 3D FE model can therefore be expressed as

$\mathbf{w}_{\text{left}} = [\ldots, \alpha_a, \alpha_b, \alpha_c, \alpha_d, \ldots]^T$ \hspace{1cm} (37)

with four balancing factors $\alpha_a$ to $\alpha_d$. Conversely, the connectivity vector for the right half is simply obtained by a change of sign,

$\mathbf{w}_{\text{right}} = -\mathbf{w}_{\text{left}} = [\ldots, -\alpha_a, -\alpha_b, -\alpha_c, -\alpha_d, \ldots]^T$ \hspace{1cm} (38)

and the total connectivity vector for all eight actuators in the 3D FE model is obtained by the assembly

$\mathbf{w}_{\text{3D}} = [\ldots, \mathbf{w}_{\text{left}}^T, \ldots, \mathbf{w}_{\text{right}}^T, \ldots]^T$ \hspace{1cm} (39)

The four balancing factors $\alpha_a$ to $\alpha_d$ are then determined by (30) for the specific actuator locations shown in Fig. 8d,e. The detailed sector-coordinate is in this example obtained numerically by the cross-sectional code described in [42]. In Table 1 the balancing factors are given relative to the actuator with gain $g_a$ in Fig. 8d.

The axial displacements of the end cross-section from the first undamped mode is seen in Fig. 10. The contribution from warping is clearly identified as the inclinations of the two flanges in Fig. 10a are not identical, which would have been the case for pure bending. With infinite gain and the chosen connectivity structure, the relative torsional warping at

<table>
<thead>
<tr>
<th>Table 1: Balancing factors.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
</tr>
<tr>
<td>$\alpha_i/\alpha_a$</td>
</tr>
</tbody>
</table>

Figure 10: (a) axial displacements for the first undamped mode and (b) infinitely damped mode.
Figure 11: Frequency loci (a,c) and damping ratio (b,d) for the first mode (a,b) and second mode (c,d) of the beam model. Red markers represent the 3D FE model and the blue square indicate the gain limit. Dashed line represent $k \to \infty$.

the four outer corners of the I-profile is restrained and leaves the axial displacements in Fig. 10b, with almost identical inclination along the flanges and a small non-linear contribution from the residual warping shown in Fig. 4b. This validates that the connectivity vector $\mathbf{w}_{3D}$ effectively extracts the collective warping component due to torsion, without affecting the bending component of the coupled vibration.

7.2. Damping efficiency

With the connectivity structure determined for the 3D FE model, the objective is now to calibrate the fictitious stiffness $k$ by (17) in order for the root locus of the beam model to initiate and terminate at the natural frequencies obtained by the full 3D FE model. The infinitely damped frequency from the 3D FE model $\omega_{3D}$ is therefore be determined by solving (32).

The effect of the position feedback filter in (12) is based on the cut-off frequency $1/\tau$. In order to damp mode $j$ sufficiently the cut-off frequency is set to two times the corresponding
Table 2: Results of analysis.

<table>
<thead>
<tr>
<th>Model</th>
<th>k</th>
<th>$(\omega_{\infty} - \omega_0)/\omega_0$</th>
<th>$\zeta_{\text{stab}}$</th>
<th>$g^\text{beam}<em>{\text{stab}}/g^\text{3D}</em>{\text{stab}}$</th>
<th>$(\sum_i \alpha_i \psi_i)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>Beam</td>
<td>$\infty$</td>
<td>0.092</td>
<td>0.189</td>
<td>4.57</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1.15 \cdot 10^8$</td>
<td>0.086</td>
<td>0.174</td>
<td>4.24</td>
</tr>
<tr>
<td></td>
<td>3D</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>Mode 2</td>
<td>Beam</td>
<td>$\infty$</td>
<td>0.136</td>
<td>0.301</td>
<td>4.57</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1.16 \cdot 10^8$</td>
<td>0.123</td>
<td>0.188</td>
<td>4.24</td>
</tr>
<tr>
<td></td>
<td>3D</td>
<td>$-$</td>
<td>0.124</td>
<td>0.189</td>
<td>$-$</td>
</tr>
</tbody>
</table>

undamped frequency,

\[
1/\tau_j = 2\omega_{0,j}
\]  

(40)

The final step is to find the gain limit by (24), which ensures a bounded response. The results of the damping analysis for both models are seen in Fig. 11. The two lowest modes are damped independently and the corresponding frequency loci are seen in Fig. 11a,c for mode 1 and 2, respectively, while the corresponding damping ratios are plotted in Fig. 11b,d. The solid lines with circles (○) indicate the beam model, the red markers (+) depict the full 3D FE model solutions, the dashed line (──) represents the beam model with $k \to \infty$ and thus no correction for the partial warping restrainment, while the blue square markers (□) represent the complex frequency at the gain limit $g_{\text{stab}}$ in both models.

As the markers representing the beam model and 3D FE model practically coincide, very good agreement between the effective beam element model and the full 3D FE model is documented. The minor discrepancies are attributed to the non-ideal boundary conditions in the 3D FE model, which only with noticeable effort may resemble the ideal pinned boundary conditions used in the beam element model. Furthermore, the small difference in $\omega_0$ is attributed to inherent cross-sectional distortion.

To the left in Fig. 11a,c the trace of the single pure imaginary eigenvalue is shown, for which instability is seen to occur for both models exactly when it reaches the origin. Thus, the beam model with fictitious stiffness $k$ exactly identifies the stability limit of the collective PPF control applied for the eight actuators on the full 3D FE model.

In Table 2 the frequency increments $(\omega_{\infty} - \omega_0)/\omega_0$ and damping ratios at instability $\zeta_{\text{stab}}$ for
Figure 12: Free response from an initial displacement showing the horizontal displacement of (a) the shear centre and (b) point B for 0.40·\(g_{stab}\) (red), 0.65·\(g_{stab}\) (blue), 0.80·\(g_{stab}\) (black) and 0.95·\(g_{stab}\) (green). (c) shows the horizontal displacement of the shear centre for 1.01·\(g_{stab}\) (solid) and 1.05·\(g_{stab}\) (dashed).

The two modes are seen. The effect of taking the partial warping restraint into account is seen to decrease the relative frequency increment and thereby the damping ratio. It is also seen that mode 2 has a larger frequency increment than mode 1, meaning that the torsional component is more pronounced, and is thus damped more than the first mode which is reflected by the damping ratios in the table. When \(k \rightarrow \infty\) the damping ratio increases indicating an overestimation of damping in the beam model, when not including the fictitious stiffness \(k\) representing partial warping restraint.

The second last column of Table 2 provides the ratio between the stability gains for the two structural models, which differ by a factor that is accurately captured by the conversion factor introduced in (36) and given in the last column of Table 2.

7.3. Free response to initial displacement

The time-response for the beam element model is determined by an initial displacement. The time response with an initial displacement corresponding to the first vibration mode, using \(10^3\) time increments with a step size of \(h = 5 \cdot 10^{-3}\) s, is obtained by the beam
element model and is solved by the ode45-routine implemented in MATLAB. The initial displacement is based on the state vector from the eigenvalue problem in state-space with $k$ as given for mode 1 in Table 2 and $g$ given as a certain fraction of the gain limit. Due to the presence of the actuator, the complex state vector will contain a velocity part arising from the phase shift as well as an initial actuator force, implemented as initial displacement $\tilde{q}_0$, velocity $\tilde{\dot{q}}_0$ and actuator force $B_{d,0}$, respectively. The eigenvector is scaled such that the initial horizontal displacement of the shear centre $A$, as shown in Fig. 8, at the middle of the beam is $\ell/100$. Fig. 12a then shows the time history of the lateral displacement in the $x_1$-direction of the shear centre, while Fig. 12b shows the equivalent displacement at the joint between the lower flange and the web, denoted as point $B$ in Fig. 8c. For point $B$ the linearized horizontal component of the rotation of the cross-section relative to the shear centre is shown, as it describes the rotational (twist) component of the response. The response is plotted for $0.40 \cdot g_{stab}$ (red), $0.65 \cdot g_{stab}$ (blue), $0.80 \cdot g_{stab}$ (black) and $0.95 \cdot g_{stab}$ (green), approximately corresponding to the mode 1 damping ratios: $\zeta_1 = 2.9\%$, $8.3\%$, $13.8\%$ and $17.4\%$ according to Fig. 11b where the $17.4\%$ is recovered in Table 2.

As the response is initiated by the eigenvector, the contributions from other modes is minimized. The red curve in Fig. 12a for $g = 0.40 \cdot g_{stab}$ is rather close to an exponentially decaying cosine function. When applying very aggressive damping by $g = 0.95 \cdot g_{stab}$, the response is almost immediately cancelled. The response of point $B$ on the lower flange is more irregular due to the presence of the torsional component. The first peak with a displacement greater than the initial displacement of point $B$ at $t = 0$ is attributed to the apparent negative stiffness part of the filter in (12). However, the response is mitigated quite rapidly when applying aggressive damping close to the limit of stability.

In a design situation a gain close to the gain limit is not recommended, as instability should be avoided at all times with a certain margin. Various uncertainties associated with the geometry of the structure, actuator and sensor dynamics (which is not taken into account) may imply a reduced gain, slightly off the theoretical estimate. The horizontal response of the shear centre when applying a gain of $g = 1.01 \cdot g_{stab}$ and $g = 1.05 \cdot g_{stab}$ is shown in Fig. 12c, represented by the solid and dashed lines, respectively. When $g < g_{stab}$ the response becomes unbounded as observed in the figure, although for a gain only slightly above the
gain limit, the increase in the response builds up more slowly over time.

8. Conclusions

Damping of coupled beam vibrations with discrete, active actuators restraining the axial warping displacements have been the objective of this paper. Emphasis has been on setting up a beam element formulation by which it is possible to perform detailed damping analyses, as verified by comparison with a large 3D FE model with isoparametric elements. On an actual structure the discrete, axial actuators only restrain warping locally, while the remaining part of the cross-section is still able to warp. This is associated with an additional flexibility, which lowers the frequency associated with infinite gain. This flexibility has been included in the beam model by a fictitious spring and it has furthermore been shown that it reduces the gain at which the combined system becomes unstable.

A numerical analysis of a beam with a single-symmetric cross-section has been used to demonstrate the accuracy of the 1D beam model and the efficiency of warping based PPF. The complex natural frequencies and corresponding damping ratios obtained by the beam model accurately reproduce the exact results from the full 3D FE model, when taking the excess flexibility from the fictitious spring into account. A key part of the analysis is the construction of the connectivity vector, which isolates the warping from torsion without affecting extension and bending. It has been demonstrated that a collective PPF control with spatial filtering imposed by the connectivity vector implies large attainable damping. The PPF control has been implemented as a simple linear filter which produces a control force that in terms of phase leads the local velocity component and therefore yields a great damping potential due to apparent negative stiffness. The combined structure and controller equation have been solved in state-space with respect to the damped frequencies and damping ratios. For the present example with an I-profile, substantial damping ratios are reached well below the system stability limit. The similarity between the two models is remarkable, suggesting that initial designs for beams with local control may successfully be carried out by a beam model with a supplemental spring that includes apparent controller flexibility. Though the gain in the two models is inherently different, it has in fact been possible to formulate a conversion factor between the two controller gains. This enables the
design of the individual actuators by the beam model results when using e.g. piezoelectric, electric or hydraulic devices to realize the control effort in the actual structure.

Appendix A. Implementation of beam elements

The differential equations governing fully coupled flexural and torsional vibrations of a beam can be derived as in [3, 4],

\[
EI_{11}\dddot{\xi}_1 + EI_{12}\dddot{\xi}_2 - \rho I_{11}\dddot{\omega}_1 - \rho I_{12}\dddot{\omega}_2 + \rho A\dddot{\xi}_1 - \rho A c\dddot{\theta} = 0
\]

\[
EI_{21}\dddot{\xi}_1 + EI_{22}\dddot{\xi}_2 - \rho I_{21}\dddot{\omega}_1 - \rho I_{22}\dddot{\omega}_2 + \rho A\dddot{\xi}_2 + \rho A c\dddot{\theta} = 0
\] (A.1)

\[
EI_\psi\dddot{\theta} - G\dddot{\theta}_\psi - \rho I_\psi\dddot{\omega} + \rho A c_1\dddot{\xi}_2 - \rho A c_2\dddot{\xi}_1 + \rho J\dddot{\theta} = 0
\]

The third and fourth terms in the top two equations in (A.1) are the rotary inertia of the cross-section, while the third term in the bottom equation in (A.1) is the warping inertia related to the axial motion of the warping. These terms are typically unimportant and therefore often discarded. However, in a finite element formulation with beam elements these effects are very easily implemented.

By discretization with the Hermitian shape functions,

\[
N = \begin{bmatrix} 2s^3 - 3s^2 + 1, s(s - 1)^2, -2s^3 + 3s^2, s^2(s - 1) \end{bmatrix}
\] (A.2)

where \( s = z/\ell \), the global stiffness and mass matrices may be written explicitly with appropriate sub-matrices from integration of the shape functions and their derivatives. This assumes a prismatic beam with identical cross-section. Integration of the shape functions
yields the following sub-matrices,

\[ E = \int_0^1 N^T N \, ds = \frac{1}{420} \begin{bmatrix} 156 & 22 & 42 \\ 22 & 54 & 13 & 156 \\ -13 & 42 & -22 & 4 \end{bmatrix} \] (A.3)

\[ F = \int_0^1 N'^T N' \, ds = \frac{1}{30} \begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & 36 \\ -36 & 3 & 36 & 4 \end{bmatrix} \] (A.4)

\[ G = \int_0^1 N''^T N'' \, ds = \begin{bmatrix} 12 & 6 & 4 \\ 6 & -12 & 12 \\ 2 & -6 & 4 \end{bmatrix} \] (A.5)

The stiffness matrix does not contain coupling elements between bending and torsion, only coupling terms relating coupled bending in the case of an unsymmetric cross-section may occur. Thus a set of \( 4 \times 4 \) matrices are defined as

\[ K_{\xi_1} = EI_{11} \ell^{-3} G \quad K_{\xi_2} = EI_{22} \ell^{-3} G \]
\[ K_{\xi_{12}} = EI_{12} \ell^{-3} G \quad K_{\theta} = EI_{\psi} \ell^{-3} G + GK \ell^{-1} F \] (A.6)

Hereby the element stiffness matrix may be written as

\[ K = \begin{bmatrix} K_{\xi_1} & K_{\xi_{12}} & 0 \\ K_{\xi_{12}}^T & K_{\xi_2} & 0 \\ 0 & 0 & K_{\theta} \end{bmatrix} \] (A.7)

If the chosen coordinate system coincides with minimum one of the principal axes the coupling matrices vanish, \( K_{\xi_{12}} = 0 \). If the elastic and shear centres do not coincide, terms in the mass matrix coupling bending and torsion arise. Thus, the diagonal terms of the mass matrix describing the inertia of bending and torsion are given as

\[ M_{\xi_1} = \rho A \ell E + \rho I_{11} \ell^{-1} F \]
\[ M_{\xi_2} = \rho A \ell E + \rho I_{22} \ell^{-1} F \]
\[ M_{\theta} = \rho J \ell E + \rho I_{\psi} \ell^{-1} F \] (A.8)
The last term in $M_\xi$ represents the rotary inertia of the cross-section and the last term in $M_\theta$ represents the warping inertia. Even though these special inertia terms in most cases are vanishing it is with a minimum of effort taken into account in this way. The off-diagonal terms consist of the coupling rotary inertia and the coupling between bending and torsion, defining the matrices,

$$M_{\xi_{12}} = \rho I_{12} \ell^{-1} F , \quad M_{\xi_{1}\theta} = -\rho A c_2 \ell^{-1} E , \quad M_{\xi_{2}\theta} = \rho A c_1 \ell^{-1} E \quad (A.9)$$

The mass matrix may then be written as

$$M = \begin{bmatrix}
M_{\xi_1} & M_{\xi_{12}} & M_{\xi_{1}\theta} \\
M_{\xi_{12}}^T & M_{\xi_2} & M_{\xi_{2}\theta} \\
M_{\xi_{1}\theta}^T & M_{\xi_{2}\theta}^T & M_{\theta}
\end{bmatrix} \quad (A.10)$$

If the cross-section is double-symmetric the mass matrix contains only diagonal entries. The axial displacements are easily included in this format, yielding $14 \times 14$ stiffness- and mass matrices. For this particular structure of the governing matrices the nodal values are collected in the nodal vector $\mathbf{q}$

$$\mathbf{q} = [\xi_1^T \quad \xi_2^T \quad \theta^T]^T \quad (A.11)$$

with the following nodal arrays,

$$\xi_1 = [\xi_{1A} \quad \ell'_{2A} \quad \xi_{1B} \quad \ell'_{2B}]^T$$
$$\xi_2 = [\xi_{2A} \quad -\ell'_{1A} \quad \xi_{2B} \quad -\ell'_{1B}]^T \quad (A.12)$$
$$\theta = [\theta_A \quad \ell\theta_A' \quad \theta_B \quad \ell\theta_B']^T$$

where the length $\ell$ ensures dimensionless sub-matrices.
References


