## Advanced accurate and computationally efficient numerical methods for wind turbine rotor blade design

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## DTU

Advanced accurate and computationally efficient numerical methods for wind turbine rotor blade design


Paola Bertolini

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Title: Advanced accurate and computationally efficient numerical methods for wind turbine rotor blade design
Department: Wind Energy

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## Project Period:

15 February 2017 - 30 June 2020

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## Preface

This thesis is submitted in partial fulfilment of the requirements for acquiring the degree of Doctor of Philosophy at the Technical University of Denmark.

The PhD project was conducted between February 2017 to June 2020 at LM Wind Power and the Structural and Composite section at the Department of Wind Energy under the supervision of Senior Researcher Martin A. Eder as main supervisor, and Researcher Ali Sarhadi as co-supervisor from the Department of Wind Energy, and Chief Engineer Torben Lindby as advisor at LM Wind Power.

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I wish to express my gratitude to my supervisor Martin Alexander Eder for supporting me in becoming a PhD candidate, for his valuable and essential contribution throughout my research. I would also express my deepest gratitude to my cosupervisor Ali Sarhadi for all our scientific and friendly conversations. Thanks to Mathias Stolpe for his guidance during the first years of the project. Finally, I would like to express my gratitude to Torben Lindby for giving me the opportunity to be part of as great company as LM Wind Power.

I would like to acknowledge Prof. Wenbin Yu for hosting and welcoming me during my external stay at Purdue University. Special thanks to my office colleague Ankit for our fruitful conversations.

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Paola Bertolini

Ris $\varnothing$ campus, Roskilde, June 2020

## Summary

The geometry of wind turbine blades is characterised by the aerodynamic lift generating surface which results in lengthwise geometrical variations (LGVs), namely tapered and twisted cross sections and precurved longitudinal axis. In particular, a tapered beam presents cross-section dimensions which smoothly vary along its longitudinal span and affects the behaviour of the structure. Hence, the stress distributions in tapered structures can be at significant variance with the ones occurring in prismatic beams. The early stages of wind turbine blade design are based on simplified beam models to reduce the computational cost otherwise entailed by 3D full finite element models. The cross-section stiffness properties required for aeroelastic analysis and the prediction of the strains/stresses for structural design and optimization purposes are provided by cross-section analysis methods. Nowadays, the available cross-section analysis methods are based on prismatic hypothesis and consequently the aforementioned taper effects are ignored, notwithstanding the available scientific literature on this matter.

The first part of the thesis sheds light on the effects of taper on the stresses in thinwalled isotropic beams with circular and rectangular cross sections. Elasticity theory is employed to derive closed-form analytical solutions which are compared to 3D finite element models for validation purposes. The analytical equations of the Cauchy stress components provide an insight into the role of taper in the beam behaviour. Indeed, taper evokes geometrical couplings which considerably affect the stress state of the beam. Particularly, shear-axial and shear-bending contributes to the definition of the in-plane shear component and significantly affect the in-plane shear both qualitatively and quantitatively. For instance, neglecting taper effect in structures such as wind turbine blades could result in underestimating the shear components in the proximity of the web adhesive joints, and, therefore, to detrimental designs. In addition, the provided closed-form solutions could be employed for validation of tapered cross-sectional analysis tools. In addition, the provided expressions could be
used for validation of tapered cross-sectional analysis tools.

The second part of the thesis investigates an alternative finite element method which suits for cross-section analysis of tapered beams. It models the beam cross-section as a tapered slice consisting of one-layered of solid finite elements. The nodal forces equivalent to axial, bending and shear are derived from the assumed surface traction acting on the two faces of the slice. In addition, constraint equations for the six rigid body modes, namely three translations and three rotations, are enforced via the Lagrange multiplier method. Parametric studies of the relation between the stresses and the magnitude of the taper angles and the thickness of the slice are conducted on a planar isotropic wedge, whose closed-form solutions in terms of stresses are known. Results reveal the ability of the slice method to predict approximately the stresses in the tapered cross-sections of the wedge.

The present study underlines the importance of the taper effects on the stress components of a tapered beam. Neglecting taper effects can result in a inaccurate stress prediction and accordingly lifetime calculation of tapered beams. The outcome of this project places the foundations for the development of a new advanced tapered cross-section analysis tool where a more accurate prediction of the stress components and lifetime of tapered structures is achieved without exploiting high computational tools.

## Resumé (Danish)

Geometrien af en vindmølle vinge er karakteriseret ved de aerodynamiske profiler, hvilket resulterer i længdevise geometriske variationer (LGV); aftrappede og twistede flader ud af planen for vingens tværsnit, samt en krum længdevis akse for vingens geometri. Dvs, at vingen er repræsenteret ved en aftrappet bjælke hvis tværsnit dimensioner varierer gradvist langs vingen. Dette påvirker strukturens opførsel. Spændingsfordelingen for en aftrappet bjælke kan variere væsentligt fra en prismatisk bjælke. De indledende stadier af vingedesignet er typisk baseret på simplificerede bjælkemodeller for at begrænse beregningstiden sammenlignet med 3D finite element modeller. Tværsnitsegenskaberne til anvendelse ved aero-elastiske beregninger og spændings/tøjnings beregningerne til styrkeoptimering er genererede ud fra tværsnits beregningsmetoder. Indtil nu, er de tilrådighedsværende tværsnitsberegningsmetoder baserede på antagelse om prismatisk geometri, og således er de førnævnte effekter af aftrappende geometri ignoreret, til trods for den tilgængelig videnskabelige litteratur på dette område. Den første del af rapporten belyser effekten af aftrapning på spændinger i tyndvæggede bjælker med isotropiske materialer og cirkulære eller rektangulære tværsnit. Elasticitetsteori er anvendt til at udlede analytiske løsninger som er sammenlignet med 3D finite-element modeller for validering. De analytiske ligninger for Cauchy spændings komponenten giver indsigt i effekten af aftrapning for en bjælke struktur. Det viser sig at at aftrapning leder til en geometrisk kobling som i væsentligt grad påvirker spændings tilstanden for bjælken. Specielt shear-axielt og shear-udbøjning indgår i definition for tværsnits-shear, både kvalitativt og kvantitativt. For eksampel, hvis effekten af aftrapning er ignoreret kan det medføre en underestimering af shear i nærheden af web samlingen, hvilket ville være en kritisk fejl i designet. Ydermere, kunne den analytiske løsning blive brugt som validering af tværsnits analyse-værktøjer. Den anden del af rapporten undersøger en alternativ metode som egner sig til tværsnitsberegninger på en aftrappet bjælke. Den modellerer tværsnittet som en aftrappet skive bestående af et-lags 3D FE elementer. Knudekræfterne repræsenterer rene aksiale kræfter, bøningsmoment og tværkræfter som optræder i
snitfladerne på skivemodellen. Desuden er randbetingelser for de 6 stivlegemeflytninger pålagt ved hjælp af Lagrange Multipliers. Parametriske studier af størrelsen af aftrapningsvinklerne og tykkelsen af den modellerede skive er udført på en plan kile som har en kendt analytisk løsning. Resultatet bekræfter metodens evne til at beregne størrelsen af spændingerne i de aftrappede tværsnit af kilen. Dette studie understreger vigtigheden aftrapnings effekten på stress komponenterne for en aftrappende bjælke. Hvis aftrapning er ignoreres, kan det medføre unøjagtige spændings beregninger med konsekvenser for levetid for bjælker som er aftrappende. Udkommet af dette projekt er er de grundlæggende komponenter til at udvikle avanceret tværsnits beregninger for af trappende bjælker med en mere præcis beregning af spændingen i en aftrappende bjælke samt dens levetid, uden brug af beregnings tunge metoder.

## Publications

List of publications appended to the thesis.
[P1] P. Bertolini, M.A. Eder, L. Taglialegne, P.S. Valvo, Stresses in constant tapered beams with thin-walled rectangular and circular cross sections, Thin-Walled Structures 137 (2019) 527-540. DOI : 10.1016/j.tws.2019.01.008 (Published)
[P2] P. Bertolini, L. Taglialegne, Analytical solution of the stresses in doubly tapered box girders, European Journal of Mechanics-A/Solids 81 (2020) 103969. DOI : 10.1016/j.euromechsol.2020.103969. (Published)
[P3] P. Bertolini, A. Sarhadi, M. Stolpe, M.A. Eder, Comparison of stress distributions between numerical cross-section analysis and 3D analysis of tapered beams, in Proceedings of the 22nd International Conference on Composite Material (ICCM22) (2019) 539-550. (Published)
[P4] P. Bertolini, M.A. Eder, A. Sarhadi, Numerical cross-section analysis of stresses in tapered slices. (To be submitted)

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## Part I

## Background

## Chapter 1

## Introduction

One of the main aims of the past decades has been the shift to clean and renewable sources of energy to reduce the carbon dioxide emission and meet the goals from the Paris Agreement, e.g. improvement of air quality by cutting greenhouse gas emissions and to mitigate the effect of climate change. Among the forms of renewable energy currently available, wind is probably the fastest-growing technology. According to the International Renewable Energy Agency (IRENA)'s latest data [33], the capacity of onshore and offshore wind farms has increased by a factor of 75 in two decades, specifically from 7.5 GW in 1997 to 564 GW in 2018. However, in order for wind energy to be competitive in the global market, improvement of both the wind turbine technology and the supply chain are necessary to further reduce its CoE (Cost of Energy). IRENA predicts a reduction in the electricity cost to less than 3 and 7 cents per kilowatt-hour [kWh] by 2050 respectively in the onshore and offshore wind farms production [33]. It is important to recall that the CoE [34] is defined as

$$
\mathrm{CoE}=\frac{\mathrm{CoT}+\mathrm{CoI}+\mathrm{CoM}}{\mathrm{PP}}
$$

where CoT is the cost of the structural parts of the turbine, i.e. foundation, tower and blades, CoI is the cost of installation and transportation, CoM is the cost of operation and maintenance during the life-time of the turbine, and PP is the power produced during the turbine life-time. Hence, it measures the overall cost of the wind turbine per kilowatt-hour produced. From the perspective of the wind turbine blade design, a reduction of the CoE can be achieved by either decreasing the nominator term CoT , e.g. reducing the amount of material employed and optimising the manufacturing process, or by increasing the denominator PP. The latter is proportional to the rotor dimension, hence to the square of the blade length. Currently, the swept area of wind


Figure 1-1: Development of the dimensions of LM Wind Power blades in the past years. The numbers below the turbines refer to the blade length and the numbers on top are the rotor diameter in meter and the rated power output from the turbine in watt. Courtesy of LM Wind Power.
turbine rotors has reached significantly greater sizes than a decade ago, especially in offshore applications. The blades manufactured by LM Wind Power are an evident proof of this trend, as illustrated in Fig. 1-1. The latest achievement in LM Wind Power is the 107 meters long blade installed on the GE's Haliade-X 12 MW in 2019 [1]. Nevertheless, the design of longer blades entails new challenges not only from a logistic point of view, i.e. manufacturing of the blades, transportation and installation, but also at a structural level. For instance, containing the gravity forces without affecting the structural reliability of the blades becomes a critical task in the design process [21]. Such a compromise is met by employing high-performance engineering materials. The main properties sought for when defining the materials for wind turbine blades are the following: high strength and fatigue resistance to endure aerodynamic and cyclic loads during the blade operating life; high stiffness-to-mass ratio to ensure the stability of the blade profile without increasing the gravity loads [51]. Moreover, the material should suit with the manufacturing of complex blade-shape structures. Composite materials fulfil all the above-mentioned requirements [49]. In addition, the stacking sequence and fibre architecture of the composite material can be designed to improve the blade behaviour. For instance, to reduce ultimately the fatigue load of the blade, the bend-twist coupling can be achieved by rotating the
fibre of the unidirectional lamina located in the spar caps [17]. Consequently, fundamental technological developments have been achieved in the past decades and, nowadays, the versatility of composite materials is exploited to meet high structural requirements at competitive prices. The methodology employed at LM Wind Power is based on glass fibre infused with polyester resin. Nonetheless, the growth of the blade dimensions in recent years has led to the development of hybrid carbon technologies which results in higher strength-to-mass ratio.

From a geometrical point of view, wind turbine blades exhibit an aerodynamic profile, which is characterised by a circular cross section at the root, and extends till the tip following an aerodynamic profile similar to the one shown in Fig. 1-2. The surface of a typical wind turbine blade presents lengthwise geometrical variations (LGVs). Two main geometrical features are taper and twist of the cross section. The former refers to the variation of the airfoil dimensions along the blade span; the latter is the rotation between the airfoil chord line and the rotor plane before deformation, and it is designed in order to increase the lift-to-drag ratio [13]. In addition, the blade longitudinal axis presents a precurved design prior deformation to increase the tower-tip clearance and, therefore, to reduce the required blade stiffness. However, the structural response occurring in straight, untwisted and prismatic beams may be affected by LGVs. Consequently, classic beam theory does not suffice for accurate stress analysis of complex structures, and entails the development of advance beam theories. Because of the rapid advancement of wind technology in the global market and the consequent increase of the costumers' demand for new blade designs in shortened time, computationally efficient blade design tool have been developed in the past decades and further investigations are required to include LGVs effects and to obtain accurate and optimal structural designs.

### 1.1 Motivation

Full 3D finite element models are able to describe the behaviour of complex structures, such as wind turbine blades and plane wings. Particularly, they can take into consideration the effects of material and geometrical couplings due to the anisotropic materials and the LGVs. Figure 1-3 shows an example of a wind turbine blade profile and its cross section. Notwithstanding the above-mentioned capabilities of 3D finite element models, they do not suit in the early stages of the design process, when conceptual aerodynamic and structural studies are performed, and thousands load cases analysed. Indeed, the high number of degrees of freedom of 3D finite element models


Figure 1-2: The 107-meter blade manufactured by LM Wind Power for GE's HaliadeX 12 MW offshore wind turbine. In the foreground, the 5.4 m diameter root and the trailing edge. Courtesy of LM Wind Power.
is several orders of magnitude more computationally expensive than simplified analyses [59]. In addition, geometrical and material details have to be defined prior to running the finite element analysis [16]. Even more, nowadays that the wind energy market is quickly expanding and blade designs are constantly changing to match the costumers' new requirements for longer and highly performing structures, the high computational demand is a killing drawback of the adopted analysis methods. On the other hand, wind turbine blades and plane wings can certainly be modelled as slender structures, given their high length-to-height and length-to-width ratios. For instance, the LM Wind Power 107 meters long blade has a slenderness ratio around 19:1 at the root which has a diameter of 5.4 meters [1]. In solid mechanics, the computational cost of a full 3D finite element analysis of a slender beam is reduced by several orders of magnitude if simplified one-dimensional beam models are employed [57]. Therefore, the design strategy of such structures typically employs simplified beam models in the early stages of the design where conceptual and optimisation studies are performed. Only when a rough optimised material and structural design are defined, a 3D finite element model of the final design is run for further studies, e.g buckling. Simplified beam models consist of cross-sectional analysis and onedimensional analysis of the global response of the beam, as illustrated in Fig. 1-4. The blade is modelled as beam elements in its longitudinal direction as shown in 1-4b. In order to represent properly the 3D structure and to derive correctly the global nodal displacements, the cross-sectional stiffness properties and the coordinates of

(b)

Figure 1-3: (a) Outer surface of an exemplary wind turbine blade where transition zone from the root to the max chord, and the longitudinal taper angle $\alpha(z)$ are highlighted. (b) Typical cross-section geometry of a wind turbine blade comprised of the lift generating airfoil made of sandwich material (black areas) and laminate (white areas). Two shear webs are adhesively connected to the spar caps forming the load-carrying box girder. The latter can be imagined as a thin-walled tapered cantilever beam with rectangular cross section. From [27].
shear and elastic centre have to be provided in the beam model. Moreover, the crosssection properties vary at each cross section of non-prismatic and anisotropic beam. Generally, cross-section analysis is defined from cross-section equilibrium equations and several methods are available in the literature. Cross-section analysis methods aim to (i) derive the cross-section stiffness and inertia properties at each cross section, and (ii) recover the cross-section stresses and strains evoked by the cross-section forces stemming from aeroelastic simulations. The focus of the present work is directed solely to cross-section analysis. Cross-section analysis methods are generally based on the assumption of small strains and displacements. Advanced cross-section analysis methods do not have any restrictions on the material or geometrical properties of the cross-section and, therefore, find application in composite design. Wang et al. classify the available methods in three groups [56]. The first group is based on 3D finite elements. These methods certainly provide accurate results in terms of stress and displacement, but at the same time, they are computationally expensive. Furthermore, supplementary post-processing methods, such as the Blade Properties Extractor (BPE), are required to obtain the cross-section properties from a FEM [41]. The second group is based on classical lamination theory (CLT). An example is the


Figure 1-4: The simplified beam model for a wind turbine rotor blade consists of (a) the cross-section analysis, where the stiffness properties of each cross section are derived, and (b) the one-dimensional beam analysis, where one-dimensional elements are employed along the longitudinal direction of the blade. Only lastly (c) the 3D finite element model of the entire structure is analysed. Figures from [10].
cross-section analysis tool Farob [39]. It assumes a thin-walled multi-cell structure, where stiffness properties of a layup are derived from CLT and collected together to obtain the properties of the entire cross-section [16]. 3D laminate theories have also been employed in cross-section analysis resulting however in an overestimated prediction of the torsional stiffness of 50-80 times [8]. The third group is based on $2 D$ finite element models of the beam cross section. Among them, it is worth recalling the Variational Asymptotic Beam Sectional analysis (VABS) and the BEam Cross-section Analysis Software (BECAS), which are both widely employed in rotor blades design given their capability to model geometrical and material couplings within the cross sections. In addition, the latter group is significantly less computationally expensive than 3D FEM, as shown in the stress analysis of a generic cross section performed by Hodges and Yu [58] via a 3D FEM (25600 brick elements) and the cross-section model in VABS (640 quadratic elements). The former required about one hour, whereas the latter only about 2 seconds. The anisotropic beam theory developed by Giavotto et al. [26] is based on the hypothesis that the warping of the cross section is decoupled from the rigid global displacement of the beam, as sketched in Fig. 1-5. In a 2D finite element context, the theory distinguishes two types of solution for the cross-section equilibrium equations, namely the homogeneous and particular solution. Employing de Saint-Venant principle, the particular solution refers to the extremities of the beam where the loads are applied, whereas the homogeneous solution is the central solutions from which the cross-section properties are derived [30]. The theory was implemented in BECAS by Blasques [10] and in the ANisotropic Beam Analysis tool


Figure 1-5: Sketch of the rigid and warping displacements in prismatic beam. The rigid displacement accounts for the rigid translation and rotation of the cross sections. The warping refers to the cross-section deformation. From Blasques [10].
(ANBA) developed by Morandini et al. [44]. As already mentioned previously, the mass and stiffness matrices are used in the one-dimensional beam analysis to run aeroelastic analyses, which will provide also the nodal forces of the beam elements. Then, the same cross-section analysis tool is employed to derive the strain and stress fields at the analysed cross section.

The cross-section commercial software VABS has been developed over the past decades by Hodges and his collaborators [15] and it is widely employed in academia and industry. It employs a variational method to calculate the stiffness cross-section properties and the stress and strain fields in a generic cross section made of anisotropic material [43]. Specifically, the Variational Asymptotic Method (VAM) by Berdichevskii [5] is exploited. In a functional depending on small parameters, the VAM represents a mathematical methodology exploited to derive its stationary points by dropping the higher order terms. Consequently, it suits with elastic problems which are stated from minimisation of the energy [30]. Order analysis has a key role in the method. To explain it, a continuous differentiable function $f(z) \in[a, b]$ whose order is named $\bar{f}$ is considered. Then, the order of $d f / d z$ is $d f / d z \bar{f} /(b-a)$. Then, the governing equation of 3D elasticity is derived from the Hamilton's extended principle $\int_{t_{1}}^{t_{2}}[\delta(K-U)+\delta \bar{W}] d t=0$, where $K$ and $U$ are the kinetic and strain energies, $\delta W$ is the virtual work resulted from the external loads [59]. In order to apply the VAM, each of the energy terms, $K, U, W$, must be all written as function of the displacement, i.e. warping and rigid body displacements. Moreover, exploiting the slenderness of the analysed beams, the expression for the Hemilton principle can be


Figure 1-6: Schematic representation of a step-wise prismatic approximation of a tapered beam. The beam is modelled as a discrete sequence of $n$ segments of length $\Delta z$ and height $H(z)$. The taper angle $\alpha$ is locally defined as the angle between the tangent at to the beam surface at $\left(x, y, z_{i}\right)$ and the longitudinal axis of the beam.
reduced to one-dimensional variational statement

$$
\begin{equation*}
\int_{t_{1}}^{t_{2}} \int_{0}^{L}[\delta(K-U)+\delta \bar{W}] d z d t=0 \tag{1.1}
\end{equation*}
$$

where $z$ is the longitudinal axis as shown in Fig. 1-3. In particular, assuming linear elastic material and small strains and local rotations, the strain energy is written per unit length. The kinetic energy term is decoupled to kinetic energy associated with warping and rigid body displacement. Using the VAM and minimising the potential energy, the warping displacements are obtained. More details of the method can be found in [59, 29, 48].

The drawback of cross-section analysis methods lies in their prismatic beam assumptions. Indeed, the analysed blade cross section is modelled as part of an equivalent prismatic beam. In other words, the 3D structures are approximated as a step-wise prismatic beam, namely as a sequence of prismatic slices characterised by the dimensions and material properties of the cross section located at the corresponding beam node, as shown in Fig. 1-6. In other words, each cross section is analysed independently from the others. Although such a simplification drastically reduces the computational time, the cross-section analysis ignores the material and geometrical couplings due to the longitudinal direction. For instance stress concentration due to ply-drops or lengthwise-taper effects are not depicted by the cross-section analysis tools in their current state. To the author's best knowledge, even though in the last decade several researchers have addressed the effects of LGVs on beam behaviour, as it will be described in the following chapters, implementation of LGVs in cross-section
analysis tools is not documented in the literature and it is licit to assume that the cross-section analysis tools are still based on step-wise prismatic models. Among the existing LGVs, i.e. taper, twist of the cross section and precurved beam axis, this work focuses on taper, which is generally defined as the variation of the cross-sectional dimensions along the beam axis as depicted in Fig. 1-3. Taper in wind turbine blades can reach amplitude up to $12^{\circ}$ at the transition zone Fig. 1-3. The main consequences of ignoring taper effects in cross-section analysis are demonstrated in [P3], where the Cauchy stresses were computed through the 2D cross-sectional analysis tool BECAS, 3D brick finite element models, and, where available, closed-form analytical solutions. The comparison sheds light on the deviation in terms of magnitude and distribution of the stresses computed with the three different methods. Details on the geometry and load conditions are available in [P3]. Particularly, the stresses computed through the cross-section analysis are not faithful in tapered structures, in fact, some of the stress components deviate in magnitude and others are not captured by the numerical cross-section analysis. Such a disagreement could have a significant impact on the fatigue life design of the composite laminate and it might lead to a structural design which is distant from the optimal one. For example, Fig. 1-7-b shows the through-thickness shear component along the web of the rectangular cross section in Fig. 1-7-a. The flanges of the cross section are made of uniaxial E-glass/epoxy laminate and the webs $\left[+45^{\circ}-45^{\circ}+45^{\circ}\right]$ laminate. The cross-section forces are shear and bending moment. The results in Fig. 1-7-b shows that BECAS predicts zero shear $\sigma_{23}$, where 0123 is the local coordinate system, whereas the results of the analytical and 3D finite element models demonstrate that it is not zero because of the lengthwise taper.

### 1.2 Outline of the thesis

The objective of the present work is to investigate efficient computational methods for tapered beams design, which find application in several engineering fields from wind energy to civil engineering. The final goal is a more accurate analysis of the structural design during the initial design phase, where conceptual studies and structural optimisation are carried out. New analytical and numerical methods are proposed which lead to a more refined prediction of the stresses acting on the cross-section of tapered beams to improve the design reliability and reduce the cost.

The thesis consists in a collection of papers and it is structured in two parts. Part $\mathbf{I}$ is an extended summary outlining the main aspects of the studies and the methods


Figure 1-7: (a) Geometry of a rectangular cross section of a tapered beam loaded at the tip by a shear force. The internal forces at the cross section are shear force and bending moment. The flanges are uniaxial E-glass/epoxy laminate and the webs are $\left[+45^{\circ}-45^{\circ}+45^{\circ}\right]$ laminate. (b) Shear stress component $\sigma_{23}$ in local csys along half of the web. From [P3].
developed during the entire project. Specifically, in chapter 2 the taper effects are analysed analytically. After a literature review on the analytical studies on tapered beams, the method to derive analytical solutions of the Cauchy stress components in 3D tapered beams with circular and rectangular cross sections is described. The method is part of [P1] and [P2]. In chapter 3 the numerical method presented in [P4] is extensively reported. The numerical cross-section analysis method consists in onelayered tapered slice of solid elements and its derivation, implementation, validation in MATLAB is described. In chapter 4 the discussion of the entire study is presented to underline the contributions and limitations of both the analytical and numerical methods. Lastly, in chapter 5 the main conclusions of this project are drawn and topics for further research are suggested.

Part II consists of four research articles. Specifically three journal articles ([P1] and [P2] published, and [P4] to be submitted), and a conference paper [P3].

## Chapter 2

## Analytical solution for the stresses in tapered beams

Slender structures are often designed in order to distribute the material based on the governing internal forces. Therefore, it is not rare to obtain beam characterised by non-prismatic geometries. Generally, beams can exhibit prior to deformation lengthwise geometrical variations (LGVs) to achieve higher aeroelastic performance in wind turbine blades or plane wings, and higher stiffness-to-mass ratio in the girder deck of bridges for example. For instance, the dimensions of the cross-section in a tapered beam as the one in Fig. 2-1-a vary as function of $\alpha(x, y, z)$, which will be defined in the next section. Beams are twisted if the principal axes of the cross section rotate around the longitudinal axis by a twist angle $\theta(z)$, as shown in Fig. 2-1-b. Beams are defined as precurved when the longitudinal axis is not straight Fig. 2-1-c. A similar classification of the beams based on their LGVs is provided in Balduzzi et al. [3]. In general, a non-prismatic beam is characterised by one or more LGVs.

Slender structures are analysed as beam models, such as the classic Euler-Bernoulli or Timoshenko beam theories [53]. When the structure exhibits LGVs, it is common practice to further simplify the forenamed beams as stepwise prismatic, i.e. as a sequence of prismatic beams with varying cross-section dimensions, as shown in Fig. 1-6. This method is based on the idea that the narrower the segments the smaller the error. This work will shed light on why prismatic beam theory does not hold in tapered beams and therefore the step-wise prismatic approach predicts strains and stresses which can be strongly at variance with the real ones.

Particularly, this section presents an analytical method to calculate the solution in terms of stress components in tapered beams. Closed-form solutions for thin-walled beams with rectangular and circular cross sections, modelled under the hypotheses of


Figure 2-1: Sketches of (a) constant tapered beam with circular cross section, (b) twisted beam, where $\theta(z)$ is the twist angle and (c) precurved beams with a not straight centre-line.
small strain and small displacements, and homogeneous, isotropic, linear elastic material behaviour are presented. Although closed-form solutions are solely valid under strict and fixed assumptions related to geometry, material and boundary conditions, they provide the researchers with a deeper and solid understanding of the beam behaviour. Moreover, they can assess the accuracy of other tapered beam models and cross-section analysis methods for in the real of tapered beams.

### 2.1 Literature review

Wind turbine blades exhibit all of the described LGVs. Taper along the trailing edge of a wind turbine blade is zero in the proximity of the root and drastically increases at the max-chord, and becomes almost constant towards the tip. The taper factor is a function of the taper angle $\alpha(x, y, z)$ highlighted in Fig. 1-6. In a beam with straight centreline taper is defined as the angle between the beam longitudinal axis and the tangent plane to the surface of the beam [7]. In the general case of a non-constant tapered beam, the taper angle varies along the beam span, $\alpha=\alpha(z)$. Otherwise, the taper angle is constant, $\alpha=\alpha_{0}$. The effects of LGVs, in particular taper, to the stress distribution have been long known. Since the beginning of the $20^{\text {th }}$ century, scientists started investigating beam theories of tapered beams and developing analytical solutions. One of the first investigations on the stresses of a tapered beam was presented by Michell [42] and Carothers [14] who employed the elasticity theory and the plane stress state to study the behaviour of an isotropic
planar infinite wedge loaded at its tip by a concentrated force and couple, respectively. Both works provided the expressions of the stress components in polar coordinate system. Exploiting the linearity of the problem and employing the superposition of the two solutions, Galerkin [24] and, later, Knops and Villaggio [38] provided the solutions of the stresses in a planar truncated wedge. Later, the hypotheses of linear elasticity behind the above-mentioned studies were exploited by other researchers, from Timoshenko and Gere [55] to more recently Bertolini [6] and Taglialegne [50], to derive the solutions also in terms of strains and displacements, restraining the wedge at the root as a cantilever planar wedge. The above-mentioned solutions show that the in-plane shear components are the one most affected by taper. On the other hand, the normal stresses are at small deviance from the ones described by the Navier equation in a prismatic beam [55]. Given the Cartesian coordinate system depicted in Fig. 2-1, Navier's equation is given by

$$
\begin{equation*}
\sigma_{z z}=\frac{F_{z}}{A}+\frac{M_{x}}{I_{x}} y \tag{2.1}
\end{equation*}
$$

where $F_{z}$ and $M_{x}$ are the internal axial force and bending moment, $A$ is the crosssection area and $I_{x}$ the second moment of area. Under the Navier hypotheses, Bleich [11] pursued an approach similar to the one employed by Jourawsky [54] to derive the well-known formula for shear stresses in prismatic beams, which is given by

$$
\begin{equation*}
\sigma_{y z}=\frac{F_{y} S_{x}}{B I_{x}} \tag{2.2}
\end{equation*}
$$

where the coordinate system in Fig. 2-1 is considered, $F_{y}$ is the internal shear force, $S_{x}$ is the first moment of area of the cross section, and $B$ the thickness of the beam. In this approach, the Cauchy equilibrium is imposed to an infinitesimal portion of a planar tapered beam where bending and shear are considered as internal tractions. Assuming the Navier linear distribution for the normal stresses, in the resulting shear stress formula shear and bending are coupled as function of the taper angle. Bleich's formula, which was derived for homogeneous isotropic beams, was extended to bilinear elastic material by Jadan [35]. Later Paglietti and Carta [45] pointed out that the maximum shear stress can occur at the centreline or at the edges in a tapered beam. Furthermore, Zhou et al. [60] pointed out that the same shear-bending coupling is selfbalanced, hence it influences the stress distribution, but not the resultant at the cross section. To strengthen the validity of the analytical and numerical solutions cited in this survey, a few experiments are presented in the literature. For example, a recent
experiment on concrete-encased beam with corrugated steel webs was performed by Zhou et al. [61] and corroborated the fact that shear stresses are affected by the introduction of taper in the lengthwise direction. In [P1] and [P2] an analytical formulation of the six Cauchy stress components in 3D tapered beams is derived and validated and it will be presented in the following sections. The method is based on linear elasticity hypothesis and assumes that cross sections remain plane after deformation. The formulation holds in straight, untwisted beams with a constant taper along their longitudinal axis and thin-walled cross sections. Although closedform solutions in continuum mechanics are applicable only to ad hoc problems, they are a valuable tool to get a deeper understanding of the beam behaviour and to validate other simplified and approximate beam models. Nonetheless, the suggested method is potentially extendable to more complex geometries. For further details, the reader is encouraged to refer to [P1] and [P2].

### 2.2 Method description

In linear isotropic prismatic beams, the analytical solution of the Cauchy stress tensor is determined through the classical linear elasticity theory. In other words, the problem is described by the Cauchy equilibrium equations, the compatibility equations, and the constitutive relations together with the definition of a consistent set of boundary conditions. A prismatic beam with homogeneous isotropic and linear elastic material is considered. The cross section has two axes of symmetry to remove couplings between the cross-section forces, which are axial force $F_{z}$ and bending moment $M_{x}$. the normal stresses of the problem above-mentioned are described by the Navier formula (2.1). Boley [12] and more recently Bennati et al. [4] state that in a tapered beam the normal stresses predicted through the Navier formula are at small variance with the real values, for moderated amplitude of the taper angles. Contrarily, the well-known Jourawsky formula Eq. (2.2) for the in-plane shear stress solution in prismatic beams, does not hold when taper is introduced. An extension of the aforesaid Jourawsky formula is derived from Cauchy equilibrium considerations on the infinitesimal portion of length $d z$ of the tapered beam in Fig. 2-2-a. The beam has length $L$ and it is characterised by a constant taper angle $\alpha(z)=\alpha$ and straight longitudinal axis. The cross sections of the beam have two axes of symmetry, hence it is defined in a principle axes coordinate system $O x y z$ with origin at the left endsection and the $z$-axis defined as the line of centroids of the cross sections. Moreover, the symmetry removes the couplings between the cross-section forces and allows to
study each case independently. The loading condition of the tapered beam results in in-plane cross-section tractions whose integral over the cross-section area is equivalent to the internal axial force, shear force and bending moment.


Figure 2-2: (a) Segment of length $d z$ of a general tapered beam. (b) Cross section of the tapered beam cut by a chord of length $c$ and local abscissa $\eta$. From [P1].

The beam is divided into two complementary portions by a chord of length $c$ whose normal abscissa $\eta$ lies on the $x y$-plane as illustrated in Fig. 2-2-b. Then, the stress equilibrium in the $z$-direction of the cut portion highlighted in Fig. 2-2-a is imposed. The stresses involved in the equilibrium are the normal stresses $\sigma_{z z}(z)$ at $z, \sigma_{z z}(z+d z)$ at $z+d z$, and the shear stress $\sigma_{z \eta}$ along the cut surface of length $d z$. Distributed axial, $p$, transverse, $q$, and couples, $m$, st the beam centreline are considered as surface integrals of the body forces $p_{z}(z)$ and $p_{y}(z)$. The equilibrium in the $z$-direction can be written as

$$
\begin{equation*}
\int_{A^{*}(z)} \sigma_{z z} d A+\sigma_{z \eta} \frac{c+(c+d c)}{2} d z-\int_{z}^{z+d z} \int_{A^{*}(z)} \frac{p_{z}}{A} d A d z=\int_{A^{*}(z+d z)}\left(\sigma_{z z}+d \sigma_{z z}\right) d A . \tag{2.3}
\end{equation*}
$$

Several mathematical artifices, which are described in details in [P1], lead to the following expression of the in-plane shear component

$$
\begin{equation*}
\sigma_{z \eta}=\frac{1}{c}\left[F_{z} \frac{d}{d z}\left(\frac{A^{*}}{A}\right)+\left(F_{y}-m_{x}\right) \frac{S_{x}^{*}}{I_{x}}+M_{x} \frac{d}{d z}\left(\frac{S_{x}^{*}}{I_{x}}\right)\right], \tag{2.4}
\end{equation*}
$$

where $m_{x}(z)$ is the distributed bending couple. Equation (2.4) is an extended version of Jourawsky's equation(2.2), where the variation of the cross-section dimensions along the longitudinal beam axis, represented by the term $d / d z$, is taken into account. In the prismatic case, i.e. $d / d z=0$, Eq. (2.4) correctly reduces to Eq. (2.2). Moreover, it is also a generalisation of the Bleich formula, since it accounts for the shear-axial coupling. The axial force and the bending moment appear as function of
$d / d z$. Specifically, the internal axial force is responsible for shear stress which varies linearly with the variation of the area of the cross section, and the bending moment causes shear stress as function of the variation of the first moment of area with respect to the second moment of area.

The expressions of the normal stress $\sigma_{z z}$ and shear stress $\sigma_{z \eta}$ work for the general case of tapered beams with doubly symmetric cross sections in the linear elastic domain. In the following part, the thin-wall theory is employed. The lateral surfaces of the beam are assumed in plane-stress state and a local reference system $O 123$ can be defined in such a way that the through-thickness stress components are zero. The local coordinate system is tangent to the middle-wall of the beam, with the 2 -axis tangent to the wall and the 3-axis outward normal to the same surface of the beam, as shown in Fig. 2-4. Given a taper angle $\alpha$, a rotation matrix $\mathbf{R}$ can be defined to transform the local coordinate systems $O 123$ to the global Oxyz. The rotation matrix $\mathbf{R}$ allows to transform the stress components from the local coordinate system to the global one through the relation $\boldsymbol{\sigma}(x, y, z)=\mathbf{R}^{T} \boldsymbol{\sigma}(1,2,3) \mathbf{R}$. The latter consists in a set of six linearly independent equations whose solution provide the expressions of the three through-thickness stress components in the global coordinate system.

Lastly, to determine the remaining stress component the following Cauchy equilibrium equations are employed:

$$
\begin{equation*}
\nabla \boldsymbol{\sigma}=0 \tag{2.5}
\end{equation*}
$$

Equation (2.5) in three-dimensional problems defined in the Cartesian reference system consists of partial differential equations. Consequently, the solutions are defined up to a constant of integration. The latter can be defined by imposing the stress equilibrium of an infinitesimal portion of length $d z$ of the cross section anew. More details are provided in the example described in the following section.

### 2.3 Application

Tapered box girders find several applications in the civil and aerospace field. The deck girders of a bridge or the main spar of a wind turbine blade are typically designed as vertically or doubly tapered thin-walled beams with rectangular or I-shaped cross section. Structures such as bridge piers or the towers of wind turbines are designed as conical beams. Therefore, the present method is applied to three specific geometries inspired by real applications. Hence, closed-form solutions for conical beams and tapered box girders are derived and compared to the results from 3D finite element
models based on linear eight-noded elements to corroborate the validity of the derived analytical solutions. The solution for the thin-walled doubly tapered cantilever beam in Fig. 2-3 presented in [P2] will be derived as an example. The conical beam and vertically tapered beam are instead treated in [P1].


Figure 2-3: (a) Side and (b) top view of a tapered cantilever box girder. The beam has length $L$ and presents a vertically and horizontally tapered profile. Specifically, constant vertical taper angle $\alpha$ and constant horizontal taper angle $\beta$. (c) Front view and (d) exemplary cross section of the doubly tapered box beam, where the projected thicknesses of the flanges and webs are highlighted. From [P2].

In order to solve the structure, support conditions are necessarily defined, but at the same time, they are not bound to the method. Thence, the beam is model as a cantilever beam with a clamped end at $z=0$ and free tip at $z=L$, where $z$ is the longitudinal beam axis and $L$ is its total length, as shown in Fig. 2-3. The box girder presents a constant vertical taper $\alpha$ and a constant horizontal taper $\beta$. The former is defined as the angle between the $x z-$ and the flange planes and the latter is the angle between the $y z-$ and webs planes. In the following, superscript $f$ and $w$ refers to the flange and web respectively. The taper angles are constant, hence the width $2 b(z)$ and the height $2 h(z)$ of the beam cross section constantly vary along the beam span decreasing from the root $(z=0)$ to the tip $z=L$. Specifically, the following
linear relations uniquely define each point of the cross section:

$$
\begin{equation*}
b(z)=b_{0}-z \tan \beta, \quad h(z)=h_{0}-z \tan \alpha . \tag{2.6}
\end{equation*}
$$

where $b_{0}$ and $h_{0}$ are the width and the height of the cross section at the root. Because of the horizontal and vertical taper, the thickness of the webs and flanges, i.e. $t_{w}^{*}$ and $t_{f}^{*}$, is projected on the $x y$-plane as

$$
\begin{equation*}
t_{w}=\frac{t_{w}^{*}}{\cos \beta}, \quad t_{f}=\frac{t_{f}^{*}}{\cos \alpha} . \tag{2.7}
\end{equation*}
$$

The external forces accounted for are the concentrated bending moment around the $x$-axis $M_{x}$ and a concentrated force $F_{e}$ arbitrarily directed. They are applied at the centroid of the cross section located at $z=L$. By exploiting the superposition principle, the concentrated force $F_{e}$ is decomposed along the $x$ - and $y$ - axes in axial $F_{z}$ and shear $F_{y}$ components. Once the beam geometry, load conditions and boundary conditions are defined, the above-described method can be applied to derive the Cauchy stresses at a specific cross section. Firstly, the normal stresses are directly determined through Navier's equation Eq. (2.1). Then, the shear stress distribution in tapered beams is given by Eq. (2.4). To apply the latter equation, a cut through the walls whose local abscissa $\eta$ is chosen moving clock-wise has to be defined. Since the cross section has two axes of symmetry, the in-plane shear component is zero at the middle of the flanges $(x=0, y= \pm h)$ for stress equilibrium. Therefore, the cut is chosen to have the origin at the centre of the flange and the area and first moment of area are given in the flanges as

$$
\begin{gather*}
A^{*}=x t_{f}, \quad S_{x}^{*}=x h(z) t_{f}  \tag{2.8}\\
A^{*}=[h(z)-y] t_{w}+b(z) t_{f}, \quad S_{x}^{*}=b(z) h(z) t_{f}+\left[h(z)^{2}-y(z)^{2}\right] \frac{t_{w}}{2} \tag{2.9}
\end{gather*}
$$

A local coordinate system $O 123$ can be defined in both the flanges and the webs, in such a way that the 2 -axis is in parallel to $x$ - and $y$-direction respectively and the 3 -axis is outward normal to the related planes, as illustrated in Fig. 2-4. Therefore, two rotation matrices should be defined to transform the stresses from the local to the global reference system. The transformation provides the throughthickness stress components in global coordinate system as functions of the normal and shear components scaled by a function of the taper angles. The solutions for the flanges and webs are obtained as functions of the taper angle and the normal and


Figure 2-4: Local coordinates systems $O 1^{w} 2^{w} 3^{w}$ and $O 1^{f} 2^{f} 3^{f}$ defined on the webs and the flanges of a doubly tapered box girder. The webs and flanges have thicknesses $t_{w}$ and $t_{f}$ respectively. The 2 -axis is parallel to $x$ - and $y$-direction respectively and the 3 -axis is outward normal to the related planes. From [P2].
shear components.

$$
\boldsymbol{R}^{f}=\left[\begin{array}{ccc}
0 & -\sin \alpha & \cos \alpha  \tag{2.10}\\
1 & 0 & 0 \\
0 & \cos \alpha & \sin \alpha
\end{array}\right], \quad \boldsymbol{R}^{w}=\left[\begin{array}{ccc}
-\sin \beta & 0 & \cos \beta \\
0 & 1 & 0 \\
\cos \beta & 0 & \sin \beta
\end{array}\right]
$$

Lastly, from the first and second Cauchy equilibrium equations given in Eq. (2.11) the two missing stress components $\sigma_{x x}^{f}$ and $\sigma_{y y}^{w}$ can be derived:

$$
\begin{equation*}
\sigma_{y y}^{w}=-\int_{0}^{y} \frac{\partial \sigma_{y z}^{w}}{\partial z} d y+\left.\sigma_{y y}^{w}\right|_{y=0}, \sigma_{x x}^{f}=-\int_{0}^{x} \frac{\partial \sigma_{z x}^{f}}{\partial z} d x+\left.\sigma_{x x}^{f}\right|_{x=0} \tag{2.11}
\end{equation*}
$$

However, two constants of integration $\left.\sigma_{y y}^{w}\right|_{y=0}$ and $\left.\sigma_{x x}^{f}\right|_{x=0}$ have to be determined. The expedient exploited to find the constants of integration is based on the equilibrium in the $x$ - and $y$-directions of the intersection between the flange and the web of infinitesimal length $d z$ which is shown in Fig. 2-5. The stress components are then compared to the ones predicted in an equivalent 3D finite element model for validation purposes. Such a comparison shows a good agreement between the two methods. Hence, the analytical method predicts with good approximation the Cauchy stresses in tapered beam within its boundaries of applicability.

The closed-form solutions of the six Cauchy stresses are then derived for a doubly tapered box girder made of homogeneous isotropic beam. The expressions are not reported for the sake of brevity, but they are available in [P1] and [P2]. The following points are worth to be discussed.

## The Navier formula

The method is developed from the key-assumption that the normal stresses in tapered beams can be described with good approximation by the Navier formula. Surely, employing the Navier formula represents a favourable expedient to derive accurate results through a handy methodology. However, it is worth underlining that Navier's formula does not retrieve the exact normal stresses distribution in tapered beams. If a planar wedge is considered, the exact normal stresses follow a quadratic distribution as demonstrated in [4] and as it will be shown numerically in chapter 3. However, the error committed in approximating it to a linear distribution is small and acceptable for moderate taper angles ( $10^{\circ}$ ) [50].

## Stresses distribution

Besides the normal stress components, which are assumed to be defined as in prismatic beams, the remaining stress components are functions of the vertical or horizontal taper angles. Among them, the in-plane shear stress given by Eq. (2.4) is mostly affected by taper. Taper induces a shear-axial coupling as function of $d / d z\left(A^{*} / A\right)$, and a shear-bending coupling proportional to $d / d z\left(S_{x}^{*} / I_{x}\right)$, which might result in a counterintuitive shear stress distribution. Indeed, if the shear-bending coupling prevails over the pure shear term, the curvature of the shear distribution is affected. For example, Figures 2-6 shows how a horizontal or vertical taper angle, which has the same amplitude of $4.3^{\circ}$, modifies the in-plane shear stress distribution $\sigma_{y z}$ along the web. An internal shear force $F_{y}$ and an internal bending moment around the $x$-direction, $M_{x}$, are assumed to occur at the analysed cross section, hence the first


Figure 2-5: Cauchy stresses in the (a) $x$ - and (b) $y$ - direction on an infinitesimal portion $d z$ of the corner between the flange and the web of a doubly tapered beam. From [P2].


Figure 2-6: In-plane shear stress distribution along half of the web of the box girder of Fig. 2-3 where (a) $\alpha=0$ and $\beta=\beta_{0} \neq 0$ (horizontal taper angle), (b) $\alpha=\alpha_{0} \neq$ and $\beta=0$ (vertical taper angle) tapered box girder. In both cases the same amplitude of the taper angle is considered $\beta_{0}=\alpha_{0}=4.3^{\circ}$. From [P2].
term in Eq. (2.4) is zero. Figure 2.4 shows the shear stress along half of the web of the box girder. In addition, the shear distribution due to the second and the third term of Eq. (2.4), i.e. "pure shear" and "pure bending" respectively, are reported separately. Figure 2-6-a refers to the horizontally tapered box girder, whereas Figure $2-6$-b to the vertically one. Both cases exhibit the same pure shear stress distribution since it is not a function of the taper angle. On the other hand, the pure bending considerably depends on the taper angle and has a stronger influence in the vertically tapered case where, because of taper, the maximum shear stress occurs at the edge $(y=500 \mathrm{~mm})$ and the minimum at the centre $(y=0)$. It is worth noting that the stress components in global coordinate system, none of which is zero in doubly tapered beams, are derived from the Cauchy equilibrium. Hence, they are expressed as functions of the normal and/or the in-plane shear and the tangent of the taper angles. In addition, since the magnitude of the internal shear-bending force varies along $z$, so does the curvature of the in-plane shear stress distribution. As an example, Fig. 2-7 shows the variation of the shear stress distribution at several cross section of a doubly tapered beam loaded by a shear force at its tip.


Figure 2-7: In-plane shear stress along the webs at seven cross sections of a tapered box girder subjected to $F_{y}=1 \mathrm{kN}$. The stresses at location $1--6$ are characterised by a convex distribution with maximum values at the edges of the webs. This is in contrast to the shear stress distribution in prismatic beam, where the shear has a concave distribution with maximum value at the centre and minimum at the edges. Only section 7, i.e. where the shear force is applied, has a concave distribution. Indeed, the shear-bending coupling evoked by taper is zero at that specific cross section. From [P2].

## Taper effects on the design

The analytical expressions derived in this chapter show that the stress components are altered by the taper, and, moreover, contrarily to what expected in prismatic beams, the stress tensor can be fully populated in a doubly tapered box girder as the one in Fig. 2-3. Therefore, if a tapered beam is approximated as a stepwise prismatic beam, the stress analysis produces results which are at strong variance with the real stresses. In particular, as already pointed out, taper evokes shear-bending coupling which is responsible for a counterintuitive distribution of the in-plane shear stress, as shown in Fig. 2-6-b. Indeed, under the hypotheses of symmetric cross-sections and homogeneous isotropic material, the maximum shear in a tapered box girder does not necessarily occur in the elastic centre, as expected in a prismatic box girder, but it might occur at the web-joints due to the shear-bending effect. Moreover, in the global coordinate system taper induces through-thickness shear components which would not occur in prismatic beams. Neglecting the effects of transverse shear components and underestimating the in-plane shear stress at the flange-web joints, could be detrimental in composite tapered box girder design, such as wind turbine blades. The outer shell and shear webs of the blades are generally assembled through adhesive bonded joints [37]. Adhesive bonded joints are prone to debonding failure and delamination [47], which are mainly governed by transverse shear. Hence, a wrong prediction of the in-plane and through thickness shear could affect the performance
of the entire structure. In addition, providing the designer with more accurate stress analysis solution, would increase the design reliability and, ultimately, lowering the safety factors. The latter would have an important impact on the blade mass and costs. In addition, transversely oriented cracks which occur in the adhesive joints of rotor blades could be caused by the through thickness stresses due to taper. Eder et al. [22] pointed out the possible relation between the cracks and LGVs.

Since taper modifies the entire stress tensor, also failure and fatigue designs are affected by it. Employing the provided analytical solutions, such effects can be evaluated by applying, for example, the von Mises failure criterion [36] to derive the equivalent stress in isotropic beams. The von Mises stress is given as

$$
\begin{equation*}
\sigma_{v M}=\sqrt{\frac{1}{2}\left[\left(\sigma_{x x}-\sigma_{y y}\right)^{2}+\left(\sigma_{y y}-\sigma_{z z}\right)^{2}+\left(\sigma_{z z}-\sigma_{x x}\right)^{2}+6\left(\sigma_{x y}^{2}+\sigma_{x z}^{2}+\sigma_{y z}^{2}\right)\right]} . \tag{2.12}
\end{equation*}
$$

If the maximum equivalent stresses in tapered and in prismatic beams are compared, as shown in Fig. 18 of [P2], taper effects are not so severe, especially in bendingdominated problems. Indeed, it is worth recalling that the normal stresses in tapered beams are approximated with the prismatic distribution given by Navier's equation (2.1). On the other hand, even a small variation of the equivalent stress leads to a strong deviance in terms of fatigue, since the stress amplitude and the number of cycles to failure are related with a power-law relation [2]. To provide the reader with an example, a box girder loaded by a shear force at its tip is vertically and horizontally tapered with a taper angle equal to $5^{\circ}$, the von Mises stress results to be the $3 \%$ lower than in a prismatic beam. However, if the same von Mises stress is employed in the fatigue design, the number of cycles to failure calculated by means of Basquin's law of fatigue [2] are about $15 \%$ lower than in the prismatic case. Consequently, the fatigue design might be underestimated.

It is licit to expect that taper evokes all the six stress components also in tapered composite beams, which are strongly susceptible to transverse stresses [46, 23]. Also in case of composite materials, a fully populated stress tensor could have detrimental effects on the failure design and the fatigue performance. Therefore, since taper induces a complex multiaxial stress state, accurate stress analysis inclusive of taper effects is necessary to avoid an overestimation of the number of cycles to failure.

## Chapter 3

## Numerical analysis for the stresses in tapered slices

The analytical studies illustrated in the previous chapter prove that taper has a key role in the analysis of tapered beams. Indeed, the variation of the dimensions in the beam lengthwise direction affects the stress distributions at each cross section. As explained in the introduction of this thesis, aircraft wings and wind turbine towers and blades are modelled as slender structures which are designed to endure cyclic aerodynamic loads to prevent failure and fatigue damage. Whilst 3D finite element models would provide an accurate description of the behaviour of such structures, simplified models are required to shorten the computational time and to allow iterative analysis. Exploiting the slenderness of the above-mentioned structures, the 3D real beam can be modelled by means of beam elements. Wind turbine blade design process consists of several phases that are executed sequentially. Given the material and geometrical properties of the airfoil and internal webs in Fig. 1-3, cross-section analysis is employed to compute the stiffness and mass matrices and the elastic and shear centres of the considered cross section. This information is necessary in aeroelastic analysis, where the entire wind turbine is modelled via beam elements [52]. The aeroelastic analysis is carried out to provide the time history of the nodal beam forces. The latter are then used in the same cross-section analysis tool to obtain the strain, stress and displacement of the cross sections. The cross-section analysis tools have therefore two scopes: (i) evaluation of the cross-section stiffness properties and (ii) recovering the stress and strain tensors in the nodes of the elements discretizing the cross section. The cross-section analysis methods employed in composite structure design have the capabilities to deal with material anisotropy and complex cross-section geometries, but they generally ignore the geometrical variations in the lengthwise direction, e.g
taper. However, as described in previous chapters, tapered beams present a different behaviour than prismatic beams. In particular, the stress components can be strongly affected and, consequently, neglecting taper effects could lead to non-accurate design. The goal of this chapter is to investigate an alternative cross-section analysis method which takes into account the lengthwise taper of the beam. The work presented here is supplementary to the manuscript [P4].

### 3.1 Literature review

To the author's best knowledge, most of the cross-section analysis tools available in the literature are based on the step-wise prismatic assumption [29], namely the cross-section material and geometrical properties are assumed to be constant along the beam longitudinal axis as shown in Fig. 1-6. Such an approximation as been long supported by the idea that narrower prismatic steps, would capture the correct tapered beam behaviour. Results shown in chapters 1 and 2 demonstrate that it is not the case since taper induces geometrical couplings in the stress components. However, including taper in the current cross-section methods is a non-trivial task, as will be explicated in the following. For instance, the theory behind BECAS, i.e. the anisotropic beam theory from Giavotto et al. [26], is considered. The theory is based on the prismatic beam assumption, i.e. the geometrical and material properties of the cross section do not vary in the longitudinal direction. In other words, the nonprismatic beam is simplified as stepwise prismatic, as shown in Fig. 1-6. The internal work of a prismatic beam is also constant in the longitudinal direction, hence it can be defined as the surface integral of strain energy per unit length as $\partial W_{\text {int }} / \partial z=\int_{A} \varepsilon^{T} \boldsymbol{\sigma} d A$. Furthermore, decoupling the rigid body motions and warping, as explained in [26], allows the application of the Euler-Bernoulli theory where the cross-section forces can be directly coupled to the cross-section displacements. Nonetheless, if LGVs are included in the formulation of the internal work, the strain energy cannot be defined per unit length but the volume integral, $W_{i n t}=\int_{V} \boldsymbol{\varepsilon}^{T} \boldsymbol{\sigma} d V$, must be considered to include the variation of the geometrical properties in the longitudinal direction. Attempts to extend the 2D cross-section analysis methods in that direction are available. It is worth mentioning two studies on tapered beams that were published from the VABS research group, where taper is introduced in the formulation [31, 32]. A 2D tapered beam model is developed through the application of the variational asymptotic method, VAM, namely the mathematical strategy applied also in VABS and introduced in section 1.1 of chapter 1. Although the importance
of including a taper-correction factor in the stress recovery and stiffness properties is underlined, the theory is not fully developed [29] nor documented in the scientific literature [48].

The current chapter lays the foundation for a cross-section analysis method which accounts for taper effects. Differently from BECAS and VABS, where the cross section is modelled as a 2D model, 3D solid finite elements are employed. Certainly, solid elements have the disadvantage of increasing the size of the finite element problem by introducing more nodes in the model. On the other hand, they allow to model lengthwise geometrical variation such as taper and to consider the geometrical couplings introduced by taper itself. Two cross-section analysis methods based on 3D solid finite element model are already available in the literature and will be addressed in the following. Ghiringhelli and Mantegazza [25] derived the cross-sectional properties of a prismatic linear, and untwisted beam from the 3D model of the cross section. The cross section is modelled using one layer in the longitudinal direction of standard solid elements and the equilibrium solution is derived following the anisotropic beam theory developed by Giavotto et al. [26]. One of the issues faced in the derivation of the solution for the 3D finite element model in [25] is related to the thickness of the model in the longitudinal direction. Nonetheless, it was concluded that the best compromise is obtained by choosing slice thicknesses equal to the averaged elements dimensions. More recently, Couturier and Krenk [19] have suggested the derivation of the cross-section stiffness properties and the stress components by modelling the cross section as a single-layered slice of solid element defined by Hermitian shape-function in the longitudinal direction. The finite element problem is defined by imposing six deformation modes corresponding to extension, torsion, and shear and bending in two directions.

The novelty of the method illustrated in this chapter is the introduction of taper in the finite element formulation, in order to include the geometrical couplings which are completely ignored otherwise. In other words, the step-wise prismatic approximation assumed in classic cross-section analysis methods and shown in Fig. 1-6 will be replaced by the step-wise tapered approximation as sketched in Fig. 3-1. The finite element model is based on serendipity linear shape-function. Three internal tractions, whose integral over the cross-section area corresponds to axial, shear and bending, are implemented. They are assumed to be defined under the Navier assumptions. Then, the advantages and disadvantages of such assumptions are probed. In addition, the six rigid-body degrees of freedom, i.e. three translations and three rotations, are constrained through the Lagrange multiplier method. Results will show the capability


Figure 3-1: Sketch of a step-wise tapered beam whose outer surface is defined by $s(z)$. The 3D tapered beam is approximated as a sequence of stepwise tapered beams of length $l$. Each step has a constant taper angle defined as the angle between the tangent at $s\left(z_{i}\right)$ and the beam longitudinal axis.
of the method to retrieve qualitative correct stresses in tapered slices. Moreover, the dependency on the slice thickness is underlined and investigated through parametric studies.

### 3.2 Method description

The cross section located at $\hat{z}$ of the tapered beam shown in Fig. 3-2 is studied in here. To define the 3D finite element model of the cross section, the slice of the tapered beam with thickness $\Delta$ is considered and consists of a single-layered slice in the $z$-direction. The mesh topology in the $x y$-plane is instead defined through standard finite element convergence studies. To develop the finite element method for


Figure 3-2: Geometry and coordinate system of a beam with constant taper and generic cross section. The highlighted cross-section slice has thickness $\Delta$ and its mid-cross section is located at the beam mid-span $\hat{z}=L / 2$.
the slice, the force equilibrium of the 3D slice is imposed. The presented formulation
is based on first-order approximation, hence geometrical and material nonlinearity cannot be considered in the current formulation. The strains and stresses at each point of the beam are expressed as six-terms vectors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$,

$$
\boldsymbol{\varepsilon}=\left[\begin{array}{lll}
\varepsilon_{x x} & \varepsilon_{y y} & \varepsilon_{z z} 2 \varepsilon_{x z} 2 \varepsilon_{y z} 2 \varepsilon_{x y}
\end{array}\right]^{T}, \quad \boldsymbol{\sigma}=\left[\begin{array}{ll}
\sigma_{x x} & \sigma_{y y}  \tag{3.1}\\
\sigma_{z z} & \sigma_{x z} \\
\sigma_{y z} & \sigma_{x y}
\end{array}\right]^{T} .
$$

Under the assumption of linear elastic material, the strain and stress vectors are related by the $6 \times 6$ material constitutive matrix $\boldsymbol{E}$ as follows

$$
\left[\begin{array}{c}
\sigma_{x x}  \tag{3.2}\\
\sigma_{y y} \\
\sigma_{z z} \\
\sigma_{x z} \\
\sigma_{y z} \\
\sigma_{x y}
\end{array}\right]=\left[\begin{array}{cccccc}
2 \mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & 2 \mu+\lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & 2 \mu+\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{array}\right]\left[\begin{array}{c}
\varepsilon_{x x} \\
\varepsilon_{y y} \\
\varepsilon_{z z} \\
2 \varepsilon_{x z} \\
2 \varepsilon_{y z} \\
2 \varepsilon_{x y}
\end{array}\right],
$$

where the Lamé constants $\mu$ and $\lambda$ are employed. The strain components at a point of the cross section are defined as

$$
\begin{equation*}
\varepsilon=\hat{\boldsymbol{B}} \boldsymbol{r} \tag{3.3}
\end{equation*}
$$

where $\boldsymbol{r}=\left[\begin{array}{lll}r_{x} & r_{y} & r_{z}\end{array}\right]^{T}$ is the displacement vector of a point of the cross section and $\hat{\boldsymbol{B}}$ is defined as [18]

$$
\hat{\boldsymbol{B}}=\left[\begin{array}{cccccc}
\frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z}  \tag{3.4}\\
0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\
0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{array}\right]^{T}
$$

In the finite element procedure, the 3D slice is discretised into $n_{e}$ elements. Therefore, the displacement vector $\boldsymbol{r}$ can be approximated through the element shape-functions $\boldsymbol{N}$ and the nodal displacements $\boldsymbol{u}\left(x_{i}, y_{i}, z_{i}\right)=\left[\begin{array}{lll}u_{x i} & u_{y i} & u_{z i}\end{array}\right]^{T}$, where $i=1 \ldots n_{\text {nodes }} \in \mathbb{N}$ as follows

$$
\begin{equation*}
\boldsymbol{r} \approx \boldsymbol{N} \boldsymbol{u}\left(x_{i}, y_{i}, z_{i}\right) \tag{3.5}
\end{equation*}
$$

Substitution of Eq. (3.5) in Eq. (3.3) provides the strain components as function of the nodal displacements

$$
\begin{equation*}
\boldsymbol{\varepsilon}=\boldsymbol{B} \boldsymbol{u}\left(x_{i}, y_{i}, z_{i}\right) \tag{3.6}
\end{equation*}
$$

where $\boldsymbol{B}=\hat{\boldsymbol{B}} \boldsymbol{N}$ is the $6 \times 3$ strain-displacement matrix. The elastic equilibrium of the cross section is enforced through the virtual work principle [43]. The variation of
the virtual work is given as

$$
\begin{equation*}
\delta W=\delta W_{i n t}+\delta W_{e x t}, \tag{3.7}
\end{equation*}
$$

where $\delta W_{\text {int }}$ and $\delta W_{\text {ext }}$ are the virtual variations of the internal and external work. The internal virtual work represents the variation in strain energy:

$$
\begin{equation*}
\delta W_{i n t}=\int_{V} \delta \varepsilon^{T} \boldsymbol{\sigma} d V \tag{3.8}
\end{equation*}
$$

where $V$ is the volume. Recasting Eq. (3.2) and (3.6) in Eq. (3.8), leads to

$$
\begin{equation*}
\delta W_{i n t}=\int_{V} \delta \boldsymbol{\varepsilon}^{T} \boldsymbol{\sigma} d V=\int_{V} \delta \varepsilon^{T} \boldsymbol{E} \boldsymbol{\varepsilon} d V=\int_{V} \delta \boldsymbol{u}^{T} \boldsymbol{B}^{T} \boldsymbol{E} \boldsymbol{B} \boldsymbol{u} d V \tag{3.9}
\end{equation*}
$$

Neglecting the volume and surface forces, the external work is defined from the surface integral of the shape-functions and the internal tractions $\boldsymbol{p}=\left[\sigma_{y y} \sigma_{y z} \sigma_{z z}\right]^{T}$ as

$$
\begin{equation*}
\delta W_{e x t}=-\int_{A} \delta \boldsymbol{r}^{T} \boldsymbol{p} d A=-\int_{A} \delta \boldsymbol{u}^{T} \boldsymbol{N}^{T} \boldsymbol{p} d A \tag{3.10}
\end{equation*}
$$

Equations (3.9) and (3.10) are substituted in Eq. (3.7). Since the virtual work must be satisfied for any virtual displacement $\delta \boldsymbol{u}$, a necessary and sufficient equilibrium conditions is that $\delta W=0$, hence $\delta W_{i n t}=-\delta W_{\text {ext }}$. By integration over the element, Eq. (3.7) becomes

$$
\begin{equation*}
\int_{V^{e}} \delta \boldsymbol{u}^{T} \boldsymbol{B}^{T} \boldsymbol{E} \boldsymbol{B} \boldsymbol{u} d V=\int_{A^{e}} \delta \boldsymbol{u}^{T} \boldsymbol{N}^{T} \boldsymbol{p} d A \tag{3.11}
\end{equation*}
$$

The element stiffness matrix $\boldsymbol{K}^{e}$ and the nodal forces vector applied by elements $\boldsymbol{f}^{e}=\left[\boldsymbol{f}_{x}, \boldsymbol{f}_{y}, \boldsymbol{f}_{z}\right]^{T}$ can be defined as

$$
\begin{equation*}
\boldsymbol{K}^{e}=\int_{V^{e}} \boldsymbol{B}^{T} \boldsymbol{E} \boldsymbol{B} d V, \quad \boldsymbol{f}^{e}=\int_{A^{e}} \boldsymbol{N}^{T} \boldsymbol{p} d A \tag{3.12}
\end{equation*}
$$

Finally, the element stiffness matrices and the element nodal forces vectors are assembled in the global stiffness matrix and global nodal force vector following the typical finite element procedure,

$$
\begin{equation*}
\boldsymbol{K}=\sum_{i=1}^{n_{e}} \boldsymbol{K}_{i}^{e}, \quad \boldsymbol{f}=\sum_{i=1}^{n_{e}} \boldsymbol{f}_{i}^{e} \tag{3.13}
\end{equation*}
$$



Figure 3-3: (a) Coordinates of the isoparametric element with centre in $\left(\xi_{0}, \eta_{0}, \zeta_{0}\right)$. (b) Real tapered element of height $H$, width $B$ and taper angle $\alpha$ with centre in $x_{0}, y_{0}, z_{0}$. The transformation between the two system is performed through the Jacobian matrix J.
where $n_{e}$ is the number of elements in the cross sections and $\sum_{e}$ stands for the assembling procedure. The weak form of the equilibrium defining a finite element problem is stated by:

$$
\begin{equation*}
\boldsymbol{K} \boldsymbol{u}=\boldsymbol{f}, \tag{3.14}
\end{equation*}
$$

where the stiffness matrix $\boldsymbol{K}$ relates the nodal forces $\boldsymbol{f}$ to the nodal displacements $\boldsymbol{u}$. The assembled stiffness matrix $\boldsymbol{K}$ obtained from (3.13) is singular hence not invertible. To solve the system in (3.14) six constrains related to the six rigid body motions have to be imposed, as illustrated in section 3.2.3.

### 3.2.1 The isoparametric formulation

The above-mentioned finite element formulation is defined in global Cartesian coordinate (Oxyz), but it is standard practice to set and solve a finite element problem in the isoparametric coordinate system $(O \xi \eta \zeta)$ where the isoparametric element in Fig. 3 -3 has dimensions $-1<\xi, \eta, \zeta<1$ in natural coordinates. The nodal coordinates are mapped from the Cartesian to the isoparametric coordinate system and vice versa through the application of the Jacobian matrix $\boldsymbol{J}$, which is defined as the gradient of the global coordinates of the element with respect to the isoparametric coordinates as shown below

$$
\boldsymbol{J}=\left[\begin{array}{lll}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi}  \tag{3.15}\\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\
\frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}
\end{array}\right] .
$$



Figure 3-4: (a) Representation and node-numbering of an eight-noded element, which has nodes located only at the corners of the element. (b) The $2 \times 2 \times 2$ Gaussian integration scheme, where the Gauss points have coordinates $\left( \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}\right)$.

Eight-noded elements have been implemented [20]. The nodal numbering and the integration scheme are based on the standard convention illustrated in Fig. 3-4. The eight-noded elements are defined by linear serendipity shape-functions [18], whose expressions are reported below in the isoparametric coordinate system.

$$
\begin{equation*}
N_{i}=\frac{1}{8}\left(1+\xi_{i} \xi\right)\left(1+\eta_{i} \eta\right)\left(1+\zeta_{i} \zeta\right), \tag{3.16}
\end{equation*}
$$

where $i=1 \ldots 8$ is the number of nodes per element.
The volume integrals in the finite element problem stated by Eq. (3.14) was solved numerically [18] using $2 \times 2 \times 2$ Gauss quadrature for the eight-noded elements. The coordinates of the Gauss points in the isoparametric coordinate system are $\left( \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}\right)$ and the weight factors are equal to one.

In addition, the Jacobian is employed in the definition of the strain-displacement matrix $\boldsymbol{B}=\hat{\boldsymbol{B}} \boldsymbol{N}$ in Eq. (3.6). Indeed, $\hat{\boldsymbol{B}}$ contains the derivatives with respect to the global coordinates $x, y, z$, whereas the shape-functions are in isoparametric coordinates $\xi, \eta, \zeta$. Therefore, the derivatives in $\hat{\boldsymbol{B}}$ are converted by means of the chain rule, as follows:

$$
\left[\begin{array}{l}
\frac{\partial N_{i}}{\partial \xi}  \tag{3.17}\\
\frac{\partial N_{i}}{\partial \eta} \\
\frac{\partial N_{i}}{\partial \zeta}
\end{array}\right]=\left[\begin{array}{lll}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\
\frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}
\end{array}\right]\left[\begin{array}{l}
\frac{\partial N_{i}}{\partial x} \\
\frac{\partial N_{i}}{\partial y} \\
\frac{\partial N_{i}}{\partial z}
\end{array}\right]=\boldsymbol{J}\left[\begin{array}{l}
\frac{\partial N_{i}}{\partial x} \\
\frac{\partial N_{i}}{\partial y} \\
\frac{\partial N_{i}}{\partial z}
\end{array}\right] .
$$

Hence, the opposite transformation, i.e from global to isoparametric, is computed by
means of the inverse of the Jacobian and, therefore, expressed as:

$$
\left[\begin{array}{l}
\frac{\partial N_{i}}{\partial x}  \tag{3.18}\\
\frac{\partial N_{i}}{\partial y} \\
\frac{\partial N_{i}}{\partial z}
\end{array}\right]=\boldsymbol{J}^{-1}\left[\begin{array}{l}
\frac{\partial N_{i}}{\partial \zeta} \\
\frac{\partial N_{i}}{\partial \eta} \\
\frac{\partial N_{i}}{\partial \zeta}
\end{array}\right] .
$$

Two critical points are encountered in the proposed finite element formulation. First, the derivation of the nodal forces vector $\boldsymbol{f}$ equivalent to the cross-section traction distributions. Second, the definition and enforcement of the boundary condition. Before probing the two points mentioned above, an investigation on the effects of taper to the Jacobian matrix is shown.

## The role of the Jacobian matrix in cross-section analysis

The Jacobian matrix is the expedient employed to map the nodal coordinates between two reference systems. The lengthwise taper of the beam enters in the finite element formulation by means of the same Jacobian matrix. Given a tapered element whose nodal coordinates are defined as function of the taper angles, i.e. $(x(\alpha, \beta), y(\alpha, \beta), z)$, the taper factor appears in the partial derivatives related to the correlated taper direction. To give an example, the 3D finite element in Fig. 3-3 is considered. It is modelled with eight-noded solid elements and it is vertically and horizontally tapered with constant taper angles $\alpha$ and $\beta$ respectively. Hence, the nodal coordinates at $z=-\Delta / 2$ are defined as $x=\hat{x}+\Delta \tan \beta$ and $y=\hat{y}+\Delta \tan \alpha$. For simplicity, the front face has fixed dimensions equal to $H=B=L=2 \mathrm{~m}$. The Jacobian matrix is modified accordingly as follows

$$
\boldsymbol{J}^{3 D}(\alpha, \beta)=\left[\begin{array}{ccc}
1-\zeta \tan (\beta) & 0 & 0  \tag{3.19}\\
0 & 1-\zeta \tan (\alpha) & 0 \\
-\xi \tan (\beta) & -\eta \tan (\alpha) & 1
\end{array}\right]
$$

The horizontal taper $\beta$ enters in the components involving a derivative of $x$ with respect to $\xi$ and $\zeta$, whereas the vertical taper $\alpha$ affects the terms related to the derivative of $y$ with respect to $\eta$ and $\zeta$. As previously mentioned, 2D cross-section analysis tools such as BECAS are based on prismatic assumptions. Consequently, the analysed cross section is assumed to be taken from a prismatic beam with constant geometrical and material properties in the longitudinal direction. As a consequence, the Jacobian matrix employed in the 2D finite element formulation presents $J(3,1)=$
$J(3,2)=J(1,3)=J(2,3)=0$. in other words:

$$
\boldsymbol{J}^{2 D}=\left[\begin{array}{ccc}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & 0  \tag{3.20}\\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & 0 \\
0 & 0 & 1
\end{array}\right]
$$

At first glance, it could be licit to assume that taper effects could be captured in the classic cross-section analysis, e.g. in BECAS, by introducing the taper factors directly in the Jacobian matrix, namely by imposing $\boldsymbol{J}^{2 D}=\boldsymbol{J}^{3 D}$. Such a conjecture was examined, but the shortcut did not succeed. Indeed, to account for taper effects, the general definition of the strain energy, i.e. as volume integral and not per unit length, is required as explained in section 3.1.

### 3.2.2 Equivalent nodal forces

The cross-section analysis aims to provide the cross-section stiffness properties which are necessary to perform aeroelastic analysis of the simplified one-dimensional beam model. The latter analysis provides the internal forces at the nodes of the beam elements, which are then included in the cross-section analysis as internal forces to recover the strains and stresses. In the finite element formulation, the right-handside of Eq. (3.14) consists of the nodal forces vector $\boldsymbol{f}$, which can be obtained from integration of the shape-functions and the internal tractions at the two cross-section faces. The issue encountered at this point is that the aeroelastic analysis gives the internal forces $\boldsymbol{\theta}=\left[F_{z}, F_{y}, F_{x}, M_{z}, M_{y}, M_{x}\right]^{T}$, whereas the distribution law of the corresponding cross-section tractions remains unknown. In other words, the internal force vector needs to be coupled to the nodal force vector: $\boldsymbol{\theta}_{1 \times 6} \Rightarrow \boldsymbol{f}_{1 \times n}$, where n is the total number of degrees of freedom.

## Nodal forces in BECAS

Cross-section analysis tools, such as BECAS, are based on favourable 2D theories which are able to solve the equilibrium equations employing only the internal forces. In BECAS, for example, the expedient employed to circumvent the derivation of the internal tractions distribution is part of the anisotropic beam theory developed by Giavotto et al. [26] and it will be illustrated in the following. The Cartesian coordinate system used in the formulation is given in Fig. 3-5.

The equilibrium equations of the finite element formulation are invoked from the


Figure 3-5: Global reference system $0 x y z$ of a prismatic beam. From Blasques [10].
virtual work principle, which is defined as

$$
\begin{equation*}
\delta \frac{\partial W}{\partial z}=\delta \frac{\partial W_{\text {int }}}{\partial z}+\delta \frac{\partial W_{e x t}}{\partial z} \tag{3.21}
\end{equation*}
$$

where $W_{\text {int }}$ and $W_{e x t}$ are the internal and external work per unit length. The entire theory is built on the hypothesis that the analysed cross section belong to a prismatic beam, namely the material and geometrical properties of the cross section do not vary along the beam span. The strain and stress vectors are named $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$, whereas the displacement and traction vectors are $\boldsymbol{r}$ and $\boldsymbol{p}$. Necessary and sufficient equilibrium condition is that $\delta W=0$, therefore Eq. (3.21) becomes

$$
\begin{equation*}
\int_{A} \delta \varepsilon^{T} \boldsymbol{\sigma} d A=\int_{A} \frac{\partial\left(\delta \boldsymbol{r}^{T} \boldsymbol{p}\right)}{\partial z} d A \tag{3.22}
\end{equation*}
$$

The displacement vector $\boldsymbol{r}$ is defined by two terms: $\boldsymbol{\psi}$ and $\boldsymbol{g}$. The former is the vector of the section strain parameters, i.e. the strain provoked by the rigid displacement, and it is defined as $\boldsymbol{\psi}=\left[\tau_{x}, \tau_{y}, \tau_{z}, \chi_{x}, \chi_{y}, \chi_{z}\right]^{T}$. The latter, $\boldsymbol{g}$, is the warping displacement vector, which can be written as $\boldsymbol{g}=\boldsymbol{N u}$ in the finite element discretization, where $\boldsymbol{N}$ is the shape functions matrix and $\boldsymbol{u}$ is the nodal displacement vector. After decoupling the warping and the rigid displacement, the variables of the system of equilibrium equations are $\boldsymbol{u}, \frac{\partial u}{\partial z}$, and $\boldsymbol{\psi}$. The reader can find the whole procedure which leads to the following system on page 11-16 of [9], which is not reported here for the sake of brevity. The final system of the equilibrium equations becomes:

$$
\left\{\begin{array}{l}
\boldsymbol{M} \frac{\partial \boldsymbol{u}}{\partial z}+\boldsymbol{C u}+\boldsymbol{L} \boldsymbol{\psi}=\boldsymbol{f}  \tag{3.23}\\
\boldsymbol{C} \frac{\partial u}{\partial z}+\boldsymbol{E u}+\boldsymbol{R} \psi=\frac{\partial \boldsymbol{f}}{\partial z} \\
\boldsymbol{L} \frac{\partial u}{\partial z}+\boldsymbol{R} \boldsymbol{u}+\boldsymbol{A} \boldsymbol{\psi}=\boldsymbol{\theta} \\
\frac{\partial \boldsymbol{\theta}}{\partial z}=\boldsymbol{T}_{\boldsymbol{r}}^{T} \boldsymbol{\theta}
\end{array}\right.
$$

where, $\boldsymbol{\theta}$ is the section forces vector, and $\boldsymbol{M}, \boldsymbol{E}, \boldsymbol{A}, \boldsymbol{C}, \boldsymbol{L}, \boldsymbol{R}$ are $6 \times 6$ stiffness matrices defined on page 15 of [9]. It is worth noting that if the warping is assumed equal to zero, $\boldsymbol{u}=0$, the third equilibrium equation in (3.23) reduces to the EulerBernoulli theory, $\boldsymbol{A} \boldsymbol{\psi}=\boldsymbol{\theta}$, where $\boldsymbol{A}$ represents the cross-section stiffness matrix. The nodal force vector $\boldsymbol{f}$ appears at the right-hand-side of the first two equations. It can be noted that, if the left- and right-hand sides of the first equation are derived with respect to $z$, and subsequently it is subtracted from the second equation, the nodal force vector disappears:

$$
\left\{\begin{array} { l } 
{ \boldsymbol { M } \frac { \partial ^ { 2 } \boldsymbol { u } } { \partial z ^ { 2 } } + \boldsymbol { C } \frac { \partial \boldsymbol { u } } { \partial z } + \boldsymbol { L } \frac { \partial \psi } { \partial z } = \frac { \partial \boldsymbol { f } } { \partial z } }  \tag{3.24}\\
{ \boldsymbol { C } \frac { \partial \boldsymbol { u } } { \partial z } + \boldsymbol { E } \boldsymbol { u } + \boldsymbol { R } \psi = \frac { \partial \boldsymbol { f } } { \partial z } } \\
{ \boldsymbol { L } \frac { \partial \boldsymbol { u } } { \partial z } + \boldsymbol { R } \boldsymbol { u } + \boldsymbol { A } \boldsymbol { \psi } = \boldsymbol { \theta } } \\
{ \frac { \partial \boldsymbol { \theta } } { \partial z } = \boldsymbol { T } _ { \boldsymbol { r } } ^ { T } \boldsymbol { \theta } }
\end{array} \Rightarrow \left\{\begin{array}{l}
\boldsymbol{M} \frac{\partial^{2} \boldsymbol{u}}{\partial z^{2}}+\left(\boldsymbol{C}-\boldsymbol{C}^{T}\right) \frac{\partial \boldsymbol{u}}{\partial z}+\boldsymbol{L} \frac{\partial \psi}{\partial z}-\boldsymbol{E} \boldsymbol{u}-\boldsymbol{R} \boldsymbol{\psi}=\mathbf{0} \\
\boldsymbol{L} \frac{\partial \boldsymbol{u}}{\partial z}+\boldsymbol{R} \boldsymbol{u}+\boldsymbol{A} \boldsymbol{\psi}=\boldsymbol{\theta} \\
\frac{\partial \boldsymbol{\theta}}{\partial z}=\boldsymbol{T}_{\boldsymbol{r}}^{T} \boldsymbol{\theta}
\end{array}\right.\right.
$$

Consequently, the nodal forces vector does not explicitly appear in the finite element formulation and its derivation is not required.

Differently, the presented 3D finite element method of the tapered slice requires the definition of the nodal forces to solve Eq. (3.14), hence the distribution of the internal tractions must be defined. A first attempt to solve the finite element problem of the tapered slice described above is investigated for the case of a rectangular cross section with constant vertical taper $\alpha$. In the derivation of the analytical method in chapter 2, it was demonstrated that a tapered beam behaviour can be described, with acceptable approximation, by the Navier hypotheses. Hence, the normal stresses in a tapered beam follow a similar distribution to the stresses in prismatic beam given by Eq. (2.1). Such an assumption allows the derivation of the whole Cauchy stress tensor by imposing the equilibrium of an infinitesimal portion of the tapered beam. Exploiting this outcome, the internal tractions of the finite element model are applied based on prismatic assumptions. Similarly, it is assumed that the pure shear traction due to the internal shear force is also defined from the Jourawsky equation (2.2) which only holds in prismatic beams. In other words, the stress distribution is assumed according to prismatic theory and the three internal load conditions, i.e. extension, shear and bending, are modelled at the two faces of the slice as represented in Fig. $3-6$. Once the traction distributions at the two faces of the slice are assumed, the shape-functions are employed to derive the equivalent nodal forces by integrating over


Figure 3-6: Assumed cross-section tractions acting at the two faces + and - of the beam slice of thickness $\Delta$ and related internal forces. The latter are the integral over the face surface of the internal tractions. Three load cases are considered, namely (a) axial force $F_{z}$, (b) shear force $F_{y}$, and (c) bending moment around the $x$-axis $M_{x}$.
the area $A$ of the related face:

$$
\begin{equation*}
\boldsymbol{f}=\int_{A} \boldsymbol{N}^{T}(\xi, \eta, \hat{\zeta}) \boldsymbol{p}(x, y, \hat{z}) d A \tag{3.25}
\end{equation*}
$$

where $\boldsymbol{p}(x, y, \hat{z})$ to the surface tractions in the global coordinate system, and $\boldsymbol{N}(\xi, \eta, \hat{\zeta})$ to the shape-functions evaluated at $\hat{\zeta}= \pm 1$, namely at the front and back face of the isoparametric element. A change of coordinates is required in Eq. (3.25), since the shape-functions are in the natural coordinates system whereas the internal traction $\boldsymbol{p}(x, y, z)$ and the area $A(x, y, z)$ are in the global coordinate system. As explained earlier, the transformation between two coordinate systems can be performed through the Jacobian matrix given in Eq. (3.15), assuming $\zeta= \pm 1$. Moreover, the Jacobian itself is responsible for introducing the taper-parameters, as shown in Eq. (3.19). Since each component of the Jacobian matrix gives the relation between global and natural coordinates, the cross-section area can be directly expressed as

$$
\begin{equation*}
d A=d x d y=|J| d \xi d \eta \tag{3.26}
\end{equation*}
$$

where $|J|$ is the determinant of the Jacobian matrix. In case of a slice with a rectangu-
lar cross section, the internal tractions are defined as functions of $y$. The $y$-coordinate has to be transformed to the natural coordinates. It can be done from the definition of the component $J(2,2)$, which gives $d y=J(2,2) d \eta$. Substitution of the latter equations into Eq. (3.25) provides the required transformation as

$$
\begin{equation*}
y=\int J(2,2) d \eta=J(2,2) \eta+c_{y} \tag{3.27}
\end{equation*}
$$

where $c_{y}$ is the integration constant. The latter can be derived by imposing the equivalence between the coordinates at the centre of the real element $\left(x_{0}, y_{0}, z_{0}\right)$ and at the centre of the isoparametric element $\xi_{0}, \eta_{0}, \zeta_{0}$. The latter coordinates are always equal to $\xi=0, \eta=0, \zeta=0$ from the isoparametric element definition as shown in Fig. 3-3. Moreover, if the slice is only vertically tapered, i.e. $\beta=0$ in Fig. 3-3, the effects of taper are introduced in the formulation through the Jacobian component $J(2,2)=1-\zeta \tan (\alpha)$.

The above-mentioned procedure can be applied to the axial, shear and bending cases and gives the expressions for the equivalent nodal forces. In the particular case of rectangular cross section they become

$$
\begin{gather*}
\text { pure axial: } f_{i}=F_{z} \int_{-1}^{1} \int_{-1}^{1} \frac{N_{i}(\xi, \eta)}{A}|J| d \xi d \eta  \tag{3.28}\\
\text { pure bending: } \quad f_{i}=F_{y} \int_{-1}^{1} \int_{-1}^{1} \frac{N_{i}(\xi, \eta)}{I_{x}}[J(2,2) \eta+\hat{y}]|J| d \xi d \eta  \tag{3.29}\\
\text { pure shear: } \quad f_{i}=M_{x} \int_{-1}^{1} \int_{-1}^{1} \frac{N_{i}(\xi, \eta)}{I_{x}}\left[\frac{H^{2}}{8}-\frac{(J(2,2) \eta+\hat{y})^{2}}{2}\right]|J| d \xi d \eta \tag{3.30}
\end{gather*}
$$

where $A, H$ and $I_{x}$ are the area, height and the second moment of area of the cross section, $F_{z}, F_{y}, M_{x}$ are the cross-section forces, $\hat{y}$ the $y$-coordinate of the centre of the element $\left(x_{0}, y_{0}, z_{0}\right)$. The surface integrals above are solved numerically through a Gauss quadrature. In case of eight-noded elements, the $2 \times 2$ schema is applied. The Gauss points coordinates and weights are the same mentioned in the previous section.

### 3.2.3 Boundary conditions

The global stiffness matrix $\mathbf{K}$ becomes singular after assembling, meaning that the system is unstable due to zero-energy modes and cannot be solved. Nonetheless, in order to solve the equilibrium system in Eq. (3.14) the stiffness matrix has to be
definite positive. Eigenvalue analysis shows that six free body modes occur and they correspond to three rigid translations and three rigid rotations. Hence, six constraints need to be enforced to remove the rigid body modes from the solution [18]. In order to apply the six rigid body motions without restraining the warping, the summation of the nodal displacements and the rotations, i.e. the average rotation of the cross section, are imposed to be zero [10]. In other words:

$$
\begin{gather*}
\sum_{i=1}^{n_{n}} u_{x}^{(i)}=0, \quad \sum_{i=1}^{n_{n}} u_{y}^{(i)}=0, \quad \sum_{i=1}^{n_{n}} u_{z}^{(i)}=0  \tag{3.31}\\
\sum_{i=1}^{n_{n}}-z_{i} u_{y}^{(i)}+y_{i} u_{z}^{(i)}=0, \quad \sum_{i=1}^{n_{n}} z_{i} u_{x}^{(i)}-x_{i} u_{z}^{(i)}=0, \quad \sum_{i=1}^{n_{n}}-y_{i} u_{x}^{(i)}+x_{i} u_{y}^{(i)}=0 . \tag{3.32}
\end{gather*}
$$

where $n_{n}$ is the total number of nodes, $\left(u_{x}^{(i)}, u_{y}^{(i)}, u_{z}^{(i)}\right)$ are the nodal displacements in the three directions $x, y, z$, and $\left(x_{i}, y_{i}, z_{i}\right)$ are the nodal coordinates expressed in the global coordinate system for the $i^{\text {th }}$ node [10].

Referring to Eqs. (3.31) and (3.32), it is noteworthy to mention that the boundary conditions are not explicit and they can be applied through the Lagrange multipliers method, which imposes the constraints directly in the virtual work principle. The homogeneous equation $\boldsymbol{C u}=\mathbf{0}$ includes the constraints given in Eqs. (3.31) and (3.32). They are collected in the constraints matrix $\boldsymbol{C}$ which is defined as follows

$$
\boldsymbol{C}=\left[\begin{array}{ccccccc}
1 & 0 & 0 & & 1 & 0 & 0  \tag{3.33}\\
0 & 1 & 0 & & 0 & 1 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 1 \\
0 & 0 & y_{1} & & 0 & 0 & y_{n_{n}} \\
0 & 0 & -x_{1} & & 0 & 0 & -x_{n_{n}} \\
-y_{1} & x_{1} & 0 & \ldots & -y_{n_{n}} & x_{n_{n}} & 0
\end{array}\right]
$$

The constraint relation can be multiplied by the row vector $\boldsymbol{\lambda}$ that comprises the Lagrange multipliers $\lambda_{i}$ in the same number as the constraints equation, i.e. six in this case:

$$
\begin{equation*}
\boldsymbol{\lambda}(\boldsymbol{C u})=0 . \tag{3.34}
\end{equation*}
$$

The potential energy can be defined and the constraints relation Eq. (3.34) can be added to it without affecting the energy [18]:

$$
\begin{equation*}
\Pi_{P}=\frac{1}{2}\left(\boldsymbol{u}^{T} \boldsymbol{K} \boldsymbol{u}-\boldsymbol{u}^{T} \boldsymbol{f}\right)+\boldsymbol{\lambda}(\boldsymbol{C u})=0 \tag{3.35}
\end{equation*}
$$

The solution of Eq. (3.35) can be obtained by minimisation of the potential energy, $\underset{u, \lambda}{\nabla} \Pi_{P}=0$ :

$$
\begin{equation*}
\frac{\partial \Pi_{P}}{\partial u} \delta \boldsymbol{u}+\frac{\partial \Pi_{P}}{\partial \lambda} \delta \boldsymbol{\lambda}=0 \tag{3.36}
\end{equation*}
$$

which must hold for any arbitrary $\delta \boldsymbol{u}$ and $\delta \boldsymbol{\lambda}$. In other words:

$$
\begin{equation*}
\frac{\partial \Pi_{P}}{\partial u}=0, \quad \frac{\partial \Pi_{P}}{\partial \lambda}=0 . \tag{3.37}
\end{equation*}
$$

Expressing the latter system in a matrix form leads to the new formulation of the equilibrium system where constraints are enforced:

$$
\left[\begin{array}{cc}
\mathbf{K} & \mathbf{C}^{T}  \tag{3.38}\\
\mathbf{C} & \mathbf{0}
\end{array}\right]\left[\begin{array}{l}
\mathbf{u} \\
\boldsymbol{\lambda}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{f} \\
\mathbf{0}
\end{array}\right]
$$

The solution to the system of equations provides the displacement row vector $\boldsymbol{u}$ which occurs under the imposed boundary conditions, and the Lagrange multipliers, which can be interpreted as forces applied to impose the constraints. Although the versatility of the Lagrange multipliers method, it increases the number of variables in the system of equation, hence higher order. In addition, since zero entries appear on the diagonal of the matrix in the left hand-side of Eq. (3.38), the matrix becomes positive semi-definite and the system of equations cannot be solved directly through e.g. the Cholesky factorisation method. Other factorisation and iterative methods can be employed [28].

At this point, the system of equations comprehensive of the constraints can be solved for the nodal displacements $\mathbf{u}$. Then, strains and stresses are obtained through numerical integration, i.e. they are evaluated at the Gauss points. For validation purposes, the numerical stresses are compared with the analytical ones. Therefore, the nodal stresses are derived through interpolation of the shape-functions as in standard procedure, and then, the interpolated values are averaged to get the stress at the centre of each element, as pointed out in Fig. 3-8.

## Pseudo code of the FE implementation

In this section, the procedure implemented in MATLAB is presented in Table 3.2.3. The number of degrees of freedom per node, $n d o f$, is equal to 3 . The number of elements of the model is $n_{e}$. The number of nodes per element is 8 and the total number of nodes is $n_{\text {nodes }}$. The dimensions of the matrices and vectors are reported in brackets.

Table 3.1: Procedure to derive the assembling matrix adof, the stiffness matrix $\mathbf{K}_{a}$, equivalent nodal forces $\mathbf{f}_{a}$

## ASSEMBLY PROCEDURE

(1) Initialise list vector of dof for the model tdof, $\left(1 \times \operatorname{ndof} \cdot \mathrm{n}_{\text {nodes }}\right)$
(2) loop over $i=1 \ldots\left(8 \cdot n d o f \cdot n_{e}\right)$
(3) loop over $j=1 \ldots\left(n d o f \cdot n_{\text {nodes }}\right)$

$$
\begin{aligned}
& \text { if } \mathrm{j}=\operatorname{tdof}(\mathrm{i}), \operatorname{adof}(\mathrm{i}, \mathrm{j})=1 \\
& \text { else }, \quad \operatorname{adof}(\mathrm{i}, \mathrm{j})=0
\end{aligned}
$$

## ELEMENT STIFFNESS MATRIX

(1) Initialise $\mathbf{K}^{a}$ ( $\mathrm{ndof} \cdot \mathrm{n}_{e}$ )
(2) Loop over elements $n_{e}$

Calculate the Jacobian matrix with differentiation of shape-functions
(3) Loop over the element nodes

Assemble the strain-displacement matrix B
(4) Loop over the Gauss points

Calculate $\boldsymbol{K}=\boldsymbol{B}^{T} \boldsymbol{E} \boldsymbol{B}|J|$
(5) Assemble to global stiffness matrix (dof $\cdot n_{\text {nodes }} \times \operatorname{dof} \cdot n_{\text {nodes }}$ )
$\boldsymbol{K}^{a}=$ adof $^{T} \boldsymbol{K}^{e}$ adof
NODAL FORCES VECTOR
(1) Loop over elements $n_{e}$

Calculate the Jacobian matrix with differentiation of shape-functions
(2) Select the load case (axial, bending, shear)
(3) Loop over the element nodes $i$ located at $\zeta= \pm 1$
(4) Loop over Gauss points

Evaluate the equivalent nodal forces in a vector $\boldsymbol{f}_{e}\left(1 \times \operatorname{dof} \cdot n_{\text {nodes }} \cdot n_{e}\right)$
$f_{i}=F_{z} / A N_{i}|J|$ when considering axial force
$f_{i}=M_{x} / I_{x} \hat{y} N(i)|J|$ when considering bending moment
$f_{i}=F_{y} / B I_{x} S_{x}|J|$ when considering internal shear force
(5) Assemble the global force vector $\boldsymbol{f}^{a}=\operatorname{adof}^{T} \boldsymbol{f}^{e}\left(\right.$ dof $\left.\cdot n_{\text {nodes }} \times 1\right)$

### 3.3 Application

The method presented above is implemented in MATLAB and uses an LU factorisation, i.e. lower upper triangular matrix. The capabilities and limitations of the method are assessed for the planar cantilever wedge clamped at the root and loaded at its tip as shown in Fig. 3-7. Analytical exact [42, 14] and approximated [50] solu-


Figure 3-7: Geometry and coordinate system of a 2D planar wedge with taper angle $\alpha$. A bending moment around the $x$-axis $M_{x}$ and a generic force $F_{e}$ are applied at the tip of the wedge. The highlighted cross-section slice has depth $\Delta$ and is located at $z=L / 2$.
tions of a planar isotropic wedge loaded at the tip with concentrated forces $F_{z}$ and $F_{y}$ and bending moment $M_{x}$ are available in the literature and they are summarised in [P4]. Therefore, the planar wedge can be used as a benchmark example to evaluate the accuracy of the tapered slice method. To avoid end-effects altering the results, the cross-section analysis is performed at a control cross section which is defined at the wedge mid-span, $z=L / 2$, where $z$ is the beam longitudinal axis and $L$ the total length of the wedge. The equivalent numerical model is the slice of the wedge defined between $(L / 2-\Delta / 2)$ and $(L / 2+\Delta / 2)$, as shown in Fig. 3-7. The slice has been modelled by means of eight-noded elements. Only one solid element is used in the longitudinal direction, namely along the thickness of the slice, whereas the number of elements in the $y$-direction is established after a standard converge study, as shown in Fig. 3-8. The finite element model used in the presented study has 30 eight-noded elements and 372 degrees of freedom.

The control cross section has fixed dimensions $H=2 \mathrm{~m}, B=0.1 \mathrm{~m}$, and the taper angle is taken equal to $\alpha=H=5.7^{\circ}$. A homogeneous isotropic material is assumed with Young modulus $E=100 \mathrm{~Pa}$ and Poisson ration $\nu=0.3$. Firstly, the method was validated against the analytical solution of a prismatic slice. Figure 3-9 shows


Figure 3-8: Model of an exemplary slice of a wedge. The coordinate system Oxyz, the dimensions and the mesh discretisation are highlighted. Moreover, the eight-noded solid elements are illustrated on the right. The internal red dot shows the location at which the average displacement, strain, and stress values are considered.
the results of the three loading cases. It is worth noting that the case of axial and shear are well predicted. However, the model of the slice based on the eight-noded elements, i.e. linear shape-functions, suffered from shear-locking in the bending case, as shown in Fig. 3-9-b. In order to mitigate the shear locking problem, the slice was modelled through twenty-noded elements defined by quadratic serendipity shapefunctions [20]. Contrarily to the eight-noded model, the twenty-noded can correctly depict the stress distribution under bending. Moreover, the solutions of the prismatic slice are independent from the dimension of the slice thickness. On the other hand, the solutions of the tapered slice are influenced by the magnitude of both the taper angle and the thickness of the slice. Whilst these parameters are interconnected, their effects are addressed separately in the following sections.

## Stresses distribution

First, the stresses evaluated in the tapered slice modelled with eight-noded elements are compared to the analytical solutions. The stress components in the finite element analysis are evaluated at the Gauss points. In order to carry out the comparison against the analytical solutions, the nodal stresses are firstly extrapolated by means of the shape-functions. Then, the values at the back and front face are averaged and applied at the centre of the element (red dot in Fig. 3-8). The cross section


Figure 3-9: (a) Normal stress due to extension, (b) bending, and (c) shear stress due to shear along half of the prismatic slice. Numerical results refer to different mesh-topology and element type and are compared to analytical solutions. Eightand twenty-noded elements are employed. The number of elements in each model is indicated in the Legend after \#. Q and L stand for quadratic and linear elements, respectively.
forces applied at the models are reported in table 3.3. Figures 3-10, 3-11 and 312 show the stresses at the control cross-section and respectively due to axial force, bending moment and shear force applied at the tip of the wedge. The presented numerical model exhibits the capability to capture taper effects. It is worth noting that, although the internal tractions of the slice model are assumed based on the prismatic formulation, the normal stresses $\sigma_{z z}$ resulting under axial, bending and shear do not exhibit the linear Navier distribution. In fact, they follow the quadratic distribution predicted by the exact analytical solutions that account for taper. In other words, the formulation retrieves the out-of-plane warping which characterises the tapered beam behaviour.

Table 3.2: Axial force, shear force and bending moment applied at the tip of the wedge and respective internal forces at the middle cross-section.

|  | Tip loads |  |  |  |  | Mid-span c.s. |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $F_{z}[\mathrm{~N}]$ | $F_{y}[\mathrm{~N}]$ | $M_{x}[\mathrm{~N} \mathrm{~m}]$ |  | $F_{z}[\mathrm{~N}]$ | $F_{y}[\mathrm{~N}]$ | $M_{x}[\mathrm{~N} \mathrm{~m}]$ |  |
|  | Shear | 10 | - | - |  | 10 | - | $-10 \mathrm{~L} / 2$ |
| Extension | - | 10 | - |  | - | 10 | - |  |
| Bending | - | - | 10 |  | - | - | 10 |  |

Figure 3-12 shows the Cauchy stresses at the control cross section of the wedge loaded at its tip by a shear force. Consequently, the internal forces are both shear and bending. Given the linearity of the problem, the superposition principle is applied to Eqs. (3.29) and (3.30). The effects of taper to the shear stress distribution, which have been widely described in chapter 2 , are depicted also in the tapered slice model. In fact, the vertical taper angle is introduced in the Jacobian matrix as the component $J(2,2)=1-\zeta \tan \alpha$, as it was described in section 3.2.2. It is also worth noting in Fig. 3-11-b that the tapered slice model, which is based on eight-noded elements, does not show the shear-locking deviation that was observed in the prismatic case Fig. 3-9-b. Hence, the taper slice was modelled only with eight-noded elements as they have advantages over the twenty-noded elements in terms of computational cost.


Figure 3-10: Cauchy stress components, namely (a) $\sigma_{y y}$, (b) $\sigma_{z z}$, and (c) $\sigma_{y z}$, in a planar tapered slice loaded by an axial tip force. The shown results are from analytical and numerical solutions. The numerical solution refers to three slice thicknesses, $\Delta=0.03,0.09,0.5 \mathrm{~m}$


Figure 3-11: Cauchy stress components, namely (a) $\sigma_{y y}$, (b) $\sigma_{z z}$, and (c) $\sigma_{y z}$, in a tapered slice loaded at its tip by a bending moment. The results refer to the analytical and numerical solutions. The latter considers three different slice thicknesses, $\Delta=0.03,0.09,0.5 \mathrm{~m}$.


Figure 3-12: Cauchy stress components, namely (a) $\sigma_{y y}$, (b) $\sigma_{z z}$, and (c) $\sigma_{y z}$, in a tapered slice loaded at its tip by a shear external force. Results are from analytical and numerical solutions. The latter considers three different slice thicknesses, $\Delta=$ $0.03,0.09,0.5 \mathrm{~m}$.

## The effects of the slice thickness

As already pointed out in Ghiringhelli and Mantegazza [25], the thickness of the slice plays an important role in defining the accuracy of the results. Even though the
distribution of the stress components is qualitatively depicted by the tapered slice analysis, the magnitude of the stresses strongly depends on the slice thickness $\Delta$. A parametric study on the error between the numerical and analytical stresses that


Figure 3-13: Normalised Cauchy's stress components (numerical-to-analytical) along the central slice of a wedge for different thickness of the slice. The values are compared at $y=0$ and $y=H / 2$. Only in (b) and (d) the stresses are zero at $y=0$ and therefore the comparison is carried out at $h=H / 3$. The results are due to axial force on the left and to bending on the right.
occurs for different slice thicknesses $\Delta$ is conducted. The stresses are evaluated at the control cross section of the wedge in Fig. 3-7. Values of the slice thickness varying between $[0,0.5 \mathrm{~m}]$ are considered. In addition, the comparison is carried out for stress values evaluated at two different points of the slice. Figure 3-13 shows the parametric study for the cases of axial force on the left and bending force on the right. The three
stress components due to extension are characterised by the same asymptotically distribution, which converges for thicker slices. Contrarily, in case of bending the thinner the slice the more accurate the numerical stresses. However, for both the axial and bending load case, the distribution of the error rapidly convergences for slice thickness $\Delta=0.1 \mathrm{~m}$ in the presented case.

## The magnitude of taper angle

The magnitude of the beam taper angle has certainly an influence on the beam behaviour. The role of taper in defining the Cauchy's stresses can be studied when closed-form analytical expressions are available, e.g. in the examples presented in Chapter 2. The analysis of such equations leads to the conclusion that various amplitude of taper differently affects both the magnitude of the stresses as well as their distribution. Similar behaviour is expected in tapered beams in general. In addition, again the analytical studies [12] show that assuming the Navier distribution of the normal stresses gives an acceptable approximation when small taper angles are considered. Since in the present method the internal tractions are imposed from Navier equation, the variation of the results with the taper angle is studied through a parametric study. Figure 3-14 shows that, as expected, the deviation increases with greater taper angles. At the same time, the deviation can be limited if the slice thickness is properly chosen.


Figure 3-14: Numerical-to-analytical stresses due to axial traction. The stresses are evaluated at the slice of the control cross-section of four wedges characterised by the following taper angles ( $\alpha=2^{\circ}, 5.7^{\circ}, 10^{\circ}, 20^{\circ}$ ). The ratio involves the stresses at $y=0$ and $y=H / 2$ of the control cross section. Several slice thicknesses are considered $\Delta=0.03,0.09,0.5 \mathrm{~m}$.

## Chapter 4

## Discussion

This work focuses on the structural behaviour of tapered beams and investigates the effects of taper in terms of Cauchy stresses from an analytical and numerical point of view. Taper in a straight beam is defined as the angle between the tangent plane to a point of the beam surface and the longitudinal beam axis. Tapered beams have been widely employed in many engineering sectors to meet architectural, structural and aerodynamic requirements. For example, in civil engineering beam framing, towers and bridges are often designed with tapered beams to reduce the mass-to-stiffness ratio and, ultimately, to distribute the material in a more optimal way. More recently, with the advancements in aeronautic and wind energy, structures as wings and blades are nonprismatic to meet aerodynamic requirements. Yet, taper significantly affects the global beam behaviour in terms of stiffness of the structure, strain and stress distributions, and deformation modes. Although taper effects are emphasised for large amplitude of the taper angle, they appear even in moderately tapered beams. Because of the many applications of tapered beams, they have been long studied in the past century and hence the effects of taper long known in scientific literature. Nonetheless, many numerical methods employed in academia and industries simplify tapered beams with stepwise prismatic models and neglecting taper effects. For instance, 2D cross-section analysis methods, e.g. BECAS and VABS, are based on theories where prismatic assumptions are made, hence they do not account for taper effects. A comparison is shown in the introduction of this thesis to highlight the deviation in terms of Cauchy stresses when a tapered beam is simplified as stepwise prismatic. The above-mentioned results clearly stress the necessity of including taper in structural design to gain a more accurate beam analysis. In the context of rotor blade design, many uncertainties are still on the table and exploiting the effects of the geometrical variations which are currently neglected in cross-section analysis, could
ultimately increase the reliability of the blades.
An analytical solution of the Cauchy stresses is developed to provide analytical closed-form equations that can be used for validation of approximated cross-section analysis methods of tapered structures. Moreover, they help understanding how taper enters in the beam formulation. The analytical approach for tapered slender beam analysis was developed for homogeneous isotropic material tapered beams with symmetric, thin-walled cross sections. Under the assumption of cross section remaining plane after deformation, the Cauchy normal stress was assumed to follow the Navier equation (2.1). This assumption was supported and shown to be acceptable for taper angles smaller than $10^{\circ}$ by previous studies available in the scientific literature $[12,4]$. The analytical method developed in this work is built on this assumption. Indeed, once the normal stresses are given, the whole stress tensor can be determined from the Cauchy equilibrium of an infinitesimal portion of the tapered beam. The analytical solutions are in good agreement with the stresses computed in 3D finite element models even though the whole method is based on the approximation of the axial stresses to the prismatic ones. In fact, the exact distribution of the axial stresses is not linear, as assumed in slender beams by Navier, but quadratic, as depicted by the numerical cross-section analysis method developed in the last part of the thesis. Differently than what expected in prismatic beams, the stress tensor can be fully populated in tapered beams. For instance, $\sigma_{x y}, \sigma_{x z}, \sigma_{x x}$ would be zero in the webs of a prismatic box girder, whereas they occur in the tapered box girder in Fig. 2-3. In other words, taper evokes through thickness, for components which instead are zero in the prismatic beams. Specifically, the through thickness stresses are derived as functions of the normal and/or the shear stress and the taper angles. Since the tangent of the taper angles enters in the equations, the magnitude of such components will always be less than the $25 \%$ of the normal/shear component for taper angles up to $15^{\circ}$. Even though these stresses are smaller than the axial and in-plane shear ones, it is important to underline that through thickness stress components have a crucial impact on the design of weak zones, such as the joints, especially in composite beams. Indeed, delamination of laminates and debonding of the adhesive joint are triggered by the through thickness stress components. It could be argued that the solutions provided are solely valid for homogeneous isotropic and symmetric problems. Nonetheless, it is licit to assume that the couplings between extension and shear, and bending and shear which characterised the in-plane shear of tapered beams, would occur independently from the material properties, hence also in composite structures. Also, the comparison illustrated in the introduction and related to [P4] shows taper
effects in composite cross sections.
An analytical analysis of more complex cross sections could be addressed with the present method after extending the Navier equation in order to account for multicell, asymmetric cross sections or even anisotropic materials. Anyway, they could be useful only for analysis of a limited number of geometries and the related expressions would not be handy. Furthermore, the analytical solutions of more complex problems might not exist at all. An analytical analysis of more complex and certainly more realistic cross sections could be addressed with the present method after extending the Navier equation in order to account for asymmetric cross sections and anisotropic materials. Nevertheless, closed-form solutions would never be derived for real complex structures such as rotor blades, but for simplified blade-like structures. Hence they would be useful to describe only a limited number of geometries, and, in addition, the expressions would not be handy. Numerical methods are therefore essential in crosssection analysis of anisotropic beams with generic cross-section geometry. Widely employed in rotor blade design are cross-section analysis methods, such as VABS and BECAS to name a few, which are based on a 2D finite element formulation. They are generally based on the prismatic assumption, which greatly simplifies the theoretical formulation. Indeed, if the material and geometrical properties do not change in the longitudinal direction, the equilibrium can be imposed from the definition of the strain energy per unit length, Eq. (3.21). In other words, any lengthwise geometrical and material variations are ignored. Exploiting the finite element model of solid tapered elements, taper was noted to enter in the finite element formulation through the Jacobian matrix $\boldsymbol{J}$, i.e. the matrix of the partial derivatives of the shape-functions. For instance, in the vertically tapered beam in Fig. 3-7 the taper angle appears in the Jacobian components involving a derivative of the $y$-coordinate. Based on such an observation, taper was directly introduced into the 2D cross-section analysis tool BECAS by modifying the Jacobian matrix given by Eq. (3.20) with its tapered version in Eq. (3.19). The results were not satisfactory. In fact, as explained above, the classic 2D formulation behind BECAS is developed from the strain energy per unit length, whereas, in order to include lengthwise taper variations, the complete strain energy has to be considered over the volume $\int_{V} \varepsilon^{T} \boldsymbol{\sigma} d V$. In other words, the whole 2D theory and not solely the Jacobian should have been re-developed to include taper and capture taper effects.

An alternative approach for cross-section analysis of tapered beams based on 3D finite element model was suggested. Contrarily to BECAS, the cross section is modelled as a one-layered tapered slice of solid elements, following the study of Ghiringhelli
and Mantegazza [25] and Couturier and Krenk [40]. Employing solid elements, i.e. eight-noded and linear shape-functions, lengthwise taper can be modelled. In the derivation and implementation of this approach, three main issues were faced: (a) imposing of boundary conditions, (b) coupling between cross-section forces and nodal forces, and (c) slice thickness dependency of the results.
(a) The finite element method provides the cross-section stiffness matrix $\boldsymbol{K}$, which results to be singular, hence the equilibrium is unstable due to the six free body modes corresponding to three rigid translation and three rigid rotations of the cross section. This issue is addressed by imposing boundary conditions to the system. In classic finite element analysis discrete boundary conditions are directly imposed. However, such expedient cannot be employed in the proposed cross-section analysis because it would restrain the cross-section warping deformation and, hence, affect the results. Therefore, the boundary conditions described as a set of constraints are imposed through the Lagrange multipliers method. It has the advantage to cancel out the six rigid body motions without employing discrete constraints, which would prevent cross-section warping deformation.
(b) Another crucial point was the coupling between the internal forces and the nodal forces, which appears on the right-hand side of the equilibrium $\boldsymbol{K} \boldsymbol{u}=\boldsymbol{f}$. The equivalent nodal forces are obtained from interpolation of the internal tractions with the shape functions by Gauss integration over the cross-section surface. Whereas the cross-section forces are given by the aeroelastic design of rotor blades, the information about the required internal tractions is not available. For instance, the 2D crosssection analysis tools BECAS is based on the anisotropic beam theory by Giavotto et al. [26] in which the energy equilibrium equations are formulated in such a way that the internal nodal tractions are not required to be explicitly determined. Nonetheless, the same approach cannot be directly extended to tapered beams because it develops from a definition of the strain energy per unit length, which holds if the cross-section properties do not change in the beam longitudinal direction, i. e. prismatic beams. Therefore, the slice method approach is employed to investigate the stress analysis of tapered cross sections. In chapter 2 the Navier equation (2.1) was supposed to hold in moderately tapered beams for the derivation of the analytical solutions. Similarly, for homogeneous isotropic and symmetric slice of the cross-sections, the bending internal tractions are assumed to follow the linear distribution given by the Navier equation. In order to evaluate the accuracy of this assumption, the method is applied to the cross section of a cantilever isotropic wedge, whose exact solutions for the Cauchy stresses were derived analytically. Indeed, although the internal tractions are assumed to
follow the Navier (2.1) and Jourawsky (2.2) equations which are valid in prismatic beams, the developed finite element analysis depicts the distribution expected from the exact analytical solutions, qualitatively. For instance, in a tapered slice subjected to bending $M_{x}$, the latter is assumed to be the integral over the surface $A$ of the Navier linear stress $\sigma_{z z}=\left(M_{x} / A\right) y$. However, the normal stresses evoked by such internal traction have the quadratic distribution predicted by the exact analytical solutions. However, the magnitude of the numerical results is affected by the slice thickness, as will be discussed later. Taper effects are depicted by means of the Jacobian matrix. Indeed, taper enters in the Jacobian matrix, as already illustrated above, and the Jacobian is employed in the derivation of the nodal forces. (c) Lastly, the role of the slice thickness in the whole formulation is investigated. It was stated in (b) that the finite element analysis of the tapered slice is able to depict the stresses distribution. However, the magnitude of the predicted stresses is strongly affected by the chosen tapered slice thickness, wheres it does not affect the results from a prismatic slice. A parametric study is therefore carried out to study how the stresses deviation changes with the various slice thickness. The study refers to extension, shear and bending separately. Both the axial and shear problems are characterised by a similar tendency and converge to the analytical values for larger slice thicknesses. On the contrary, if a slice is subjected to bending, the thinner the slice thickness, the little the deviation. However, all the three load cases converge at the proximity of slice thickness equal to the other element dimensions. Hence, it would be worth investigating the effects of the aspect ratio. Lastly, it is worth mentioning that, as expected, the 3D prismatic slice modelled with eight-noded elements suffers from shear locking under bending.

## Chapter 5

## Conclusions

The present thesis contributes to cross-section analysis of tapered beams. In the first part, analytical closed-form solutions of the Cauchy stresses were developed for homogeneous isotropic, symmetric thin-walled tapered beams. The analytical studies shed light on the role of taper in the structural behaviour of tapered beams and on the potential implications of taper on structural design. In the second part, a novel crosssection analysis method for tapered slices is presented and represents an advancement of the existing stepwise prismatic methods. The derivation of the method, namely imposition of Dirichlet boundary conditions, the coupling of the cross-section forces to the cross-section slice and its implementation in MATLAB were presented. The method represents a first attempt to model tapered 3D slices in the realm of crosssection analysis. The main conclusions of the work are summarised in the following:

- Even a small taper angle affects the stresses distribution, which, therefore, differ from the ones derived in equivalent prismatic beams.
- In homogeneous isotropic slender beams with symmetric cross section, the Navier formula for normal stresses represents a good approximation in moderately tapered beams $\left(<10^{\circ}\right)$.
- The Navier approximation allows to obtain the remaining Cauchy stress tensor components in tapered beams with good agreement.
- The in-plane shear stress component is mostly affected by taper. Indeed, taper may evoke shear-axial and shear-bending couplings which do not exist in prismatic beams. In other words, axial force or bending moment provokes shear stresses in tapered beams.
- In global coordinate system, through thickness stress components are derived as functions of the normal and/or shear stress components scaled by the taper angle. Therefore, they always occur in tapered beams.
- Taper evokes through thickness shear components which play a crucial role in the design of tapered beams. Especially in composite structures which are susceptible to delaminate in multiaxial stress states, or in adhesive joints where the prevention of debonding has a key role, taper effects must be considered for accurate stress analysis leading to reliable designs.

From the stress analysis of the 3D tapered slice, the following conclusions are drawn:

- A step-wise prismatic approach is generally adopted in 2D cross-section analysis employed in academia and industries. Because of this approach, lengthwise taper cannot be modelled and, consequently, taper effects are not depicted.
- Cross-section analysis methods aim to (1) provide the cross-section stiffness properties, and (2) to evaluate the strains and stresses. Neglecting taper effects is reflected in an inaccurate prediction of the cross-section behaviour, in particular in terms of stress and strain.
- 3D formulation of the finite element cross-section analysis is necessary to consider taper in the lengthwise direction. Hence, the stepwise prismatic approximation could be replaced by a stepwise tapered simplification to provide a better approximation.
- It was demonstrated that assuming the prismatic stress distribution as internal traction in a 3D tapered slice, the cross-section analysis is capable of predicting taper effects in the stress analysis, qualitatively.
- The larger the taper angle, the larger the error induced by the assumption of prismatic internal traction.
- Also the thickness of the modelled slice strongly affects the discrepancy between the numerical and analytical results. In the bending load case the thinner the slice the smaller the error, whereas it is vice versa in shear and axial load cases.

The above-mentioned conclusions shed light on the potential of the 3D cross section analysis of the tapered slice, which, differently from the methods currently available
in the literature, is able to predict the stress and the strain tensor and the warping in the cross-section of tapered beams. Indeed, the presented work places the foundation for further developments in the fields of analytical tapered beam analysis and crosssection analysis of tapered slices.

## Opportunities for future work

In the author's opinion, the following open questions deserve further attention and, therefore, are recommended for future studies. The analytical solution described in chapter 2 refers solely to the Cauchy stress components, but it could be extended to the strain and displacement components by employing elasticity relations. The current version of the method develops from the Navier equation and, consequently, it is restricted to symmetric cross-section. However, extension of the Navier equation to non-symmetric cross sections could then be similarly employed in the analytical method presented and provides the stress analysis of tapered beam with non-symmetric cross sections. In addition, all the geometrical properties, i.e. walls thickness or taper angles, could be implemented as non-constant variables in the lengthwise direction without any significant modification in the method. Lastly, the hypothesis of isotropic material could be also removed by generalising the Navier formula in such a way that the material couplings are taken into account.

The stress analysis performed through the cross-section analysis of the 3D tapered slice analysis shows the potential of the method, which paves the way for further studies in tapered cross-section analysis. As mentioned before, the main limitation associated with the tapered slice method was the requirement for calculation of the nodal forces, $\mathbf{f}$. However, to solve the taper problem, the virtual work equation for 3D tapered geometry should be solved where an approach similar to [26] could be employed to cancel out the nodal forces from the equilibrium equations. It requires overcoming mathematical complexity associated with the 3D taper solution. Currently, the tapered slice method is validated for the case of a planar wedge and the internal forces are coupled to the internal tractions by assuming a prismatic stress distribution of the tractions. More complex benchmark examples can be provided to validate the slice method. For instance, the thin-walled tapered beams with rectangular and circular cross-sections, whose analytical solutions were derived in chapter 2 could be employed for further validation of the tapered slice method. The validity of assuming prismatic internal tractions should be investigated for non-symmetric cross sections. Also, nonsymmetric tapered beams should be studied. The present method
is based on eight-noded elements and linear shape-functions. Higher order elements could be implemented to account for the effects of non-constant taper angles $\alpha(z)$, hence curvature of the beam surface.

## Bibliography

[1] Turbines of the year 2019: Rotor blades. www.windpowermonthly.com/article/ 1669245/turbines-year-2019-rotor-blades.
[2] M. F. Ashby, R. W. Messler, R. Asthana, E. P. Furlani, R. E. Smallman, A. H. W. Ngan, R. J. Crawford, and N. Mills. Engineering materials and processes desk reference. Butterworth-Heinemann, 2009.
[3] G. Balduzzi, M. Aminbaghai, E. Sacco, J. Füssl, J. Eberhardsteiner, and F. Auricchio. Non-prismatic beams: a simple and effective timoshenko-like model. International Journal of Solids and Structures, 90:236-250, 2016.
[4] S. Bennati, P. Bertolini, L. Taglialegne, and P. S. Valvo. On shear stresses in tapered beams. In Proceedings of the GIMC-GMA 2016-21 ${ }^{\text {st }}$ Italian Conference on Computational Mechanics and $8^{\text {th }}$ Meeting of the AIMETA Materials Group, pages 83-84, Lucca, 2016.
[5] V. L. Berdichevskii. Variational-asymptotic method of constructing a theory of shells. Journal of Applied Mathematics and Mechanics, 43(4):664-687, 1979.
[6] P. Bertolini. Structural analysis of wind turbine blades: a study of the effects of tapering on shear stresses. Master's thesis, University of Pisa, 2016.
[7] P. Bertolini, M. A. Eder, L. Taglialegne, and P. S. Valvo. Stresses in constant tapered beams with thin-walled rectangular and circular cross sections. ThinWalled Structures, 137, 2019.
[8] G. S. Bir. User's guide to precomp (pre-processor for computing composite blade properties). Technical report, National Renewable Energy Lab.(NREL), Golden, CO (United States), 2006.
[9] J. P. Blasques. A cross section analysis tool for anisotropic and inhomogeneous beam sections of arbitrary geometry. Risø DTU-National Laboratory for Sustainable Energy Technical University of Denmark. Frederiksborgvej, 399, 1785.
[10] J. P. Blasques. User's manual for BECAS - a cross section analysis tool for anisotropic and inhomogeneous beam sections of arbitrary geometry. Technical report, DTU Wind Energy, Technical University of Denmark, Roskilde, 2012.
[11] F. Bleich. Stahlhochbauten, Vol. 1. Springer, Berlin, 1932.
[12] B. A. Boley. On the Accuracy of the Bernoulli-Euler Theory for Beams of Variable Section, 1963.
[13] M. Capellaro and P. W. Cheng. An iterative method to optimize the twist angle of a wind turbine rotor blade. Wind Engineering, 38(5):489-497, 2014.
[14] S. D. Carothers. Plane strain in a wedge, with applications to masonry dams. Proceedings of the Royal Society of Edinburgh, 33:292-306, 1914.
[15] C. E. S. Cesnik and D. H. Hodges. VABS: a new concept for composite rotor blade cross-sectional modeling. Journal of the American helicopter society, 42(1):2738, 1997.
[16] H. Chen, W. Yu, and M. Capellaro. A critical assessment of computer tools for calculating composite wind turbine blade properties. (December 2009):497-516, 2010.
[17] J. Chen, X. Shen, X. Zhu, and Z. Du. Study on composite bend-twist coupled wind turbine blade for passive load mitigation. Composite Structures, 213:173189, 2019.
[18] R. D. Cook et al. Concepts and applications of finite element analysis. John Wiley \& sons, 2007.
[19] P. J. Couturier and S. Krenk. Wind turbine cross-sectional stiffness analysis using internally layered solid elements. AIAA Journal, 54(7):2149-2159, 2016.
[20] G. Dhondt. Method for Three-dimensional Method for Three-dimensional Thermomechanical Applications. John Wiley \& Sons, Ltd, 2004.
[21] N. K. Dimitrov. Structural Reliability of Wind Turbine Blades: Design Methods and Evaluation. PhD thesis, Technical University of Denmark, 2013.
[22] M. A. Eder, R. Bitsche, and F. Belloni. Effects of geometric non-linearity on energy release rates in a realistic wind turbine blade cross section. Composite Structures, 132:1075-1084, 2015.
[23] M. A. Eder, S. Semenov, and M. Sala. Multiaxial stress based high cycle fatigue model for adhesive joint interfaces. In Hiroshi Okada and Satya N. Atluri, editors, Computational and Experimental Simulations in Engineering, pages 621-632, Cham, 2020. Springer International Publishing.
[24] B. G. Galerkin. On the problem of stresses in dams and retaining walls with trapezoidal profile. Sbornik Leningr Inst. Inzh. Putei. Soobshch, (99):147-170, 1929.
[25] G. L. Ghiringhelli and P. Mantegazza. Linear, straight and untwisted anisotropic beam section properties from solid finite elements. Composites Engineering, 4(12):1225-1239, 1994.
[26] V. Giavotto, M. Borri, P. Mantegazza, G. L. Ghiringhelli, V. Carmaschi, G. C. Maffioli, and F. Mussi. Anisotropic beam theory and applications. Computers §3 Structures, 16(1-4):403-413, 1983.
[27] J. Z. Hansen and R. Østergaard. The effects of fibre architecture on fatigue life-time of composite materials. DTU Wind Energy, 2013.
[28] J. Herskovits. Advances in structural optimization, volume 25. Springer Science \& Business Media, 2012.
[29] J. C. Ho, D. H. Hodges, and W. Yu. Energy Transformation to Generalized Timoshenko Form for Nonuniform Beams. AIA A Journal, 48(6):1268-1272, 2010.
[30] Dewey H Hodges. Nonlinear composite beam theory. American Institute of Aeronautics and Astronautics, 2006.
[31] Dewey. H. Hodges, Jimmy C. Ho, and W. Yu. The effect of taper on section constants for in-plane deformation of an isotropic strip. Journal of Mechanics of Materials and Structures, 3(3):425-440, 2008.
[32] D.y H. Hodges, A. Rajagopal, J. C. Ho, and W. Yu. Stress and strain recovery for the in-plane deformation of an isotropic tapered strip-beam. Journal of Mechanics of Materials and Structures, 5(6):963-975, 2010.
[33] IRENA. Renewable capacity statistics 2019, International Renewable Energy Agency (IRENA), Abu Dhabi. 2019.
[34] T. K. Jacobsen. Materials technology for large wind turbine rotor blades-limits and challenges. In 32nd Risø International Symposium on Material Science, Composite materials for structural performance: towards higher limits, Roskilde, Denmark, volume 5, 2011.
[35] D. K. Jadan. Analytical Solution of Tapered Bimodular Beams. Anbar Journal for Engineering Sciences, pages 79-100, 2012.
[36] R. M. Jones. Mechanics of composite materials. CRC press, 1998.
[37] J. B. Jørgensen. Adhesive Joints in Wind Turbine Blades. PhD thesis, Technical University of Denmark, 2017.
[38] R. J. Knops and P. Villaggio. Recovery of stresses in a beam from those in a cone. Journal of elasticity, 53(1):65-75, 1998.
[39] H. J. T. Kooijman. Bending-Torsion Coupling of a Wind Turbine Rotor Blade, 1996.
[40] S. Krenk and P. J. Couturier. Equilibrium-based nonhomogeneous anisotropic beam element. AIAA Journal, 55(8):2773-2782, 2017.
[41] D. J. Malcolm and D. L. Laird. Extraction of equivalent beam properties from blade models. Wind Energy, 10(2):135-157, 2007.
[42] J. H. Michell. The stress in an æolotrophic elastic solid with an infinite plane boundary. Proceedings of the London Mathematical Society, s1-32(1):247-257, 1900.
[43] S. Minera, M. Patni, E. Carrera, M. Petrolo, Paul M. Weaver, and A. Pirrera. Three-dimensional stress analysis for beam-like structures using Serendipity Lagrange shape functions. International Journal of Solids and Structures, 141-142(February):279-296, jun 2018.
[44] M. Morandini, M. Chierichetti, and P. Mantegazza. Characteristic behavior of prismatic anisotropic beam via generalized eigenvectors. International Journal of Solids and Structures, 47(10):1327-1337, 2010.
[45] A. Paglietti and G. Carta. Remarks on the current theory of shear strength of variable depth beams. Open Civil Engineering Journal, 3(1):28-33, 2009.
[46] H. Pörtner. Multi-axial fatigue models for composite lightweight structures. Master's thesis in applied mechanics department of applied mechanics division of material and computational mechanics, Chalmers University of Technology, Göteborg, Sweden, 2014.
[47] R. Quispe Rodríguez, W. P. De Paiva, P. Sollero, M. R. Bertoni Rodrigues, and É. L. De Albuquerque. Failure criteria for adhesively bonded joints. International Journal of Adhesion and Adhesives, 37(September):26-36, 2012.
[48] A. Rajagopal. Advancements in rotor blade cross-sectional analysis using the variational-asymptotic method. PhD thesis, Georgia Tech, 2014.
[49] A. R. Stäblein, M. H. Hansen, and D. R. Verelst. Modal Properties and Stability of Bend-Twist Coupled Wind Turbine Blades. Wind Energy Science Discussions, pages 1-27, 2016.
[50] L. Taglialegne. Stress fields in wind turbine blades with thin-walled variable cross sections. PhD thesis, International Doctorate "Civil and Environmental Engineering", Universities of Florence, Perugia and Pisa - TU C.W. Braunschweig, 2018.
[51] L. Thomas and M. Ramachandra. Advanced materials for wind turbine bladeA Review. Materials Today: Proceedings, 5(1):2635-2640, 2018.
[52] W. A. Timmer. Aerodynamic characteristics of wind turbine blade airfoils at high angles-of-attack. In 3rd EWEA Conference-Torque 2010: The Science of making Torque from Wind, Heraklion, Crete, Greece, 28-30 June 2010. European Wind Energy Association, 2010.
[53] S. Timoshenko. Bending stresses in curved tubes of rectangular cross section. Transactions of the American Society of Mechanical Engineers, 45:135-140, 1923.
[54] S. Timoshenko and J. N. Goodier. Theory of Elasticity. McGraw-Hill book Company, 1951.
[55] S. P. Timoshenko and J. M. Gere. Mechanics of Materials. Van Nostrand Reinhold, New York, 1972.
[56] L. Wang, X. Liu, L. Guo, N. Renevier, and M. Stables. A mathematical model for calculating cross-sectional properties of modern wind turbine composite blades. Renewable Energy, 64(April 2014):52-60, 2014.
[57] W. Yu. A unified theory for constitutive modeling of composites. Journal of Mechanics of Materials and Structures, 11(4):379-411, 2016.
[58] W. Yu and D. H. Hodges. Generalized timoshenko theory of the variational asymptotic beam sectional analysis. Journal of the American Helicopter Society, 50(1):46-55, 2005.
[59] W. Yu, D. H. Hodges, and J. C. Ho. Variational asymptotic beam sectional analysis - An updated version. International Journal of Engineering Science, 59:40-64, 2012.
[60] M. Zhou, H. Fu, and L. An. Distribution and properties of shear stress in elastic beams with variable cross section: Theoretical analysis and finite element modelling. KSCE Journal of Civil Engineering, 24:1240-1254, 2020.
[61] M. Zhou, Z. Liu, J. Zhang, and H. Shirato. Stress analysis of linear elastic nonprismatic concrete-encased beams with corrugated steel webs. Journal of Bridge Engineering, 22(6):1-13, 2017.

## Part II

## Articles

## Paper 1

P. Bertolini, M. A. Eder, L. Taglialegne, P. Valvo, Stresses in constant tapered beams with thin-walled rectangular and circular cross sections, Thin-Walled Structures 137 (2019) 527 - 540.

Full length article

# Stresses in constant tapered beams with thin-walled rectangular and circular cross sections 

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#### Abstract

Tapered beams are widely employed in efficient flexure dominated structures. In this paper, analytical expressions are derived for the six Cauchy stress components in untwisted, straight, thin-walled beams with rectangular and circular cross sections characterised by constant taper and subjected to three cross-section forces. These expressions pertain to homogeneous, isotropic, linear elastic materials and small strains. In fact, taper not only alters stress magnitudes and distributions but also evokes stress components, which are zero in prismatic beams. A parametric study shows that increasing taper decreases the von Mises stress based fatigue life, suggesting that step-wise prismatic approximations entail non-conservative designs.


| Nomenclature |  |  |
| :--- | :--- | :--- |
| Symbol | Unit | Description <br> $A$ |
| $A^{*}$ | $\mathrm{~m}^{2}$ | area of cross section <br> $b$ |
| $\mathrm{~m}^{2}$ | m | area of hatched cross section |
| half width of box girder cross section |  |  |


| $p_{y}, p_{z}$ | $\mathrm{~N} / \mathrm{m}$ | distributed loads per unit length |
| :--- | :--- | :--- |
| $R$ | m | radius of conical beam cross section |
| $R_{0}$ | m | radius of conical beam cross section at $z=0$ |
| $S_{x}^{*}$ | $\mathrm{~m}^{3}$ | first moment of area of hatched cross section |
| $t$ | m | thickness of conical beam wall |
| $t_{f}$ | m | thickness of box beam flange |
| $t_{p}$ | m | projected wall thickness |
| $t_{w}$ | m | thickness of box beam web |
| $V$ | $\mathrm{~m}^{3}$ | volume |
| $x, y, z$ |  | Cartesian coordinates <br> $1,2,3$ |
| $\alpha$ | rad | local reference axes <br> angle of taper |
| $\eta$ | m | local abscissa |
| $\theta$ | rad | polar angle <br> $\nu$ |
| $\sigma_{i j}$ | Pa | Poisson's ratio <br> stress tensor components |
| $\sigma_{v M}$ | Pa | equivalent von Mises stress |
| $\sigma_{v M}^{n o r m}$ |  | normalised von Mises stress (cone over cylinder) |
| $\sigma_{v M}^{\alpha}$ |  | normalised von Mises stress (negative over positive angle) |

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this paper denoted as taper - refers to a variation of the cross-section height and/or width along the beam axis according to the governing internal force distribution. Some prominent examples of tapered beams are bridge girders at intermediate supports, frame/truss structures for industrial halls and hangars, aircraft wings, wind turbine towers, and wind turbine rotor blades [1-3].

In a beam with a straight centreline and variable cross section, the angle of taper, $\alpha$, can be defined as the angle enclosed by the local tangent plane to the beam lateral surface and the beam axis i.e. the lengthwise gradient of the lateral surface with respect to the beam axis. In general, the angle of taper will be a pointwise function. In the simplest case - herein referred to as constant taper - the angle of taper does not change along the beam axis [4].

It is well-known in literature that beams with variable cross sections show a significantly different behaviour in contrast to prismatic beams. Variable cross-section beams exhibit a non-trivial stress distribution. In particular, the shear stresses evoked are counter intuitive and hardly predictable by the classical theory for prismatic beams [5,6].

The effects of taper on the shear stress distribution in simple planar non-prismatic beams were already investigated by Timoshenko [7]. Bleich [8] derived a closed-form solution showing that in tapered beams shear stresses are induced not only by shear forces, but also by axial forces and bending moments. Unfortunately, Bleich was misled by the analogy with prismatic beams and referred to the centreline as the locus of maximum shear stresses. Later, Paglietti and Carta [9] demonstrated that the maximum shear stress does not necessarily occur at the elastic centre of the cross-section.

Atkin proposed a different approach based on classical elasticity, defining proper stress functions for specific aeronautical problems [10,11]. Subsequently, Krahula [12] compared the predictions of Bleich's formula - even though not citing directly [8] but referring to Timoshenko and Gere [13] - and the solution of a two-dimensional elasticity problem for a tapered cantilever beam loaded by a concentrated shear force at its free end. Elasticity theory has been used to model tapered beams also by Knops and Villaggio [14], and more recently by Trahair and Ansourian [15].

The behaviour of tapered beams under bending and torsion was extensively investigated by Lee and Szabo [16] and Lee et al. [17].

Chong et al. [18] showed by means of simplified mechanical models that the shear stress in the webs of I- and box girders strongly depends on both the sign of the taper, i.e. the positive or negative slope, and the direction of the shear force.

With the rise of structural optimisation in the past decades, research increasingly focused on the development of computationally efficient semi-analytical numerical methods for tapered elastic 2D beams with solid rectangular cross sections. Hodges et al. [19,20] and Rajagopal [21] developed the variational-asymptotic method, which is capable of providing a full elasticity solution in terms of stresses, strains, and displacements for 2D beams with constant taper subjected to shear and axial forces, and bending moments. Balduzzi et al. [22] derived a nonprismatic planar beam element from a 2D elastic solution. More recently, Balduzzi et al. [23] extended the approach also to multilayer planar non-prismatic beams.

Taglialegne [6] derived an exact analytical elastic solution for a tapered planar beam subject to shear and axial forces, and bending moment based on the solution of the wedge proposed by Michell [24] and Carothers [25]. Bennati et al. [4,26] showed that the shear stress distribution - also numerically predicted by Balduzzi et al. [22] - may be a satisfactory approximation of the exact solution.

It is noteworthy to mention that the optimisation of tapered beams comprising of complex thin-walled single- or multi-cellular hollow sections gained a lot of attention, especially in the industry. Topology optimisation techniques rely on computationally efficient stress analysis tools which usually exclude the use of computationally demanding 3D finite element models. As a remedy, the so-called cross-sectional analysis tools, such as BECAS [27] and VABS [28], have been developed
for the efficient analysis of slender beam-type structures. However, many cross-sectional analysis tools as well as beam elements intended for the use of modelling tapered hollow sections approximate the tapered beams to step-wise prismatic beams.

This paper provides analytical expressions for the six Cauchy stress tensor components occurring in tapered beams with thin-walled rectangular and circular cross-sections, shortly referred to in the following as tapered box beam and thin-walled conical beam, respectively. The derivation is carried out in the hypothesis of homogeneous, isotropic, linear elastic material behaviour and first-order Euler-Bernoulli beam theory. First, an extension of Jourawski's formula for shear stresses is deduced for straight and untwisted beams with doubly-symmetric variable cross sections, subjected to distributed loads producing axial force, shear forces, and bending moment (in a symmetry plane of the cross section). The deduction assumes that Navier's formula yields a good approximation of the normal stresses in variable cross-section beams with moderate taper angles. (for a deeper discussion of this issue the reader is referred to Boley [29]). Subsequently, the extended Jourawski's formula is specialised to the tapered box beam and thinwalled conical beam, under the assumed absence of distributed loads. The remaining stress components are obtained through the assumption of plane stress in the thin cross-section walls by integration of the Cauchy equilibrium differential equations.

The analytical solutions are validated against 3D finite element analyses for constant-taper cantilever beams clamped at one end (the root section) and loaded at the free end (the tip section). The numerical results show that already small taper angles (in the order of few degrees) not only considerably alter the stress distributions obtained from prismatic beam theory, but can evoke stress components which are zero otherwise. The potential implications of the effects of taper on beam designs have to date not been investigated to the best knowledge of the authors. Therefore, the analytically derived solutions were used in a comprehensive parametric study to shed light on the effect of constant taper on the von Mises stress and consequently on the fatigue life of thin-walled conical beams.

## 2. Analytical solution

### 2.1. Extended shear formula

A variable cross-section beam of length $L$, having a straight centreline and a doubly symmetric cross section (Fig. 1) is considered. A global Cartesian reference system $O x y z$ is fixed with the origin $O$ located in the centre of one of the end sections; the $x$ - and $y$-axes are aligned with the cross-section principal axes of inertia (coincident with the symmetry axes), while the z-axis is coincident with the beam centreline. Here, it is stipulated that the beam is not twisted, i.e. that the principal directions of inertia of each cross section are parallel to the $x$ - and $y$ axes.

The beam is subjected to distributed loads, $p_{y}(z)$ and $p_{z}(z)$, acting in the $y$ - and $z$-directions, respectively, and to a distributed bending


Fig. 1. (a) Beam with variable cross section subjected to distributed axial and transverse loads, and bending couple. (b) Generic cross section with two axes of symmetry.
couple, $m_{x}(z)$. With the above assumptions, the internal forces acting on each cross section will be the shear force, $F_{y}(z)$, axial force, $F_{z}(z)$, and bending moment, $M_{x}(z)$. Local equilibrium demands that:
$\frac{\mathrm{d} F_{z}(z)}{\mathrm{d} z}+p_{z}(z)=0, \frac{\mathrm{~d} F_{y}(z)}{\mathrm{d} z}+p_{y}(z)=0, \frac{\mathrm{~d} M_{x}(z)}{\mathrm{d} z}+m_{x}(z)=F_{y}(z)$

Global equilibrium of the beam requires also concentrated forces $F_{y}^{0}=F_{y}(0), F_{z}^{0}=F_{z}(0)$, and $M_{x}^{0}=M_{x}(0)$ to be applied at $z=0$ and $F_{y}^{L}=F_{y}(L), F_{z}^{L}=F_{z}(L)$, and $M_{x}^{L}=M_{x}(L)$ at $z=L$.

In a prismatic beam, under the assumptions that plane cross sections remain plane after deformation and that the material is homogeneous, isotropic, and linearly elastic [30], Navier's formula furnishes the normal stresses on cross sections notably:
$\sigma_{z z}=\frac{F_{z}}{A}+\frac{M_{x}}{I_{x}} y$
where $A$ and $I_{x}$ respectively are the area and second moment of area with respect to the $x$-axis of the cross section. Here and in the following, the dependency upon $z$ will be omitted if not strictly necessary.

Here, it is assumed that Navier's Eq. (2) holds also for beams of variable cross section. The validity of this assumption has been investigated by Boley [29], where a good approximation is obtained for moderate taper angles: for example, if $\alpha=10^{\circ}$, the error is around $7.5 \%$.

Firstly, the infinitesimal segment of a variable cross-section beam slice between two cross sections located at $z$ and $z+\mathrm{d} z$ as shown in Fig. 1(a), is considered. Fig. 2(a) illustrates the uniform and linearly variable normal stress distributions respectively induced by the axial force and bending moment acting on the infinitesimal beam segment. Secondly, a generic straight chord of length $c$ is introduced, which subdivides the cross section into two complementary parts. Furthermore, a local abscissa denoted as $\eta$ aligned orthogonal to the chord direction is introduced as shown in Fig. 2(b). Consequently, the infinitesimal beam segment itself turns out to be subdivided into two parts. The ensuing focus is put on the 'hatched' part associated with the side of positive $\eta$. The corresponding cross-section area is denoted as $A^{*}$.

In order to determine an expression for the shear stresses acting on the cross section in the direction orthogonal to the chord, $\sigma_{z \eta}$, the equilibrium is imposed to the 'hatched' part of the beam segment depicted in Fig. 3. Assuming that the axial loads, $p_{z}$, are uniformly distributed on the cross section, the equilibrium in the $z$-direction can be written as:

$$
\begin{align*}
& \int_{A^{*}(z)} \sigma_{z z} \mathrm{~d} A+\sigma_{z \eta} \frac{c+(c+\mathrm{d} c)}{2} \mathrm{~d} z-\int_{z}^{z+\mathrm{d} z} \int_{A^{*}(z)} \frac{p_{z}}{A} \mathrm{~d} A \mathrm{~d} z \\
& =\int_{A^{*}(z+\mathrm{d} z)}\left(\sigma_{z z}+\mathrm{d} \sigma_{z z}\right) \mathrm{d} A \tag{3}
\end{align*}
$$

Following Taglialegne [6], Eq. (3) can be expanded and higherorder infinitesimal terms neglected. Hence,


Fig. 3. Equilibrium in the $z$-direction of the 'hatched' part of the infinitesimal beam segment of length $\mathrm{d} z$.
$\sigma_{z \eta}=\frac{1}{c} \int_{A^{*}} \frac{\mathrm{~d} \sigma_{z z}}{\mathrm{~d} z} \mathrm{~d} A+\frac{1}{c} \frac{p_{z}}{A} \int_{A^{*}} \mathrm{~d} A$
By substituting Eqs. (1) and (2) into (4), after simplification, the general solution for the cross-section shear stress component is obtained:
$\sigma_{z \eta}=\frac{1}{c}\left[F_{z} \frac{\mathrm{~d}}{\mathrm{~d} z}\left(\frac{A^{*}}{A}\right)+\left(F_{y}-m_{x}\right) \frac{S_{x}^{*}}{I_{x}}+M_{x} \frac{\mathrm{~d}}{\mathrm{~d} z}\left(\frac{S_{x}^{*}}{I_{x}}\right)\right]$
where $S_{x}^{*}$ is the first moment of area of the 'hatched' part of the cross section with respect to the $x$-axis.

It is worth noting that Eq. (5) can be considered as a generalisation of a similar formula that Bleich [8] derived for a beam with constant width and variable height for the specific case that $\mathrm{d} F_{z} / \mathrm{d} z=0$.

### 2.2. Vertically tapered box beam

The extended shear formula Eq. (5) can be specialised to the vertically tapered box beam shown in Fig. 4. The cantilever beam studied in this paper is exemplary. However, the assumed support conditions do not pose any restriction to the validity of these specialised equations. The cross-section width, $2 b$, is constant, while the cross-section height, $2 h(z)$, varies linearly with the $z$-coordinate according to:
$h(z)=h_{0}-z \tan \alpha$
where $h_{0}$ denotes the half height of the root section and $\alpha$ is the angle of taper. The thicknesses of the flanges, $t_{f}$, and webs, $t_{w}$, are assumed to be constant along the $z$-coordinate and small with respect to the crosssection dimensions. According to the defined geometry, the Cartesian coordinates of the thin wall mid-surface vary within the following limits:
$-b \leq x \leq b,-h(z) \leq y \leq h(z), \quad 0 \leq z \leq L$
The flange thickness projected in the cross-section plane is
$t_{p}=\frac{t_{f}}{\cos \alpha}$


Fig. 2. (a) Infinitesimal beam segment of length $\mathrm{d} z$. (b) Cross section with generic chord of length $b$ and local abscissa $\eta$.


Fig. 4. (a) Side and (b) front views of a thin-walled vertically tapered cantilever box beam of length $L$ and taper angle $\alpha$. (c) Arbitrary cross section, where the projected flange thickness, $t_{p}$, and web thickness, $t_{w}$, are highlighted.

Hence, the cross-section area and second moment of area with respect to the $x$-axis are respectively:
$A=4\left[b t_{p}+t_{w} h(z)\right]$
$I_{x}=4\left[b t_{p} h(z)^{2}+\frac{1}{3} t_{w} h(z)^{3}\right]$
Utilising symmetry, the solution can be reduced to a quarter of the cross section. In what follows, the stress fields are evaluated separately in the half flange and half web of the positive quadrant, $x \geq 0$ and $y \geq 0$.

### 2.2.1. Stress components in the flange

The half flange defined by $0 \leq x \leq b$ and $y=h(z)$ is considered. Here, the normal stress component is directly determined by substituting Eqs. (9) and (10) into (2):
$\sigma_{z z}^{f}=\frac{1}{4}\left[\frac{F_{z}}{b t_{p}+t_{w} h(z)}+\frac{3 M_{x}}{3 b t_{p} h(z)+t_{w} h(z)^{2}}\right]$
The area and first moment of area with respect to the $x$-axis of the 'hatched' portion of cross section on the flange (see Fig. 5) with $\eta=b-x$, can be written as follows:
$A^{*}=x t_{p}$
$S_{x}^{*}=x t_{p} h(z)$
Subsequently, by substituting Eqs. (9), (10), (12) and (13) into (5), with $c=t_{p}$, the shear stress component on the flange is obtained:

$$
\begin{align*}
\sigma_{z x}^{f}= & -\frac{1}{4} x\left\{\frac{F_{z} t_{w} \tan \alpha}{\left[b t_{p}+t_{w} h(z)\right]^{2}}+\frac{3 F_{y}}{3 b t_{p} h(z)+t_{w} h(z)^{2}}\right. \\
& \left.+\frac{3 M_{x} \tan \alpha\left[3 b t_{p}+2 t_{w} h(z)\right]}{h(z)^{2}\left[3 b t_{p}+t_{w} h(z)\right]^{2}}\right\} \tag{14}
\end{align*}
$$

To determine the remaining stress components, first a local coordinate system 123 is defined such that the 2 -axis is parallel to the $x$ axis and the 3 -axis is the outward normal to the flange as illustrated in Fig. 6.

The stress components in the local and global reference systems can


Fig. 5. Hatched portion of cross section on the flange.


Fig. 6. Quarter of an infinitesimal segment of the vertically tapered box beam with global, $x y z$, and local, 123, reference systems.
be related by introducing a rotation matrix as follows:
$\left[\begin{array}{lll}\sigma_{11}^{f} & \sigma_{12}^{f} & \sigma_{13}^{f} \\ \sigma_{21}^{f} & \sigma_{22}^{f} & \sigma_{23}^{f} \\ \sigma_{31}^{f} & \sigma_{32}^{f} & \sigma_{33}^{f}\end{array}\right]=\left[\begin{array}{ccc}0 & -\sin \alpha & \cos \alpha \\ 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha\end{array}\right]\left[\begin{array}{ccc}\sigma_{x x}^{f} & \sigma_{x y}^{f} & \sigma_{x z}^{f} \\ \sigma_{y x}^{f} & \sigma_{y y}^{f} & \sigma_{y z}^{f} \\ \sigma_{z x}^{f} & \sigma_{z y}^{f} & \sigma_{z z}^{f}\end{array}\right]\left[\begin{array}{ccc}0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \\ \cos \alpha & 0 & \sin \alpha\end{array}\right]$

Accordingly, under the assumption of a plane stress state in the flange, the local stress component condition $\sigma_{31}^{f}=\sigma_{32}^{f}=\sigma_{33}^{f}=0$ must hold. Consequently, Eqs. (15) yield:
$\sigma_{y y}^{f}=\sigma_{z z}^{f} \tan ^{2} \alpha$
$\sigma_{x y}^{f}=-\sigma_{z x}^{f} \tan \alpha$
$\sigma_{y z}^{f}=-\sigma_{z z}^{f} \tan \alpha$
The last unknown stress component is obtained by integrating the first Cauchy equilibrium equation, Eq. (A.1), assuming null body forces:
$\sigma_{x x}^{f}=-\int_{0}^{x} \frac{\partial \sigma_{z x}^{f}}{\partial z} \mathrm{~d} x+\left.\sigma_{x x}^{f}\right|_{x=0}$
where $\left.\sigma_{x x}^{f}\right|_{x=0}$ represents an integration constant which can be determined after deduction of the solution for the stresses in the web.

### 2.2.2. Stress components in the web

In the following the half web, defined by $x=b$ and $0 \leq y \leq h(z)$, is considered. The normal stress component is directly determined by substituting Eqs. (9) and (10) into (2):
$\sigma_{z z}^{w}=\frac{1}{4}\left[\frac{F_{z}}{b t_{p}+t_{w} h(z)}+\frac{3 M_{x} y}{3 b t_{p} h(z)^{2}+t_{w} h(z)^{3}}\right]$


Fig. 7. Hatched portion of cross section on the web.

On the web, the 'hatched' portion of cross section includes the half flange and a part of the web, with $\eta=y$, as illustrated in Fig. 7. The area and moment of inertia with respect to the $x$-axis are respectively:
$A^{*}=b t_{p}+t_{w}[h(z)-y]$
$S_{x}^{*}=b t_{p} h(z)+\frac{t_{w}}{2}\left[h(z)^{2}-y^{2}\right]$
By substituting Eqs. (9), (10), (21) and (22) into (5), with $c=t_{w}$, the shear stress component on the web is obtained as follows:
$\sigma_{y z}^{w}=\frac{A_{1}(y, z)}{4} F_{z}+\frac{3 A_{2}(y, z)}{8 t_{w}} F_{y}+\frac{3 A_{3}(y, z)}{8 t_{w}} M_{x}$
where

$$
\begin{aligned}
A_{1}(y, z)= & -\frac{t_{w} y \tan \alpha}{\left[b t_{p}+t_{w} h(z)\right]^{2}} \\
A_{2}(y, z)= & \frac{2 b t_{p} h(z)+t_{w}\left[h(z)^{2}-y^{2}\right]}{h(z)^{2}\left[3 b t_{p}+t_{w} h(z)\right]} \\
A_{3}(y, z)= & \frac{\tan \alpha}{\left[3 b t_{p}+t_{w} h(z)\right]^{2} h(z)^{3}}\left\{6 b^{2} t_{p}^{2} h(z)\right. \\
& \left.+2 b t_{p} t_{w}\left[2 h(z)^{2}-3 y^{2}\right]+t_{w}^{2} h(z)\left[h(z)^{2}-3 y^{2}\right]\right\}
\end{aligned}
$$

The webs are assumed to be in a plane-stress state. Consequently, $\sigma_{x x}^{w}=\sigma_{x y}^{w}=\sigma_{x z}^{w}=0$. The last unknown stress component is obtained by integration of the second Cauchy equilibrium equation, Eq. (A.2), assuming null body forces:
$\sigma_{y y}^{w}=-\int_{0}^{y} \frac{\partial \sigma_{y z}^{w}}{\partial z} \mathrm{~d} y+\left.\sigma_{y y}^{w}\right|_{y=0}$
where $\left.\sigma_{y y}^{w}\right|_{y=0}$ is an integration constant.

### 2.2.3. Equilibrium conditions on edges

The solution for the stress distribution in the box beam is already completely determined, except for the two integration constants, $\left.\sigma_{x x}^{f}\right|_{x=0}$ and $\left.\sigma_{y y}^{w}\right|_{y=0}$. The latter can be calculated by imposing the equilibrium of

(a)
an infinitesimal edge portion of length $\mathrm{d} z$ connecting the flange and web as shown in Fig. 8. Neglecting higher-order infinitesimal terms, equilibrium in the $x$-direction gives:
$\left.t_{f} \sigma_{x x}^{f}\right|_{x=b} \frac{\mathrm{~d} z}{\cos \alpha}+\left.t_{w} \sigma_{x y}^{w}\right|_{y=h} \mathrm{~d} z+\left.t_{w} \sigma_{z x}^{w}\right|_{y=h} \tan \alpha \mathrm{~d} z=0$
and equilibrium in the $y$-direction gives:
$\left.t_{f} \sigma_{x y}^{f}\right|_{x=b} \frac{\mathrm{~d} z}{\cos \alpha}+\left.t_{w} \sigma_{y y}^{w}\right|_{y=h} \mathrm{~d} z+\left.t_{w} \sigma_{y z}^{w}\right|_{y=h} \tan \alpha \mathrm{~d} z=0$.
By substituting Eqs. (19) and (24) into (25) and (26) respectively, results in:
$\left.\sigma_{x x}^{f}\right|_{x=0}=\int_{0}^{b} \frac{\partial \sigma_{z x}^{f}}{\partial z} \mathrm{~d} x-\frac{t_{w}}{t_{f}}\left(\left.\cos \alpha \sigma_{x y}^{w}\right|_{y=h}+\left.\sin \alpha \sigma_{z x}^{w}\right|_{y=h}\right)$
$\left.\sigma_{y y}^{w}\right|_{y=0}=\int_{0}^{h} \frac{\partial \sigma_{y z}^{w}}{\partial z} \mathrm{~d} y-\left.\tan \alpha \sigma_{y z}^{w}\right|_{y=h}-\left.\frac{t_{f}}{t_{w} \cos \alpha} \sigma_{x y}^{f}\right|_{x=b}$
Due to the complexity of the mathematical expressions involved, it is convenient to solve Eqs. (27) and (28) numerically at each cross section to determine the values of the integration constants and complete the determination of the stress distribution in the beam.

### 2.3. Thin-walled conical beam

Fig. 9 shows a thin-walled conical cantilever beam. The radius defining the wall mid-surface can be written as:
$R(z)=R_{0}-z \tan \alpha$
where $\alpha$ is the taper angle and $R_{0}$ is the root radius. The polar angle, $\theta$, is measured counterclockwise from the $x$-axis.

The wall thickness, $t$, is assumed to be constant with the $z$-coordinate and small with respect to the cross-section radius. The projected wall thickness, cross-sectional area and second moment of area with respect to the $x$-axis respectively are:
$t_{p}=\frac{t}{\cos \alpha}$
$A=2 \pi t_{p} R(z)$
$I_{x}=2 \int_{0}^{\pi} t_{p} \sin ^{2} \theta R^{3}(z) \mathrm{d} \theta=\pi t_{p} R^{3}(z)$
The normal stress on the cross section is directly obtained by substituting Eqs. (31) and (32) into (2) with $y=\sin \theta R(z)$ :
$\sigma_{z z}=F_{z} \frac{\cos \alpha}{2 \pi t R(z)}+M_{x} \frac{\cos \alpha \sin \theta}{\pi t R(z)^{2}}$
The area and first moment of area of the 'hatched' part of the conical

(b)

Fig. 8. Stresses acting in the (a) $x$ - and (b) $y$-directions on an infinitesimal edge portion between the flange and web of the box beam.


Fig. 9. (a) Side and (b) frontal views of a thin-walled conical cantilever beam of length $L$ and taper angle $\alpha$. (c) Arbitrary cross section, where the projected wall thickness $t_{p}$ is highlighted.


Fig. 10. Hatched portion of conical cross section.
cross section defined by $\eta=\theta R(z)$ and illustrated in Fig. 10, are respectively given by:
$A^{*}=\int_{\theta}^{\pi / 2} t_{p} R(z) \mathrm{d} \theta=(\pi-\theta) t_{p} R(z)$
$S_{x}^{*}=\int_{\theta}^{\pi / 2} t_{p} \sin \theta R^{2}(z) \mathrm{d} \theta=t_{p} \cos \theta R^{2}(z)$
The circumferential shear stress is determined by substituting Eqs. (31), (32), (34) and (35) into (5), with $c=t_{p}$ :
$\sigma_{z \theta}=F_{y} \frac{\cos \alpha \cos \theta}{\pi t R(z)}+M_{x} \frac{\sin \alpha \cos \theta}{\pi t R(z)^{2}}$

Fig. 11 shows the three different coordinate systems (CSYS) defined in a conical beam segment: the global CSYS, $x y z$, the cylindrical CSYS, $r \theta z$, and the local CSYS, 123. The latter is oriented such that the 12 plane is tangent to the thin wall mid-surface with the 2 -axis opposite to $\theta$ and with the 3 -axis pointing in the outer normal direction.

Thus, the stress tensor components in the cylindrical and local reference systems can be related as follows:


Fig. 11. Global coordinates $x, y, z$, cylindrical coordinates $r, \theta, z$, and local coordinates $1,2,3$ in a conical beam.

$$
\begin{align*}
& {\left[\begin{array}{lll}
\sigma_{r r} & \sigma_{r \theta} & \sigma_{r z} \\
\sigma_{\theta r} & \sigma_{\theta \theta} & \sigma_{\theta z} \\
\sigma_{z r} & \sigma_{z \theta} & \sigma_{z z}
\end{array}\right]} \\
& \quad=\left[\begin{array}{lll}
-\sin \alpha & 0 & \cos \alpha \\
0 & -1 & 0 \\
\cos \alpha & 0 & \sin \alpha
\end{array}\right]\left[\begin{array}{lll}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{array}\right]\left[\begin{array}{lll}
-\sin \alpha & 0 & \cos \alpha \\
0 & -1 & 0 \\
\cos \alpha & 0 & \sin \alpha
\end{array}\right] \tag{37}
\end{align*}
$$

Under the assumption of plane stress in the wall, the local stress components $\sigma_{31}, \sigma_{32}$, and $\sigma_{33}$ are identically null. Consequently, Eqs. (37) produce a set of three linear equations for the unknowns $\sigma_{r r}, \sigma_{r \theta}$, and $\sigma_{r z}$ :
$\left\{\begin{array}{l}\sigma_{31}=2 \sigma_{z r} \cos 2 \alpha-\left(\sigma_{r r}-\sigma_{z z}\right) \sin 2 \alpha=0 \\ \sigma_{32}=\sigma_{r \theta} \cos \alpha+\sigma_{\theta z} \sin \alpha=0 \\ \sigma_{33}=\sigma_{r r} \cos ^{2} \alpha+\sigma_{z r} \sin 2 \alpha+\sigma_{z z} \sin ^{2} \alpha=0\end{array}\right.$
The solution of the linear set of equations is:
$\sigma_{r r}=\sigma_{z z} \tan ^{2} \alpha$
$\sigma_{r \theta}=-\sigma_{\theta z} \tan \alpha$
$\sigma_{z r}=-\sigma_{z z} \tan \alpha$
Finally, the hoop stress component, $\sigma_{\theta \theta}$, is immediately derived from the first of the local equilibrium equations for a hollow thin conical element, whose derivation is presented in Appendix B. Substituting Eqs. (41) and (37) into (B.5) leads to $\sigma_{\theta \theta}=0$ for all the loading conditions considered.

## 3. Numerical analysis

The analytical solutions of the two described geometries were compared with finite element models for verification. Two cantilever beams, namely the rectangular beam and the conical beam, were modelled inside the commercial finite element package Abaqus 2017 [31]. The mesh topology of these models is depicted in Fig. 12.

A control section perpendicular to the beam $z$-axis at the mid-span cross section was used for validation. Following Saint-Venant's principle the control section was chosen sufficiently far away from the root and tip sections in order to avoid boundary effects affecting the far-field solutions derived. The geometrical properties of the two models are listed in Table 1.

Homogeneous, isotropic, linear elastic material properties for steel were assigned with an elastic modulus of $E=210 \mathrm{GPa}$ and a Poisson's ratio of $v=0.3$. The models were discretized by enriched eight-noded solid elements (Abaqus element type C3D8R) as given by Table 2. The wall was discretised with two elements through the thickness.

A convergence study of different mesh densities was performed and the numerical results presented are obtained from sufficiently discretized models.

The kinematic (rigid) coupling constraints applied to the nodes of both ends of the beam, were coupled to a master node located in the elastic centre of the cross sections. The model was loaded at the master node located at the tip through application of concentrated forces, $F_{y}^{L}$,


Fig. 12. 3D finite element model of (a) the tapered rectangular beam, and (b) the cone. In both figures the loads and the boundary conditions are applied at a reference point, which is linked through a rigid coupling constraint to the tip or root cross section. In Figure (a) the detail of the mesh refinement at the corner is illustrated.

Table 1
Geometrical properties of the rectangular and conical beam models.

|  | Beam |  |  |  |  | Mid-span cross section |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\begin{aligned} & L \\ & {[\mathrm{~m}]} \end{aligned}$ | $\alpha$ <br> [deg.] | $\begin{aligned} & t_{f} \\ & {[\mathrm{~mm}]} \end{aligned}$ | $\begin{aligned} & t_{w} \\ & {[\mathrm{~mm}]} \end{aligned}$ | $\begin{aligned} & t \\ & {[\mathrm{~mm}]} \end{aligned}$ | $\begin{aligned} & 2 h \\ & {[\mathrm{~m}]} \end{aligned}$ | $\begin{aligned} & 2 b \\ & {[\mathrm{~m}]} \end{aligned}$ | $\begin{aligned} & 2 R \\ & {[\mathrm{~m}]} \end{aligned}$ |
| Rectangular | 10.0 | 4.0 | 10.0 | 10.0 | - | 1.0 | 1.0 | - |
| Conical | 10.0 | 4.0 | - | - | 10.0 | - | - | 1.0 |

Table 2
Mesh discretisation and model size parameters of the two numerical models. The fourth and fifth columns refer to the largest and smallest transverse element sizes measured in the cross-section plane. The rectangular model has a higher mesh density because of the higher resolution required at the corners.

|  | \# of el. | \# of nodes | $\max$ el. size [m] | min el. size [m] |
| :--- | :--- | :--- | :--- | :--- |
| Rectangular | 134,400 | 202,104 | $49.25 \times 10^{-3}$ | $5.00 \times 10^{-3}$ |
| Conical | 128,000 | 216,720 | $27.14 \times 10^{-3}$ | $5.43 \times 10^{-3}$ |

Table 3
Load cases applied to both models in order to investigate the effect of taper under shear-bending, pure axial force and pure bending, and cross-section forces at mid-span. The applied bending moment is comparable to the resulting shear bending.

| Case | Tip loads |  |  | Internal forces at the mid-span |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $F_{y}[\mathrm{~N}]$ | $F_{z}[\mathrm{~N}]$ | $M_{x}[\mathrm{Nm}]$ |  | $F_{y}[\mathrm{~N}]$ | $F_{z}[\mathrm{~N}]$ | $M_{x}[\mathrm{~N} \mathrm{~m}]$ |
| Shear | 1000 | - | - | 1000 | - | -5000 |  |
| Extension | - | 1000 | - | - | 1000 | - |  |
| Bending | - | - | 5000 | - | - | 5000 |  |

$F_{z}^{L}$, and/or bending moment, $M_{x}^{L}$. Table 3 lists the single load cases used in this study. All six degrees of freedom of the master node of the root section were restrained such that the beam was fully clamped. A direct solution strategy was used (Abaqus linear perturbation). Results were extracted in global coordinates along node-paths located in the wall mid-surface at the control cross section.

## 4. Results

### 4.1. Stress analysis

The six components of the stress field produced by the axial and shear loads, and bending moment were evaluated via both the analytical solutions derived in Section 2 and the numerical finite element analysis for the two geometries previously described. Because of symmetry in the $x$ and $y$ direction, the stresses were evaluated along the centre line of a quarter of the control cross section. In Figs. 13-20, the label $A n$ indicates the stresses evaluated through the analytical solutions, whereas the label Num indicates the FE results. Moreover, the deviation between the numerical and the analytical results is given by
the Normalised Mean Square Error (NMSE) [32] in Tables 4-6.

### 4.1.1. Vertically tapered box beam

The plots concerning the rectangular beams show the stresses evaluated along the mid-surface of half flange $(0 \leq x \leq b)$ and half web $(0 \leq y \leq h)$. Figs. 13-15 depict the stresses distribution along the flange, while Figs. 16 and 17 along the web. The stress components $\sigma_{x x}$, $\sigma_{x y}$, and $\sigma_{x z}$ are zero along the web and have therefore been omitted. The stress singularity induced by the sharp corner in the FE models can be noted in Figs. 14-16 when $x$ and $y$ approach 0.5 m . The singularity effect of the corner is not considered in the analytical expressions which explains the deviations between the latter and the numerical predictions when approaching the corner.

### 4.1.2. Thin-walled conical beam

Figs. 18-20 compare the five analytically and numerically obtained stress components in cylindrical coordinates. The stresses were evaluated in the first quadrant, $0 \leq \theta \leq \pi / 2$, utilising the cross section symmetry. Moreover, the numerical stress field was extracted after transformation from global Cartesian to cylindrical coordinates in Abaqus. The numerical results of the hoop stress component are not reported, since it is equal to zero along the cross section wall.

### 4.2. Parametric study

In order to highlight the impact of the taper on design applications, a parametric study of the taper affecting the equivalent stress of a beam under different loads condition is presented. Since a linear isotropic material is considered, the equivalent stress is evaluated through the von Mises criterion, $\sigma_{\nu M}=\sqrt{3 J_{2}}$, where $J_{2}$ is the second deviatoric stress invariant [13]. It considers all stress components and it is invariant with respect to rotation of the coordinates system: in this way the global stress tensor could directly be used without the necessity of transformation into the local material stress tensor. The parametric study compares a thin-walled cylinder of radius $2 R=1 \mathrm{~m}$ to a set of thinwalled cones having a fixed control section of radius $2 R=1 \mathrm{~m}$. The loads are introduced through the eccentricity, which is defined as the ratio between the bending moment and the shear force at the cross section. The study also includes cones with negative taper, as the one shown in Fig. 21: the analytical solution derived in Section 2 is still valid and only the stress components which are function of $\sin \alpha$ and $\tan \alpha$ change when the taper is negative. The variables of this study are chosen as the taper angle, $\alpha \in\left[-15^{\circ} ; 15^{\circ}\right]$, and the eccentricity parameter $e \in[-15 \mathrm{~m} ; 15 \mathrm{~m}]$. Furthermore, the von Mises criterion, here employed in cylindrical coordinates, is used to evaluate the variation of the number of cycles in fatigue, when the taper effects are not neglected. The number of cycles to failure $N$ can be determined by employing the well-known Basquin law [33]. The material used in this study is steel, whose Basquin's constants, defined from the ultimate tensile strength and the endurance limit of the material, are $C=4.56 \times 10^{30}$, and $n=-9.84$ [34].


Fig. 13. (a) $\sigma_{x x}$ and (b) $\sigma_{z z}$ stress distributions along the flange.

### 4.2.1. Von Mises stress

The von Mises stresses were evaluated along the control cross section for $0 \leq \theta \leq \pi / 2$, and the maximum stress was subsequently normalised with the maximum von Mises stress in the cylindrical beam.

Fig. 22 (a) shows the variation of the normalised von Mises stresses within the eccentricity for two different taper angles, $0.5^{\circ}, 15^{\circ}$. Four zones are highlighted in the legend. Zone 1 presents constant lines: the von Mises stresses are not a function of the eccentricity and they increase quadratically with the taper angle, as shown by the continuous line in Fig. 22 (b). In zone 2 only negative eccentricities are involved and the von Mises stresses augment when $e \rightarrow 0$. The sensitivity of the von Mises stress towards the eccentricity is proportional to the taper angle. The dashed line in Fig. 22 (b) shows that in a cone with a taper angle of $\alpha=15^{\circ}$, an eccentricity $e=-1 \mathrm{~m}$ causes an increase of $20 \%$ of the equivalent stress. Between zone 2 and zone 3 a local maximum occurs. Zone 3 includes positive and negative eccentricities and in here the equivalent stresses decrease to a local minimum at $e=0.5 \mathrm{~m}$. In the range $0 \leq e \leq 0.5 \mathrm{~m}$, the taper reduces the von Mises stress in the cylinder, as shown by the point-dashed line in Fig. 22 (b). Finally, in zone 4 the von Mises stresses increase again.

### 4.2.2. Fatigue life

The aforementioned maximum equivalent stresses were employed to investigate the relation between the taper angle variation and the fatigue behaviour. In this case, the number of cycles to failure was evaluated for the set of cones previously defined and then normalised to the number of cycles to failure in the respective cylindrical beam. Since the number of cycles to failure is inversely proportional to the equivalent stress [33], in Fig. 23 (a) the same four zones can be distinguished. Zone 1 is not a function of the eccentricity and the
normalised number of cycles decreases with the increase of the taper, as shown by the continuous line in Fig. 23 (b). Zone 2 extends in the negative eccentricities and presents a decrement of $\tilde{N}$ with the eccentricity. As expected, the smaller the taper is, the smaller the eccentricity has to be to affect $\tilde{N}$. For example, in Fig. 23 (b), for $e=-0.2 \mathrm{~m}$ and $\alpha=5^{\circ}, \tilde{N}$ decreases already of approximately $10 \%$. In zone $3, \tilde{N}$ moves from a local minimum in the negative eccentricity zone, to a local maximum at $e=0.5 \mathrm{~m}$. In the range $0 \leq e \leq 0.5 \mathrm{~m}$ the number of cycles to failure significantly increases. Fig. 23 (b) shows that $\tilde{N}$ doubles when $e=0.2 \mathrm{~m}$ and $\alpha=5^{\circ}$. Finally, $\tilde{N}$ decreases in zone 4.

### 4.2.3. Negative taper

Lastly, the effects of the sign of the taper on the maximum equivalent stress that occur in a cross section of a conical beam were studied. The normalised parameter $\sigma_{v M}^{\alpha}$ was defined as ratio between the maximum von Mises stress in a negatively and a positively tapered beam. In Fig. 24 (a) the variation of the equivalent stress with the eccentricity is shown for $\alpha= \pm 0.5^{\circ}$ and $\alpha= \pm 15^{\circ}$ and four different zones are distinguished in the legend. In zone 1 , the sign of the taper does not affect the maximum von Mises stress. Zone 2 extends in the negative eccentricities. In here, a positive taper causes a higher stress than the same negative taper. For example, Fig. 24 (b) shows that for $e=0.5 \mathrm{~m}$, a taper $\alpha=5^{\circ}$ produces a von Mises stress $20 \%$ higher than $\alpha=-5^{\circ}$. Furthermore, the less pronounced the taper is, the smaller the eccentricity has to be to have the sign affecting the equivalent stress. Zone 3 extends from the local maximum in the negative eccentricities to the local minimum in the positive eccentricities, passing by $\sigma_{\nu M}^{\alpha}=1$ where the eccentricity is zero. Negative tapers have a worst effect when positive eccentricities occur. Zone 4 has the same properties as zone 2, but it refers to positive eccentricities and negative tapers.


Fig. 14. (a) $\sigma_{y y}$ and (b) $\sigma_{y z}$ stress distributions along the flange.


Fig. 15. (a) $\sigma_{x y}$ and (b) $\sigma_{x z}$ stress distributions along the flange.

## 5. Discussion

Although the analytical solutions presented in this article are an approximation due to the adoption of Navier's equation (i.e. strictly speaking Navier does not hold in tapered cross-sections), good agreement was found between the analytically and numerically predicted stress components. It is noteworthy that the validity of the provided solutions is restricted to moderately tapered cross sections in sufficiently slender beams where the axial stress component $\sigma_{z z}$ follows a quasi-linear distribution.

In the case of the vertically tapered box beam, all six Cauchy stress components are induced in the flange. Fig. 13 a shows the $\sigma_{x x}$ parabolic distribution with zero magnitude at the corners i.e. $x= \pm b$, and maximum at $x=0$ : shear load and bending result in a concave distribution whereas axial loading causes a convex distribution. The $\sigma_{z z}$ component is not a function of the taper angle, therefore it has the same distribution as in a classic prismatic beam. Interestingly, the components $\sigma_{y z}$ and $\sigma_{y y}$ in the flange are proportional to the $\sigma_{z z}$ component and has, therefore, a constant distribution, as shown in Figs. 14. In Fig. 15 b, the component $\sigma_{x z}$ is not only induced by the shear load, as in a prismatic beam, but also by bending moment and axial load. In all cases, the stress exhibits a linear distribution with a zero transition at $x=0$. The in-plane shear stress component $\sigma_{x y}$ in Fig. 15 a presents a similar behaviour. It is worth noting that the maximum stress varies with the sign of the applied forces. In the web of a vertically tapered beam it was found that only the components $\sigma_{z z}, \sigma_{y z}$ and $\sigma_{y y}$ are non-zero. The normal stress in Fig. 15 a has the same linear distribution as evident in the prismatic case. The out-of-plane shear stress component $\sigma_{y z}$ in


Fig. 17. $\sigma_{y y}$ stress distribution along the web.

Fig. 15 b, follows a quadratic distribution under shear load and bending moment, and a linear distribution under axial load. In a tapered beam under shear load, $\sigma_{y z}$ follows the classic concave quadratic distribution, with its maximum at $y=0$; conversely, the application of a bending moment can result in a convex stress distribution. In the latter case, it is possible that the shear stress distribution attains its maximum at $y= \pm h$. Fig. 15 shows that the axial and shear loads induce a convex quadratic $\sigma_{y y}$ distribution whereas bending causes a concave cubic stress distribution.

In the case of a conical beam only the hoop stress component $\sigma_{00}$ is zero. The axial stress component $\sigma_{z z}$ in a conical has the same


Fig. 16. (a) $\sigma_{z z}$ and (b) $\sigma_{y z}$ distributions along the web.


Fig. 18. (a) $\sigma_{r r}$ and (b) $\sigma_{r z}$ stress distributions along a quarter of a cone.
distribution than in a cylindrical beam as stipulated in the derivation. Both the shear stress $\sigma_{z r}$ and the through thickness $\sigma_{r r}$ in Fig. 18 are proportional to the axial stress component $\sigma_{z z}$. In contrast to the cylindrical beam, the shear stress $\sigma_{z \theta}$, and $\sigma_{r \theta}$, are functions of all the three cross section forces. Their distributions are trigonometric and both of which attain their maximum at $\theta=0$ and their zero values at $\theta= \pm \pi / 2$, as shown in Fig. 19.

The parametric study shows that neglecting the taper leads to an over- or underestimation of the stresses in the cross section. The equivalent stress always increases when a cylindrical beam is tapered, except in the range $0<e<0.5 \mathrm{~m}$, as shown in Fig. 19 a. As previously mentioned, the study is based on the comparison between the maximum equivalent stress that occurs in a conical beam and the one in a cylindrical beam. It is important to highlight that, when the eccentricity approaches zero, the maximum von Mises stresses occur at different locations of the cross sections in the cylindrical beam and in the cone, resulting in the four behaviours of the equivalent stress ratio highlighted in Fig. 22. In zone 1, the maximum von Mises stress occurs at $\theta=90^{\circ}$; consequently, in the set of Eqs. (35)-(43) all the terms which are multiplied by $\cos \theta$ vanish, and the ratio of the equivalent stresses between cone and cylinder are no longer a function of the eccentricity. In zone 2 , the maximum equivalent stress occurs at $\theta=0^{\circ}$ in the conical beams and $\theta=90^{\circ}$ in the cylinder for negative $e$; otherwise for positive $e$. Lastly, the drop in zone 3 is caused by the maximum equivalent stress occurring at $\theta=0^{\circ}$ in both conical and cylindrical beams. The maximum von Mises stress moves from $\theta=90^{\circ}$ to $\theta=0^{\circ}$ in zone 4.

Eventually it can be concluded that neglecting the taper effects leads


Fig. 20. $\sigma_{z z}$ stress distribution along a quarter of a cone.
to an overestimation of the number of cycles to failure, except for the known range $0 \leq e \leq 0.5 \mathrm{~m}$. When a conical beam has a negative taper, both $\sigma_{r \theta}$ and $\sigma_{z \theta}$ are affected since they are functions of $\sin \theta$. The sign of the taper, together with the loading direction leads to critical designs. In particular, when a negative eccentricity is applied, a negative taper halves the equivalent stress caused by the equivalent positive taper; vice-versa when a positive eccentricity is applied. Nonetheless, the equivalent stress is not affected by the orientation of the taper in zone 1.


Fig. 19. (a) $\sigma_{r \vartheta}$ and (b) $\sigma_{\ni z}$ stress distributions along a quarter of a cone.

Table 4
NMSE along the flange of a vertically tapered beam. It was evaluated after excluding the three points closer to the cross section corner, where a singularity occurs.

| Flange | $\sigma_{x x}$ | $\sigma_{x y}$ | $\sigma_{x z}$ |
| :--- | :--- | :--- | :--- |
| $F_{z}$ | $1.31 \times 10^{-5}$ | $6.76 \times 10^{-6}$ | $7.33 \times 10^{-6}$ |
| $F_{y}$ | $1.35 \times 10^{-4}$ | $7.43 \times 10^{-5}$ | $7.47 \times 10^{-5}$ |
| $M_{x}$ | $9.88 \times 10^{-5}$ | $7.70 \times 10^{-5}$ | $1.85 \times 10^{-5}$ |
|  | $\sigma_{y z}$ | $\sigma_{y y}$ | $\sigma_{z z}$ |
| $F_{z}$ | $2.77 \times 10^{-5}$ | $9.66 \times 10^{-5}$ | $2.61 \times 10^{-6}$ |
| $F_{y}$ | $1.05 \times 10^{-5}$ | $2.00 \times 10^{-5}$ | $5.92 \times 10^{-6}$ |
| $M_{x}$ | $1.75 \times 10^{-4}$ | $5.91 \times 10^{-4}$ | $1.65 \times 10^{-5}$ |

Table 5
NMSE along the web of a vertically tapered beam. It was evaluated after excluding the three points closer to the cross section corner, where a singularity occurs.

| Web | $\sigma_{z z}$ | $\sigma_{y z}$ | $\sigma_{y y}$ |
| :--- | :--- | :--- | :--- |
| $F_{z}$ | $1.11 \times 10^{-5}$ | $7.37 \times 10^{-6}$ | $1.84 \times 10^{-4}$ |
| $F_{y}$ | $4.30 \times 10^{-6}$ | $1.32 \times 10^{-6}$ | $2.16 \times 10^{-6}$ |
| $M_{x}$ | $3.80 \times 10^{-5}$ | $1.44 \times 10^{-5}$ | $2.31 \times 10^{-5}$ |

Table 6
NMSE along a quarter of a conical beam.

|  | $\sigma_{z z}$ | $\sigma_{r z}$ | $\sigma_{z \theta}$ | $\sigma_{r \theta}$ | $\sigma_{r r}$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $F_{z}$ | $1.31 \times 10^{-5}$ | $4.07 \times 10^{-6}$ | - | - | $1.97 \times 10^{-6}$ |
| $F_{y}$ | $2.70 \times 10^{-3}$ | $2.79 \times 10^{-3}$ | $3.08 \times 10^{-3}$ | $3.06 \times 10^{-3}$ | $2.85 \times 10^{-3}$ |
| $M_{x}$ | $2.70 \times 10^{-3}$ | $2.58 \times 10^{-3}$ | $2.56 \times 10^{-3}$ | $2.56 \times 10^{-4}$ | $1.97 \times 10^{-6}$ |



Fig. 21. Conical cantilever beam tapered by a constant negative angle, $\alpha \in\left[0^{\circ}\right.$ ; $15^{\circ}$ ].

## 6. Conclusions

The following conclusions can be drawn from the analytical stress solutions and the parametric studies conducted:
i. The introduction of taper into beams can potentially evoke all six stress components already at seemingly small taper angles.
ii. In principle all three cross section forces are highly coupled to the stress components in tapered beams. That is to say, a decoupling of shear force from bending and axial force as in prismatic beams is not per-se admissible in tapered beams.
iii. Application of prismatic solutions such as Jourawski's shear stress formula to tapered beams can lead to results which are significantly at variance with the real stress states. In contrast to the widely and erroneously established assumption in engineering practice, the stress state in tapered beams cannot be obtained by pure stress tensor transformation of the prismatic solution into a local coordinate system.
iv. Ignoring the taper can lead to an underestimation of the von Mises stress representing a non-conservative assumption. In the present study of a conical cantilever beam - considering static loading conditions - the von Mises stress increased less than $1 \%$ for $\alpha \leq 8^{\circ}$ and exceeded $10 \%$ for $\alpha \geq 25^{\circ}$.
v. Ignoring the taper can lead to a significant overestimation of the fatigue life. In the present study of a conical beam the number of cycles to failure decreased by $10 \%$ for $\alpha=5^{\circ}$ and decreased by $40 \%$ for $\alpha=25^{\circ}$.
vi. The analytical solutions and the parametric study in this article were based on isotropic material properties. However, it is deemed that additional stress components similarly to those presented will also be evoked in tapered beams with anisotropic material behaviour, such as fibre reinforced composites. Bearing in mind that composite materials are highly susceptible to failure owing to interfibre stress components, emphasises the importance of an accurate prediction of the stress components especially in tapered beams.

Possible future developments of the present work include the derivation of closed-form solutions in terms of strains and displacements for the analysed cases, as well as the solution for stresses in horizontally tapered and doubly tapered box girders. Moreover, the extended Jourawski's formula could be further extended to beams with nonsymmetric or non-homogeneous (e.g. laminated) cross sections.


Fig. 22. (a) Normalised von Mises stresses variation (conical over cylindrical) with eccentricity for different taper angles; (b) Normalised von Mises stresses variation (conical over cylindrical) with $\alpha$ for different eccentricities.


Fig. 23. (a) Normalised number of cycles to failure variation with the eccentricity for different taper angles; (b) normalised number of cycles to failure variation with the taper angle for different eccentricities.


Fig. 24. The parameter $\sigma_{v M}^{\alpha}$ shows the effects of negative $\alpha$ in comparison with positive $\alpha$. (a) Its variation with $\alpha$ is shown for different eccentricities; (b) Its variation with $e$ is shown for different $\alpha$.

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## Appendix A. Cauchy's equilibrium equations in Cartesian coordinates

The well-known Cauchy's differential equations expressing local equilibrium in Cartesian coordinates are [30]:
$\frac{\partial \sigma_{x x}}{\partial x}+\frac{\partial \sigma_{x y}}{\partial y}+\frac{\partial \sigma_{x z}}{\partial z}+b_{x}=0$
$\frac{\partial \sigma_{y x}}{\partial x}+\frac{\partial \sigma_{y y}}{\partial y}+\frac{\partial \sigma_{y z}}{\partial z}+b_{y}=0$
$\frac{\partial \sigma_{z z}}{\partial x}+\frac{\sigma_{z y}}{\partial y}+\frac{\sigma_{z z}}{\partial z}+b_{z}=0$
where $b_{x}, b_{y}, b_{z}$ are the body forces in Cartesian coordinates.

## Appendix B. Cauchy equilibrium equations for a thin-walled conical element

The volume of an infinitesimally small thin-walled cone element (Fig. B.25) can be obtained - under consideration of small angles d $\theta$ - by the trapezoidal equation as follows:
$\mathrm{d} V=\frac{1}{2} t_{f}[R \mathrm{~d} \theta+(R+\mathrm{d} R) \mathrm{d} \theta] \frac{\mathrm{d} z}{\cos \alpha}=R t_{p} \mathrm{~d} \theta \mathrm{~d} z$
Translatory equilibrium is imposed in the three main directions as follows. In the $r$ direction:


Fig. B.25. Infinitesimally small element of a thin-walled cone.

$$
\begin{align*}
& -\sigma_{r z} R t_{p} \mathrm{~d} \theta+\left(\sigma_{r z}+\mathrm{d} \sigma_{r z}\right)(R+\mathrm{d} R) t_{p} \mathrm{~d} \theta \\
& -\left(\sigma_{\theta \theta}+\mathrm{d} \sigma_{\theta \theta}\right) t_{p} \sin \frac{\mathrm{~d} \theta}{2} \mathrm{~d} z+\left(\sigma_{r \theta}+\mathrm{d} \sigma_{\mathrm{r} \theta}\right) t_{p} \cos \frac{\mathrm{~d} \theta}{2} \mathrm{~d} z \\
& -\sigma_{\theta \theta} t_{p} \sin \frac{\mathrm{~d} \theta}{2} \mathrm{~d} z-\sigma_{r \theta} t_{p} \cos \frac{\mathrm{~d} \theta}{2} \mathrm{~d} z+b_{r} R \quad t_{p} \mathrm{~d} \theta \mathrm{~d} z=0 \tag{B.2}
\end{align*}
$$

## In $\theta$ direction:

$-\sigma_{\theta \theta} t_{p} \cos \frac{\mathrm{~d} \theta}{2} \mathrm{~d} z+\sigma_{r \theta} t_{p} \sin \frac{\mathrm{~d} \theta}{2} \mathrm{~d} z+\left(\sigma_{\theta \theta}+\mathrm{d} \sigma_{\theta \theta}\right) t_{p} \cos \frac{\mathrm{~d} \theta}{2} \mathrm{~d} z-\sigma_{\theta z} R t_{p} \mathrm{~d} \theta$
$+\left(\sigma_{r \theta}+\mathrm{d} \sigma_{r \theta}\right) t_{p} \sin \frac{\mathrm{~d} \theta}{2} \mathrm{~d} z+\left(\sigma_{\theta z}+\mathrm{d} \sigma_{\theta z}\right)(R+\mathrm{d} R) t_{p} \mathrm{~d} \theta+b_{\theta} R t_{p} \mathrm{~d} \theta \mathrm{~d} z=0$
In $z$ direction:

$$
\begin{gather*}
-\sigma_{z z} R t_{p} \mathrm{~d} \theta+\left(\sigma_{z z}+\mathrm{d} \sigma_{z z}\right)(R+\mathrm{d} R) t_{p} \mathrm{~d} \theta-\sigma_{\theta z} t_{p} \mathrm{~d} z \\
+\left(\sigma_{\theta z}+\mathrm{d} \sigma_{\theta z}\right) t_{p} \mathrm{~d} z+b_{z} R t_{p} \mathrm{~d} \theta d z=0 \tag{B.4}
\end{gather*}
$$

After some manipulation of the set of Eqs. (B.2), (B.4) and by using Eq. (B.1) the Cauchy equilibrium equations for a conical beam in cylindrical coordinates are the following:
$\frac{1}{R}\left(-\sigma_{\theta \theta}+\sigma_{r z} \frac{\mathrm{~d} R}{\mathrm{~d} z}+\frac{\partial \sigma_{r \theta}}{\partial \theta}\right)+\frac{\partial \sigma_{r z}}{\partial z}+b_{r}=0$
$\frac{1}{R}\left(\sigma_{r \theta}+\sigma_{\theta z} \frac{\mathrm{~d} R}{\mathrm{dz}}+\frac{\partial \sigma_{\theta \theta}}{\partial \theta}\right)+\frac{\partial \sigma_{\theta z}}{\partial z}+b_{\theta}=0$
$\frac{1}{R}\left(\sigma_{z z} \frac{\mathrm{~d} R}{\mathrm{~d} z}+\frac{\partial \sigma_{\theta z}}{\partial \theta}\right)+\frac{\partial \sigma_{z z}}{\partial z}+b_{z}=0$

## References

[1] W. McGuire, G. Winter, Steel Structures, Prentice-Hall, Englewood Cliffs, 1968.
[2] D. Peery, Aircraft Structures, McGraw-Hill, New York, 1950.
[3] E. Hau, Wind Turbines: Fundamentals, Technologies, Application, Economics, 3rd edition, Springer, New York, 2013.
[4] S. Bennati, P. Bertolini, L. Taglialegne, P.S. Valvo, On stresses in tapered beams, Meccanica (2019) (Submitted for publication).
[5] G. Balduzzi, G. Hochreiner, J. Füssl, Stress recovery from one dimensional models for tapered bi-symmetric thin-walled I beams: deficiencies in modern engineering tools and procedure, Thin-Walled Struct. 119 (1) (2017) 934-944.
[6] L. Taglialegne, Analytical Study of Stress Fields in Wind Turbine Blades (Ph.D. Thesis, International Doctorate "Civil and Environmental Engineering"),
Universities of Florence, Perugia and Pisa - TU Braunschweig, Florence, Italy, 2018.
[7] S. Timoshenko, Bending stresses in curved tubes of rectangular cross section, Trans. Am. Soc. Mech. Eng. 45 (1923) 135-140.
[8] F. Bleich, Stahlhochbauten, 1 Springer, Berlin, 1932 (Language: German).
[9] A. Paglietti, G. Carta, Remarks on the current theory of shear strength of variable depth beams, Open Civ. Eng. J. 3 (1) (2009) 28-33.
[10] E.H. Atkin, Tapered beams: suggested solutions for some typical aircraft cases (Parts I \& II), Aircr. Eng. Aerosp. Technol. 10 (11) (1938) 347-351.
[11] E.H. Atkin, Tapered beams: suggested solutions for some typical aircraft cases (Part III), Aircr. Eng. Aerosp. Technol. 10 (12) (1938) 371-374.
[12] J.L. Krahula, Shear formula for beams of variable cross section, AIAA J. 13 (10) (1975) 1390-1391.
[13] S.P. Timoshenko, J.M. Gere, Mechanics of Materials, Van Nostrand Reinhold, New York, 1972.
[14] R.J. Knops, P. Villaggio, Recovery of stresses in a beam from those in a cone, J. Elast. 53 (1999) 65-75.
[15] N.S. Trahair, P. Ansourian, In-plane behaviour of web-tapered beams, Eng. Struct. 108 (2016) 47-52.
[16] G. Lee, B. Szabo, Torsional response of tapered I-girders, J. Struct. Div. 93 (5) (1967) 233-252.
[17] G. Lee, M. Morrell, R.L. Ketter, Design of tapered members, Weld. Res. Counc. Bull. 173 (1972) 1-32.
[18] K. Chong, W. Swanson, R. Matlock, Shear analysis of tapered beams, J. Struct. Div. 102 (9) (1976) 1781-1788.
[19] D.H. Hodges, J.C. Ho, W. Yu, The effect of taper on section constants for in-plane deformation of an isotropic strip, J. Mech. Mater. Struct. 3 (3) (2008) 425-440.
[20] D.H. Hodges, A. Rajagopal, J.C. Ho, W. Yu, Stress and strain recovery for the inplane deformation of an isotropic tapered strip-beam, J. Mech. Mater. Struct. 5 (6) (2010) 963-975.
[21] A. Rajagopal, Advancements in Rotor Blade Cross-Sectional Analysis Using the Variational-Ssymptotic Method (Ph.D. Thesis), Georgia Institute of Technology, 2014.
[22] G. Balduzzi, M. Aminbaghai, E. Sacco, J. Füssl, J. Eberhardsteiner, F. Auricchio, Non-prismatic beams: a simple and effective timoshenko-like model, Int. J. Solids Struct. 90 (2016) 236-250.
[23] G. Balduzzi, M. Amindaghai, F. Auricchio, J. Füssl, Planar Timoshenko-like model for multilayer non-prismatic beams, Int. J. Mech. Mater. Des. 14 (1) (2018) 51-70.
[24] J. Michell, The stress in an æolotrophic elastic solid with an infinite plane boundary, Proc. Lond. Math. Soc. s1-32 (1) (1900) 247-257.
[25] S. Carothers, Plane strain in a wedge, with applications to masonry dams, Proc. R. Soc. Edinb. 33 (1914) 292-306.
[26] S. Bennati, P. Bertolini, L. Taglialegne, P.S. Valvo, On shear stresses in tapered
beams, in: Proceedings of the GIMC-GMA 2016-21st Italian Conference on Computational Mechanics and 8th Meeting of the AIMETA Materials Group, Lucca, 2016.
[27] J.P. Blasques, User's Manual for BECAS - A Cross Section Analysis Tool for Anisotropic and Inhomogeneous Beam Sections of Arbitrary Geometry (Tech. Rep.), DTU Wind Energy, Technical University of Denmark, Roskilde, 2012.
[28] C.E. Cesnik, D.H. Hodges, VABS: a new concept for composite rotor blade crosssectional modeling, J. Am. Helicopter Soc. 42 (1) (1997) 27-38.
[29] B.A. Boley, On the accuracy of the bernoulli-euler theory for beams of variable section, J. Appl. Mech. 30 (3) (1963) 373-378.
[30] S. Timoshenko, J.N. Goodier, Theory of Elasticity, McGraw-Hill book Company, New York, 1951.
[31] ABAQUS 2017, Documentation, Dassault Systemes Simulia Corporation.
[32] A.A. Poli, M.C. Cirillo, On the use of the normalized mean square error in evaluating dispersion model performance, Atmos. Environ. Part A. Gen. Top. 27 (15) (1993) 2427-2434.
[33] O.H. Basquin, The exponential law of endurance tests, Am. Soc. Test. Mater. 10 (1910) 625-630.
[34] C. Li, W. Dai, F. Duan, Y. Zhang, D. He, Fatigue life estimation of medium-carbon steel with different surface roughness, Appl. Sci. 7 (4) (2017) 338.

## Paper 2

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# Analytical solution of the stresses in doubly tapered box girders 

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#### Abstract

Lengthwise geometrical variations such as taper, are widely employed in engineering structures, e.g. in wind turbine blades, aircraft wings and bridges. Notwithstanding the effects of taper on the mechanical behaviour of beams, it is common design procedure to model such structures as prismatic beams. Indeed, the derivation of analytical solution of the stress components in tapered beams is not a trivial task. This work provides the above-mentioned analytical solution of the stresses in tapered box girders, whose width and height are varying along the longitudinal axis of the beam. The proposed solution deals with any arbitrarily oriented external forces and it is based on the hypothesis of homogeneous isotropic material. The analytical solution shows that a combination of even small vertical and horizontal tapers highly influences the stress distribution. A parametric study of the relation between taper and the von Mises stress was investigated. Furthermore, it is evinced that disregarding taper effects might lead to an overestimation of the fatigue lifetime of the structures up to $12 \%$.


## 1. Introduction

Non-prismatic beams are commonly used in modern design to respond to structural, aerodynamic, and architectural requirements. For example, to reduce the stiffness-to-mass ratio, beams with tapered webs are often employed in the deck of steel and concrete bridges (Zevallos et al., 2016), and in metal or glued laminated beam framing. In aerospace and wind energy, aircraft structures and wind turbine blades are designed as slender structures with aerodynamic profiles. These beams have lengthwise geometrical variations, such as twisted and tapered cross-sections, and curved longitudinal axis. The variation of the cross-section dimensions along the beam-span can be identified through taper angles. The latter are defined in a straight beam as the angles between the tangent of the beam lateral surface and the longitudinal axis (Bertolini et al., 2019). This paper addresses the general case of doubly tapered beams, characterised by vertical and horizontal taper. Vertical taper $\alpha$ refers to the angle enclosed between the flange and the longitudinal axis, whereas the horizontal taper $\beta$ to the one between the web and the longitudinal axis. Real structures such as wind turbine blades, are often characterised by both vertical and horizontal taper angles, which can vary from $0^{\circ}$ at the root to $4^{\circ}$ and $12^{\circ}$ at max-chord respectively (Bak et al., 2012). Nowadays that wind turbine blades have impressive dimensions, e.g. the 107 metre long wind turbine blade designed by LM Wind Power, the
demand for more accurate stress analysis is crucial in their structural design.

Stress components which are zero in prismatic beams, arise in tapered beams under the same loading conditions. This fact has been long known in the scientific literature. For example, Timoshenko (1923) derived the in-plain shear distribution in a 2D truncated wedge subjected to a shear force. Bleich (1932) derived a closed-form solution for the in-plane shear stress in tapered beams under not only shear force, but also axial force and bending moment. Some of the effects due to the lengthwise geometrical variations are taken into account in the analysis of 3D finite elements models. Nonetheless, they are ignored in the early stages of the design, when the structure is simplified as a step-wise prismatic beam. In the past decade, many researchers, e.g. Paglietti and Carta (2009), showed how such an approximation could lead to an incorrect stress analysis. Therefore, new advance analytical theories, which are inclusive of the effects lengthwise geometrical variations need to be developed.

The variation of the stresses distribution in linearly tapered planar beams has been studied by Bennati et al. (2016), Balduzzi et al. (2016), and Hodges et al. (2010). Taglialegne (2018) derived an extension to Bleich's shear stress equation. Bertolini et al. (2019) provided closed-form solutions of isotropic vertically tapered beams with thinwalled rectangular and circular cross-sections. The latter presents a

[^2]

Fig. 1. (a) Side and (b) top view of a thin-walled tapered cantilever box beam of length $L$ with vertical taper angle $\alpha$ and horizontal taper angle $\beta$.

| Nomenclature |  |
| :---: | :---: |
| Oxyz | Cartesian coordinates (-) |
| O123 | Local coordinates system (-) |
| $h(z)$ | Half height of the cross section (m) |
| $h_{0}$ | Half height of the root cross section (m) |
| $b(z)$ | Half width of the cross section (m) |
| $b_{0}$ | Half width of the root cross section (m) |
| $\alpha, \beta$ | Vertical and horizontal taper angles (deg) |
| $L$ | Total length of the beam (m) |
| $t_{w}, t_{f}$ | Projected thickness of the web and the flange (m) |
| $t_{w}^{*}, t_{f}^{*}$ | Thickness of the web and the flange (m) |
| $A(z)$ | Area of the cross section ( $\mathrm{m}^{2}$ ) |
| $I_{x}$ | Second moment of area of the cross section ( $\mathrm{m}^{4}$ ) |
| $\sigma_{i j}$ | Stress tensor components (Pa) |
| $F_{y}, F_{z}$ | Internal shear and axial forces ( N ) |
| $M_{x}$ | Internal bending moment ( Nm ) |
| $S_{x}$ | First moment of area ( $\mathrm{m}^{3}$ ) |
| R | Rotation matrix (-) |
| E | Young modulus (Pa) |
| $v$ | Poisson ratio (-) |
| $N, Q$ | Axial and shear forces ( N ) |
| M | Bending moment ( N m) |
| C, $n$ | Basquin constants (-) |
| $\sigma_{e q}$ | Von Mises stress (Pa) |

comprehensive overview of the state-of-the-art of analysis of tapered beams.

The case of beams with tapered flanges and webs, named doubly tapered beams, has been investigated by Kuś (2015) for the analysis of lateral-torsional buckling. Here, it was concluded that the critical moment can be increased by tapering the flanges rather than the webs or both simultaneously. To the authors' best knowledge, the latter is the only investigation on doubly tapered beams.

This paper is a generalisation of the method proposed in Bertolini et al. (2019). In particular, it provides closed-form solution of the stresses in doubly tapered beams subjected to forces with arbitrary directions. The previous work addressed the analytical solution for vertically tapered beams subjected to restricted load cases. The method is based on the Euler-Bernoulli hypotheses. The validity of the provided analytical solutions is proven by comparison to 3D finite element models of tapered box girders clamped at one end loaded at the other one. The closed-form solutions are employed in a parametric study between the taper angles and the equivalent von Mises stress and between the taper angles and the fatigue lifetime of similar structures.

(a)

(b)

Fig. 2. (a) Front view of a thin-walled tapered cantilever box beam and (b) arbitrary cross section, where the projected web thickness, $t_{w}$, and flange thickness, $t_{f}$, are highlighted.

## 2. Analytical solutions of the stress components

The full stress Cauchy's tensor is determined for the homogeneous isotropic doubly tapered thin-walled beam shown in Figs. 1 and 2. The cantilever support condition is chosen as an example and it does not limit the validity of the method.

The width and the height of the cross sections are defined from the centre line of the flange and web. They are written as functions of the longitudinal coordinate $z$, the vertical taper angle $\alpha$, and the horizontal taper angle $\beta$ as follows
$h(z)=h_{0}-z \tan \alpha, \quad b(z)=b_{0}-z \tan \beta$,
where $h_{0}$ and $b_{0}$ are respectively half of the height and of the width of the root section. The Cartesian coordinates of the wall mid-surface are defined within the limits
$-b(z) \leq x(z) \leq b(z), \quad-h(z) \leq y(z) \leq h(z), \quad 0 \leq z \leq L$.
Two local coordinate systems, named $O 1^{f} 2^{f} 3^{f}$ and $O 1^{w} 2^{w} 3^{w}$, are defined in the flanges and webs respectively, as illustrated in Fig. 3. In both cases, the 3-axis is the outer normal to the flange and web, and the 2-axis lies on the $x y$-plane. Thus, the 1 -axis is defined in accordance with those assumptions.

The thickness of the web and flange projected onto the cross-section plane are
$t_{w}=\frac{t_{w}^{*}}{\cos \beta}, \quad t_{f}=\frac{t_{f}^{*}}{\cos \alpha}$.
The cross-sectional area, $A(z)$, and the second moment of area with respect to the $x$-axis, $I_{x}(z)$, are
$A(z)=4\left[h(z) t_{w}+b(z) t_{f}\right], \quad I_{x}(z)=4\left[b(z) t_{f} h(z)^{2}+\frac{1}{3} t_{w} h(z)^{3}\right]$.
Shear force $F_{y}(z)$, axial force $F_{z}(z)$, and bending moment $M_{x}(z)$ are considered as internal forces acting at each cross section. The dependency on $z$ will be omitted for brevity. Assuming that any plane section remains plane after deformation and recalling Navier formula (Timoshenko and Goodier, 1951), the normal stresses are determined with


Fig. 3. Quarter of an infinitesimal segment of the tapered box beam with global, $O x y z$, and local, $1^{f} 2^{f} 3^{f}$ in the flange and $1^{w} 2^{w} 3^{w}$ in the web, reference systems.

Eq. (2). The latter is valid also in moderately tapered beams, as verified by Boley (1963).
$\sigma_{z z}=\frac{F_{z}}{A}+\frac{M_{x}}{I_{x}} y$.
Under the above-mentioned assumptions, the extended in-plane shear expression derived in Eq. 5 in Bertolini et al. (2019) is valid. Assuming null distributed forces and moments, the equation simplifies to
$\sigma_{z i}=\frac{1}{t}\left[F_{z} \frac{\mathrm{~d}}{\mathrm{~d} z}\left(\frac{A^{*}}{A}\right)+F_{y} \frac{S_{x}^{*}}{I_{x}}+M_{x} \frac{\mathrm{~d}}{\mathrm{~d} z}\left(\frac{S_{x}^{*}}{I_{x}}\right)\right], \quad i=x, y$.
where $t$ is the thickness of the cross section, $A^{*}$ and $S_{x}^{*}$ is the area and first moment of area with respect to the $x$-axis of the portion of the cross section where the stress is analysed.

Because of the symmetry of the analysed cross section, this study refers to half of the flange ( $0 \leq x \leq b(z), y=h(z)$ ), and half of the web ( $0 \leq y \leq h(z), x=b(z)$ ).

### 2.1. Stress components in the flange

The normal stress component in the flange, $\sigma_{z z}^{f}$, is directly determined by substituting Eqs. (1) into Eq. (2), resulting in
$\sigma_{z z}^{f}=\frac{1}{4}\left[\frac{F_{z}}{t_{f} b+t_{w} h}-\frac{3 M_{x}}{\left(3 b t_{f}+h t_{w}\right) h}\right]$.
The out-of-plane component $\sigma_{z x}$ in the web is derived from Eq. (3), imposing the equilibrium of the 'hatched' portion on the flange in Fig. 4, where $\eta=b-x$. The area and first moment of area with respect to the $x$-axis of such 'hatched' area are

$$
\begin{equation*}
A^{*}=x t_{f}, \quad S_{x}^{*}=x h t_{f} \tag{5}
\end{equation*}
$$

Then, after substituting Eqs. (5) in Eq. (3), it is obtained that

$$
\begin{align*}
\sigma_{x z}^{f}= & \frac{x}{4}\left\{\frac{F_{z}\left(t_{f} \tan \beta+t_{w} \tan \alpha\right)}{\left(t_{f} b+t_{w} h\right)^{2}}+\frac{3 F_{y}}{\left(3 t_{f} b+t_{w} h\right) h}\right. \\
& \left.+\frac{3 M_{x}\left[\left(3 t_{f} b+2 t_{w} h\right) \tan \alpha+3 t_{f} h \tan \beta\right]}{\left(3 t_{f} b+t_{w} h\right)^{2}}\right\} . \tag{6}
\end{align*}
$$

It is worth noting how both taper angles contribute to the stress distribution.

When $\alpha \neq 0$ and $\beta=0$, i.e. when the beam is vertically tapered, Eq. (6) becomes identical to Eq. (16) in Bertolini et al. (2019). When $\alpha=0$ and $\beta \neq 0$, i.e. when the beam is horizontally tapered, Eq. (6) reduces to
$\left.\sigma_{x z}^{f}\right|_{\alpha=0}=\frac{x}{4}\left[\frac{F_{z} t_{f} \tan \beta}{\left(t_{f} b+t_{w} h\right)^{2}}+\frac{3 F_{y}}{\left(3 t_{f}^{*} b+t_{w} h\right) h}+\frac{9 M_{x} t_{f}^{*} h \tan \beta}{\left(3 t_{f}^{*} b+t_{w} h\right)^{2}}\right]$.


Fig. 4. Hatched portion of cross section on the flange.


Fig. 5. Hatched portion of cross section on the web.

Moreover, when $\alpha=\beta=0$ the beam is prismatic and Eq. (6) reduces to the classic Jourawski's equation (Timoshenko and Goodier, 1951)
$\left.\sigma_{x z}^{f}\right|_{\alpha=\beta=0}=\frac{F_{y} x}{4\left(t_{f}^{*} b h+t_{w}^{*} \frac{h^{2}}{3}\right)}$.
The stresses in the global reference system are rotated into the flange local CSYS through the rotation matrix $\mathbf{R}^{f}$ as follows
$\left[\begin{array}{ccc}\sigma_{11}^{f} & \sigma_{12}^{f} & \sigma_{13}^{f} \\ \sigma_{21}^{f} & \sigma_{22}^{f} & \sigma_{23}^{f} \\ \sigma_{31}^{f} & \sigma_{32}^{f} & \sigma_{33}^{f}\end{array}\right]=\left[\begin{array}{ccc}0 & -\sin \alpha & \cos \alpha \\ 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha\end{array}\right]\left[\begin{array}{ccc}\sigma_{x x}^{f} & \sigma_{x y}^{f} & \sigma_{x z}^{f} \\ \sigma_{y x}^{f} & \sigma_{y y}^{f} & \sigma_{y z}^{f} \\ \sigma_{z x}^{f} & \sigma_{z y}^{f} & \sigma_{z z}^{f}\end{array}\right]\left[\begin{array}{ccc}0 & -\sin \alpha & \cos \alpha \\ 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha\end{array}\right]^{T}$
Under the hypothesis of thin-walled beam, the flange is in plane stress state in local coordinate system, meaning that $\sigma_{33}^{f}=\sigma_{13}^{f}=\sigma_{23}^{f}=0$. Expanding and manipulating the system of Eqs. 7, leads to the following expressions for the stresses
$\sigma_{y y}^{f}=\sigma_{z z}^{f} \tan ^{2} \alpha$,
$\sigma_{x y}^{f}=-\sigma_{z x}^{f} \tan \alpha$,
$\sigma_{y z}^{f}=-\sigma_{z z}^{f} \tan \alpha$.
The first Cauchy's equation ( $\partial \sigma_{x x, x}+\partial \sigma_{x y, y}+\partial \sigma_{x z, z}=0$ ) is used to determine the missing component $\sigma_{x x}^{f}$
$\sigma_{x x}^{f}=-\int_{x}^{b(z)} \frac{\partial \sigma_{z x}^{f}}{\partial z}+\frac{\partial \sigma_{x y}^{f}}{\partial y} d x+\left.\sigma_{x x}^{f}\right|_{x=b}$
The term $\left.\sigma_{x x}^{f}\right|_{x=b}$ is a constant of integration. It can be determined by imposing the horizontal equilibrium at the corner of the rectangular cross section once the stress field of the web is determined.

### 2.2. Stress components in the web

The normal stress component $\sigma_{z z}^{w}$ in the web is directly determined by substituting Eqs. (1) into (2) which gives
$\sigma_{z z}^{w}=\frac{1}{4}\left[\frac{F_{z}}{t_{f} b+t_{w} h}+\frac{3 M_{x} y}{h^{2}\left(3 t_{f} b+t_{w} h\right)}\right]$.

The 'hatched' portion of cross section in Fig. 5 in the web includes the half flange and a part of the web, with $\eta=y$. The area and the first moment of area of the hatched part with respect to the $x$-axis are
$A^{*}=(h-y) t_{w}+b t_{f}, \quad S_{x}^{*}=b h t_{f}+\frac{t_{w}}{2}\left(h^{2}-y^{2}\right)$.
Therefore, substituting Eqs. (13) in Eq. (3) results in

$$
\begin{align*}
\sigma_{z y}^{w}= & -\frac{F_{z}\left(t_{f} \tan \beta+t_{w} \tan \alpha\right) y}{4\left(t_{f} b+t_{w} h\right)^{2}}+\frac{3 F_{y}\left[2 t_{f} h b+t_{w}\left(h^{2}-y^{2}\right)\right]}{8 t_{w} h^{2}\left(3 t_{f} b+t_{w} h\right)}  \tag{14}\\
& +\frac{3 M_{x}}{8 h^{3} t_{w}\left(3 t_{f} b+h t_{w}\right)^{2}}\left\{h t_{f} t_{w}\left(h^{2}-3 y^{2}\right) \tan \beta\right. \\
& \left.+\left[h\left(h^{2}-3 y^{2}\right) t_{w}^{2}+2 t_{f} t_{w} b\left(2 h^{2}-3 y^{2}\right)+6 t_{f}^{2} b^{2} h\right] \tan \alpha\right\} .
\end{align*}
$$

Once again, both the horizontal and vertical taper affect the shear stress distribution.

When $\alpha \neq 0$ and $\beta=0$, i.e. when the beam is vertically tapered, Eq. (14) is identical to Eq. (25) in Bertolini et al. (2019). When $\alpha=0$ and $\beta \neq 0$, i.e. in a horizontally tapered beam, Eq. (14) becomes

$$
\begin{align*}
\left.\sigma_{z y}^{w}\right|_{\alpha=0}= & -\frac{F_{z} t_{f}^{*} y \tan \beta}{4\left(t_{f}^{*} b+t_{w} h\right)^{2}}+\frac{3 F_{y}\left[2 t_{f}^{*} h b+t_{w}\left(h^{2}-y^{2}\right)\right]}{8 t_{w} h^{2}\left(3 t_{f}^{*} b+t_{w} h\right)}  \tag{15}\\
& +\frac{3 M_{x} h t_{f}^{*} t_{w}\left(h^{2}-3 y^{2}\right) \tan \beta}{8 h^{3} t_{w}\left(3 t_{f}^{*} b+h t_{w}\right)^{2}}
\end{align*}
$$

In addition, when $\alpha=\beta=0$, i.e. in a prismatic beam, Eq. (6) reduces to Jourawski equation
$\left.\sigma_{z y}^{w}\right|_{\alpha=\beta=0}=\frac{3 F_{y}\left[2 t_{f}^{*} h b+t_{w}^{*}\left(h^{2}-y^{2}\right)\right]}{8 t_{w}^{*} h^{2}\left(3 t_{f}^{*} b+t_{w}^{*} h\right)}$.
The stress components on the web can be rotated from the global reference system to the local one by means of the rotation matrix $\mathbf{R}^{\mathbf{w}}$ as follows
$\left[\begin{array}{ccc}\sigma_{11}^{w} & \sigma_{12}^{w} & \sigma_{13}^{w} \\ \sigma_{21}^{w} & \sigma_{22}^{w} & \sigma_{23}^{w} \\ \sigma_{31}^{w} & \sigma_{32}^{w} & \sigma_{33}^{w}\end{array}\right]=\left[\begin{array}{ccc}-\sin \alpha & 0 & \cos \alpha \\ 0 & 1 & 0 \\ \cos \alpha & 0 & \sin \alpha\end{array}\right]\left[\begin{array}{ccc}\sigma_{x x}^{w} & \sigma_{x y}^{w} & \sigma_{x z}^{w} \\ \sigma_{y x}^{w} & \sigma_{y y}^{w} & \sigma_{y z}^{w} \\ \sigma_{z x}^{w} & \sigma_{z y}^{w} & \sigma_{z z}^{w}\end{array}\right]\left[\begin{array}{ccc}-\sin \alpha & 0 & \cos \alpha \\ 0 & 1 & 0 \\ \cos \alpha & 0 & \sin \alpha\end{array}\right]^{T}$
In local coordinates, the web is assumed to be in plane stress state, i.e. $\sigma_{13}^{w}=\sigma_{23}^{w}=\sigma_{33}^{w}=0$. Therefore, Eqs. 16 reduces to
$\sigma_{x x}^{w}=\sigma_{z z}^{w} \tan ^{2} \beta$,
$\sigma_{x z}^{w}=-\sigma_{z z}^{w} \tan \beta$,
$\sigma_{x y}^{w}=-\sigma_{z y}^{w} \tan \beta$.
Lastly, the missing stress component $\sigma_{y y}^{w}$ is determined from integration of the second Cauchy's equation $\left(\partial \sigma_{y x, x}+\partial \sigma_{y y, y}+\partial \sigma_{y z, z}=0\right)$
$\sigma_{y y}^{w}=-\int_{y}^{h(z)} \frac{\partial \sigma_{x y}^{w}}{\partial x}+\frac{\partial \sigma_{z y}^{w}}{\partial z} d y+\left.\sigma_{y y}^{w}\right|_{y=h}$.
where $\left.\sigma_{y y}^{w}\right|_{y=h}$ is a constant of integration. It is determined from the vertical equilibrium of the stresses at the corner of the cross section.

### 2.3. Vertical and horizontal equilibrium at the corners

To determine the constants of integration $\left.\sigma_{x x}^{f}\right|_{x=b}$ in Eq. (11) and $\left.\sigma_{y y}^{w}\right|_{y=h}$ in Eq. (20), the horizontal and vertical equilibrium of an infinitesimal portion of the corner of the beam is imposed, as shown in Fig. 6. The equilibrium in the $x$-direction and in the $y$-direction are respectively given by
$\left.t_{f} \sigma_{x x}^{f}\right|_{x=b} d z+\left.t_{f} \sigma_{z x}^{f}\right|_{x=b} d z \tan \beta+\left.t_{w} \sigma_{x z}^{w}\right|_{y=h} d z \tan \alpha+\left.t_{w} \sigma_{x y}^{w}\right|_{y=h} d z=0$
$\left.t_{w} \sigma_{y y}^{w}\right|_{y=h} d z+\left.t_{w} \sigma_{y z}^{w}\right|_{y=h} d z \tan \alpha+\left.t_{f} \sigma_{x y}^{f}\right|_{x=b} d z+\left.t_{f} \sigma_{y z}^{f}\right|_{x=b} d z \tan \beta=0$

Table 1
Values of the external loads applied at one end of the box girder together with the resulting internal forces at the control section. The external forces are the axial load $N$, the shear load $Q$ and the bending moment $M$. The bending moment is chosen in such a way that it is comparable to the resulting shear-coupled bending.

| Case | External loads |  |  | Internal forces at the mid-span |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $N[\mathrm{~N}]$ | $Q[\mathrm{~N}]$ | $M[\mathrm{Nm}]$ |  | $F_{y}[\mathrm{~N}]$ | $F_{z}[\mathrm{~N}]$ | $M_{x}[\mathrm{Nm}]$ |
| Shear | 1000 | - | - |  | 1000 | - | -5000 |
| Extension | - | 1000 | - |  | - | 1000 | - |
| Bending | - | - | 5000 | - | - | 5000 |  |

Table 2
Geometrical dimensions of the beam model used for comparison between analytical and numerical stress solution.

| $L[\mathrm{~m}]$ | $\alpha[\mathrm{deg}]$ | $\beta[\mathrm{deg}]$ | $h[\mathrm{~m}]$ | $b[\mathrm{~m}]$ | $t_{f}[\mathrm{~mm}]$ | $t_{w}[\mathrm{~mm}]$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 10.0 | 5.0 | 2.0 | 0.5 | 0.55 | 10.0 | 10.0 |

Table 3
Mesh discretisation and element sizes. The third and fourth columns refer to the largest and smallest transverse element size measured in the cross-section plane.

| \# of el. | max el. size $[\mathrm{m}]$ | min el. size $[\mathrm{m}]$ |
| :--- | :--- | :--- |
| 103200 | $92.54 \times 10^{-3}$ | $5.00 \times 10^{-3}$ |

The equations above can be directly solved with Maple (Maple 2017,2018 ) to determine the constants of integration and, consequently, the stress components $\sigma_{x x}^{f}$ and $\sigma_{y y}^{w}$. The expressions of the stresses can be found in the Appendix as Eqs. (A.1) and (A.3).

## 3. Finite element model

The correctness of the presented method was evaluated by comparison against finite element models. The cantilever beam in Fig. 7 was modelled inside the commercial finite element package ABAQUS (2017). The comparison between analytical and numerical solutions was performed at the 'control cross section', which is located the midspan cross section. Such a control section is perpendicular to the beam $z$-axis and located at $z=L / 2$, in order to avoid the influence of boundary effects at the clamped root and at the loaded tip. The applied external loads and the internal forces at the control cross section, are listed in Table 1. The material of the beam has Young modulus $E=$ 210 GPa , and Poisson ratio $v=0.3$. The geometrical dimensions of the entire beam and of the control cross section are summarised in Table 2.

The finite element model was discretised by enriched eight-noded solid elements (Abaqus element type C3D8R) as described in Table 3. The mesh is finer at the proximity of the control section, and two elements were imposed through the thickness of the walls. A convergence study of different mesh densities justifies the discretisation of the model.

To reproduce the clamped conditions the master node of the root section had all the six degrees of freedom restrained. A direct solution strategy was used (Abaqus linear perturbation). Results were extracted in global coordinates along node-paths located in the wall mid-surface at the control cross section.

## 4. Results and discussion

The stresses at the control section of the finite element model are compared to the ones derived analytically to evaluate the accuracy of the presented method. Specifically, Figs. 8-10 show all six Cauchy stress distributions along half of the flange ( $0 \leq x \leq b$ ), and Figs. 11-13 along half of the flange, $0 \leq y \leq h$. The label 'An' and 'Num' indicate respectively the results from the analytical and numerical analysis. The deviation between the two methods is evaluated as relative percentage error for each stress component. In addition, the Normalised Mean Square Error (NMSE) (Poli and Cirillo, 1993) is calculated and reported in Tables 4 and 5.


Fig. 6. (a) Horizontal and (b) vertical equilibrium of the stresses acting in the web and flange of an infinitesimal portion of the corner of a tapered beam.


Step: Step-1
Increment 1: Step Time $=2.2200 \mathrm{E}-16$

Fig. 7. 3D finite element model (Abaqus) loaded at the master node located at the tip through application of concentrated forces, $F_{y}, F_{z}$, and bending moment, $M_{x}$. The loads and the boundary conditions are applied at a reference point located in the elastic centre of the cross sections, which is linked through a kinematic (rigid) coupling to the tip or root cross section. The mesh of the cross section is refined at its corners, as illustrated in the detail.

### 4.1. Stress components in the flange and web

The normal stress distributions in both the flange and the web are shown in Figs. 8(a) and 11(a). The good agreement between numerical and analytical distributions supports the correctness of assuming Navier equation in slightly tapered beams.

In the flange the stress components $\sigma_{y y}^{f}$ in Fig. 8(b), and $\sigma_{y z}^{f}, \sigma_{x y}^{f}$ in Fig. 9 vanish only when $\alpha$ is zero. In the web the stress components $\sigma_{x x}^{w}$ in Fig. 11(b), and $\sigma_{x z}^{w}, \sigma_{x y}^{w}$ in Fig. 12 appear when $\beta$ is not zero. Consequently, a constant stress component $\sigma_{y z}^{f}$ and $\sigma_{x z}^{w}$ always arises through the thickness in thin-walled tapered beams.

The normal stress component $\sigma_{x x}^{f}$ in Fig. 10(b) has a quadratic distribution under all the three load conditions. The maximum stress occurs at the corner $(x=b)$ under shear force and the stress changes sign under pure-bending and axial force.

In the web, the normal component $\sigma_{y y}^{w}$ in Fig. 13(b) arises to guarantee the equilibrium. Under shear load, the maximum value of $\sigma_{y y}^{w}$ occurs at the corner, meanwhile under pure bending the maximum is located at 400 mm from the mid-plane of the web.

It is worth noting that when the axial force is considered, the stress $\sigma_{x x}^{f}$ and $\sigma_{y y}^{w}$ approach zero at circa 220 mm from the centre of the flange and web respectively. The relative errors at these points reach the peak of $10 \%$. However, because of the small values of the stresses at these locations, the absolute errors are negligible, as shown also in Tables 4, 5.

The in-plane shear stress $\sigma_{y z}^{w}$ in the web is shown in Fig. 13(a). It follows a linear distribution with zero stress at $y=0$ under axial force, and a parabolic distribution in the other cases. It is worth noting that under shear force the maximum stress occurs at $y=h$, and not at $y=0$ as expected from the classic Jourawski's formula.

Table 4
NMSE along the flange evaluated after excluding the three points closer to the cross section corner, where a singularity occurs.

| Flange | $\sigma_{x x}$ | $\sigma_{z z}$ | $\sigma_{x z}$ |
| :--- | :--- | :--- | :--- |
| $F_{z}$ | $0.24 \times 10^{-3}$ | $0.72 \times 10^{-3}$ | $2.17 \times 10^{-6}$ |
| $F_{y}$ | $1.58 \times 10^{-4}$ | $1.33 \times 10^{-5}$ | $4.61 \times 10^{-6}$ |
| $M_{x}$ | $7.04 \times 10^{-5}$ | $9.19 \times 10^{-5}$ | $2.34 \times 10^{-5}$ |
|  | $\sigma_{y y}$ | $\sigma_{x y}$ | $\sigma_{y z}$ |
| $F_{z}$ | $6.89 \times 10^{-6}$ | $3.23 \times 10^{-6}$ | $6.42 \times 10^{-6}$ |
| $F_{y}$ | $1.35 \times 10^{-5}$ | $4.58 \times 10^{-6}$ | $1.33 \times 10^{-5}$ |
| $M_{x}$ | $9.69 \times 10^{-5}$ | $2.71 \times 10^{-5}$ | $8.48 \times 10^{-5}$ |

Table 5
NMSE along the web evaluated after excluding the three points closer to the cross section corner, where a singularity occurs.

| Web | $\sigma_{x x}$ | $\sigma_{z z}$ | $\sigma_{x z}$ |
| :--- | :--- | :--- | :--- |
| $F_{z}$ | $1.99 \times 10^{-4}$ | $0.74 \times 10^{-3}$ | $5.73 \times 10^{-5}$ |
| $F_{y}$ | $2.51 \times 10^{-5}$ | $6.88 \times 10^{-6}$ | $6.91 \times 10^{-6}$ |
| $M_{x}$ | $1.04 \times 10^{-3}$ | $1.13 \times 10^{-4}$ | $1.79 \times 10^{-4}$ |
|  | $\sigma_{y y}$ | $\sigma_{x y}$ | $\sigma_{y z}$ |
| $F_{z}$ | $1.56 \times 10^{-5}$ | $1.32 \times 10^{-4}$ | $1.03 \times 10^{-6}$ |
| $F_{y}$ | $1.83 \times 10^{-7}$ | $3.92 \times 10^{-6}$ | $4.14 \times 10^{-6}$ |
| $M_{x}$ | $3.16 \times 10^{-5}$ | $6.82 \times 10^{-4}$ | $2.57 \times 10^{-5}$ |

### 4.1.1. In-plane shear stress distributions along the span of a beam

Seven cross sections of the tapered box girder shown in Fig. 1 are considered. They are located at a distance of 1.6 m from each other and are defined in the plane perpendicular to the $z$-axis of the beam. The in-plane shear stresses evoked by the axial and shear forces, and bending moment are evaluated by means of the analytical solutions and of the FE model already described in Section 3. Figs. 14(a), 15(a), 16(a) show the stress distribution along the flange of the beam, whereas the others along the web. It is worth noting that the analytical solutions are in good agreement with the numerical results in the central cross sections, namely the ones far from the boundaries. As expected, the two methods deviates at sections 1 , root of the beam, and 7 , its tip, since the boundary conditions and the external forces are applied at those locations. The stresses at the root and tip sections are differently scaled for readability.

### 4.1.2. In-plane shear stress: a comparison between vertical and horizontal taper

Vertical and horizontal taper influence the in-plane shear behaviour differently. Two homogeneous isotropic beams with rectangular cross sections and subjected to a shear load $Q$ at their tips are compared. The vertically tapered beam is Case $a$, whereas the horizontally tapered beam is Case b. Their geometrical properties are listed in Table 6 and have been chosen in such a way that the two control cross-sections have the same dimensions.

Fig. 17 shows the in-plane shear distribution along the web. The contributions of the pure shear and the pure bending are isolated to clarify the different behaviours. The dashed-lines, representative of the

 control cross section. The three cases of axial force, shear force and bending moment are shown.

 control cross section. The three cases of axial force, shear force and bending moment are shown.

Table 6
Geometrical dimensions of the vertically $a$ and the horizontally $b$ tapered beams.

| Case | $L[\mathrm{~m}]$ | $\alpha[\mathrm{deg}]$ | $\beta[\mathrm{deg}]$ | $h[\mathrm{~mm}]$ | $b[\mathrm{~mm}]$ | $t_{f}[\mathrm{~mm}]$ | $t_{w}[\mathrm{~mm}]$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $a$ | 10.0 | 4.0 | 0.0 | 500 | 500 | 10.0 | 10.0 |
| $b$ | 10.0 | 0.0 | 4.0 | 500 | 500 | 10.0 | 10.0 |

pure-shear, have the same distribution and values in both cases. On the contrary, the shear-bending (point-dashed-lines) has much higher values in case $a$ that the maximum stress occurs at the corner of the cross section and not at the centre.

### 4.2. Parametric study of the equivalent stress

A parametric study on the variation of the equivalent stress in tapered beams is carried out. The von Mises criterion is used (Timoshenko and Goodier, 1951). Such parametric study is carried out by
analysing a series of different tapered beams. Such beams present the same material, load conditions, beam length, and dimensions of the control cross section, which is located at $L / 2$. Specifically, $L=10 \mathrm{~m}$, $2 h=2 b=1 \mathrm{~m}$. Then, for each beam, the number of cycles to failure is calculated using Basquin's law (Basquin, 1910). The material chosen in this example is steel with Basquin's constants $C=4.56 \cdot 10^{30}$ and $n=-9.84$ (Li et al., 2017).

Fig. 18(a) shows the variation of the maximum von Mises stress for beams with $\alpha$ and $\beta$ varying in the range $\left[-10^{\circ},+10^{\circ}\right]$ with respect to the prismatic case. When $\alpha=0^{\circ}$ and $\beta>0^{\circ}$, the equivalent stress increases. Otherwise when $\beta<0^{\circ}$. When $\beta=0^{\circ}$ and the beam is vertically taper, the equivalent stress increases independently of the sign of $\alpha$. Note that, even if the von Mises stress increases of maximum $4 \%$ when $\alpha= \pm 10^{\circ}$ and $\beta=8^{\circ}$, the number of cycles to failure reduces up to $20 \%$ as shown in Fig. 18(b).


 zero at $x=230 \mathrm{~mm}$.


Fig. 11. Stress distributions along the web (a) $\sigma_{z z}$ and (b) $\sigma_{x x}$ together with the relative error between the analytical and numerical values.

## 5. Conclusions

This paper presents an extension of the method derived in Bertolini et al. (2019) to doubly tapered beams. Results show the accuracy of the solutions to predict the Cauchy's stress field of an arbitrary homogeneous isotropic tapered beam with rectangular cross-section. These closed-form solutions could be used for stress analysis in structural design of similar cross sections. Particularly, the following conclusions are drawn:

1. The classic Jourawski's formula for prismatic beams is not valid for arbitrary tapered beams. Taper causes a redistribution of the in-plane shear stress which could lead to non-optimal design.
2. Through-thickness shear stress appears in beams with vertical and/or horizontal taper. Specifically, this shear component is
given from the product of the normal stress and the tangent of the taper angle. Consequently, the more the beam is tapered, the higher the component becomes.
3. The proposed solution is also applicable to the case of shear force and bending moment applied in any direction within the cross section. This can be achieved through the superposition principle, by decomposing the force and the moment along the principal directions.
4. The provided analytical solution for stress tensor analysis guarantees an accurate prediction of the fatigue lifetime. For example in a beam with vertical and horizontal taper of 4 degrees, the number of cycle to failure reduces up to $12 \%$ than in a prismatic beam of comparable size.
The current method can be directly applied to any tapered thinwalled beams with symmetric cross-section. Minor modifications to


Fig. 12. Stress distributions along the web (a) $\sigma_{x z}$ and (b) $\sigma_{x y}$ together with the relative error between the analytical and numerical values.


 error is negligible.
the current approach will provide the solution for beams with variable thickness along the $z$-direction. Further extension will regard variable thickness within the cross section. Moreover, a generalisation of Navier's formula will allow to derive the solution of beams with asymmetric cross-section. Also the introduction of anisotropic and inhomogeneous materials will be a great benefit.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Analytical expressions for the normal stress components $\sigma_{x x}^{f}$ in the flange and $\sigma_{y y}^{w}$ in the web

The expressions for the normal stress components determined in Section 2.3 by imposing the vertical and horizontal equilibrium of the stresses acting at an infinitesimal part of the corner of the cross section are presented in here for completeness.

In the flange, Eq. (21) is substituted in Eq. (11). Then, after some manipulation, the stress components $\sigma_{x x}^{f}$ become


Fig. 14. In-plane shear stress $\sigma_{y z}$ along (a) the flange and (b) the web of seven cross sections of a doubly tapered box girders. The axial force $F_{z}=1 \mathrm{kN}$ is applied at the tip. Legend: $x=$ numerical solution; $-=$ analytical solution.

(a)

(b)
 Legend: $x=$ numerical solution; $-=$ analytical solution.

$$
\begin{align*}
\sigma_{x x}^{f}= & \frac{F_{z} t_{w}}{4\left(b t_{f}+h t_{w}\right)^{3}}\left\{x^{2}\left(t_{f} \tan \beta+t_{w} \tan \alpha\right)^{2}\right.  \tag{A.1}\\
& {\left.\left[\left(h^{2} t_{w}+2 b h t_{f}\right) \tan \beta^{2}-2 \tan \beta \tan \alpha b^{2} t_{f}-b^{2} t_{w} \tan \alpha^{2}\right] t_{w}\right\} } \\
& -\frac{3 F_{y}}{4 h^{2}\left(3 b t_{f}+h t_{w}\right)^{2}}\left\{x^{2}\left[3 h t_{f} \tan \beta+\left(3 b t_{f}+2 h t_{w}\right) \tan \alpha\right]\right. \\
& \left.+b\left(3 b t_{f}+2 h t_{w}\right)(h \tan \beta-b \tan \alpha)\right\} \\
& -\frac{3 M_{x}}{4 h^{2}\left(3 b t_{f}+h t_{w}\right)^{2}}\left\{x ^ { 2 } \left[\frac{\tan \alpha^{2}\left(3 b t_{f}+h t_{w}\right)}{h}\right.\right.  \tag{A.2}\\
& \left.+\frac{3 t_{f} \tan \beta+t_{w} \tan \alpha}{\left(3 b t_{f}+h t_{w}\right)}\left(3 t_{f} \tan \beta \tan \alpha+t_{w} \tan \alpha^{2}+3 h t_{f} \tan \beta\right)\right] \\
& -\frac{1}{h\left(3 b t_{f}+h t_{w}\right)}\left[h^{4} t_{w}^{2} \tan \beta^{2}+3\left(3 b^{4} t_{f}^{2}\right.\right.
\end{align*}
$$

$$
\begin{aligned}
& \left.+3 b^{3} h t_{f} t_{w}+b^{2} h^{2} t_{w}^{2}\right) \tan \alpha^{2} \\
& \left.\left.-9 h b\left(b^{2} t_{f}^{2}+h b t_{f} t_{w}+\frac{4}{9} h^{2} t_{w}^{2}\right) \tan \alpha \tan \beta\right]\right\}
\end{aligned}
$$

Note that when $\alpha=\beta=0$, i.e. when the beam is prismatic, Eq. (A.1) reduces to zero. Moreover, when $\beta=0$ and $\alpha \neq 0$ it simplifies to

$$
\begin{aligned}
\left.\sigma_{x x}^{f}\right|_{\beta=0} & =\frac{\left(x^{2}-b^{2}\right) \tan \alpha}{4}\left[\frac{F_{z} t_{w}^{2} \tan \alpha}{\left(b t_{f}+h t_{w}\right)^{3}}-\frac{3 F_{y}\left(3 b t_{f}+2 h t_{w}\right)}{h^{2}\left(3 b t_{f}+h t_{w}\right)^{2}}\right. \\
& \left.-\frac{3 M_{x}\left(3 t_{f}^{2} b^{2}+3 h b t_{f} t_{w}+h^{2} t_{w}^{2}\right) \tan \alpha}{h^{3}\left(3 b t_{f}+h t_{w}\right)^{3}}\right]
\end{aligned}
$$

which corresponds to Eq. 27 in Bertolini et al. (2019).


Fig. 16. In-plane shear stress $\sigma_{y z}$ along (a) the flange and (b) the web of seven cross sections of a doubly tapered box girders. The bending moment $M_{x}=5 \mathrm{kN} \mathrm{m}$ is applied at the tip. Legend: $x=$ numerical solution; $-=$ analytical solution.


Fig. 17. $\sigma_{y z}$ distribution along the web of a (a) vertically and (b) horizontally tapered cantilever beam. The $1 \mathrm{~m} \times 1 \mathrm{~m}$ cross section is located at 5 m from the root and the tip. In both cases the taper angle is equal to 4.3 deg.


Fig. 18. Ratio with respect to a prismatic beam of (a) the von Mises stress distribution and (b) the number of cycle to failure in cantilever beams with vertical taper angle $\alpha$ and horizontal taper $\beta$.

Along the web, the missing stress component $\sigma_{y y}^{w}$ is obtained after substituting Eq. (22) in Eq. (20)

$$
\begin{align*}
\sigma_{y y}^{w}= & \frac{3 F_{y} y}{4 h^{3} t w\left(3 b t_{f}+h t_{w}\right)^{2}}\left\{h t_{f} t_{w}\left(h^{2}-y^{2}\right) \tan \beta\right.  \tag{A.3}\\
& \left.+\left[t_{w}^{2}\left(h^{3}-h y^{2}\right)+2 b t_{f} t_{w}\left(2 h^{2}-y^{2}\right)+6 b^{2} h t_{f}^{2}\right] \tan \alpha\right\} \\
& +\frac{F_{z}}{4 t_{w}\left(b t_{f}+h t_{w}\right)^{3}}\left[t_{w}^{2}\left(t_{w} y^{2}+2 b h t_{f}\right)+t_{w} t_{f}^{2}\left(b^{2}-h^{2}+y^{2}\right)\right. \\
& +\frac{t_{w} t_{f}^{2}\left(h^{2}-y^{2}\right)}{\cos \beta^{2}}-\left(\frac{t_{w}}{\cos \alpha^{2}}+2 t_{f} \tan \alpha \tan \beta\right) \\
& \left.\left(b^{2} t_{f}^{2}+2 b h t_{f} t_{w}+t_{w}^{2} y^{2}\right)\right] \\
& -\frac{3 M_{x} y}{4 h^{4} t_{w}\left(3 b t_{f}+h t_{w}\right)^{3}}\left\{2 h t_{f} t_{w}\left[3 b t_{f} y^{2}+h t_{w}\left(2 y^{2}-h^{2}\right)\right]\right.
\end{align*}
$$

$\tan \alpha \tan \beta$

$$
\begin{aligned}
& +3 t_{f}^{2} t_{w}\left[h^{4}+h^{2}\left(6 b^{2}-y^{2}\right)-3 b^{2} y^{2}\right]+\frac{1}{\cos \alpha^{2}}\left[h^{2} t_{w}^{3}\left(2 y^{2}-h^{2}\right)\right. \\
& +h^{2} t_{w}^{3}\left(h^{2}-2 y^{2}\right)+2 h b t_{f} t_{w}^{2}\left(3 h^{2}-4 y^{2}\right)+18 h b^{3} t_{f}^{3} \\
& -\frac{3 h^{2} t_{f}^{2} t_{w}\left(h^{2}-y^{2}\right)}{\cos \beta^{2}} \\
& \left.\left.-2 h b t_{f} t_{w}^{2}\left(3 h^{2}-4 y^{2}\right)-9 b^{2} t_{f}^{2} t_{w}\left(2 h^{2}-y^{2}\right)-18 b^{3} h t_{f}^{3}\right]\right\} .
\end{aligned}
$$

When $\alpha=\beta=0$ the stress component $\sigma_{y y}^{w}$ is zero, as expected in prismatic beams. When $\beta=0$ and $\alpha \neq 0$, i.e. when the beam is vertically tapered, this expressions simplifies to Eq. (A.4), which corresponds to Eq. 28 in Bertolini et al. (2019).

$$
\begin{aligned}
\sigma_{y y}^{w}= & \frac{3 F_{y} y \tan \alpha}{4 h^{3} t w\left(3 b t_{f}+h t_{w}\right)^{2}} \\
& {\left[t_{w}^{2}\left(h^{3}-h y^{2}\right)+2 b t_{f} t_{w}\left(2 h^{2}-y^{2}\right)+6 b^{2} h t_{f}^{2}\right] } \\
& +\frac{F_{z}}{4 t_{w}\left(b t_{f}+h t_{w}\right)^{3}}\left(b^{2} t_{f}^{2}+2 b h t_{f} t_{w}+t_{w}^{2} y^{2}\right)\left(1-\frac{1}{\cos \alpha^{2}}\right) \\
& -\frac{3 M_{x} y}{4 h^{4} t_{w}\left(3 b t_{f}+h t_{w}\right)^{3}}\left\{3 t_{f}^{2} t_{w}\left[h^{4}+h^{2}\left(6 b^{2}-y^{2}\right)-3 b^{2} y^{2}\right]\right. \\
& +\frac{1}{\cos \alpha^{2}}\left[h^{2} t_{w}^{3}\left(2 y^{2}-h^{2}\right)-2 h b t_{f} t_{w}^{2}\left(3 h^{2}-4 y^{2}\right)\right. \\
& \left.-9 b^{2} t_{f}^{2} t_{w}\left(2 h^{2}-y^{2}\right)-18 b^{3} h t_{f}^{3}\right]-3 h^{2} t_{f}^{2} t_{w}\left(h^{2}-y^{2}\right) \\
& +3 t_{w} t_{f}^{2}\left(h^{4}+h^{2}\left(6 b^{2}-y^{2}\right)-3 b^{2} y^{2}\right) \\
& \left.+2 h b t_{f} t_{w}^{2}\left(3 h^{2}-4 y^{2}\right)+h^{2} t_{w}^{3}\left(h^{2}-2 y^{2}\right)+18 b^{3} h t_{f}^{3}\right\} .
\end{aligned}
$$

## References

ABAQUS, 2017. Dassault systemes simulia corporation.
Bak, C., Bitsche, R., Yde, A., Kim, T., Hansen, M.H., Zahle, F., Gaunaa, M., Blasques, J.P.A.A., Døssing, M., Heinen, J.-J.W., et al., 2012. The 10 MW reference wind turbine. In: EWEA 2012-European Wind Energy Conference \& Exhibition. European Wind Energy Association (EWEA).
Balduzzi, G., Aminbaghai, M., Sacco, E., Füssl, J., Eberhardsteiner, J., Auricchio, F., 2016. Non-prismatic beams: a simple and effective timoshenko-like model. Int. J. Solids Struct. 90, 236-250.
Basquin, O.H., 1910. The exponential law of endurance tests. Amer. Soc. Testing Mater. 10, 625-630.
Bennati, S., Bertolini, P., Taglialegne, L., Valvo, P.S., 2016. On shear stresses in tapered beams. In: Proceedings of the GIMC-GMA 2016-21 ${ }^{\text {St }}$ Italian Conference on Computational Mechanics and $8^{T h}$ Meeting of the AIMETA Materials Group, Lucca.
Bertolini, P., Eder, M.A., Taglialegne, L., Valvo, P.S., 2019. Stresses in constant tapered beams with thin-walled rectangular and circular cross sections. Thin-Walled Struct. 137, 527-540.
Bleich, F., 1932. Stahlhochbauten, Vol. 1. Springer, pp. 76-85.
Boley, B.A., 1963. On the accuracy of the Bernoulli-Euler theory for beams of variable section. J. Appl. Mech. 30 (3), 373-378.
Hodges, D.H., Rajagopal, A., Ho, J.C., Yu, W., 2010. Stress and strain recovery for the in-plane deformation of an isotropic tapered strip-beam. J. Mater. Struct. 5 (6), 963-975.
Kuś, J., 2015. Lateral-torsional buckling of steel beams with tapered flanges and web. Steel Compos. Struct. 19 (4), 897-916.
Li, C., Dai, W., Duan, F., Zhang, Y., He, D., 2017. Fatigue life estimation of medium-carbon steel with different surface roughness. Appl. Sci. 7 (4), 338.
Maple 2017, 2018. Maplesoft, Waterloo, Ontario.
Paglietti, A., Carta, G., 2009. Remarks on the current theory of shear strength of variable depth beams. Open Civ. Eng. J. 3 (1), 28-33.
Poli, A.A., Cirillo, M.C., 1993. On the use of the normalized mean square error in evaluating dispersion model performance. Atmos. Environ. Part A Gen. Top. 27 (15), 2427-2434.

Taglialegne, L., 2018. Analytical Study of Stress Fields in Wind Turbine Blades (Ph.D. thesis). Universities of Florence, Perugia and Pisa - TU Braunschweig, International Doctorate "Civil and Environmental Engineering".
Timoshenko, S., 1923. Bending stresses in curved tubes of rectangular cross section. Trans. Amer. Soc. Mech. Eng. 45, 135-140.
Timoshenko, S., Goodier, J.N., 1951. Theory of Elasticity. McGraw-Hill book Company.
Zevallos, E., Hassanein, M., Real, E., Mirambell, E., 2016. Shear evaluation of tapered bridge girder panels with steel corrugated webs near the supports of continuous bridges. Eng. Struct. 113, 149-159.

## Paper 3

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# COMPARISON OF STRESS DISTRIBUTIONS BETWEEN NUMERICAL CROSS-SECTION ANALYSIS AND 3D ANALYSIS OF TAPERED BEAMS 

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Keywords: Tapered beam, Cross-section analysis, Stress distribution, Finite element model


#### Abstract

While the demand for new and longer wind turbine blades is increasing, time for their design and production is shortening. 3D finite element models generally have the capability of providing a complete and detailed analysis of the behaviour of such structures. Nonetheless, computationally efficient and accurate cross-section analysis tools are required to improve the efficiency of the workflow in the conceptual design phase. Several cross-section analysis software, such as BEam Cross-Section Analysis Software BECAS, have been developed for analysis of prismatic beams. As a result, structures which are tapered along their longitudinal axis, i.e. aircraft wings or wind turbine blades, are modelled as step-wise prismatic beams and the known effects on stresses due to taper are ignored. This study provides a numerical comparison of the Cauchy's stress components evaluated with both 3D finite element and cross-section analysis consisting of linear elastic isotropic and anisotropic materials. Results highlight how every cross-section formulation that relies on the stepwise prismatic assumption lacks the capability to correctly recover the stresses in tapered beams.


## 1 INTRODUCTION

Nowadays, aerospace and wind energy industries are facing new challenges related to an upscaling tendency of aircrafts and wind turbine rotors, where low costs of production and optimal structural designs must be assured. In such cases, advanced composite materials and lengthwise geometrical variations (LGVs) play a key role. Aircraft structures and wind turbine blades are typically made from glass/carbon-fibre polymer composite beams and comprise of lengthwise geometrical variations LGVs such as pre-curved beam axis and twist and variation of the dimensions of their cross section from the root to the tip, i.e. taper [1]. Material anisotropy and LGVs induce material and geometry coupling, respectively. This study is concerned only with geometrical coupling due to taper.

Among the LGVs, taper is defined as the variation of the dimensions of the cross section along the longitudinal axis of the beam [2]. The effects of taper in beams have been noticed since the last century, when studies on a planar isotropic wedge loaded at its tip showed a non-trivial shear stress distribution [3, 4]. Recently, Bennati et al. [5] focused on the analysis of a planar truncated isotropic beam under axial and shear load as well as bending moment, highlighting how the maximum shear stress does not necessarily occur at the cross-section centre-line in tapered beams. Bertolini et al. [2] derived a methodology to describe the full Cauchy's stress field in thin-walled tapered beams. Balduzzi et al. [6] studied the variation of the stress distribution in planar anisotropic tapered beams. The aforementioned studies revealed that taper effects are not negligible in beam design and that they are not captured in classic beam models.

In wind turbine manufactory industries, the blade designs are constantly and rapidly changing to answer to the costumers' demand. For this reason, accurate and computationally efficient analysis tools to perform aero-elastic and structural analysis are required. Even though 3D finite element analyses are able to model LGVs, they are too computationally expensive to be employed in the above-mentioned analyses. Since wind turbine blades have a high length-to-height and length-to-width ratio, they can be modelled as slender beams.

Nowadays the above-mentioned cross-section analysis can be accomplished by different design tools, such as VABS [7], which is based on the variational asymptotic method, ANBA [8] and BECAS [9], which are both based on the anisotropic beam theory developed by Giavotto et al. [10]. They provide the six-by-six stiffness matrix of the cross-section of a slender linear elastic anisotropic prismatic beam for small strains and displacements used for aeroelastic analysis. On the other hand, they can be used to recover stresses and strains in cross sections when the internal forces are known e.g. from aeroelastic analysis. The aforementioned cross-section analysis tools use the definition of the energy per unit beam length under the assumption of prismatic beam geometries. Therefore, tapered beams are approximated as step-wise prismatic beams and the recovered stress field lacks the effects of taper previously mentioned.

The objective of the present work is to investigate how cross-section analysis could over/underestimate the stress field in a tapered beam. Tapered homogeneous isotropic and anisotropic beams with four different cross-sections and subjected to an external shear force are modelled with 3D finite element and cross-section analyses to compute the Cauchy's stress components. The numerical comparison between the two solutions highlight how the latter fails to capture some of the stress components as well as to predict the stress distribution correctly.

## 2 METHOD

A 3D finite element model of a tapered cantilever beam loaded by an external shear force is created to extract the stresses at its middle section. The results are numerically compared with the ones evaluated by cross section analysis in the cross section at the mid-span. The latter is subjected to the internal forces induced by externally applied load. The software BECAS developed at DTU Wind Energy is used for cross-section analysis and the commercial finite element package Abaqus [11] for the numerical analysis of the 3D finite element model.


Figure 1: Side and top view of the analyzed doubly tapered beams. The beam presents a constant vertical taper $\alpha$ and horizontal taper $\beta$. The DOFs in the plane at $z=0$ are constrained and the load is applied at $\mathrm{z}=\mathrm{L}$.

Four tapered cantilever beams are considered. They are represented in the coordinate system $O x y z$ as illustrated in Fig. 1. The beams have length L and constant vertical and horizontal taper angles, named $\alpha$ and $\beta$ respectively. As demonstrated in Bertolini et al. [2], a thin-walled tapered homogeneous isotropic beam exhibits taper effects under axial or shear force or bending moment. Aircraft wings and wind turbine blades experience a combination of aerodynamic-gravitational-inertial loads during operation life [1]. Bending in the direction perpendicular to the blade axis (flapwise) is one of the main load cases. Therefore, for simplicity, this paper analyses cross sections which are subjected only to an internal shear force and bending moment. Small taper angles (see Table 2) have been used to allow the comparison of the isotropic thin-walled rectangular model with the analytical solution provided in [12].

The analysed tapered beams involve four different cross sections, as described in the following. Two homogeneous isotropic tapered beams with solid rectangular cross section (ISR) and thin-walled rectangular cross section (ITWR); two anisotropic tapered beams with thin-walled rectangular cross section (CTWR) and with blade-like cross section (CB). The latter is a combination of a semicircle and a trapezoid.

In order to solely investigate the geometrical coupling caused by taper, the fibre directions in the rectangular cross section are chosen in such a way to eliminate any source of anisotropy material
coupling. Moreover, analytical solutions for ITWR tapered beams are available in the literature [2, 12] and they will be used for comparison in this study. The blade-like cross section is chosen to analyse a more realistic example. Aluminum, which is widely used in aircraft industries, and E-glass/epoxy lamina, which is typically used in wind turbine blades, are employed to investigate the relation between taper and linear isotropic and linear anisotropic materials, respectively. The properties of the two materials are listed in Table 1. For simplicity, the flanges of CTWR as well as the flanges and the leading edge of CB are uniaxial E-glass/epoxy laminate and the webs of both models are $\left[+45^{\circ}-45^{\circ}\right.$ $\left.+45^{\circ}\right]$ laminate.

| Material | $\mathrm{E}_{11}$ <br> $[\mathrm{GPa}]$ | $\mathrm{E}_{22}=\mathrm{E}_{33}$ <br> $[\mathrm{GPa}]$ | $\mathrm{G}_{12}=\mathrm{G}_{13}$ <br> $[\mathrm{GPa}]$ | $\mathrm{G}_{23}$ <br> $[\mathrm{GPa}]$ | $v_{12}=v_{13}$ | $v_{23}$ | $\rho$ <br> $\left[\mathrm{Kg} / \mathrm{m}^{3}\right]$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Aluminium | 70 | 70 | 26 | 26 | 0.3 | 0.3 | 2700 |
| E-glass/ epoxy | 39.5 | 12.10 | 4.54 | 4.54 | 0.275 | 0.333 | 1845 |

Table 1: Properties of the isotropic (aluminium) and of the composite materials (E-glass/ epoxy [1]).
In order for the model from BECAS to be comparable to the one from Abaqus, a reference cross section located at the midspan of each cantilever beam is considered. Being the reference cross section sufficiently far from the clamped root and the loaded tip, the influence of the boundary effects becomes negligible. Moreover, the reference cross sections are chosen perpendicular to the beam axis and located at $\mathrm{z}=5 \mathrm{~m}$. Their geometrical dimensions are given in Table 2.

| Model | L <br> $[\mathrm{m}]$ | B <br> $[\mathrm{mm}]$ | H <br> $[\mathrm{mm}]$ | t <br> $[\mathrm{mm}]$ | $\alpha$ <br> $[\mathrm{deg}]$ | $\beta$ <br> $[\mathrm{deg}]$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| ITWR | 10 | 1000 | 1200 | 24 | 2.5 | 1.5 |
| CTWR | 10 | 1000 | 1200 | 24 | 2.5 | 1.5 |
| ISR | 10 | 1000 | 1175 | - | 2.5 | 1.5 |
| CB | 10 | 300 | 600 | 24 | 1.2 | 3.4 |

Table 2: Geometrical dimensions of the four beams.
The mesh discretization of the four models is shown in Fig. 2 and 3. The number of elements in each part of the cross section and the total number of elements of the Abaqus and BECAS models are given in Table 3.


Figure 2: Finite element mesh of the cross sections (a) ISR and (b) ITWR. The local directions $x$ and $y$ indicates the directions of the paths along which the stresses are extracted.


Figure 3: Finite element mesh of the cross sections (a) CTWR and (b) CB. The coloured webs have direction $\left[+45^{\circ}-45^{\circ}+45^{\circ}\right]$. The local directions $\mathrm{x}, \mathrm{y}, \mathrm{s}_{\mathrm{f}}$, and $\mathrm{s}_{1}$ indicates the directions of the paths along which the stresses are extracted.

|  | ISR |  | ITWR |  | CTWR |  | CB |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | \# el | bias-ratio | \# el | bias-ratio | \# el | bias-ratio | \# el | bias-ratio |
| Flange | 40 | 3 | 40 | 2 | 40 | 2 | 50 | 2 |
| Flatback | 40 | 3 | 40 | 2 | 40 | 2 | 20 | 1 |
| Internal web | - | - | - | - | - | - | 50 | 2 |
| Leading edge | - | - | - | - | - | - | 80 | 2 |
| Length | 180 | 4 | 180 | 4 | 180 | 4 | 120 | 2 |
| Total \#el in Abaqus | 80000 |  | 51600 |  | 51600 |  | 46080 |  |
| Total \#el in BECAS | 1600 |  | 516 |  | 516 |  | 576 |  |

Table 3: Number of elements and ratio of the size of the coarsest element to the size of the finest element (bias-ratio) in each part of the cross sections. 'Length' refers to the number of elements along the span of the beam. The total number of elements in the two models is also given.

### 2.1 3D finite element model

The 3D finite element models were discretised with incompatible mode eight-noded brick elements (Abaqus element type C3D8I). Convergence studies were performed by refining the mesh until a mesh-independent solution was achieved. They are omitted in this paper for the sake of brevity. As shown in Fig. 2 and 3, the mesh of the cross sections was refined close to the corners to increase the accuracy of the evaluated stress field in the proximity of the singularities. For the two examples CTWR and CB, each laminate was modelled from partitioning the wall thickness in the three laminae and by assigning the material orientation as designed. The materials properties are defined in Table 1.

The external load and the clamped condition were applied through a reference point located at the geometrical centroid of the tip and the root. These reference points were then constrained to the surfaces at the tip and the root using kinematic couplings. The Abaqus linear perturbation solver was used. After defining a nodal path, the stresses were extracted in global coordinates in ISR and ITWR, and in the material coordinates in CTWR and CB, as defined in the Results Section.

### 2.2 Cross section model

The software BECAS provides the six-by-six cross-section stiffness matrix, the shear and elastic centres of any anisotropic cross-section and arbitrary geometry. Given as input the geometry of a cross section, the properties of the materials and the internal forces, a finite element model of the cross section was created and analysed. The cross sections of this study were modelled with four-noded 2D plane elements (Q4), coherently with the chosen linear elements used in the Abaqus model. The applied internal forces were equivalent to the ones caused by the tip load applied in the models in Abaqus and they are given in Table 4.

| External force | Internal forces |  |
| :---: | :---: | :---: |
| Shear force | Shear force | Bending moment |
| 1000 N | 1000 N | -5000 Nm |

Table 4: Internal forces at the reference cross sections due to the external load $\mathrm{F}_{\mathrm{y}}$ applied at the tip.

## 3 RESULTS AND DISCUSSION

The Cauchy's stresses were evaluated at the centroid of each element in the 3D FEM from Abaqus and in the model from BECAS. To validate the models from BECAS, each cross section was modelled as prismatic 3D finite element in Abaqus. The stress components in the three different models provide a numerical comparison of the effects of LGVs.

The cross sections of ISR, ITWR, CTWR have two axes of symmetry, therefore the stress distributions are symmetric, and it suffices to show to show them along a quarter of the cross section ( $0<x<B / 2$ and $0<y<H / 2$ ). Results from CB are reported only for half of the cross-section $(0<y<$ H/2). In Fig. 4-13 the label 'BECAS CS' refers to the results from BECAS, '3D CSP' and '3D CST' to the ones from the prismatic and tapered models in Abaqus respectively.

### 3.1 ISR: a solid rectangular cross-section with homogeneous isotropic material

A doubly tapered cantilever beam with solid rectangular cross section was modelled and analysed. The comparison in Fig. 4 and 5 refers to a vertical and to a horizontal path located at $\mathrm{x}=0.49 \mathrm{~m}$ and y $=0.57 \mathrm{~m}$ respectively. Stresses are evaluated in the global coordinate system Oxyz.

In a cross section of a homogeneous isotropic prismatic beam with solid rectangular cross section subjected to internal shear and shear-bending, only axial and in-plane shear stresses arise. Figures 4 and 5 show that if the same beam is slightly tapered none of the stress components is zero.

As expected [13], the axial stress component $\sigma_{z z}$ in both the vertical and horizontal path is not affected by taper. The other stresses $\sigma_{x x}, \sigma_{x y}, \sigma_{x z}, \sigma_{y y}$ are strongly at variance and they increase when moving toward the edge of the cross section. For example, the shear stress distribution along the vertical path of the prismatic beam reaches its maximum value at the mid-span, whereas it occurs at the extremity of the cross section in the tapered beam, as experienced in homogeneous isotropic thinwalled cross sections [2].

### 3.2 ITWR: a thin-walled rectangular cross-section with homogeneous isotropic material

The stresses along the web and flange of the current cross section are evaluated in the global coordinate system Oxyz. Three different methods were applied: the cross-section analysis BECAS, the 3D finite element model in Abaqus, and the closed-form analytical solutions ('Analytical') provided in Bertolini and Taglialegne [12]. Analytical solutions and Abaqus models of the tapered beams are in good agreement and they are both able to capture the effects of taper. As in the previous example, the axial stress component $\sigma_{z z}$ in the flange and the web is hardly affected by small taper angles [13]. On the other hand, the shear stress distributions are wrongly predicted in BECAS because the shearbending coupling caused by the taper is not depicted. It reduces the shear stress $\sigma_{y z}$, but it also shifts its
maximum from the mid-span to the corner, as shown in Fig. 7-e. In addition, the stress components $\sigma_{\mathrm{xx}}, \sigma_{\mathrm{xz}}, \sigma_{\mathrm{yy}}$ are not zero even for the small taper angles used in this model.


Figure 4: Stress distributions along a horizontal path located in the proximity of the edge of model ISR under an external shear force. Legend: --- BECAS CS, - 3D CST, ….. 3D CSP.


Figure 5: Stress distributions along a vertical path located in the proximity of the edge of model ISR under shear force. Legend: --- BECAS CS, - 3D CST, … 3D CSP.


Figure 6: Stress distributions along the flange of model ITWR under shear force.
Legend: --- BECAS CS, — 3D CST, … 3D CSP, $\cdots$ - Analytical.


Figure 7: Stress distributions along the web of model ITWR under shear force.
Legend: --- BECAS CS, — 3D CST, - $\cdots$... 3D CSP, - - - Analytical.

### 3.3 CTWR: a thin-walled rectangular cross-section with composite material

The same cross section analysed in the previous section is now composed by three laminae of Eglass/epoxy. In the flanges, the fibres are oriented along the beam direction $\left(0^{\circ}\right)_{3}$ and in the webs, the plies have direction angles $\left[+45^{\circ}-45^{\circ}+45^{\circ}\right]$. The stresses are therefore evaluated in the material coordinate system $O 123$, where the axis corresponding to the 1 -direction is aligned with the fibres and the axis corresponding to the 3 -direction is the outward normal. Figure 8 shows the stresses in half of
the internal lamina of the flange. It is worth noting that the axial stress $\sigma_{11}$ and the-out-of-plane components are not affected by taper. The in-plane shear stress has opposite sign and it is three times overestimated in BECAS.

Figure 9 shows the stresses in the internal lamina of the web where the fibres have direction $-45^{\circ}$. In this case, BECAS correctly predicts the in-plane shear stress $\sigma_{12}$, whereas it overestimates the inplane stress components $\sigma_{11}$ and $\sigma_{22}$ by three-times in comparison to Abaqus. In addition, taper induces the out-of-plane stress components, which are zero in the prismatic case. Out-of-plane stresses have a key role in failure design because they are involved in delamination of the laminate. It is worth noting that even if $\sigma_{13}$ and $\sigma_{23}$ are 10 times smaller than $\sigma_{12}$, the transverse tensile strength is typically 20 times smaller than the longitudinal tensile strength in unidirectional laminae [14]. Therefore, the ratios stresses-to-strength in the transverse and longitudinal direction are comparable.

### 3.4 CB: a wind turbine blade cross sectional geometry of composite material

The last example is a tapered cantilever beam with the blade-like cross section given in Fig. 3-b. The airfoil is made of three-layer uniaxial laminate and the webs of three-layers laminate with ply angles of $\left[+45^{\circ}-45^{\circ}+45^{\circ}\right]_{3}$. Stresses are computed in the material coordinate system O123, which is defined in such a way that the fibres follow the 1-direction and the 3-direction is outward normal to the laminate. The numerical comparison refers only to the stresses along the upper flange in Fig. 10, the leading edge in Fig. 11, and half of the webs in Fig. 12 and 13 are presented.

Results from BECAS and the prismatic FEM in Fig. 10 show that only axial and in-plane shear stresses occur in the upper flange. When the beam is tapered, the axial stress $\sigma_{11}$ slightly increases and the shear stress $\sigma_{12}$ is strongly reduced and has a different distribution with maximum value $\mathrm{s}_{\mathrm{f}}=0$ and quasi-zero stress at the external corner ( $\mathrm{s}_{\mathrm{f}} \rightarrow 0.8 \mathrm{~m}$ ). Among the remaining components, the out-ofplane stresses are not zero in the tapered model and the maximum value of $\sigma_{23}$ has the same order of magnitude as $\sigma_{12}$. The combination of such a high transverse shear stress in Fig. 10-e and transverse tensile stress in Fig. 10-f , might be critical in delamination of the laminate or debonding of the adhesive joint, as already pointed out in the previous section.


Figure 8: Stress distributions along half of the flange of model CTWR loaded by a shear force. The stresses are from the internal lamina $\left(0^{\circ}\right)$.

Legend: --- BECAS CS, - 3D CST, ….. 3D CSP.


Figure 9: Stress distribution along half of model CTWR loaded by a shear force. The stresses from the internal lamina $\left(-45^{\circ}\right)$. Legend: --- BECAS CS, - 3D CST, ….. 3D CSP.

Figure 11 refers to half of the leading edge. Only axial and in-plane shear stresses arise in the prismatic case. In the tapered model the axial stress does not change, whereas the in-plane shear stress halves and $\sigma_{12}=0$ at $\mathrm{s}_{1}=0.3 \mathrm{~m}$. As observed in the flange of CB , the out-of-plane shear stress $\sigma_{23}$ in Fig. 11-e has values comparable to the in-plane shear stress at the intersection between the leading edge, the flange and the internal web. Delamination could have a driving role in the failure design. The distribution of $\sigma_{13}$ shown in Fig. 13-d does not provide sufficiently accurate results. Indeed, results from BECAS deviate from the prismatic model in Abaqus and therefore a deeper investigation is needed.


Figure 10: Stress distribution along the upper flange of model CB under a shear force. The stresses are extracted in the internal lamina $\left(0^{\circ}\right)$. Legend: --- BECAS CS, - 3D CST, $\cdots \cdots 3$ C. ${ }^{\text {CSP }}$.

Figure 12 shows the stresses in the web along the external lamina $\left(+45^{\circ}\right)$. The in-plane shear stress $\sigma_{12}$ from the three models is the same, the axial component $\sigma_{11}$ is overestimated in BECAS, whereas the $\sigma_{22}$ is underestimated. As in the previous cases, the out-of-plane stresses are not zero when the beam is tapered. In the beam analysed in this study, the transverse shear components in Fig. 12-d and Fig. 12-e are circa four or six times smaller than the in-plane shear component, but as explained before, delamination could be crucial.

Similar observations can be drawn for the flatback, whose stress components are shown in Fig. 13. The component $\sigma_{12}$ is not affected by the taper, whereas $\sigma_{11}$ and $\sigma_{22}$ are overestimated in BECAS. Delamination failure may be critical also in this region, since the out-of-plane shear stresses in Fig. 13d and Fig. 13-e are not zero.


Figure 11: Stress distribution along the leading edge of model CB under a shear force. The stresses are extracted in the internal lamina $\left(0^{\circ}\right)$. Legend: --- BECAS CS, - 3D CST, $\cdots \cdots 3$. ${ }^{\text {CSP }}$.


Figure 12: Stress distribution along the internal web of model CB under a shear force. The stresses are extracted in the external lamina ( $+45^{\circ}$ ). Legend: --- BECAS CS, - 3D CST, $\cdots \cdots 3$. ${ }^{\text {. }}$.


Figure 13: Stress distribution along the flatback of model CB under a shear force. The stresses are extracted in the internal lamina $\left(-45^{\circ}\right)$. Legend: --- BECAS CS, - 3D CST, … 3D CSP.

## 9 CONCLUSIONS

In this paper the Cauchy stress distributions in tapered cantilever beams subject to concentrated shear loads are compared with stress distributions recovered from cross-section analysis induced by equivalent internal cross section forces. In this study three different cross section geometries are investigated where both isotropic and anisotropic material behaviour is considered. The aim of this study is to shed light on the deviation between the stress fields in tapered 3D finite element models and those provided by stepwise prismatic cross-section analysis models. The following conclusions can be drawn from the results:

1. The numerically predicted stress distributions obtained by both, 3D analysis and cross-section analysis agreed well in all prismatic cases. The analytically obtained stress distributions agreed well with those predicted by 3D analysis in the designated tapered cases.
2. The stepwise prismatic approach adopted in the cross-section analysis methods incorrectly predicts the stresses provided by tapered 3D finite element models. The stress distributions in tapered cross sections were found to be significantly at variance where the magnitude of deviation is a strong function of the taper angle.
3. The deviations occur for any taper angle different from zero irrespective of isotropic or anisotropic material behaviour. The taper can significantly augment the stress distributions prevailing in prismatic cases whence the deviations are typically counterintuitive.
4. Particularly relevant for fibre-polymer composites it was demonstrated that taper is prone to induce through thickness transverse tensile stress components. Such peeling stress components have the propensity to significantly affect the fatigue life of composites.

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## REFERENCES

[1] C. Bak, F. Zahle, R. Bitsche, A. Yde, L.C. Henriksen, A. Nata and M.H. Hansen, The DTU 10MW reference wind turbine, Danish Wind Power Research 2013, 2013.
[2] P. Bertolini, M.A. Eder, L. Taglialegne and P.S. Valvo, Stresses in constant tapered beams with thin-walled rectangular and circular cross sections, Thin-Walled Structures, 119, 2019, pp. 527540.
[3] S.P. Timoshenko, J.M. Gere, Mechanics of materials, Van Nostrand Reinhold, New York, 1972.
[4] J.H. Michell, Elementary distributions of plane stress, Proceedings of the London Mathematical Society, s1-32, 1900, pp. 247-257.
[5] S. Bennati, P. Bertolini, L. Taglialegne and P.S. Valvo, On stresses in tapered beams (submitted).
[6] G. Balduzzi, M. Aminbaghai F. Auricchio and J. Füssl, Planar Timoshenko-like model for multilayer non-prismatic beams, International Journal of Mechanics and Materials in Design, 14, 2017, pp.51-70.
[7] C.E. Cesnik and D.H. Hodges, VABS: a new concept for composite rotor blade cross-sectional modeling, Journal of the American Helicopter Society, 42, 1997, pp. 27-38.
[8] M. Morandini, M. Chierichetti and P. Mantegazza, Characteristic Behavior of Prismatic Anisotropic Beam Via Generalized Eigenvectors, International Journal of Solids and Structures, 47, 2010, pp. 1327-1337.
[9] P. Blasques, R. Bitsche, V. Fedorov and M.A. Eder, Applications of the BEam Cross-section analysis Software (BECAS), Proceedings of the 26th Nordic Seminar on Computational Mechanics, 2013, pp. 46-49.
[10] V. Giavotto, M. Borri, P. Mantegazza and G. Ghiringhelli, Anisotropic beam theory and applications, Computer \& Structures, 16, 1983, pp. 403-413.
[11] ABAQUS 2017, Dassault Systemes Simulia Corporation, 2018.
[12] P. Bertolini and L. Taglialegne, Analytical expressions in doubly tapered beams (submitted).
[13] B.A. Boley, On the accuracy of the Bernoulli-Euler theory for beams of variable section, Journal of Applied Mechanics, 30, 1963, pp. 373-378.
[14] E.J. Barbero, Composite Materials Design, CRC press, Boca Raton, 2017.

## Paper 4

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# Numerical cross-section analysis of stresses in tapered slices 

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#### Abstract

The field of cross-section analysis has a key role in determining the stiffness cross-section properties and for recovering the strain and stress components in complex structures. Most of the 2D cross-section analysis methods available in the literature are based on stepwise prismatic assumption and, consequently, the well-known effects of lengthwise taper to the beam behaviour are not taken into account. The six boundary conditions are imposed through the Lagrange multipliers method, and the nodal forces are derived from the equivalent internal tractions. The method is validated against analytical solutions of the stresses in a cantilever wedge. The results show the capability of the method to capture the taper effects on the stresses and displacements. This work paves the way for a new advance cross-section analysis method based on one-layered tapered slice of the cross section.


Keywords: Finite element method, Tapered beams, Cross-section analysis, Solid elements

## 1. Introduction

Nonprismatic beams are widely applied in the civil and aeronautic sectors. For instance, wind turbine blades and wings are characterised by a tapered and twisted airfoil to increase the stiffness-to-mass ratio and the lift-to-drag ratio. 3D finite element analysis of nonprismatic structures would provide an accurate analysis of the beam behaviour. Nevertheless, the high computational cost of 3D finite element analysis does not suit with the required conceptual aeroelastic investigation nor with structural optimisation routines. Since blades are long and slender structures, they are modelled with
one-dimensional beam models, which provides results with the same accuracy of 3D finite element analysis, when the hypotheses are correctly formulated, in a much shorter time. To perform 1D beam analysis, the cross-section stiffness properties are necessary. Several cross-section analysis methods are available in the literature, as described in Hodges [1]. The most common methods are based on the finite element discretization of the cross-section. Among them it is worth recalling the Variational Asymptotic Beam Sectional Analysis (VABS) [2] and the BEam Cross-sectional Analysis Software (BECAS) [3]. The former is based on the theory developed by Cesnik and Hodges [2] and further developed by Yu et al. [4]. The latter is an implementation of the anisotropic beam theory derived by Giavotto et al. [5]. Both methods rely on a 2D finite element analysis of the cross-section, which has no restrictions in the geometrical nor material properties in the cross-section plane, but it is developed under prismatic hypothesis. In other words, lengthwise geometrical variations are simplified as stepwise prismatic and, consequently, the potential effects of geometrical variations are neglected.

Tapered beams are characterised by a variation of the cross-section dimensions along the longitudinal span. A parameter, taper, can be defined in a straight beam as the angle between the longitudinal axis and the plane tangent to the lateral surface [6]. Tapering a beam implies a counterintuitive stresses re-distribution under any external load. Indeed, the error committed in calculating the stresses in the cross section of a tapered beam by means of 2D cross-section analysis tools was reported by Bertolini et al. [7]. Particularly, the Cauchy stresses in several tapered beams evaluated with analytical solutions, when available, or 3D FEM are compared with the ones evaluated with the cross-section analysis tool BECAS. For example, it was pointed out that the current formulation of the cross-section analysis method cannot depict the through thickness stress components which are evoked in tapered beams. An incorrect prediction of the stress field may cause an inaccurate structural design. In addition, the scientific literature has several studies, from Bleich [8] to Balduzzi et al. [9] and Bennati et al. [10], where the nontrivial behaviour of tapered beams is described and analysed analytically.

An attempt to include taper in VABS is demonstrated by the study published by Hodges et al. [11], where a tapered beam model is derived by means of the variational asymptotic method. Nonetheless, further developments nor implementation are not documented in the literature.

Two scientific studies suggest to model the cross section as one-layered slice of solid elements in the longitudinal direction. Ghiringhelli and Man-
tegazza [12] extended the anisotropic beam theory developed by Giavotto et al. [5] to model a 3D slice of a linear, straight, and untwisted beam. Solid elements with linear shape-functions in the longitudinal direction were employed, but LGVs are neglected. Recently, Couturier and Krenk [13] studied the stiffness properties and stress distribution of a cross-section slice. One-single element with cubic shape-function in the longitudinal direction is employed. The Lagrange multipliers method is employed to impose the displacements at the two faces of the slice, representing the six independent deformations modes corresponding to extension, torsion, bending and shear.

The present work paves the way for a tapered slice cross-section analysis method based on solid finite elements. The boundaries conditions are applied via the Lagrange multipliers and three load conditions, axial, shear, and bending, are considered. The nodal forces are derived assuming the distribution of the internal tractions from the classic prismatic theories. Results show that the slice model can depict the taper effects to the stresses.

## 2. Method

A 3D finite element formulation of a tapered slice is derived and implemented in MATLAB to derive the cross-section stiffness properties and to recover the strain and stress components. The formulation is based on the coordinate system shown in Fig. 1. The strains and stresses at each point of


Figure 1: Geometry and coordinate system of a 2D planar wedge. The highlighted crosssection slice has thickness $\Delta$ and is located at $z=L / 2$.
the beam are expressed as six-terms vectors $\boldsymbol{\varepsilon}=\left[\varepsilon_{x x} \varepsilon_{y y} \varepsilon_{z z} 2 \varepsilon_{x z} 2 \varepsilon_{y z} 2 \varepsilon_{x y}\right]^{T}$ and $\boldsymbol{\sigma}=\left[\sigma_{x x} \sigma_{y y} \sigma_{z z} \sigma_{x z} \sigma_{y z} \sigma_{x y}\right]^{T}$. Under the assumption of linear elastic material, the constitutive relation between the strain and stress vectors is
given as $\boldsymbol{\varepsilon}=\boldsymbol{E} \boldsymbol{\sigma}$, where $E$ is the $6 \times 6$ constitutive matrix. The strain components at a generic point of the cross section are defined as function of the displacement vector $\boldsymbol{r}=\left[\begin{array}{lll}r_{x} & r_{y} & r_{z}\end{array}\right]^{T}$ through the relation $\hat{\boldsymbol{\varepsilon}}=\boldsymbol{B} \boldsymbol{r}$, where $\boldsymbol{B}$ is defined as follows

$$
\hat{\mathbf{B}}=\left[\begin{array}{cccccc}
\frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z}  \tag{1}\\
0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\
0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{array}\right]^{T} .
$$

The continuum structure is discretised into $n_{e}$ eight-noded elements and $n_{\text {nodes }}$ is total number of nodes. The nodal displacements are defined as $\boldsymbol{u}\left(x_{i}, y_{i}, z_{i}\right)=\left[u_{x i} u_{y i} u_{z i}\right]^{T}$, where $i=1 \ldots n_{\text {nodes }}$. The displacement vector $\boldsymbol{r}$ is approximated to $\boldsymbol{r} \approx \boldsymbol{N} \boldsymbol{u}\left(x_{i}, y_{i}, z_{i}\right)$, where $\boldsymbol{N}$ is the matrix of the shape-functions. Substituting the latter relation into Eq. (1), the following expression is obtained:

$$
\begin{equation*}
\boldsymbol{\varepsilon}=\boldsymbol{B} \boldsymbol{u}\left(x_{i}, y_{i}, z_{i}\right), \tag{2}
\end{equation*}
$$

where $\boldsymbol{B}=\hat{\boldsymbol{B}} \boldsymbol{N}$ the $6 \times 3$ strain-displacement matrix [14]. The variation of the virtual work used to enforce the elastic equilibrium of the cross section is

$$
\begin{equation*}
\delta W=\delta W_{\text {int }}+\delta W_{e x t} \tag{3}
\end{equation*}
$$

The internal virtual work $W_{\text {int }}$ is the work done by the stresses moving through the virtual strain and, based on the finite element discretization, becomes

$$
\begin{equation*}
\delta W_{i n t}=\int_{V} \delta \varepsilon^{T} \boldsymbol{\sigma} d V=\int_{V} \delta \varepsilon^{T} \boldsymbol{E} \boldsymbol{\varepsilon} d V=\int_{V} \delta \boldsymbol{u}^{T} \boldsymbol{B}^{T} \boldsymbol{E} \boldsymbol{B} \boldsymbol{u} d V \tag{4}
\end{equation*}
$$

where $V$ is the volume. Neglecting the volume and surface forces, the external work is instead defined from the shape-functions and the vector of the internal tractions $\boldsymbol{p}=\left[\sigma_{y y}, \sigma_{y z}, \sigma_{z z}\right]^{T}$ :

$$
\begin{equation*}
\delta W_{e x t}=-\int_{A} \delta \boldsymbol{r}^{T} \boldsymbol{p} d A=-\int_{A} \delta \boldsymbol{u} \boldsymbol{N}^{T} \boldsymbol{p} d A \tag{5}
\end{equation*}
$$

Equations (4) and (5) are substituted in Eq. (3). Since the virtual work must be satisfied for any virtual displacement $\delta \boldsymbol{u}$, a necessary and sufficient equilibrium conditions is that $\delta W=0$, hence

$$
\begin{equation*}
\int_{V} \delta \boldsymbol{u}^{T} \boldsymbol{B}^{T} \boldsymbol{E} \boldsymbol{B} \boldsymbol{u} d V=\int_{A} \delta \boldsymbol{u}^{T} \boldsymbol{N}^{T} \boldsymbol{p} d A \tag{6}
\end{equation*}
$$

The element stiffness matrix and the element nodal force vector $\boldsymbol{f}^{e}=$ $\left[\boldsymbol{f}_{x}, \boldsymbol{f}_{y}, \boldsymbol{f}_{z}\right]^{T}$ are expressed from Eq. (6) as

$$
\begin{equation*}
\boldsymbol{K}^{e}=\int_{V^{e}} \boldsymbol{B}^{T} \boldsymbol{E} \boldsymbol{B} d V, \quad \boldsymbol{f}^{e}=\int_{A^{e}} \boldsymbol{N}^{T} \boldsymbol{p} d A \tag{7}
\end{equation*}
$$

Exploiting a standard assembly procedure, the element matrix and vector are transformed in global stiffness matrix and nodal forces vector, $\boldsymbol{K}=\sum_{e} \boldsymbol{K}^{e}$, $\boldsymbol{f}=\sum_{e} \boldsymbol{f}^{e}$. Finally, the weak form of the equilibrium is given as

$$
\begin{equation*}
\boldsymbol{K} \boldsymbol{u}=\boldsymbol{f} \tag{8}
\end{equation*}
$$

where the stiffness matrix $\boldsymbol{K}$ relates the nodal forces $\boldsymbol{f}$ to the nodal displacements $\boldsymbol{u}$.

### 2.1. Boundary conditions

The stiffness matrix $\mathbf{K}$ is singular, hence not invertible. A set of six constraints related to the six rigid body motions, i.e. three rotations and three translations, are imposed. To avoid restraining the warping displacements, the Lagrange multipliers method is adopted. It imposes that the summation of the nodal displacements and the average rotation of the cross section are zero [5]:

$$
\begin{gather*}
\sum_{e=1}^{n e} u_{x \mid e}=0, \quad \sum_{e=1}^{n e} u_{y \mid e}=0, \quad \sum_{e=1}^{n e} u_{z \mid e}=0  \tag{9}\\
\sum_{e=1}^{n}-z_{e} u_{y \mid e}+y_{e} u_{z \mid e}=0, \quad \sum_{e=1}^{n} z_{e} u_{x \mid e}-x_{e} u_{z \mid e}=0, \quad \sum_{e=1}^{n}-y_{e} u_{x \mid e}+x_{e} u_{y \mid e}=0
\end{gather*}
$$

where $n$ is the total number of nodes, $\left(u_{x \mid e}, u_{y \mid e}, u_{z \mid e}\right)$ and $\left(x_{e}, y_{e}, z_{e}\right)$ are the nodal displacements and coordinates in global coordinates [3].

### 2.2. Nodal forces

The nodal forces vector $f$ is given by surface integration of the shapefunction matrix $\boldsymbol{N}$ and the cross-section tractions vector $\boldsymbol{p}$ :

$$
\begin{equation*}
\boldsymbol{f}_{i}=\int_{A} \boldsymbol{N}_{i}^{T}(\xi, \eta, \hat{\zeta}) \boldsymbol{p}(x, y, \hat{z}) d A \tag{10}
\end{equation*}
$$

where $A$ is the area of the slice face, $i=1 \ldots n_{e}$ is the element node, $\boldsymbol{p}(x, y, \hat{z})$ the surface tractions in the global coordinate system and $N_{i}(\xi, \eta, \hat{\zeta})$ the


Figure 2: Internal forces and corresponding traction acting the two faces of the slice having thickness $\Delta$. (a) Axial force and pressure, (b) shear force and shear traction, (c) bending moment and axial traction.
shape-functions evaluated at $\hat{\zeta}= \pm 1$, namely at the front and back faces of the isoparametric element. Generally, the cross-section forces are given from the aeroelastic analysis in case of blade design, whereas information on the cross-section tractions is unknown. Under the Navier assumptions, analytical studies on tapered beams $[6,15,16]$ have shown that the normal stresses in tapered beam can be approximated with the Navier formula with good agreement. Therefore, under the same hypotheses, the internal tractions in the present method are modelled from classic prismatic beam analysis. In other words, as shown in Fig. 2, the axial and bending tractions are assumed by the Navier equation [17]

$$
\begin{equation*}
\sigma_{z z}=\frac{F_{z}}{A}+\frac{M_{x} y}{I_{x}} \tag{11}
\end{equation*}
$$

and the shear tractions by the Jourawsky equation [17]

$$
\begin{equation*}
\sigma_{y z}=\frac{F_{y} S_{x}}{I_{x} B} \tag{12}
\end{equation*}
$$

where $A$ is the cross-section area, $S_{x}$ and $I_{x}$ are the first and second moment of area and $B$ the width of the cross-section.

A change of coordinates is required to solve Eq. (10), since the shapefunctions are in the isoparametric reference system and the area and internal traction in the global one. The transformation is done via the $3 \times 3$ Jacobian matrix $\boldsymbol{J}$, which maps from the element in the global reference system to the


Figure 3: (a) Coordinates of the isoparametric and (b) real tapered element of height $H$, width $B$ and taper angle $\alpha$.
element in the isoparametric reference system and vice versa. It is defined as

$$
\boldsymbol{J}=\left[\begin{array}{lll}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi}  \tag{13}\\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\
\frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}
\end{array}\right] .
$$

It is worth mentioning that the taper factor, $\alpha$, appears in the formulation through the Jacobian matrix. The infinitesimal value of the face surface becomes $d A=d x d y=|J| d \xi d \eta$, where $|J|$ is the determinant of the Jacobian. Moreover, the transformation between $y$ and $\eta$ is performed through the component $J(2,2)$ of the Jacobian matrix, and it can be written as

$$
\begin{equation*}
y=\int J(2,2) d \eta=\bar{F}(\xi, \eta)+c_{y} \tag{14}
\end{equation*}
$$

where $c_{y}$ is the constant of integration which is evaluated by imposing the equivalence of the coordinates of the centre of the elements in the natural and the real domains, i.e. $\left(\xi_{0}, \eta_{0}\right)$ and $\left(x_{0}, y_{0}\right)$, as shown in Fig. 3. In other words:

$$
\begin{equation*}
c_{y}=y_{0 e}-\bar{F}\left(\xi_{0}, \eta_{0}\right), \quad c_{x}=x_{0 e}-\bar{F}\left(\xi_{0}, \eta_{0}\right) \tag{15}
\end{equation*}
$$

The derivation of the nodal forces in the specific cases of axial, bending and shear are described in the following paragraphs.

Nodal forces equivalent to cross-section axial force. Based on Navier's assumption, the case of extension is described by a constant pressure $\hat{p}=F_{z} / A$
distributed on the two faces of the slice, as shown in Fig. 2-a. Therefore, in Eq. (10), the traction is moved out from the integral and the nodal forces are straightforwardly derived as

$$
\begin{equation*}
f_{i}=F_{z} \int_{-1}^{1} \int_{-1}^{1} \frac{N_{i}(\xi, \eta)}{A}|J| d \xi d \eta \tag{16}
\end{equation*}
$$

Nodal forces equivalent to cross-section bending moment. Bending is described from the Navier's linear stress distribution given by the second term of Eq. (11) and shown in Fig. 2-c. In this case, $\hat{p}(y)$ is a linear function of the global coordinate $y$. Consequently, recasting Eq. (14) in Eq. (11) and, consequently in Eq. (10), results in :

$$
\begin{equation*}
f_{i}=\frac{M_{x}}{I_{x}} \int_{-1}^{1} \int_{-1}^{1} N_{i}^{T}(\eta, \xi)\left[\bar{F}+\left(y_{0 e}-\bar{F}\left(\xi_{0}, \eta_{0}\right)\right)\right]|J| d \xi d \eta \tag{17}
\end{equation*}
$$

where $i$ is the number of nodes of the cross-section,
Nodal forces equivalent to cross-section pure shear. The pure shear traction is given from Eq. (12), as shown in Fig. 2-b. The second moment of inertia of the cross-section is function of the $y$-coordinate through the first moment of area and in the specific case of rectangular cross section, it is defined as:

$$
\begin{equation*}
S_{x}=\int_{-B / 2}^{B / 2} \int_{\tilde{y}}^{H / 2} y d y d x=B\left(\frac{H^{2}}{8}-\frac{y^{2}}{2}\right) . \tag{18}
\end{equation*}
$$

The same relation previously derived between $y$ and $\eta$ can be employed in this case and the final expression of the equivalent nodal forces becomes

$$
\begin{equation*}
f_{i}=\frac{F_{y}}{2 I_{x}} \int_{-1}^{1} \int_{-1}^{1} N_{i}^{T}(\eta, \xi)\left(\frac{H^{2}}{4}-\left[\bar{F}(\xi, \eta)+\left(y_{0 e}-\bar{F}\left(\xi_{0}, \eta_{0}\right)\right)\right]^{2}\right)|J| d \xi d \eta \tag{19}
\end{equation*}
$$

## 3. Application

In the scientific literature, the exact and approximate analytical solutions of the Cauchy stresses are available for the planar cantilever isotropic wedge of length $L$ and taper angle $\alpha$ shown in Fig. 1 and loaded at its tip by concentrated force $\mathbf{F}_{\mathbf{e}}$, whose components are shear $F_{y}$ and axial force $F_{z}$, and bending moment $\mathbf{M}_{\mathbf{x}}$. The exact solutions are derived from the elasticity

Table 1: Expressions of $g_{k}(\alpha)$ and $\hat{G}_{i j k}$ of the exact analytical equations in Eq. (20) of a wedge.

|  |  | Axial force |  | Bending moment |
| :---: | :---: | :---: | :---: | :---: |
| $\mathbf{g}_{\mathbf{k}}$ | $\sigma_{i j}(\alpha+\sin \alpha \cos \alpha)$ | $(2 \alpha \cos 2 \alpha-\sin 2 \alpha)$ | $(\alpha-\sin \alpha \cos \alpha)$ |  |
| $\hat{\mathbf{G}}_{\mathbf{i j k}}$ | $\sigma_{z z}$ | $\left(L_{0}-z\right)^{3}$ | $\frac{x y\left[3 x^{2}-y^{2}-\left(x^{2}+y^{2}\right) \cos 2 \alpha\right]}{\left[y^{2}+(L 0-z)^{2}\right]}$ | $y\left(L_{0}-z\right)^{2}$ |
|  | $\sigma_{y y}$ | $y^{2}\left(L_{0}-z\right)$ | $\frac{x^{4}+y^{4}-6 x^{2} y^{2}-\left(x^{4}-y^{4}\right) \cos 2 \alpha}{\left[y^{2}+(L 0-z)^{2}\right]}$ | $y^{3}$ |
|  | $\sigma_{y z}$ | $y\left(L_{0}-z\right)^{2}$ | $\frac{x y\left[x^{2}-3 y^{2}-\left(x^{2}+y^{2}\right) \cos 2 \alpha\right]}{\left[y^{2}+(L 0-z)^{2}\right]}$ | $y^{2}\left(L_{0}-z\right)$ |

theory, hence they satisfy the set of equilibrium, constitutive and compatibility equations $[18,16]$. The wedge is assumed in plane-stress state, namely the through thickness stress components are zero. The exact solutions of the Cauchy stresses are summarised in Eq. (20):

$$
\begin{equation*}
\sigma_{i j}=\frac{\hat{P}_{k}}{g_{k}(\alpha)} \frac{\hat{G}_{i j k}}{\left[y^{2}+(L-z)^{2}\right]^{2}}, \tag{20}
\end{equation*}
$$

where $i, j=y, z$, and $f_{k}(\alpha)$ and $\hat{P}_{k}$ are defined in Table 1. The details of the derivation of Eq. (20) are described in Taglialegne [16]. A more handy solution for the Cauchy stresses in a planar wedge was derived in Bennati et al. [10] approximating the normal stresses with the Navier equation Eq. (11). Thereafter, imposing the stress equilibrium on an infinitesimal portion $d z$ of a tapered beam with generic cross-section provides a generalisation of the classic shear stress formula Eq. (12). The generalised shear stress equation neglects high-order infinitesimal quantities, therefore the solution is approximate. The set of approximate equations for the Cauchy stress components is given in Eq. (21)-(22)

$$
\begin{align*}
\sigma_{z y} & =\frac{1}{B}\left(-\frac{y \tan \alpha}{2 h^{2}} F_{z}+\frac{3}{4} \frac{h^{2}-y^{2}}{h^{3}} F_{y}+\frac{3}{4} \frac{h^{2}-3 y^{2}}{h^{4}} \tan \alpha M_{x}\right),  \tag{21}\\
\sigma_{y y} & =\frac{3 y}{2 B}\left[\frac{y \tan \alpha^{2}}{3 h^{3}} F_{z}+\frac{\left(y^{2}-h^{2}\right) \tan \alpha}{h^{4}} F_{y}+\frac{\left(2 y^{2}-h^{2}\right) \tan \alpha^{2}}{h^{5}} M_{x}\right] . \tag{22}
\end{align*}
$$

Equations (20) to (22) will be used to benchmark the stresses from the tapered slice method of a wedge cross section.

### 3.1. Geometry description

The present method was applied to a 2D wedge clamped at its root. Specifically, the control cross-section located at $z=L / 2$ was chosen for validation purposes. The axial and shear forces, and bending moment listed in Table 2 were applied at the tip. Linear elastic material properties were assigned to the models with an elastic modulus of $E=100 \mathrm{~Pa}$ and Poisson ratio of $\nu=0.3$. The dimensions of the wedge were chosen in such a way that the width and the height of the middle cross-section are $B=0.10 \mathrm{~m}$ and $H_{0}=2.00 \mathrm{~m}$ respectively, whereas the taper angle $\alpha$ and the length of the wedge can vary as shown in Table 3.

Table 2: Axial force, shear force and bending moment applied at the tip of the wedge and respective internal forces at the middle cross-section.

|  | Tip loads |  |  |  |  | Mid-span c.s. |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $F_{z}[\mathrm{~N}]$ | $F_{y}[\mathrm{~N}]$ | $M_{x}[\mathrm{~N} \mathrm{~m}]$ |  | $F_{z}[\mathrm{~N}]$ | $F_{y}[\mathrm{~N}]$ | $M_{x}[\mathrm{~N} \mathrm{~m}]$ |  |
|  | Shear | 10 | - | - |  |  | 10 | - |

Table 3: Geometrical dimensions of the analysed wedges.

| $\alpha\left[^{\circ}\right]$ | 2.0 | 5.7 | 10.0 | 20.0 |
| :---: | :---: | :---: | :---: | :---: |
| $\mathrm{~L}[\mathrm{~m}]$ | 57.27 | 20.04 | 11.34 | 5.49 |

In the same way, the numerical model is defined from the control crosssection, considering the slice of thickness $\Delta$ highlighted in Fig. 1. The finite element model of the slice consists of one layer of solid elements in the $z$-direction and the $x$-direction, whereas 30 elements are employed in the $y$-direction. A convergence study was performed to establish the optimal number of elements along the height of the slice. The exemplary topology of the slice is shown in Fig. 4. Eight-noded elements with the linear serendipity shape-functions in Eq. (23) are employed.

$$
\begin{equation*}
N_{n}=\frac{1}{8}\left(1+\xi_{n} \xi\right)\left(1+\eta_{n} \eta\right)\left(1+\zeta_{n} \zeta\right) \tag{23}
\end{equation*}
$$



Figure 4: Model of an exemplary slice of a wedge. The coordinate system $O x y z$, the dimensions and the mesh discretisation are highlighted. Moreover, the examples of an eight-noded linear solid element and a twenty-noded quadratic solid element are given on the right. The red dot show the location at which the average displacement, strain, and stress values are considered.
where $(\xi \eta \zeta)$ are the isoparametric coordinates and $i$ refers to the element nodes. The integrals in the finite element analysis are solved numerically with the $2 \times 2$ Gauss quadrature [19], which is characterised by eight Gauss points located at $( \pm 1 / \sqrt{3}, \pm 1 / \sqrt{3}, \pm 1 / \sqrt{3})$.

## 4. Results

The Cauchy stresses derived from the finite element slice analysis are compared to the analytical ones. After validating the method for a prismatic slice, the tapered case in Fig. 4 is considered. The numerical and analytical results are extrapolated at the control-section in order to avoid boundary effects. The finite element model evaluates the strain and stress solutions with the Gauss quadrature, hence at the Gauss points. Therefore, the element stress at the centre is evaluated by averaging the nodal stresses located at the back and front face of the element, as indicated by the red dot in Fig. 4.

### 4.1. Prismatic slice

A planar prismatic beam of dimensions $H=2 \mathrm{~m}$ and $B=0.1 \mathrm{~m}$ is clamped at $z=0$ and loaded at the tip with the forces in Table 2. For the prismatic case, also the twenty-noded element with quadratic serendipity shape functions [19] is implemented. The Cauchy stresses at the control
cross section are evaluated with both the slice method and the classic solid mechanic, Eq. (11) and (21) where $\alpha=0$. The slice thickness is arbitrarily chosen as $\Delta=0.01 \mathrm{~m}$ and its value does not interfere with the prismatic results. Figure 5 shows the stress distributions along half of the slice height due respectively to the three analysed external forces.


Figure 5: Normal stress due to (a) extension, (b) bending, and (c) shear stress due to shear along half of the prismatic slice. Different mesh-topology and both eight-noded (L) and twenty-noded elements $(Q)$ are considered. The number of elements used in each model is indicated in the Legend after \#.

### 4.2. Tapered slice

The effects of the amplitude of the taper angle are evaluated, but only the results for the axial force are reported for the sake of brevity. Four different wedges, $\alpha=2^{\circ}, 5.7^{\circ}, 10^{\circ}, 20^{\circ}$, are considered. Figure 6 shows the ratio between the numerical and the exact analytical stresses for the four analysed wedges. The stresses at the centre and the edge of the slice are considered for comparison.

Nonetheless, in the following study, the taper angle is fixed to $\alpha=5.7^{\circ}$ and only the slice thickness is allowed to vary in the range $\Delta \in[0.01 \mathrm{~m}$, 0.5 m ].

### 4.2.1. Extension of the slice

The slice in Fig. 2-a subjected to a constant distributed traction $\hat{p}=$ $F_{z} / A$ is considered. The stress components resulting from the finite element


Figure 6: Numerical-to-analytical stresses at the control cross-section of four wedges subjected to extension. The ratio involves the stresses at the centre $y=0$ and the edge $y=H / 2$ of the central slice.
analysis of the tapered slice having $\Delta=0.03,0.09,0.5 \mathrm{~m}$, and from both the exact and approximate analytical formulas are evaluated and reported in Fig. 7. To investigate the relation between the magnitude of the stresses and the


Figure 7: Cauchy stress components, namely (a) $\sigma_{y y}$, (b) $\sigma_{z z}$, and (c) $\sigma_{y z}$, in a tapered slice subjected to axial traction. Results from analytical and numerical solutions. The latter considers three different thickness of the slice.
slice thickness, the parametric study in Fig. 8 is conducted. It compares the ratio between the numerical and analytical stresses evaluated at the centre $y=0$ and at the edge $y=H / 2$ of the slices.

### 4.2.2. Bending of the slice

Bending follows the Navier distribution $\hat{p}=\left(M_{x} y\right) / I_{x}$ and it is applied at both faces of the slice as shown in Fig. 2-c, considering several slice thicknesses. The numerical stress components are compared to the analytical solutions in Fig. 9. It is worth noting that, contrarily to the axial case, the deviation of the results in the linear models increases with the thickness of the slice.

### 4.2.3. Shear of the slice

The wedge is now subjected to an external shear force, which results in cross-section shear force and cross-section bending moment around the $x$-axis, as shown in Table 2. Therefore, given the linearity of the problem, the internal tractions are derived from superposition of the pure shear


Figure 8: Case of extension. Cauchy stresses at the centre and the edge of slices with different thicknesss $\Delta$. Specifically (a) $\sigma_{y y}$, (b) $\sigma_{z z}$, (b) $\sigma_{y z}$.


Figure 9: Cauchy stress components, namely (a) $\sigma_{y y}$, (b) $\sigma_{z z}$, and (c) $\sigma_{y z}$, in a tapered slice subjected to bending traction. Results from analytical and numerical solutions. The latter considers three different thickness of the slice, $\Delta=0.03,0.09,0.5 \mathrm{~m}$.


Figure 10: Case of bending. Cauchy stresses at the centre and the edge of slices with different thicknesss $\Delta$. Specifically, (a) $\sigma_{y y}$, (b) $\sigma_{z z}$, (b) $\sigma_{y z}$.
traction in Fig. 2-b and pure bending traction from Fig. 2-c. Figure 11 shows the stresses occurring in a tapered slice under the effects of shear and shear-bending.


Figure 11: Cauchy stress components, namely (a) $\sigma_{y y}$, (b) $\sigma_{z z}$, and (c) $\sigma_{y z}$, in a tapered slice subjected to shear traction. Results from analytical and numerical solutions. The latter considers three different thickness of the slice, $\Delta=0.03,0.09,0.5 \mathrm{~m}$.

## 5. Discussion

A new tapered slice finite element model is presented for analysis of crosssection properties. One single layer of solid element is employed in the longitudinal direction. The rigid body constraints are imposed by means of the Lagrange multipliers and the nodal forces derived from the assumption that the internal tractions follow the prismatic stress distribution. A convergence study and validation is performed considering a prismatic slice. Figure 5 shows a good agreement between the results for the axial and shear cases. The eight-noded model under bending is at strong deviation as shown in Fig. 5 -b, because the eight-noded elements are based on linear shape-functions and suffer of shear-locking. On the other hand, the model based on the twenty-noded element shows a good agreement also under bending. The solution is strongly affected by two main factors, i.e. the taper angle and the slice thickness. The results of the prismatic slice reveal that the slice thickness does not affect the prismatic model. Figure 6 shows that the deviation
between the numerical and analytical stresses increases with larger taper angles. Indeed, it is worth recalling that the internal tractions are defined from the Navier formula, which gives a good approximation of the stresses only in moderately taper angles, as it was shown in the literature [20]. On the other hand, it is worth noting that by changing the slice thickness, the deviation can be contained.

Figures 7, 9, 11, shows the capability of the method to depict the correct stresses distribution. It is worth noting in Fig. 7-b and in Fig. 9-b that the normal stresses depicted by the slice analysis follow the quadratic distribution corresponding to the exact analytical solutions. Hence, it deviates from the constant Navier solution, which is predicted by the approximate analytical solution. In other words, even though the internal tractions are approximate, the finite element formulation can depict taper effects. Moreover, in the model based on the eight-noded elements the shear locking trend, which was observed in the prismatic model, does not appear in the slice model. Therefore, the twenty-noded model was not considered for computational time efficiency.

Yet, the magnitude of the stresses is at variance with the expected exact values. The parameter $\Delta$, namely the thickness of the slice, is responsible for the deviation. Figures 8 and 10 show a parametric study between the numerical and analytical stresses for a slice thickness varying from $0-0.5 \mathrm{~m}$. It is worth noting that the cases of axial and shear force converge for larger slice thicknesses. Contrarily, the bending problem approaches the analytical values for smaller thicknesses.

## 6. Conclusions

This work suggests to perform cross-sections analysis from its 3D finite element model. The cross-section is represented as a tapered slice with a single layer of elements in the longitudinal direction. The internal tractions are assumed to follow the prismatic distributions. The following conclusions can be drawn from the present investigation, which is limited to a planar isotropic wedge:
(i) Finite element analysis of a tapered slice can depict the taper effects to the Cauchy stress components.
(ii) Modelling the internal tractions under the Navier hypothesis results in a good approximation of the stress components allows to predict taper
effects in the stress analysis, for moderate taper angles.
(iii) The thickness of the modelled slice has a strong effect on the numerical results. In case of axial and shear internal force the accuracy increases for thicker slices, however it is vice versa in the bending load case.

The present work is a first attempt to introduce taper in cross-section analysis and sheds light on the capability of 3D tapered slice models. Further benchmark examples whose analytical solutions are available should be analysed, e.g. thin-walled tapered box girders. In addition, the remaining internal forces, $F_{x}, M_{y}, M_{x}$ can be implemented. Furthermore, higher order elements could be used to model the effects of curvature.

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## References

[1] D. H. Hodges, Nonlinear composite beam theory, American Institute of Aeronautics and Astronautics, 2006.
[2] C. E. Cesnik, D. H. Hodges, Vabs: a new concept for composite rotor blade cross-sectional modeling, Journal of the American helicopter society 42 (1) (1997) 27-38.
[3] J. P. Blasques, User's manual for BECAS - a cross section analysis tool for anisotropic and inhomogeneous beam sections of arbitrary geometry, Tech. rep., DTU Wind Energy, Technical University of Denmark, Roskilde (2012).
[4] W. Yu, D. H. Hodges, J. C. Ho, Variational asymptotic beam sectional analysis-an updated version, International Journal of Engineering Science 59 (2012) 40-64.
[5] V. Giavotto, M. Borri, P. Mantegazza, G. L. Ghiringhelli, V. Carmaschi, G. Maffioli, F. Mussi, Anisotropic beam theory and applications, Computers \& Structures 16 (1-4) (1983) 403-413.
[6] P. Bertolini, M. A. Eder, L. Taglialegne, P. S. Valvo, Stresses in constant tapered beams with thin-walled rectangular and circular cross sections, Thin-Walled Structures 137 (2019).
[7] P. Bertolini, A. Sarhadi, M. Stolpe, M. A. Eder, Comparison of stress distributions between numerical cross-section analysis and 3d analysis of tapered beams, ICCM22 2019 (2019) 539-550.
[8] F. Bleich, Stahlhochbauten, Vol. 1, Springer, Berlin, 1932.
[9] G. Balduzzi, M. Aminbaghai, E. Sacco, J. Füssl, J. Eberhardsteiner, F. Auricchio, Non-prismatic beams: a simple and effective timoshenkolike model, International Journal of Solids and Structures 90 (2016) 236-250.
[10] S. Bennati, P. Bertolini, L. Taglialegne, P. S. Valvo, On shear stresses in tapered beams, in: Proceedings of the GIMC-GMA 2016-21st Italian Conference on Computational Mechanics and $8^{\text {th }}$ Meeting of the AIMETA Materials Group, Lucca, 2016, pp. 83-84.
[11] D. H. Hodges, J. C. Ho, W. Yu, The effect of taper on section constants for in-plane deformation of an isotropic strip, Journal of Mechanics of Materials and Structures 3 (3) (2008) 425-440.
[12] G. L. Ghiringhelli, P. Mantegazza, Linear, straight and untwisted anisotropic beam section properties from solid finite elements, Composites Engineering 4 (12) (1994) 1225-1239.
[13] P. J. Couturier, S. Krenk, Wind turbine cross-sectional stiffness analysis using internally layered solid elements, AIAA Journal 54 (7) (2016) 2149-2159.
[14] R. D. Cook, et al., Concepts and applications of finite element analysis, John Wiley \& sons, 2007.
[15] P. Bertolini, L. Taglialegne, Analytical solution of the stresses in doubly tapered box girders, European Journal of Mechanics-A/Solids 81 (2020) 103969.
[16] L. Taglialegne, Stress fields in wind turbine blades with thin-walled variable cross sections, Ph.D. thesis, International Doctorate "Civil and Environmental Engineering", Universities of Florence, Perugia and Pisa TU C.W. Braunschweig (2018).
[17] S. Timoshenko, J. N. Goodier, Theory of Elasticity, McGraw-Hill book Company, 1951.
[18] P. Bertolini, Structural analysis of wind turbine blades: a study of the effects of tapering on shear stresses, Master's thesis, University of Pisa (2016).
[19] G. Dhondt, Method for Three-dimensional Method for Threedimensional Thermomechanical Applications, John Wiley \& Sons, Ltd, 2004.
[20] B. A. Boley, On the Accuracy of the Bernoulli-Euler Theory for Beams of Variable Section (1963).


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