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Closed form second order energy release rate in a cracked elastic beam

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Abstract

This study presents a closed form, second order, Mode-I fracture mechanics solution for a cracked elastic beam subject to axial compressive load. The solution considers a crack with a symmetric parabolic geometric imperfection in the beam and provides the strain energy release rate (SERR) as a continuous function of the axial load level. The effect of the elastic interface is considered by means of an additional equivalent beam length determined from a Winkler beam model. The closed form solution is corroborated through numerical non-linear fracture analyses for different geometries with isotropic and transverse isotropic material properties. The results show that the analytical models provide accurate predictions with an accuracy ranging between 2% and 4%. It is analytically adduced that the second order SERR in a cracked beam subject to axial loading is stable inasmuch the SERR decreases with increasing crack length. Moreover, the results show a significant influence of the elastic interface on the SERR.

Keywords: closed form solution; second order effect; parabolic imperfection; compressive load; accurate; strain energy release rate

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>SI Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_0(x) )</td>
<td>[-]</td>
<td>Auxiliary function of the general solution</td>
</tr>
<tr>
<td>( b_1(x) )</td>
<td>[m]</td>
<td>Auxiliary function of the general solution</td>
</tr>
<tr>
<td>( b_2(x) )</td>
<td>[m²]</td>
<td>Auxiliary function of the general solution</td>
</tr>
<tr>
<td>( h )</td>
<td>[m]</td>
<td>Constant cross-section height</td>
</tr>
<tr>
<td>( i, j, k )</td>
<td>[-]</td>
<td>Indices ( \in \mathbb{N} ); or elastic embedment modulus ( (k) )</td>
</tr>
<tr>
<td>( l_i )</td>
<td>[m]</td>
<td>Length parameter; subscript refers to member number</td>
</tr>
<tr>
<td>( q )</td>
<td>[N/m]</td>
<td>Uniformly distributed load</td>
</tr>
<tr>
<td>( w, w_0 )</td>
<td>[m]</td>
<td>Beam displacement; imperfection amplitude</td>
</tr>
<tr>
<td>( w_{imp} )</td>
<td>[m]</td>
<td>Imperfection distribution i.e. geometrical deviation of the straight beam axis</td>
</tr>
<tr>
<td>( x )</td>
<td>[m]</td>
<td>Local axial beam coordinate in the deformed configuration</td>
</tr>
<tr>
<td>( B )</td>
<td>[m]</td>
<td>Constant cross-section width</td>
</tr>
<tr>
<td>( C_{1,2,3,4} )</td>
<td>[m]</td>
<td>Four constants of Winkler beam ODE</td>
</tr>
<tr>
<td>( E, E_1, E_2 )</td>
<td>[Pa]</td>
<td>Elastic modulus (isotropic), in fibre direction and transverse fibre direction.</td>
</tr>
<tr>
<td>( F_x )</td>
<td>[N]</td>
<td>External axial load</td>
</tr>
<tr>
<td>( G_{AB} )</td>
<td>[J/m²]</td>
<td>Strain energy release rate in crack tips A and B</td>
</tr>
<tr>
<td>( I )</td>
<td>[m⁴]</td>
<td>Moment of inertia or second moment of area</td>
</tr>
<tr>
<td>( L )</td>
<td>[m]</td>
<td>Total length of the elastic beam</td>
</tr>
<tr>
<td>( N_{crit} )</td>
<td>[N]</td>
<td>Critical buckling load of a single beam member</td>
</tr>
<tr>
<td>( Q )</td>
<td>[N]</td>
<td>Shear force in the local (deformed) beam CSYS</td>
</tr>
<tr>
<td>( R )</td>
<td>[N]</td>
<td>Transverse force in the global (undeformed) beam CSYS</td>
</tr>
<tr>
<td>( U_M )</td>
<td>[J]</td>
<td>Strain energy, induced by bending moment, induced by axial force</td>
</tr>
</tbody>
</table>
1. Introduction

The effects of buckling induced delamination of thin films on substrates or of laminates in composite materials as well as adhesive joints is an important fracture mechanics problem in engineering practice. Early works in this field were conducted by Sallam and Simitses [1, 2] and Williams [3] who investigated the SERR of cracked composite beams and cylindrical shells at the instance of instability failure. To this end, first, the homogeneous solution of the governing ordinary differential equation (ODE) formulated in the deformed reference frame was solved. Second, the particular solution of the ODE was obtained for the prevailing load case. With the knowledge of the general displacement function (pertaining to the first Eigenmode), the section forces are readily obtained by utilising the well-known beam kinematics. Sallam and Simitses [1, 2] used the J-integral while Williams [3] adopted the variational approach to eventually obtain the SERR at the instant of instability failure. Thouless [4] investigated the driving force for 2D thin film delamination cracks and found that cracking of the buckled film enables crack tunnelling in compressively stressed films. Sleight and Wang [5] studied the buckling driven through-width debonding of sandwich face skins under compressive loading. Their model is based on an elastically embedded beam in which the debonding region is simulated by setting the embedment modulus to zero. The displacement function is approximated with the Rayleigh-Ritz method as a sum of sines. The critical buckling load is obtained by solving the homogeneous ODE by means of the finite difference method for various debonded lengths and sandwich foam stiffness configurations. Fraternali [6] modelled the compressive load induced delamination of layered elastic orthotropic composite beams. He adopted laminate theory where each laminae is modelled as a separate beam with an elastic foundation using lateral and transverse springs. A penalty parameter affecting the embedment modulus is used to model the delamination. The non-linear problem is solved by finite element approximation using an arc-length solution strategy. The SERR is obtained by the variation of the total potential.

Cotterell and Chen [7] modelled the buckling delamination of a thin film plate under consideration of the substrate deformation. First the compliances of the delaminated and buckled portion of the thin film are obtained and compared against finite element (FE) simulations. Previously derived solutions for the Mode-I/II stress intensity factor are used as a function of the Dundurs parameters. The model follows the work of [4] but assumes that the buckling load is transferred to the substrate via the bottom edge rather than the neutral axis. This work was later extended by Yu and Hutchinson [8] by studying a straight-sided thin film delamination buckle under compressive loading. Their model predicted the average SERR at the curved end of a tunneling blister crack. Kinawi and Butler [9] derived an analytical model for a thin sub-laminate perfectly bonded onto a thick substrate subject to pure bending. A displacement solution is assumed as a function of the unknown delamination beam end-rotations and the delamination buckle displacement at peak. An sine-square imperfection function is defined according to the first term of the general displacement function. The problem is condensed into a set of two differential equations, which are solved for the unknown rotations and displacements. Subsequently, the displacement and cross section forces are used to compute the total potential whose variation renders the SERR of their mixed-mode fracture problem.

Chattopadhyay and Jain [10] modelled the linear buckling and post-buckling behaviour of a delamination located in an axially loaded isotropic beam. First the prevailing 4th order ODE according to equilibrium in the deformed configuration is solved using a general ansatz. The integration constants were obtained from boundary conditions assuming rotationally restrained beam ends. The critical buckling load was obtained by solving the homogeneous ODE. The J-integral method was used to obtain the delamination buckling...
induced SERR. Naghinejad and Ovsey [11] used a variational energy approach to obtain the energy release rate in a through-with delamination crack caused by a buckled delamination. Their work is an extension of previous approaches using higher order shear deformation theory. A general trigonometric displacement function is assumed. An eigenvalue problem is used to obtain the critical buckling load by utilization of compatibility and constitutive equations. The authors validate their solution using the FE-method and state that the solution is lengthy. Yazdani et al. [12] developed a semi-analytical strain energy based method to model cracked media. In this method, only the boundaries of problems are discretized using specific higher-order subparametric elements and higher-order Chebyshev mapping functions.

It can be asserted from the literature [1-12] that second order effects and the consideration of large deformations in fracture mechanics to date lead to the discovery of new interesting effects such as the effect of the delamination depth and transverse shear on the SERR, the crack tunnelling behaviour of thin films as a function of the buckling mode, the stability of buckling driven crack growth, and the buckling load of delamination cracking depending on the substrate stiffness, to name a few. The investigation of these effects by means of analytical models provides a valuable approach to gain insights into the fracture behaviour complimentary to numerical simulations.

In this work a closed form second order solution for the SERR in a cracked elastic transverse isotropic beam under compressive load is presented. The model is developed for compressive load levels way below the critical buckling load as they typically occur in high cycle fatigue situations. The model considers the effects of a symmetric parabolic imperfection in the beam axes as well as the effects of the elastic interface along the bonded regions of the model. A variational energy approach is used to derive the SERR. While the formulation of the problem follows along the lines of other available problems in the literature, the incentive of this work was to provide a concise closed form solution for this second order fracture problem by avoiding numerical solution schemes. The analytical solution was greatly simplified by converting the elastically embedded parts of the model (c.f. Section 3) into an equivalent free beam length with an equivalent stiffness. In this way the coupled problem could be described with one single ODE considering equilibrium in the deformed configuration.

The analytical model presented in this study opens manifold application opportunities in different engineering disciplines, e.g., adhesive joint cracks in the trailing edge bond line of wind turbine blades [13, 14, 15, 16], see supplementary materials, pre-cracked superconductors [17] or concrete beams [18]. Analytical closed form solutions, such as the one presented in this work, are particularly useful for continuous optimization problems in which analytical sensitivities (gradients) of the model parameters are essential for accuracy and computational efficiency.

2. Fundamentals

2.1 Governing differential equation

In this work, the second order theory refers to planar beam bending problems where equilibrium is satisfied in the deformed configuration [19, 20, 21]. The governing ODE is hence, obtained by consideration of equilibrium on an infinitesimal beam element in its deformed state as depicted in Fig.1
Figure 1 Equilibrium - in the deformed state - of infinitesimal Euler-Bernoulli beam element of length $dx$ subject to a distributed load with internal (in-plane) section forces on the linearised displacement configuration $w(x)$. It is assumed, that the deformation magnitude is small and that the axial force $N = c_{st}$ does not vary along the beam length. It is noteworthy that the transverse force denoted as $R$ is different to the shear force $Q$ as it acts in the undeformed CSYS.

In a geometric non-linear formulation, the change of the infinitesimal beam length component $dx$ defined in the undeformed configuration (c.f. Fig. 1) due to: (i) the extension/shortening under the axial force and (ii) the shortening due to the horizontal displacement component needs to be considered. That is to say, neither the beam length nor its components are preserved during deformation. While geometrically nonlinear problem formulations are rigorous, their solution usually requires non-linear solution strategies. In order to obtain an analytical solution, the displacement $w(x)$ is linearised inasmuch the length of the infinitesimal beam element $dx$ is assumed to remain unchanged under deformation, such that the effect of the change in length of the beam axis in the deformed configuration is neglected (c.f. Fig. 1). In other words, in this work equilibrium in the deformed configuration is indeed considered albeit, with the assumption of (i) an infinite extensional stiffness $E_A = \infty$ and (ii) small bending displacements and small cross section rotations. The implications become more apparent when inspecting the second order relation between the transverse force in the undeformed configuration $R$ and the shear force $Q$ in the deformed configuration given as $Q = R - N \left( \frac{dw}{dx} + \frac{dw_{imp}}{dx} \right)$. The latter expression shows that due to the assumed small cross-section rotations $N$ can be decoupled from $w$. These simplifications distinguish this second order formulation from the general large deformation theory with the advantage of offering a closed form solution, which would otherwise be cumbersome to obtain.

It is noteworthy to mention that in the expression for $Q$ mentioned above, $w_{imp}$ refers to an imperfection stipulated in this work as the geometrical deviation from the ideally straight beam axis in the unloaded condition. This imperfection pertains to the beam axis geometry and should not be confused with a flaw or the crack itself.

The equation of bending moment equilibrium in the deformed configuration after omitting second order derivatives (i.e. linearisation as mentioned above) can be written as follows:

$$dM - R dx + N(dw + dw_{imp}) = 0$$

(1)

It is now possible to specialise Eq.1 by assigning a symmetric (quadratic) parabolic imperfection $w_{imp}$ to the beam axis with an imperfection amplitude defined as $w_0$. Ignoring shear deformation, the governing ODE for second order bending derived from Eq. 1 can now be specialised into the following form:

$$\frac{d^2M}{dx^2} - \frac{N}{EI} M = \frac{8Nw_0}{l^2} = q$$

(2)
The expression $8Nw_0/l^2$ derives from the thrust-line for a uniformly distributed load which is the locus of all geometrical points at which the internal bending moment vanishes for a given load. The assumed parabolic imperfection constitutes the thrust-line for a uniformly distributed load; thus, the effect of the axial force $N$ on a parabolic imperfection is equivalent to the effect of a uniformly distributed transverse load $q$. For the analytical model presented in Section 3, the general solution framework for Eq. 2 established by Rubin [20] is adopted to obtain the homogeneous and particular solution for the bending moment distributions which is given as follows:

$$M = c_0b_0 + c_1b_1 + \frac{8Nw_0}{l^2}b_2$$

(3)

$$b_0(x) = \cos fx$$

(4)

$$b_1(x) = \frac{\sin fx}{f}$$

(5)

$$b_2(x) = \frac{b_0(x)-1}{K}$$

(6)

where $c_0$ and $c_1$ are integration constants to be determined by the boundary conditions and $K = -\frac{N}{EI}$ with $f = \sqrt{|K|}$.

2.2 Assumptions

The following assumptions have been made for the presented model:

- linear elastic material properties
- Euler-Bernoulli beam theory applies i.e. $h \ll l$
- body forces are not considered
- the influence of shear force on the SERR is neglected
- the axial force $N = \text{cst.}$ is not a function of the deformation $w$
- the axial stiffness is infinite $EA = \infty$ i.e. no shortening due to axial load

3. A cracked linear elastic beam model

3.1 Model description

Figure 2 (a) depicts the geometry of a slender elastic beam comprising of a lengthwise crack in the centre. The crack faces feature a second order parabolic imperfection of amplitude $w_0$ on both sides along the length $l_2$. A sharp crack with inverse square root singularity (crack tips located in points $A$ and $B$) is present in both extensions along the length $l_1$. The beam is fully fixed on the left hand side (LHS) beam end where rotations and lateral displacements are restrained on the right hand side (RHS) end. The beam is loaded with a concentrated compressive axial force $F_x$ applied on the RHS beam end. Figure 2 (b) shows that the problem can be modelled as a pair of beams with elastic interfaces along the bonded region hatched in grey. The embedment modulus $k$ corresponds to the elastic modulus in the transverse direction of the beam. In this section a closed form solution for the the SERRs at crack tips $A$ and $B$ as a function of the axial load level is derived.
3.2 Equivalent beam length along the elastic interface

Figure 2 (b) shows that the problem constitutes two different coupled ODEs, one describing the embedded beam part and the other one describing the second order free beam with the parabolic imperfection. In order to avoid impractically lengthy solutions the problem is decoupled by computing a free beam length $l_c$ for the embedded part along the elastic interface. The rotational stiffness of this free beam length is equivalent to the rotational stiffness of the elastically embedded part. Consequently, an effective beam length can be considered using $l_{1,\text{eff}} = l_1 + l_c$ which allows the formulation of the problem as an equivalent beam model that can be described with a single ODE. Figure 2 (b) shows that a unit bending moment $M_0$ is applied to the tip of the embedded beam to compute the corresponding cross section rotation $\theta_0$ in the same point. In order to compute $l_c$, the fourth order ODE defining the elastically embedded Euler-Bernoulli beam region ($q = 0$) given by Eq. 7 needs to be solved.

$$EI \frac{d^4 w}{dx^4} + kw = 0 \tag{7}$$

An auxiliary constant material parameter is defined as $\kappa = \left(\frac{E_{22}}{4E_{11}}\right)^{\frac{1}{2}}$ where $E_{11}$ and $E_{22}$ represent the elastic modulus in the beam direction and the transverse direction respectively. Subsequently, the general solution of Eq. 7 [22] can be written as follows:
$$w(x) = e^{\kappa x} (C_1 \sin \kappa x + C_2 \cos \kappa x) + e^{-\kappa x} (C_3 \sin \kappa x + C_4 \cos \kappa x)$$  \hspace{1cm} (8)$$

Utilizing the well-known moment flexure and shear force curvature kinematics for an Euler-Bernoulli beam, the bending moment distribution as well as the shear force distribution in the embedded beam is given below:

$$M(x) = -2EI\kappa^2 e^{\kappa x} \cos \kappa x C_1 + 2EI\kappa^2 e^{\kappa x} \sin \kappa x C_2 + 2EI\kappa^2 e^{-\kappa x} \cos \kappa x C_3 - 2EI\kappa^2 e^{-\kappa x} \sin \kappa x C_4$$  \hspace{1cm} (9)

$$Q(x) = 2EI\kappa^3 e^{\kappa x} (\sin \kappa x - \cos \kappa x) C_1 + 2EI\kappa^3 e^{\kappa x} (\sin \kappa x + \cos \kappa x) C_2 - 2EI\kappa^3 e^{-\kappa x} (\sin \kappa x + \cos \kappa x) C_3 + 2EI\kappa^3 e^{-\kappa x} (\sin \kappa x - \cos \kappa x) C_4$$  \hspace{1cm} (10)

The four constants can be obtained from the von Neumann boundary conditions $M(x = 0) = M_0$, $Q(x = 0) = 0$, $M(x = l_e) = 0$ and $Q(x = l_e) = 0$. A specialisation of Eq. 9 and 10 for the boundary conditions renders a system of four linearly independent equations which can be written as $A \{C_1, C_2, C_3, C_4\}^T = \{M_0, 0, 0, 0\}^T$. The latter can be solved directly which leads to the following coefficient vector:

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} M_0(2\cos(\kappa l_e)^2-2\cos \kappa L \sin \kappa L + e^{-2\kappa l_e-3}) \\ 4EI\kappa^2(2\cos(\kappa L)^2+\cosh 2\kappa L -3) \\ M_0(2\cos(\kappa L)^2+2\cos \kappa L \sin \kappa L -e^{-2\kappa l_e-1}) \\ 4EI\kappa^2(2\cos(\kappa L)^2+\cosh 2\kappa L -3) \\ M_0(2\cos(\kappa L)^2+2\cos \kappa L \sin \kappa L +e^{2\kappa l_e-3}) \\ 4EI\kappa^2(2\cos(\kappa L)^2+\cosh 2\kappa L -3) \\ M_0(2\cos(\kappa L)^2-2\cos \kappa L \sin \kappa L -e^{2\kappa l_e-1}) \\ 4EI\kappa^2(2\cos(\kappa L)^2+\cosh 2\kappa L -3) \end{pmatrix}$$  \hspace{1cm} (11)

The rotation of the beam-end due to $M_0$ is obtained by substitution of the constants (Eq.11) into the first derivative of Eq. 8. The additional length is defined as the ratio of the member bending stiffness over the equivalent rotational spring stiffness as follows

$$l_c = \frac{EI}{1/\beta_0} = \frac{4 \cos(\kappa l_e) \sin(\kappa l_e) - e^{-2\kappa l_e} + e^{2\kappa l_e}}{\kappa(4 \cos(\kappa l_e)^2 + e^{-2\kappa l_e} + e^{2\kappa l_e} - 6)}$$  \hspace{1cm} (12)

3.3 Equivalent beam model

Now, with the equivalent beam length $l_c$ in place the coupled model shown in Fig. 2 (b) can be converted into a single equivalent beam model as depicted in Fig. 2 (c). In order to comply with the linearisation assumption mentioned in Section 2, the imperfection amplitude should be $w_0 \ll l_2$ and in a similar vein, the bending deformation magnitude in member 2 should remain in the order of the imperfection amplitude. Members 1 and 3 are considered according to the first order theory where member 2 bearing the imperfection is modelled according to the second order theory. Symmetry allows a further reduction of the beam model into the subsystem shown in Fig. 2 (d) in which the equivalent length of members 1 and 3 is converted into a couple of linear rotational springs with stiffness $\overline{\epsilon}$. A set of auxiliary parameters is defined below, where the axial force $N > 0 \in \mathbb{R}$ is stipulated positive for compression.

$$\varepsilon = l_2 \sqrt{\frac{N}{EI}}$$  \hspace{1cm} (13)

$$\overline{\beta} = \frac{\varepsilon \sin \overline{\varepsilon}^{-1}}{\varepsilon^2}$$  \hspace{1cm} (14)
\[ \mu = \frac{l_2 \tan(\epsilon) \varepsilon^2}{(l_{1,eff} \varepsilon^2 + l_2) \tan(\epsilon) - l_2 \varepsilon} \]  

The second order bending moments in the nodes \( b \) and \( c \) (c.f. Fig. 2 (c)) can be written with the auxiliary parameters given by Eq.13 – 16 as follows:

\[ M_b = M_c = -\frac{1 - \mu \hat{b}}{1 - \mu^2 \hat{b}^2} \mu \hat{b} 8 N w_0 = \frac{4w_0(-2 \sin(\epsilon) + \epsilon(\cos(\epsilon) + 1)) N l_2}{(l_2 \sin(\epsilon) + l_{1,eff} \epsilon(\cos(\epsilon) + 1)) \varepsilon^2} \]  

In order to obtain the continuous function of the bending moment in member 2 it is necessary to obtain the second order shear force in the beam node \( b \) given as

\[ Q_{bc} = \frac{b_1(l_2) (-b_0(l_2)M_b + M_c) + b_2(l_2) 8 N w_0}{l_2} = \frac{4(\cos(\epsilon) - 1)(2 (\cos(\epsilon) + 1) l_{1,eff} E l \varepsilon^3 + 2E l \sin(\epsilon) l_2 \varepsilon^2 + (\cos(\epsilon) + 1)(l_2^2 N e - 2N \sin(\epsilon) l_2^3)w_0}{el_2^2 \sin(\epsilon) (l_2 \sin(\epsilon) + l_{1,eff} \epsilon(\cos(\epsilon) + 1))} \]  

The bending moment distribution in member 1 including the \( P\Delta \)-effect is given as follows:

\[ M_1(x) \equiv \int \frac{M_{aN}}{E l} x \, dx = \frac{M_{aN}}{2E l} x^2 + c = M_b + \frac{M_{bN}}{2E l} (l_{1,eff}^2 - x^2) \]  

The second order bending moment distribution in member 2 can be written as

\[ M_2(x) = M_b b_0(x) + Q_{bc} b_1(x) + 8 N \frac{w_0}{l_2} b_2(x) \]  

3.4 Energy release rate

In the assumed slender beam model the portion of the shear strain energy is small in comparison to the contribution from bending and therefore neglected (see assumptions). Only the strain energy induced by the bending moment contributes to the SERR since the axial force is constant in all members. Therefore, members 4 and 5 are not contributing to the SERR. The bending moment distributions given by Eq. 19 and 20 can now be used to compute the strain energy.

Next, the equivalent uniformly distributed load induced by the axial force and the parabolic imperfection is obtained through the expression \( q = \frac{8Nw_0}{l_2^2} \). Using the equivalent load, the strain energy in the total system can subsequently be written as follows:

\[ U_M = \frac{2}{2E l} \int_0^{l_2} M_2(x)^2 \, dx + \frac{4}{2E l} \int_0^{l_{1,eff}} M_b^2 \, dx = \frac{2M_b^2 l_{1,eff}}{E l} + \frac{1}{2E l K^2 \varepsilon^3} \left(-2Kf Q_{bc}(KM_b - q) \cos(\epsilon)^2 + \left(-Q_{bc} K^2 + f^2(KM_b - q)^2 \sin(\epsilon) - 4Q_{bc} Kf q \right) \cos(\epsilon) + 4f \left(qf (KM_b - q) \sin(\epsilon) + \frac{K^2 l_2^2 Q_{bc}}{4} + \frac{K(KM_b + q) Q_{bc}}{2} + f^2 \left(\frac{K^2 l_2 M_b}{4} - \frac{K l_2 Q_{bc}}{2} + \frac{3K^2 l_2^4}{4}\right)\right)\right) \]  

The Mode-I SERR rate in each crack tip is equal and its value eventually obtained by the total derivative of the internal energy:

8
\[ G_A = G_B = \frac{1}{2B} \frac{dU_M}{d\varepsilon_{1,eff}} \left( -f \left( (KM_b - q)Q_{bc} + Q_{bc}M_b K \right) \cos(\varepsilon) + \left( -Q_{bc}Q_{bc}K + \dot{M}_b f (KM_b - q) \sin(\varepsilon) - 2Q_{bc}f \dot{q} \cos(\varepsilon) + 2\dot{M}_b f q \sin(\varepsilon) + (Q_{bc}K l_2 + KM_b + q) Q_{bc} + (Q_{bc}K + f^2 l_2 (KM_b - q)) \dot{M}_b + 2K f^2 M_b^2 \right) f \right) \] (22)

The total derivatives of the shear force and bending moment in node \( b \) used in Eq. 22 are as follows:

\[ \frac{dQ_{bc}}{d\varepsilon_{1,eff}} = \dot{Q}_{bc} = \frac{4Nw_0 (\sin(\varepsilon) \varepsilon + 2 \cos(\varepsilon) - 2)}{(l_{1,eff} \varepsilon - l_2^2) \cos(\varepsilon) + l_{2,eff} \varepsilon^2 + 2 \sin(\varepsilon) l_{1,eff} l_2 \varepsilon + l_2^2} \] (23)

\[ \frac{dM_b}{d\varepsilon_{1,eff}} = \dot{M}_b = -\frac{4Nw_0 l_2 (\cos(\varepsilon) \varepsilon - 2 \sin(\varepsilon) + \varepsilon)}{\varepsilon (l_{1,eff} \varepsilon^2 - l_2^2) \cos(\varepsilon) + l_{2,eff} \varepsilon^2 + 2 \sin(\varepsilon) l_{1,eff} l_2 \varepsilon + l_2^2} \] (24)

**4. Numerical analysis**

The predicted SERR by Eq. 22 is compared against geometric non-linear numerical analysis using a two-dimensional plane stress finite element model. Numerical analyses were conducted with the commercial FE software package Abaqus [23]. In this representative example, the problem (c.f. Fig.2 (a)) was discretized with four-node plane stress linear elements with reduced integration points, CPS4R. \( l_1 = 2 \) m and \( l_2 = 10 \) m was used with \( h = 0.5 \) m and \( B = 1 \) m. The material properties \( E = 2.1 \times 10^{11} \) Pa and \( v = 0.3 \) were adopted. The characteristic element size was 0.05 m x 0.05 m, resulting in a total number of around 4100 and 8100 elements for the model with \( h = 0.25 \) m and 0.50 m, respectively. The Virtual Crack Closure Technique (VCCT) [24] was used to provide the SERRs at the crack tip in the beam. According to the analytical modelling assumptions, one end of the beam is fully fixed while the other end is only allowed to deform in the \( x \) direction for load introduction. Multi-point constraints (MPC) were used to distribute the concentrated force applied in a reference point to the nodes of the loading surface. A contact pair is assigned to the interface with frictionless tangential behaviour and ‘hard’ contact normal behaviour to prevent surface overclosure. The Abaqus type ‘general static solver’ (a Newton-Raphson method) was chosen where geometric nonlinearity was enabled. A comparative study with quadratic elements, Abaqus type CPS8R, showed a SERR difference of around 1.2 % compared to the linear elements. A mesh size sensitivity was conducted and it is asserted that the adopted element size renders a sufficiently discretized domain with a negligible discretization error. The Abaqus model used in this study is provided in the supplementary material.

![Finite element model](image)

*Figure 3* Finite element model for a representative case showing the load, boundary conditions, initial imperfection and the crack tip and contact pair in the cracked beam. A contact pair shown in white and brown dashed lines is assigned to the interface and VCCT as shown in a red dashed line is used to compute the SERR at the crack tip. Due to symmetry, the contact pair and VCCT
features are only modelled in the LHS beam. A fixed boundary condition is applied to the RHS beam end and only the horizontal displacement degree of freedom is released in the LHS beam end for axial loading.

5. Results and discussion

Figure 4 compares the analytically predicted SERR with the numerical fracture analysis for different beam geometries and isotropic material properties. The SERR is normalised with the critical buckling load defined as $N_{cr} = \frac{4 \pi^2 E l}{(2l_1 + l_2)^2}$. Figure 4 shows good agreement between the analytically and numerically predicted SERR showing a quadratic increase with the axial load level.

Figure 4 (a) and (c) show that Eq. 22 overestimates the SERR by $\approx 10\%$ if the elastic interface is ignored since the beam end rotation in the crack roots is different from zero. Conversely, Fig. 4 (b) and (d) show that consideration of the additional compliance by the elastic interface decreases the deviation of the SERR significantly. Comparison of Fig. 4 (a, b) with (c, d) shows that the agreement improves with increasing beam slenderness - agreeing more closely with the Euler-Bernoulli assumption.

![Graph comparison](image)

*Figure 4* Comparison of normalised analytical SERR namely $\frac{G\beta}{N_{cr}}$ (continuous graph) with numerically predicted SERR (discrete markers) versus the normalised axial load level in the member evaluated for different member lengths $l_1$ for (a) $h = 0.25$ m without consideration of the elastic interface, (b) $h = 0.25$ m with consideration of the elastic interface, (c) $h = 0.50$ m without consideration of the elastic interface, (d) $h = 0.50$ m with consideration of the elastic interface. $w_0 = l/250$ in all models.
Figure 5 shows good agreement between the analytical numerical SERR for transverse isotropic material properties with a material contrast resembling that of glass fibre- and carbon fibre epoxy composites, respectively. The deviation between the analytical- and numerical results for the cases in which the elastic interface was considered ranged between 2% to 4%. Unlike the first order solution, the deviation of the second order solution is a strong function of the load level. Linearisation of the displacements and the assumption of deformation invariant axial forces of the second order solution leads to a linear load versus displacement response proportional to the Dischinger factor $[21] \eta(N)$ according to $M'' \propto \eta M'$. Figure 4 and 5 show that the deviation between the full non-linear numerical response and that of the analytical second order prediction increases with increasing load levels. The analytical solution has the tendency to over-predict the SERR at higher axial load levels. The analytical solutions are therefore only valid for axial force magnitudes that are small compared to the critical buckling load. The study presented in this work shows that good approximations are obtained at load levels of $N \leq N_{cr}/100$.

Figure 5 Comparison between normalised analytical and numerical SERR namely $\frac{G\beta}{N_{cr}}$ for transverse isotropic material properties with $E_1/E_2=4$ (a) and 16 (b) representing typical anisotropy ratios in unidirectional glass fibre and carbon fibre laminate, respectively. $w_0 = l_2/250$ in all models.

Fig. 6 (a) shows Eq. 12 as a function of the material stiffness contrast for four different embedded beam lengths. It shows that the equivalent beam length for isotropic materials is in the order of $l_c \approx h$. Conversely, for transverse isotropic materials with a stark stiffness contrast, the additional length can be multiples of $h$. The equivalent length asymptotically approaches a limit curve for increasing embedded beam lengths. That is, already at $l_e = 4.5h$ the equivalent length $l_c$ curve shown in Fig. 6 (a) does not change anymore with further increased $l_e$. Figure 6 (b) shows that the SERR increases in absolute values with increasing $l_2$ since the imperfection amplitude was kept as a fraction of the member length which increased the strain energy accordingly. Furthermore, for a fixed $l_2$ the SERR decreases proportional to $1/l_{1,eff}^2$ with increasing $l_{1,eff}$ where $\frac{\partial G}{\partial l_{1,eff}} < 0$ representing a stable fracture equilibrium state [19]. The gradient of the decrease depends on the length ratio of $l_{1,eff}$ to $l_2$. That is, the gradient increases with increasing $l_2$. 
Figure 6 (a) Equivalent free beam length $l_e$ normalised against the cross-section height $h$ as a function of the elastic modulus contrast for different embedded lengths $l_e$ and (b) Normalised SERR as a function of the characteristic parameter of member 1 $\varepsilon_1 = l_{1,eff} \sqrt{\frac{N}{EI}}$ for four different characteristic parameters of member 2 $\varepsilon_2 = l_{2,eff} \sqrt{\frac{N}{EI}}$; all cases loaded with an axial force of $N = 1 \times 10^6 N$ and an imperfection amplitude of $w_0 = \frac{l_2}{250} = \text{cst}$.

6. Concluding remarks

The following conclusions can be conflated from this work:

- The deviation of the SERR between the analytical solution and the numerical predictions ranged between 2% and 4%.
- The consideration of the elastic interface is important, particularly in the case of orthotropic materials.
- The SERR predicted by Eq.22 is a linear function of the imperfection amplitude and a quadratic function of the axial force.
- The analytical solution shows that $G \propto \frac{1}{l_{1,eff}^2}$ whence crack growth below the critical buckling load in this model is stable.

The analytical solution developed in this work provides deep insights into the complex fracture behaviour of large-scale structures such as the wind turbine rotor blade exhibited in Fig. 7. The latter shows the nonlinear buckling driven crack opening in the trailing edge induced by fatigue loading observed during a testing campaign reported in [13]. The blade is loaded in such a way, that the cracked trailing edge panels are subject to axial compressive stress; this consequently induces a cyclic geometrically nonlinear Mode-I crack-opening mode that is closely resembled by the cracked elastic beam model presented in this work. Our analytical solution is successfully used to evaluate fracture behaviour of such large-scale modes. Furthermore, it facilitates the development of computationally efficient semi-analytical sub-modelling approaches for real-time structural health assessment required for realisation of wind turbine blade digital twins e.g. developed in the research project ReliaBlade [25].
Figure 7. Two adjacent cracks in the trailing edge of a full-scale wind turbine blade tested under fatigue loading. The bending induced axial compressive stress field in the trailing edge panels entails geometric nonlinear Mode-I crack breathing, i.e., cyclic opening and closing in an imperfect and cracked beam subject to low stress levels – this failure mode from experimental observation can be assessed by the model presented in this study. See also the supplementary video.

In the second order formulation applied in this work the axial cross-section force is posited to remain unaltered in the undeformed and deformed configuration. This assumption makes the model unsuitable for very large deformation formulations as they might occur in the post buckling regime. On the other hand the solution is shown to be a good approximation under the proviso of small deformations relative to the beam length and crack lengths that are large relative to the beam height as e.g. the case shown in Fig.7.

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Data accessibility. Matlab codes and FE models are provided as supplementary materials.

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Competing interests. The authors declare to have no competing interests.

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