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Simulation of three-dimensional nonlinear water waves using a pseudospectral volumetric method with an artificial boundary condition

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Abstract

This paper presents a pseudospectral method for the simulation of nonlinear water waves described by potential flow theory in three spatial dimensions. The method utilizes an artificial boundary condition that limits the vertical extent of the fluid domain, and it is found that the reduction in domain size offered by the boundary condition enables the solution of the Laplace problem with roughly half the degrees of freedom compared to another spectral method in the literature for (wave number times water depth) $kh \geq 2\pi$. Moreover, it is found that the location of the artificial boundary condition can be chosen once and for all at the beginning of simulations such that the size of the fluid domain is reduced substantially, even when the lowest point of the free surface elevation varies significantly with time. The method is tested by simulating steady nonlinear wave trains, the development of crescent waves from a steady nonlinear wave train, a nonlinear focusing event and a Gaussian hump which is initially at rest. It is shown that in all but the most nonlinear cases the method is capable of obtaining accurate results.

Keywords:

Nonlinear water waves, Spectral methods, Potential flow, Artificial boundary condition, Accuracy, Stability.

1 Introduction

In their 1996-review, Tsai & Yue [1] concluded that the numerical methods for simulation of nonlinear potential flow problems at that time were reasonably mature, although large problems were still out of reach due to lack of robustness and efficiency of the methods. While intense research efforts and massive hardware improvements certainly have advanced the numerical simulations quite significantly since then, the search for robust and efficient methods continues. Many of the currently available nonlinear free surface potential flow solvers suffer from unstable time integration and combat this problem using artificial damping. The typical picture is that for waves whose degree of nonlinearity is not too large, artificial damping may be used to obtain a stable and accurate time integration. For steep waves this strategy, however, fails to deliver satisfying results. With that being said, the different methods definitely do vary in their robustness. Among the more robust methods we find the volumetric methods which map the fluid domain to a time-independent rectangular box through a stretching of the vertical coordinate and solve the transformed Laplace problem by discretizing the entire box. The research into these methods presumably began with the work of Li & Fleming [2], who developed a finite difference-based multigrid method for three-dimensional wave fields, and since then, many different authors have used a volumetric approach for wave simulations. For example, Bingham & Zhang [3], Engsig-Karup et al. [4], Christiansen et al. [5], Yates & Benoit [8], Raoult et al. [7] and Klahn et al. [10] have all employed volumetric methods, and as a consequence it is now well established that this approach can handle variable bathymetry without approximations and that it leads to a stable computation of surface velocities as well as velocity profiles. These attractive features come at the prize of a substantially more involved computational procedure than, for example, that of the classic High-Order Spectral method derived independently by Dommermuth & Yue [14] and West et al. [15]. Although the HOS method is not capable of dealing with very steep waves (see e.g. Klahn et al. [11] for an investigation of its range of applicability), it is as a starting point much more efficient than the volumetric methods when the degree of nonlinearity is not too large. In fact, Ducroz et al. [16] have shown that their implementation of the HOS method has an execution time which is typically two orders of magnitude smaller

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than that of the finite difference method of Ensig-Karup et al. [4] when dealing with long-time propagation of regular wave trains and requiring the final phase error to be less than 0.1 degrees (see Figure 13 of Ducroz et al.).

For that reason, efforts have been made in order to increase the efficiency of the volumetric methods. Ensig-Karup et al. [17] have, for example, been able to increase the computational performance of their method substantially by executing it in parallel on a graphics processing unit. Another interesting attempt has been made by Nicholls [12], who showed that an artificial boundary condition could be used to reduce the vertical extent of the domain on which the Laplace equation must be solved. As a consequence, the boundary condition enabled the computation of the surface velocities to a given accuracy with fewer degrees of freedom in the vertical direction than if the whole fluid domain was discretized, giving rise to large computational savings. Nicholls did not test this strategy on any “real” wave problem, however, but instead used the method of manufactured solutions to test the solution of a single, time independent Laplace problem with (wave number times water depth) $kh = 1000$. As such at least two different aspects of the boundary condition remain to be clarified. Since the effect on water waves by a change in the water depth is practically insignificant if $kh > 2\pi$ because the velocity is expected to decrease exponentially with depth in that case (see e.g. the proof of this fact for two-dimensional wave fields given by Longuet-Higgins [18]), simulations with finite water depth are usually carried out with $kh \leq 2\pi$. It is hence interesting to know if noteworthy savings in the vertical resolution can also be made for these water depths. Moreover, it is interesting to know if it is possible in practice to choose the vertical location of the artificial boundary condition once and for all at the beginning of a simulation, such that at all times 1) it limits the vertical extent of the fluid domain substantially and 2) the lowest point of the free surface remains above the location of the boundary condition. Here the interest is motivated by the fact that several computational quantities, which we will mention explicitly at a later stage, must be recomputed if the location of the boundary condition is changed during a simulation. While a good position for the boundary is easily chosen a priori in the case of steady waves, the positioning of the boundary is much harder when simulating irregular wave fields, since their lowest trough varies with time.

The aim of this work is to shed light on the above mentioned aspects of the artificial boundary condition when using a pseudospectral method in three spatial dimensions. To be specific, the goal of the investigation is to find out whether the location of the artificial boundary condition can be chosen such that it meets the requirements 1) and 2) above and how large computational savings can potentially be realized under these conditions. Along the way, we test the accuracy, stability and efficiency of the method. The method that we will describe assumes periodic boundary conditions in the two horizontal directions and employs the Fourier collocation method and the classical fourth order Runge-Kutta method to discretize the free surface boundary conditions in space and time, respectively. Moreover, the solution of the Laplace problem is based on the artificial boundary condition used by Nicholls, in combination with a discretization using the Fourier collocation method in the horizontal directions and the Legendre collocation method in the vertical direction.

The remainder of this paper is organized as follows: Section 2 gives an introduction to the physical system considered, assumptions made and governing equations. Section 3 gives a detailed description of the numerical method, while Sections 4-7 present the results that we have obtained using it to simulate steady nonlinear waves, the formation of crescent waves, a nonlinear focusing event and a Gaussian hump initially at rest. In section 8 we discuss the computational efficiency of the numerical method before conclusions are finally drawn in Section 9.

2 Physical system and governing equations

We consider the motion of an inviscid, incompressible and irrotational fluid with a free surface in three dimensions (x , y and z) and assume the motion to be spatially periodic in the horizontal x and y directions with periods L_x and L_y , respectively. In the vertical z direction, the fluid is taken to be bounded from below by an impermeable flat seabed at $z = -h$, to have its still water level at $z = 0$ and to be bounded from above by its free surface $\eta(x, y, t)$. We note here that this way of representing the free surface limits our description to non-overtopping waves.

For this physical system, the equations of motion are the irrotational Euler equations, which take their simplest form when written in terms of the velocity potential Φ . As Zakharov [13] proved now more than 50 years ago, the state of the system is uniquely determined by the free surface elevation and the velocity potential at the free surface,

$\Phi_s = \Phi|_{z=\eta}$, and we follow his work and write the governing equations as

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{for } -h < z < \eta(x, y, t), \quad (1a)$$

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{for } z = -h, \quad (1b)$$

$$\frac{\partial \eta}{\partial t} = \left(1 + \left(\frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial y} \right)^2 \right) w_s - \frac{\partial \eta}{\partial x} \frac{\partial \Phi_s}{\partial x} - \frac{\partial \eta}{\partial y} \frac{\partial \Phi_s}{\partial y}, \quad (1c)$$

$$\frac{\partial \Phi_s}{\partial t} = -g\eta - \frac{1}{2} \left(\left(\frac{\partial \Phi_s}{\partial x} \right)^2 + \left(\frac{\partial \Phi_s}{\partial y} \right)^2 \right) + \frac{1}{2} \left(1 + \left(\frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial y} \right)^2 \right) w_s^2, \quad (1d)$$

where $w_s = \partial_z \Phi|_{z=\eta}$ is the vertical velocity of the fluid at the free surface and g denotes the gravitational acceleration for which we shall use the value 9.81 m/s² throughout this work. The physical meaning of these equations is that the Laplace equation (1a) expresses conservation of mass, while the boundary condition (1b) states that the seabed is impermeable. Moreover, the kinematic boundary condition (1c) follows from the fact that a fluid particle cannot pass through the free surface, and the dynamic boundary condition (1d) states that the pressure along the free surface is constant.

3 Numerical methods

This section is devoted to the description of the numerical method that we employ to integrate (1) in time. This system of equations constitutes an initial value problem for the quantities η and Φ_s in the sense that if η and Φ_s are initially known, then w_s can be computed from the Laplace problem consisting of (1a) and (1b), supplemented with the boundary condition $\Phi|_{z=-h} = \Phi_s$; η and Φ_s can then in turn be stepped forward in time using (1c) and (1d). Due to this structure of the problem, we will in Section 3.1 describe how the boundary conditions at the free surface are integrated in time under the assumption that w_s is known, before we present the method with which we compute w_s from η and Φ_s in Section 3.2. In both parts of the method we seek approximations of the relevant quantities in the form of series expansions, for the simple reason that for non-overturning waves these quantities will always be smooth functions of x , y and possibly z . As such, the series representations will converge at a rate faster than any algebraic rate (see e.g. Kopriva [21]), meaning that only few degrees of freedom are needed to obtain accurate approximations. Moreover, we want to emphasize that neither of the two parts employ any kind of dealiasing. We have made this choice as we have found that dealiasing substantially decreases the method's computational performance both in terms of computation time and memory requirements while seeming to give no improvement of the method in terms of accuracy and stability.

3.1 Time integration of the free surface boundary conditions

To integrate the free surface boundary conditions in time we employ a method of lines approach and discretize the spatial and temporal dimensions separately. Owing to the periodicity assumptions in the x and y directions, we discretize the spatial part of equations (1c) and (1d) using the Fourier collocation method. By this we mean that we satisfy (1c) and (1d) at the set of grid points

$$\left\{ (x_{n_x}, y_{n_y}) = \left(\frac{L_x n_x}{2N_x}, \frac{L_y n_y}{2N_y} \right) \mid 0 \leq n_x < 2N_x \text{ and } 0 \leq n_y < 2N_y \right\} \quad (2)$$

with the underlying assumption that the functions η , Φ_s and w_s are given by the truncated Fourier series

$$\eta(x, y, t) = \sum_{n_x=-N_x}^{N_x-1} \sum_{n_y=-N_y}^{N_y-1} \hat{\eta}_{n_x, n_y} \exp(i \mathbf{k}_{n_x, n_y} \cdot \mathbf{r}) \quad (3a)$$

$$\Phi_s(x, y, t) = \sum_{n_x=-N_x}^{N_x-1} \sum_{n_y=-N_y}^{N_y-1} \hat{\Phi}_{n_x, n_y} \exp(i \mathbf{k}_{n_x, n_y} \cdot \mathbf{r}) \quad (3b)$$

$$w_s(x, y, t) = \sum_{n_x=-N_x}^{N_x-1} \sum_{n_y=-N_y}^{N_y-1} \hat{w}_{n_x, n_y} \exp(i \mathbf{k}_{n_x, n_y} \cdot \mathbf{r}) \quad (3c)$$

where $\mathbf{k}_{n_x, n_y} = (2\pi n_x/L_x, 2\pi n_y/L_y)$ and $\mathbf{r} = (x, y)$. Since this method has been used extensively in the existing literature on water waves (see for example the different methods of Dommermuth & Yue [14], Fructus et al. [19] and Christiansen et al. [5]) we are at this point content with stating that our computation of the various derivatives of η and Φ_s is based on the fast Fourier transform and that we compute all nonlinear products in physical space without any dealiasing procedure.

The spatial discretization leaves a set of ordinary differential equations that describe the time evolution of the values of η and Φ_s at the grid points. We integrate these equations in time using the classical fourth order Runge-Kutta method with a constant step size Δt . Anticipating the course of events, we have found that the time integration becomes unstable in a number of situations, and to overcome this instability we use artificial damping. We will at later stages explicitly mention if damping is used, and when applied, it consists of multiplying the (n_x, n_y) th Fourier coefficients of η and Φ_s by the number

$$D_{n_x, n_y} = \begin{cases} 1 & \text{if } \kappa_{n_x, n_y} \leq \alpha \kappa_{N_x, N_y} \\ \exp\left(-36 \left(\frac{\kappa_{n_x, n_y}}{\kappa_{N_x, N_y}}\right)^{12}\right) & \text{otherwise} \end{cases} \quad (4)$$

after every time step. Here $\kappa_{n_x, n_y} = |\mathbf{k}_{n_x, n_y}|$ and $0 < \alpha < 1$ is a parameter to be specified. We note that this way of damping the time integration is a special case of the damping described in Section 5 in the textbook by Hesthaven & Warburton [20].

3.2 Solving the Laplace equation and computing w_s

Our strategy to solve the Laplace equation is based on a volumetric approach, and because this can be a rather costly affair in terms of computation time, we have used an artificial boundary condition as proposed by Nicholls [12]. The idea behind this artificial boundary condition is to reduce the size of the fluid domain such that fewer degrees of freedom are needed. We here duly note that the idea has been used by several authors previously and we refer to the paper by Nicholls and the references therein. Very briefly, our method consists of the following steps: First we introduce the artificial boundary condition to limit the vertical extent of the fluid domain. Second, we map the reduced fluid domain to a time independent box using a simple stretching of the vertical coordinate. As it is non-conformal, this change of coordinates transforms the Laplace equation, and in the third step of our method we solve this transformed Laplace equation using a pseudospectral method.

Since solving the Laplace equation does not involve time, we will suppress the time dependence of the functions η , Φ and w_s for notational convenience in this section.

3.2.1 Reducing the vertical extent of the fluid domain by using an artificial boundary condition

To introduce the artificial boundary condition we start by letting b denote a positive real number such that the level $z = -b$ lies between the seabed and the lowest point of the surface elevation. As will be explained in the following, the vertical position of the artificial boundary condition is taken as $z = -b$, and formally we choose b such that $-h < -b < \min \eta(x, y)$ where the minimum is taken over the rectangle $[0, L_x] \times [0, L_y]$. Dividing the fluid domain into an upper part with $-b < z < \eta(x, y)$ and a lower part with $-h < z < -b$, the Laplace problem consisting of (1a) and (1b) supplemented with the boundary condition $\Phi|_{z=\eta} = \Phi_s$ can be restated as Laplace problems on the upper and lower parts of the domain together with a matching condition at $z = -b$ (see Nicholls [12]). We write the Laplace problem on the upper part of the domain as

$$\frac{\partial^2 \Phi_U}{\partial x^2} + \frac{\partial^2 \Phi_U}{\partial y^2} + \frac{\partial^2 \Phi_U}{\partial z^2} = 0 \quad \text{for } -b < z < \eta(x, y), \quad (5a)$$

$$\Phi_U = \Phi_s \quad \text{for } z = \eta(x, y), \quad (5b)$$

the Laplace problem on the lower part of the domain as

$$\frac{\partial^2 \Phi_L}{\partial x^2} + \frac{\partial^2 \Phi_L}{\partial y^2} + \frac{\partial^2 \Phi_L}{\partial z^2} = 0 \quad \text{for } -h < z < -b, \quad (6a)$$

$$\frac{\partial \Phi_L}{\partial z} = 0 \quad \text{for } z = -h, \quad (6b)$$

and the matching condition at $z = -b$ as

$$\Phi_U = \Phi_L \quad \text{for } z = -b, \quad (7a)$$

$$\frac{\partial \Phi_U}{\partial z} = \frac{\partial \Phi_L}{\partial z} \quad \text{for } z = -b. \quad (7b)$$

The point of doing this domain decomposition is that we can solve (6) analytically, if we for the moment assume the function $\phi = \Phi_U|_{z=-b}$ to be known. In the case where the horizontal part of Φ_U is approximated by a truncated Fourier series with $2N_x \times 2N_y$ coefficients, we may write

$$\phi(x, y) = \sum_{n_x=-N_x}^{N_x-1} \sum_{n_y=-N_y}^{N_y-1} \hat{\phi}_{n_x, n_y} \exp(i\mathbf{k}_{n_x, n_y} \cdot \mathbf{r}) \quad (8)$$

and it immediately follows that the unique solution of (6) is given by

$$\Phi_L(x, y, z) = \sum_{n_x=-N_x}^{N_x-1} \sum_{n_y=-N_y}^{N_y-1} \hat{\phi}_{n_x, n_y} \exp(i\mathbf{k}_{n_x, n_y} \cdot \mathbf{r}) \frac{\cosh(\kappa_{n_x, n_y}(h+z))}{\cosh(\kappa_{n_x, n_y}(h-b))} \quad (9)$$

which in turn implies that

$$\frac{\partial \Phi_L}{\partial z}(x, y, -b) = \sum_{n_x=-N_x}^{N_x-1} \sum_{n_y=-N_y}^{N_y-1} \hat{\phi}_{n_x, n_y} \kappa_{n_x, n_y} \tanh(\kappa_{n_x, n_y}(h-b)) \exp(i\mathbf{k}_{n_x, n_y} \cdot \mathbf{r}). \quad (10)$$

If we now define the operator T by the action

$$T[\phi] = \sum_{n_x=-N_x}^{N_x-1} \sum_{n_y=-N_y}^{N_y-1} \hat{\phi}_{n_x, n_y} \kappa_{n_x, n_y} \tanh(\kappa_{n_x, n_y}(h-b)) \exp(i\mathbf{k}_{n_x, n_y} \cdot \mathbf{r}) \quad (11)$$

and combine this with the matching condition (7b), we can write the Laplace problem on the upper part of the domain as

$$\frac{\partial^2 \Phi_U}{\partial x^2} + \frac{\partial^2 \Phi_U}{\partial y^2} + \frac{\partial^2 \Phi_U}{\partial z^2} = 0 \quad \text{for } -b < z < \eta(x, y), \quad (12a)$$

$$\Phi_U = \Phi_s \quad \text{for } z = \eta(x, y), \quad (12b)$$

$$\frac{\partial \Phi_U}{\partial z}(x, y, -b) - T[\Phi_U(x, y, -b)] = 0 \quad (12c)$$

which is a problem for Φ_U whose vertical range is not the original interval $[-h, \eta(x, y)]$, but rather the interval $[-b, \eta(x, y)]$. Since b in many cases might be chosen much smaller than h - in particular on deep water - this offers us the possibility of using fewer collocation points in the vertical dimension than if we had attempted to solve the original problem (1a) directly. We also note, however, that the operator T depends on b , meaning that if b is changed during a simulation, then the matrix representing the action of T must be recalculated.

2.2 Mapping the reduced fluid domain to a time independent rectangular box

In order to solve the Laplace problem (12) we map the reduced fluid domain to a time independent rectangular box through the coordinate transformation $(x, y, z) \mapsto (x, y, s)$ where s is given by

$$s(x, y, z) = \frac{2z + b - \eta(x, y)}{b + \eta(x, y)}. \quad (13)$$

As z varies between $-b$ and $\eta(x, y)$ it follows that s lies in the interval $[-1, 1]$. If we now define the function F by $\Phi_U(x, y, s(x, y, z)) = \Phi_U(x, y, z)$ it follows from (12a), the chain rule and the definition of s that F must satisfy the equation

$$0 = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \left(\left(\frac{\partial s}{\partial x} \right)^2 + \left(\frac{\partial s}{\partial y} \right)^2 + \left(\frac{\partial s}{\partial z} \right)^2 \right) \frac{\partial^2 F}{\partial s^2} + 2 \frac{\partial s}{\partial x} \frac{\partial^2 F}{\partial x \partial s} + 2 \frac{\partial s}{\partial y} \frac{\partial^2 F}{\partial y \partial s} + \left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} \right) \frac{\partial F}{\partial s} \quad (14)$$

and after inserting the derivatives of s and multiplying both sides of the equation by $(b + \eta)^2$ it becomes

$$\begin{aligned} 0 = & (b + \eta)^2 \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right) + \left(4 + (1 + s)^2 \left(\left(\frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial y} \right)^2 \right) \right) \frac{\partial^2 F}{\partial s^2} \\ & - 2(b + \eta)(1 + s) \left(\frac{\partial \eta}{\partial x} \frac{\partial^2 F}{\partial x \partial s} + \frac{\partial \eta}{\partial y} \frac{\partial^2 F}{\partial y \partial s} \right) \\ & + (1 + s) \left(2 \left(\left(\frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial y} \right)^2 \right) - (b + \eta) \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) \right) \frac{\partial F}{\partial s}. \end{aligned} \quad (15)$$

In addition to this equation, F must satisfy boundary conditions at $s = 1$ and $s = -1$. The boundary condition on F at $s = 1$ can be deduced from (12b) and is simply

$$F|_{s=1} = \Phi_s \quad (16)$$

while the boundary condition on F at $s = -1$ follows from (12c) and reads

$$\frac{2}{b+\eta} \frac{\partial F}{\partial s}(x, y, -1) - T[F(x, y, -1)] = 0. \quad (17)$$

2.3 Discretization and solution of the transformed Laplace equation

To solve the problem consisting of (15), (16) and (17) we employ a pseudospectral method and seek to determine the values of F on the grid

$$\left\{ (x_{n_x}, y_{n_y}, s_{n_s}) \mid 0 \leq n_x < 2N_x, \ 0 \leq n_y < 2N_y \text{ and } 0 \leq n_s < N_s + 1 \right\} \quad (18)$$

where x_{n_x} and y_{n_y} are given by (2) and s_{n_s} is the n_s th point of the Legendre-Gauss-Lobatto quadrature of order N_s ordered such that $s_0 = -1$. We require the boundary condition (16) to be satisfied at the set of points that have $n_s = N_s$ and the boundary condition (17) to be satisfied at the set of points with $n_s = 0$. At the remaining $4N_x N_y (N_s - 1)$ points we require the differential equation (15) to be satisfied. We compute all nonlinear products without any dealiasing procedure, and to compute the derivatives of F on the grid, we assume F to be of the form

$$F(x, y, s) = \sum_{n_x=-N_x}^{N_x-1} \sum_{n_y=-N_y}^{N_y-1} \sum_{n_s=0}^{N_s} \hat{F}_{n_x, n_y, n_s} \exp(i\mathbf{k}_{n_x, n_y} \cdot \mathbf{r}) l_{n_s}(s) \quad (19)$$

where

$$l_{n_s}(s) = \prod_{\substack{p=0 \\ p \neq n_s}}^{N_s} \frac{s - s_p}{s_{n_s} - s_p} \quad (20)$$

is the n_s th Lagrange polynomial of order N_s . With F of this form, its derivatives in the x - and y -dimensions can be computed using fft-based differentiation, while its derivatives in the s -dimension require the derivatives of l_{n_s} which can be computed using the procedure outlined in Section 3.5.2 in the book by Kopriva [21].

The above method of satisfying the boundary conditions and the differential equation at the set of points (18) generates a system of $4N_x N_y (N_s + 1)$ linear equations for the values of F on the grid (18), and we will denote these equations collectively by $\mathcal{A}\mathbf{x} = \mathbf{b}$ where $\mathcal{A}$ is a matrix and $\mathbf{x}$ and $\mathbf{b}$ are column vectors. This system may be solved in a number of different ways and we have chosen to use the iterative method GMRES (see Saad & Schultz [22]) with relative tolerance ϵ , mainly because it does not need the matrix $\mathcal{A}$ to be explicitly constructed but only a means of computing the product of $\mathcal{A}$ with a vector. Despite this advantage, the computational cost of GMRES can still be prohibitive if no preconditioner is applied to the problem. In order to construct a preconditioner, we follow the idea of Fuhrman & Bingham [23] and consider the problem consisting of (15), (16) and (17) with $\eta = 0$, i.e.

$$0 = b^2 \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right) + 4 \frac{\partial^2 F}{\partial s^2} \quad (21a)$$

$$F(x, y, 1) = \Phi_s(x, y) \quad (21b)$$

$$\frac{2}{b} \frac{\partial F}{\partial s}(x, y, -1) - T[F(x, y, -1)] = 0. \quad (21c)$$

In terms of efficiency, this problem is attractive as a preconditioning problem because it is easily solved for the expansion coefficients of F , and because it is time independent, meaning that it suffices to factor it once. Regarding the resemblance of the original problem, we note that (21) is obtained from the original problem in the limit of negligible wave steepness and therefore should provide a good approximation. In fact, we have found the iteration count to be around 10 or less even for the most nonlinear cases, when combining the preconditioning strategy with using an appropriate linear combination of solutions to the Laplace problem at the three previous time steps as the initial guess.

Figure 1: The sparsity patterns of the matrices (a) $\mathcal{L}$ and (b) $\mathcal{U}$ with $N_s = 4$.

Now, to actually apply (21) as a preconditioning operation, we let F be of the form (19) and insert it into (21). By using the uniqueness of Fourier series, as well as the fact that $l_{n_s}(s_p) = \delta_{n_s,p}$ (where $\delta_{n_s,p}$ is the Kronecker delta) we arrive at the set of linear equations

$$0 = -(\kappa_{n_x, n_y} b)^2 \hat{F}_{n_x, n_y, p} + 4 \sum_{n_s=0}^{N_s} \hat{F}_{n_x, n_y, n_s} \frac{d^2 l_{n_s}}{ds^2}(s_p) \quad \text{for } 1 \leq p < N_s \quad (22a)$$

$$\hat{F}_{n_x, n_y, N_s} = \hat{\Phi}_{n_x, n_y} \quad (22b)$$

$$0 = \frac{2}{b} \sum_{n_s=0}^{N_s} \hat{F}_{n_x, n_y, n_s} \frac{dl_{n_s}}{ds}(s_0) - \hat{F}_{n_x, n_y, 0} \kappa_{n_x, n_y} \tanh(\kappa_{n_x, n_y} (h - b)) \quad (22c)$$

which we denote by $\mathcal{M}\hat{\mathbf{x}} = \hat{\mathbf{b}}$. The matrix $\mathcal{M}$ happens to have a very sparse and ordered LUP-decomposition, $\mathcal{M} = \mathbf{P}^T \mathcal{L} \mathcal{U}$, with $\mathbf{P}$ a permutation matrix and $\mathcal{L}$ and $\mathcal{U}$ lower and upper triangular matrices, respectively. The sparsity patterns of $\mathcal{L}$ and $\mathcal{U}$ are shown for $N_s = 4$ in figure 1 which illustrates the fact that $\mathcal{L}$ consists of N_s bands of width 1 while $\mathcal{U}$ consists of $N_s + 1$ bands of width 1. Due to their structure, we compute $\mathbf{P}$, $\mathcal{L}$ and $\mathcal{U}$ once at the beginning of a simulation and apply the preconditioner consisting of (21) in the following three steps:

1. Given a vector of function values, $\mathbf{x}$, compute the vector of coefficients $\hat{\mathbf{x}}$ using the two-dimensional discrete Fourier transform at every s -point.
2. Compute $\mathcal{M}^{-1}\hat{\mathbf{x}}$ by solving the equation $\mathcal{L}\mathcal{U}\mathbf{v} = \mathbf{P}\hat{\mathbf{f}}$ for $\mathbf{v}$ using forward and backward substitution.
3. Compute the function values of the solution to the preconditioning problem by using the inverse two-dimensional discrete Fourier transform at every s -point.

Before going on we note that $\mathcal{M}$ depends on the vertical position of the artificial boundary condition, $-b$, and hence must be reconstructed and factored again if the value of b is changed.

3.2.4 Computing w_s from F

Once the values of F are known on the grid (18), computing the values of w_s on the grid (2) becomes straight forward. From the definitions of w_s , F and s it follows that w_s is given by

$$w_s(x, y) = \frac{2}{b + \eta(x, y)} \frac{\partial F}{\partial s}(x, y, 1) \quad (23)$$

which is the formula that we use to compute the values of w_s . To avoid any potential doubt, we again emphasize that we compute the product in the formula point wise without any dealiasing procedure.

4 Simulation of steady nonlinear waves

As a first test of our numerical method we consider steady nonlinear waves. Besides testing the method's ability to compute the vertical surface velocity, w_s , from η and Φ_s , and how well it can compute the vertical profile of the

horizontal velocity, $u = \partial_x \Phi$, below the wave crest, we also investigate the stability and accuracy of the time integration with and without artificial damping. Despite the fact that steady nonlinear waves are essentially two-dimensional, they provide an interesting test case because their time evolution is known (numerically) exactly. In addition, they represent a great computational challenge as they may contain a sustained degree of nonlinearity hardly found anywhere else. Taking the wavelength, λ , to be fixed, we characterize the waves by their relative water depth kh and their relative height $H/H_{\max}$, where $k = 2\pi/\lambda$ is the fundamental wave number, H is the vertical distance from the trough to the crest and $H_{\max}$ is the (h and λ dependent) maximal wave height corresponding to wave breaking. According to Battjes [24], the maximum steepness is given by

$$\frac{H_{\max}}{\lambda} = 0.1401 \tanh(0.8863kh) \quad (24)$$

which is the formula we use to calculate the values of $H_{\max}$. For completeness, values for the wave steepness H/λ will also be reported. Throughout this section we will only consider a single wavelength and hence take $L_x = \lambda$. The value of L_y can be chosen arbitrarily and in this work we take $L_y = \lambda$. Moreover, we take $N_y = 1$ and initialize η and Φ_s such that $\eta(x, y_0) = \eta(x, y_1)$ and $\Phi_s(x, y_0) = \Phi_s(x, y_1)$ for all values of x . Finally, if not explicitly stated, we choose the location of the artificial boundary condition by setting $b = \max \eta(x, y)$ where the maximum is taken over all grid points. We note that this choice makes sense, because for steady nonlinear waves the crest is further away from the zero level than the trough.

Regarding the initialization of the surface quantities, there is a rich literature of accurate methods for computing steady nonlinear waves, among which the method of Rienecker & Fenton [25] is probably the most widely used. Here we have chosen to use the method of Clamond & Dutykh [26] for initializing η and Φ_s as well as for providing the target solution for the vertical surface velocity, w_s^T . When executed in double precision, this method is capable of computing the surface quantities of all but the highest waves correct to double precision in a very small amount of time, but it gives the values of the surface quantities on a grid which is not equidistant. For that reason we use the method of Clamond & Dutykh to produce the values of η , Φ_s and w_s^T at a very large number of non-equidistant points (typically around 100,000), and using interpolation we compute the values of η and Φ_s on an equidistant grid with $2N_x$ points while we compute the values of w_s^T on an equidistant grid with $2N_T + 1$ points. Here N_T is the smallest integer such that w_s^T is accurate to double precision, by which we mean that $\hat{w}_1^T / \hat{w}_{N_T}^T = O(10^{-14})$ where $\hat{w}_n^T$ is the n th Fourier coefficient of w_s^T . In all cases we will have $N_T > N_x$.

When computing w_s from η and Φ_s with our method in the sections below, we assess the accuracy of the computation by measuring the error of w_s relative to the target velocity, w_s^T . Having computed w_s we measure its error using the L^2 -norm, i.e.

$$\text{Error}[w_s] = \sqrt{\frac{\sum_{n_x=-N_T}^{N_T} |\hat{w}_{n_x,0} - \hat{w}_{n_x}^T|^2}{\sum_{n_x=-N_T}^{N_T} |\hat{w}_{n_x}^T|^2}} \quad (25)$$

where it is understood that w_s has been zero-padded such that it contains $2N_T + 1$ coefficients.

3.1 Accuracy of w_s as a function of N_x

To begin with, we assess our numerical method by testing the convergence of the computed w_s to the numerically exact result as a function of N_x for fixed values of N_s and with $\epsilon = 10^{-14}$. We perform the test in shallow water with $kh = 0.3$, at an intermediate water depth with $kh = 1$ and on deep water with $kh = 2\pi$, and for each value of kh we consider $H/H_{\max} = 0.1, 0.5, 0.9$ and 0.99 . The values of N_T and H/λ for each of the cases are listed in tables 1 and 2, respectively, and we note that with the choice of b , the domain size is reduced roughly by the factor $1 - b/h$ which is about 90% or more in the deep water case, about 55% or more in the intermediate case and about 33% or more in the shallow water case.

The results of the computation are shown in Figure 2 and from these we conclude that if N_x and N_s are chosen large enough, our numerical method is capable of computing w_s to about 12 significant digits in a stable way for all the chosen values of kh and $H/H_{\max}$. We emphasize that this is a property not shared by many numerical methods for water wave. For example Dommermuth & Yue [14] showed that with their HOS method, which we note is widely used today, the computation of w_s for steady nonlinear waves in deep water becomes unstable when the steepness reaches roughly $H/H_{\max} \approx 0.8$. In this connection it should be noted that computing w_s correct to 12 digits for a steady wave in deep water with $H/H_{\max} = 0.8$ is a much easier task than for a wave with $H/H_{\max} = 0.99$, since much larger N_x and N_s must be used in the latter case.

Figure 2: The convergence of w_s to the exact result as a function of N_x for different values of N_s . Starting from the top, the rows correspond to $H/H_{\max} = 0.1, 0.5, 0.9$ and 0.99 , respectively, while the columns from left to right correspond to the shallow, intermediate and deep water cases, respectively. For the figures with no legend, the legend of the figure in the same row and first column applies.

kh	$H/H_{\max}$			
	0.1	0.5	0.9	0.99
0.3	25	80	310	2000
1	15	35	120	500
2π	12	30	100	500

Table 1: The values of N_T used for the convergence study of w_s as a function of the relative water depth and the relative wave height.

kh	$H/H_{\max}$				
	0.1	0.5	0.9	0.99	1
0.3	0.0036	0.0182	0.0327	0.0360	0.0364
1	0.0099	0.0470	0.0894	0.0984	0.0994
2π	0.0140	0.0700	0.1260	0.1387	0.1401

Table 2: The values of H/λ as a function of the relative water depth and the relative wave height.

4.2 Smallest N_x and N_s to achieve w_s to a certain accuracy

In the previous section we provided evidence that our numerical method can compute w_s correct up to 12 significant digits for steady nonlinear waves at virtually all values of $H/H_{\max}$ and kh , provided that we are willing to use fine enough resolution. For most engineering problems 12 digits of accuracy are, however, much more than what is actually needed, and hence a significant amount of time may be saved by using just the right amount of collocation points to meet the accuracy requirements. For that reason we devote this section to finding the smallest values of N_x and N_s such that $\text{Error}[w_s] < \tau$ where τ is some small number. We will denote these smallest values of N_x and N_s by N_x^* and N_s^* , respectively, and compute them for the cases where $\tau = 10^{-4}$, 10^{-5} and 10^{-6} . In order to make sure that the error on w_s is not affected by the GMRES method, we take $\epsilon = 10^{-14}$.

Now, suppose that we are given τ and want to find N_x^* and N_s^* . From the convergence results given in figure 2 we conclude that $N_s = 24$ is large enough that w_s can be computed with a relative error less than 10^{-6} in all cases, and our algorithm for finding N_x^* and N_s^* is therefore the following:

1. Let $H/H_{\max}$ and kh be given and set $N_s = 24$. Moreover, set $N_x = 2$.
2. Compute $\text{Error}[w_s]$ for these values of N_x and N_s . If $\text{Error}[w_s] \leq \tau$ then $N_x^* = N_x$ and we continue to the next step. If on the other hand $\text{Error}[w_s] > \tau$, we increase N_x by 1 and repeat this step with the new value of N_x .
3. Having found N_x^* we set $N_s = 2$ and compute $\text{Error}[w_s]$ using this value of N_s together with N_x^* . If $\text{Error}[w_s] \leq \tau$ we have $N_s^* = N_s$. If, however, $\text{Error}[w_s] > \tau$ we increase N_s by 1 and repeat this step until we have found the smallest value of N_s .

We have executed this algorithm for $kh = 0.3, 1$ and 2π with $0.01 \leq H/H_{\max} \leq 0.99$, and the obtained values of N_x^* and N_s^* are shown as functions of the wave height in figure 3 and figure 4 for the three different values of τ . Moreover, the values of N_x^* and N_s^* when computed with $\tau = 10^{-5}$ and $\tau = 10^{-6}$ relative to the values of N_x^* and N_s^* when computed with $\tau = 10^{-4}$ are shown in the figures. We conclude that if one is only interested in computing w_s with a relative error less than 10^{-4} , then in many situations, and almost regardless of kh , the required horizontal resolution can be reduced by about 25% or more compared to when computing w_s with a relative error of 10^{-5} , and by about 40% or more compared to when computing w_s with a relative error of 10^{-6} . In addition we conclude that the required vertical resolution can be reduced by roughly the same amount. Besides the possible computational savings it is interesting to note that the value of N_x^* grows exponentially as $H/H_{\max}$ approaches 1. In practice this means that a Fourier series description of regular waves is not of much use for the very steepest waves simply because too many degrees of freedom are needed to obtain accurate results.

Besides comparing the present method to itself, it is of course also interesting to compare it to other methods in order to see the resolution needed to achieve w_s with a certain accuracy. For that reason we have computed N_x^* and N_s^* using our recent Galerkin method (see Klahn et al. [10]) which discretizes the entire fluid domain instead of using an artificial boundary condition as done here but otherwise may be thought of as an alias-free two-dimensional version of the present method. We have found that, despite the fact that the Galerkin method is free of aliasing errors, the required horizontal resolution is almost exactly the same as for the present method for all water depths.

Figure 3: First row: N_x^* as a function of the relative wave height for different values of the relative water depth kh and the tolerance τ . Second row: The value of N_x^* with tolerance τ relative to N_x^* with tolerance 10^{-4} as a function of the relative wave height for different values of the relative water depth kh . The legends apply rowwise.

Regarding the required vertical resolution, we have found that for $kh = 0.3$ and $kh = 1$ the two methods again require almost the same resolution. For $kh = 2\pi$ our results show that the present method only requires about half the vertical resolution compared to the Galerkin method, and we therefore conclude that substantial savings may be obtained using the artificial boundary condition for deep water problems, while practically nothing is gained by using the boundary condition for shallow to intermediately deep water problems.

4.3 The accuracy of w_s as a function of ϵ

So far we have taken $\epsilon = 10^{-14}$ in all our calculations in order to test the behavior of $\text{Error}[w_s]$ when the spatial resolution is varied. Having found N_x^* and N_s^* needed to compute w_s such that $\text{Error}[w_s] < \tau$, an obvious question is whether we can use a larger (i.e. less restrictive) value of ϵ without losing accuracy and if so, if we can reduce the number of GMRES iterations by using the larger value of ϵ .

To find out, we have computed the error of w_s and monitored the number of GMRES iterations as a function of ϵ for different values of τ , kh and $H/H_{\max}$ when using $N_x = N_x^*$ and $N_s = N_s^*$. Figure 5 shows our results for the case where $kh = 0.3$ and $H/H_{\max} = 0.95$ (i.e. $H/\lambda = 0.0346$), and from the figure it is clear that $\text{Error}[w_s]$ stagnates long before $\epsilon = 10^{-14}$, meaning that a larger value of ϵ can be used while still achieving $\text{Error}[w_s] < \tau$. Moreover, the figure illustrates that the number of iterations keeps increasing when ϵ decreases, even though the error of w_s becomes constant. Combined, these two results thus imply that the computation of w_s may be executed significantly faster without any loss of accuracy simply by choosing a larger value for ϵ .

The results in Figure 5 of course only constitute a single example and are computed without utilizing earlier solutions to the Laplace problem to form an initial guess. We have, however, found the results for this case to be representative for many others, including small, intermediate and large values of kh and $H/H_{\max}$ as well as during time integration where an initial guess is used. Given τ , choosing $\epsilon = \tau/10^3$ seems in general to be a small enough tolerance to avoid errors from the GMRES method, and for relatively small heights ϵ can typically be chosen significantly larger, for example $\epsilon = \tau/50$.

4.4 Computation of velocity profiles

The calculation of the horizontal velocity, $u = \partial_x \Phi$, as a function of z below the wave crest of steep regular waves is a demanding problem that has only been attempted by a very limited number of authors. We mention here Madsen et al. [27], who used a Boussinesq-type model, and Engsig-Karup et al. [9], who used a nodal spectral element method. Madsen et al. did their test with $kh = 10$ and $H/H_{\max} = 0.96$ (i.e. $H/\lambda = 0.135$) and found that their result was

Figure 4: First row: N_s^* as a function of the wave steepness relative to the maximum steepness (24) for different values of the relative water depth kh and the tolerance τ . Second row: The value of N_s^* with tolerance τ relative to N_s^* with tolerance 10^{-4} as a function of the wave steepness relative to the maximum steepness (24) for different values of the relative water depth kh . The legends apply row wise.

Figure 5: (a): The error of w_s as a function of the relative tolerance, ϵ , used in GMRES for different values of τ with $kh = 0.3$ and $H/H_{\max} = 0.95$ corresponding to $H/\lambda = 0.0346$. (b): The number of GMRES iterations as a function of the relative tolerance ϵ for different values of τ and the same water depth and wave height as in (a). The legend applies to both figures.

kh	$H/H_{\max}$	
	0.95	0.99
0.3	7.5×10^{-6}	1.8×10^{-3}
1	1.5×10^{-5}	3.0×10^{-4}
10	6.4×10^{-6}	6.9×10^{-4}

Table 3: The relative difference between the results for u/c at the crest of our method and the stream function method for different values of kh and $H/H_{\max}$. The results have been obtained using the parameters $N_x = 400$, $N_s = 100$, $\epsilon = 10^{-14}$ and $b = \max |\eta(x, y)|$ for $kh = 0.3$ and $kh = 1$ and $b = 10 \times \max |\eta(x, y)|$ for $kh = 10$.

in good agreement with the result obtained from the stream function method of Rienecker & Fenton [25] close to the crest while their result was visibly inaccurate close to the seabed. Engsig-Karup et al. considered the cases $kh = 1$ and $kh = 10$ with $H/H_{\max} = 0.95$ (corresponding to $H/\lambda = 0.10$ and $H/\lambda = 0.133$, respectively). For both cases they found their method to be most accurate around the seabed and least accurate around the wave crest. Reading off their figure 8, it may be seen that the error (when compared to the result obtainable by the method of Rienecker & Fenton) of u/c at the wave crest, with c the wave celerity, was roughly 4×10^{-3} for the intermediate water depth and 15×10^{-3} for the deep water case.

In this work we consider the computation of $u(z)$ below the wave crest of waves with $H/H_{\max} = 0.95$ and $H/H_{\max} = 0.99$ for the three relative water depths $kh = 0.3$, $kh = 1$ and $kh = 10$ to test how well our method can calculate the horizontal velocity for steep waves. In the upper part of the domain, i.e. $-b < z < \eta$, we have calculated $\partial_x \Phi$ from the solution to the transformed Laplace equation (15), F , as

$$\frac{\partial \Phi}{\partial x} = \frac{\partial F}{\partial x} - \frac{1}{b + \eta} (1 + s) \frac{\partial \eta}{\partial x} \frac{\partial F}{\partial s} \quad (26)$$

while in the lower part of the domain, $-h < z < -b$, we have used the analytical solution (9) to calculate $\partial_x \Phi$ as

$$\frac{\partial \Phi}{\partial x} = \sum_{n_x=-N_x}^{N_x-1} \sum_{n_y=-N_y}^{N_y-1} i \frac{2\pi n_x}{L_x} \hat{\phi}_{n_x, n_y} \exp(i \mathbf{k}_{n_x, n_y} \cdot \mathbf{r}) \frac{\cosh(\kappa_{n_x, n_y}(h + z))}{\cosh(\kappa_{n_x, n_y}(h - b))}. \quad (27)$$

We recall that $\hat{\phi}_{n_x, n_y}$ denotes the (n_x, n_y) th Fourier coefficient of the function $\phi = F|_{s=-1}$, where $s = -1$ corresponds to $z = -b$. In all cases we have used the parameters $N_x = 400$, $N_s = 100$ and $\epsilon = 10^{-14}$ and while we have taken $b = \max |\eta(x, y)|$ for the shallow and intermediately deep water cases, we have taken $b = 10 \times \max |\eta(x, y)|$ for the deep water case for the simple reason that the cosh-functions in (27) easily overflow for large arguments when working in double precision. Our results for $u(z)$ below the wave crest are shown in figure 6 together with the best results we could obtain using the stream function method of Rienecker & Fenton. The figure shows a close match between the two methods for all values of kh and $H/H_{\max}$, and we note that besides our own recent work (see Klahn et al. [10]), seemingly nobody has attempted or been able to compute the profiles for the waves with $H/H_{\max} = 0.99$. The relative difference between the results for u/c at the crest of our method and the stream function method is shown in table 3 and from these we conclude that the difference between our method and the stream function method is smaller than the difference between the method of Engsig-Karup et al. and the stream function method by about a factor 1000 when $kh = 1$ and about a factor 625 when $kh = 10$.

4.5 Stability of the time integration of η and Φ_s without artificial damping

The stability of the time integration of the present method inevitably varies from case to case. The general rule seems to be that the more nonlinear the wave is and the finer it is resolved, the earlier the simulation breaks down. To illustrate this general rule, we here test the method's ability to integrate a steady nonlinear wave train in time for 50 periods without blowing up for different values of $H/H_{\max}$ and kh as well as varying resolutions. We define the blowup time, t_b , as the smallest t for which

$$\max \{ \eta(x, y, t) \} > 2 \max \{ \eta(x, y, 0) \} \quad (28)$$

where the maximum is taken over all grid points. We choose N_x and N_s such that we initially have $\text{Error}[w_s] < \tau$ and take the time step to be $\Delta t = T/(4N_x)$, where T is the temporal period of the wave, such that the wave moves half the distance between two adjacent grid points per time step corresponding to the Courant number $C = c\Delta t/\Delta x = 0.5$

Figure 6: Vertical profiles of the horizontal velocity u for different values of kh and $H/H_{\max}$ (given by the numbers to the right of the figures). Our results are shown as the blue lines and have been computed with the parameters $N_x = 400$, $N_s = 100$, $\epsilon = 10^{-14}$ and $b = \max |\eta(x, y)|$ for $kh = 0.3$ and $kh = 1$ and $b = 10 \times \max |\eta(x, y)|$ for $kh = 10$. The red circles are the best results that we could obtain using the stream function method of Rienecker & Fenton [25].

where $c = L/T$ is the wave celerity. The results of the test are given in figure 7 which shows the quantity $\min\{t_b/T, 50\}$ for $kh = 0.3, 1$ and 2π as well as $\tau = 10^{-4}$ and 10^{-6} . We note that t_b starts to decrease rapidly around $H/H_{\max} \approx 0.5$, with the exact value of $H/H_{\max}$ depending on kh and τ . Integrating the waves for 50 periods is of course an arbitrary choice and for the waves not blowing up within the first 50 periods, the figure does not say much. For that reason we have tried integrating the seemingly stable waves for longer times and found that these in general blow up much later, typically after more than 1000 periods.

At this point we note that temporal instabilities experienced with nonlinear potential flow models are sometimes believed to be due to aliasing errors which build up over time (see e.g. Section 4.4 of Engsig-Karup et al. [4]). For two reasons we are, however, confident that this is not the case for the present model. Firstly, it is well known that aliasing errors are solely a result of lack of resolution, meaning that if aliasing was the problem then the simulations with $\tau = 10^{-6}$ should be more stable than the simulations with $\tau = 10^{-4}$. The opposite is found to be the case. Secondly, when we compare the stability performance of the present model to the performance of our recent Galerkin model (see Klahn et al. [10]), which in essence is a two-dimensional version of the present model that treats all nonlinear products in a completely aliasing-free fashion, we see almost no difference in stability properties. At this stage, we do not know what causes the instability for increasing nonlinearity and resolution. In this regard it is interesting to note that unexplained stability issues with nonlinear potential flow models have been observed since at least 1976 when Longuet-Higgins & Cokelet [30] reported instabilities in their boundary integral method and concluded that “the cause of the instability is unknown”.

4.6 Stabilizing the time integration of steep waves by artificial damping

As a practical solution to the stability problem of the time integration of steep waves, we resort to artificial damping as mentioned in Section 3.1. In order to test the accuracy of the damped time integration, we have again tested how well it can propagate steep waves for 50 periods. Figure 8 shows our results for the surface elevation at $y = 0$ for $kh = 2\pi$ and different values of $H/H_{\max}$ when using the parameters $N_x = 32$, $N_s = 20$, $\Delta t = T/100$, $\epsilon = 10^{-9}$ and $\alpha = 0.5$. While $\eta(x, 50T)$ matches $\eta(x, 0)$ very well for $H/H_{\max} = 0.85$, a slight difference is seen for $H/H_{\max} = 0.90$. For $H/H_{\max} = 0.95$ the difference is clearly seen, and for $H/H_{\max} = 0.99$ the final surface elevation is not only out of phase with the exact result but also has a visibly smaller height. These results give an illustration of the fact that even

Figure 7: The quantity $\min\{t_b/T, 50\}$ as a function of $H/H_{\max}$ for different values of kh and τ . The legend applies to all figures.

With the sophisticated numerical methods of today, accurate time integration of extremely nonlinear waves remains out of reach, thus motivating the search for stable algorithms. When taking $kh = 1$ we have obtained similar results using the same parameters, but in shallow water with $kh = 0.3$ we have had significantly more trouble integrating steep waves accurately in time even when using damping. Our experiments with different parameter choices show that for $kh = 0.3$ we could get a match similar to the one in figure 8 (a) for wave heights up to about $H/H_{\max} = 0.8$ and to obtain this, we had to use less damping ($\alpha = 0.9$) and a significantly smaller time step ($\Delta t = T/400$) than used for $kh = 2\pi$. In comparison, we have recently shown (see Klahn et al. [11]) that for $kh = 0.3$ the HOS method of Dommermuth & Yue [14] only allows stable time integration of waves with $H/H_{\max} \lesssim 0.3$ if a comparable accuracy is to be achieved. When seen in this light, the time integration of the present method may therefore be considered to be quite robust indeed.

Simulation of crescent waves

As a first truly three-dimensional test, we here test the method's ability to simulate the formation of crescent waves. The crescent pattern occurs due to quintet instabilities of steep steady deep-water nonlinear wave trains, and the test thus indicates whether the method is capable of handling a three-dimensional case with a high degree of nonlinearity.

Historically, the research into crescent waves can be traced back to the beginning of the 1980s starting with the work of Saffman & Yuen [31]. Working with an equation valid for weakly nonlinear waves on deep water due to Zakharov [13], they showed that for a given pair of numbers (p, q) , the branch of solutions corresponding to steady two-dimensional waves would at some steepness bifurcate into solution branches corresponding to the steady two-dimensional waves being modulated by a wave of wave numbers $(k_p, k_q) = (1 + p, q)k$ where $k = 2\pi/\lambda$ is the fundamental wave number of the steady two-dimensional wave train. Shortly thereafter, Meiron et al. [?] confirmed by numerical calculations that this bifurcation also takes place in the unapproximated equations of potential flow theory, and Su [33] found experimentally that if steep enough, steady two-dimensional wave trains indeed develop a three-dimensional shape similar to that predicted by Meiron et al. Denoting the patterns that repeat themselves every second, third and fourth wavelength of the original steady nonlinear wave by L2, L3 and L4, respectively, Su found that the steady nonlinear waves would develop the L2, L3 and L4 patterns in roughly 90%, 10% and 1% of the experiments. The explanation for this finding is generally accepted to be a combination of the facts that the waves in Su's experiments contained small random disturbances including a mixture of the unstable disturbances found by McLean [34], and that for the steepness considered by Su, the disturbances leading to the L2 pattern have significantly larger rates of growth than the disturbances leading to the L3 and L4 patterns. Although the theoretical understanding of the formation of crescent waves has been in place for about 40 years, it was first around the start of this millennium that numerical simulations of these waves based on the full potential flow equations appeared, and to the best of our knowledge, Xue et al. [35] were the first to achieve this milestone. Fuhrman et al. [36], Fructus et al. [19] and Xu & Guyenne [29] soon followed meaning that several authors have now successfully simulated the phenomena.

In this work we will confine ourselves to the numerical simulation of the most unstable L2 pattern. To be able to compare our results to those of Su [33], we consider steady two-dimensional waves of steepness $H/\lambda = 0.105$ and take the relative water depth to be $kh = 2\pi$. For these parameters McLean's analysis predicts that the wave numbers of the most unstable disturbance leading to the L2 pattern correspond to $(p, q) = (0.5, 1.23)$ and we therefore initialize

Figure 8: The free surface elevation at $t = 0$ and $t = 50T$ for (a) $H/H_{\max} = 0.85$ corresponding to $H/\lambda = 0.1191$, (b) $H/H_{\max} = 0.90$ corresponding to $H/\lambda = 0.1261$, (c) $H/H_{\max} = 0.95$ corresponding to $H/\lambda = 0.1331$ and (d) $H/H_{\max} = 0.99$ corresponding to $H/\lambda = 0.1387$. The figures have been produced using the parameters $N_x = 32$, $N_s = 20$, $\Delta t = T/100$, $\epsilon = 10^{-9}$ and $\alpha = 0.5$, and the legend applies to all four figures.

η and Φ_s in our simulations as steady two-dimensional waves perturbed by

$$\tilde{\eta}(x, y) = \varepsilon H \sin(k_p x) \cos(k_q y) \quad (29a)$$

$$\tilde{\Phi}_s(x, y) = \frac{\varepsilon H \sqrt{g}}{(k_p^2 + k_q^2)^{1/4}} \cos(k_p x) \cos(k_q y) \exp\left((k_p^2 + k_q^2)^{1/2} \eta(x, y, 0)\right) \quad (29b)$$

which corresponds to the dominant Fourier modes of the unstable disturbance. In (29) ε is some (in principal infinitely) small dimensionless number and $\eta(x, y, 0)$ is the surface elevation of the steady two-dimensional wave train. Moreover, we take $L_x = 4\lambda$ and $L_y = 4\pi/k_q$ such that the L2 pattern is repeated twice in both the x and y directions and we perform the simulations using $N_x = 64$, $N_y = 16$, $N_s = 8$ and $\epsilon = 10^{-9}$ (not to be confused with the amplitude of the perturbation, ε .) We employ artificial damping in our simulations using $\alpha = 0.5$ and take $b = 1.1 \times \max\{\eta(x, y, 0)\}$, where the maximum is taken over all grid points in the (x, y) -plane, giving a reduction in domain size by roughly 93%. Finally we take the time step to be $\Delta t = T/50$.

Qualitatively our results are in good agreement with the experiments of Su and the above mentioned numerical studies. This point is illustrated by figure 9 which shows the evolution of the perturbed steady two-dimensional waves into the L2 pattern for $\varepsilon = 10^{-4}$. Like Xue et al. and Fuhrman et al. we have found that from a qualitative point of view, the value of ε is only important for the time scale on which the crescent pattern forms by which we mean that larger values of ε lead to a similar shape but at an earlier time. On the other hand, we have found the value of ε to be quite important from a quantitative point of view. In the literature it has become common practice to assess the accuracy of simulations of the L2 pattern by comparing the wavelength and wave height ratios λ_2/λ_1 , h_{11}/h_{12} , h_{11}/h_{21} and h_{21}/h_{22} as well as the maximum steepness ($\max \partial_x \eta$) all evaluated at $y = \pi/k_q$ to the experimental results of Su where the quantities λ_1 , λ_2 , h_{11} , h_{12} , h_{21} and h_{22} are defined in figure 10. Our results for the wavelength and wave height ratios are shown in figure 11 for different values of ε up to the point in time where the damping changes the shape of the crescent pattern significantly, and we conclude that the local variation of the quantities is $O(\varepsilon)$. For that reason we find a comparison to Su's results most meaningful when ε is chosen smaller than 0.01, since his results are given with two digits after the decimal point. Table 4 shows our results for the above mentioned quantities as a function of time when using $\varepsilon = 10^{-4}$, and it is clear that at no instant in time do we reach a perfect agreement with the results of Su. On the other hand, we do get a reasonable agreement at e.g. $t = 43T$ where our results for h_{11}/h_{12} , h_{21}/h_{22} , λ_2/λ_1 and h_{11}/h_{21} are about 1%, 1%, 12% and 9% off those of Su, and our result for $\max \partial_x \eta$ is about 32% and 16% off the result of Su depending on which value we compare to. We note that these deviations are similar to those obtained in the simulations by Xue et al., Fuhrman et al. and Xu & Guyenne, and on average somewhat larger than those obtained in the simulations by Fructus et al. who, however, used $H/\lambda = 0.095$ as the initial steepness of the steady two-dimensional waves. It should be noted that for this steepness, the instability leading to the L2 pattern is not the strongest one and is in fact dominated by two-dimensional Benjamin-Feir type instabilities.

Another way to check the accuracy of the simulation is to compare the growth rate of the initial disturbance to the growth rate predicted by the analysis of McLean [34]. At any instant in time the free surface elevation may be written in the form (3a) and the amplitude of the initial disturbance can be directly measured by the absolute value of the coefficient $\hat{\eta}_{6,2}$ since the disturbance repeats itself six times in the x -direction and two times in the y -direction over the chosen domain. The time evolution of $|\hat{\eta}_{6,2}|$ when using the parameters $\varepsilon = 10^{-4}$, $N_x = 64$, $N_y = 16$, $N_s = 8$, $\epsilon = 10^{-9}$, $\alpha = 0.5$ and $\Delta t = T/50$ is shown in figure 12 together with the prediction of McLean, namely, that $|\hat{\eta}_{6,2}|$ is proportional to $\exp(0.0316\sqrt{gk}t)$. The results are seen to match well at all times, except at the very beginning of the simulation. This early discrepancy may be explained by the fact that the initial disturbance (29) corresponds to a linear combination of eigenfunctions of McLean rather than just the most unstable eigenfunction, and for such a linear combination the time evolution will only resemble that of the most unstable eigenfunction for large enough times. Because the results of McLean in principle only predict the time evolution of the steady wave train immediately after it has been perturbed, it is interesting to note that our results show that the disturbance grows at the same rate as that predicted by McLean at all times during the formation of the L2 pattern. This finding differs from the results of Fuhrman et al. [36], who found the growth to accelerate during the formation. At the present time we believe the reason for this disagreement to be that Fuhrman et al. measured the rate of growth differently than we have done in this work.

6 Simulation of a nonlinear wave focusing event

As a further validation of the method, we employ it to simulate a focusing event in an irregular wavefield. Our strategy for simulating such an event is to initialize the wavefield at time $t_0 < 0$ such that linear theory predicts the event to occur with maximum crest elevation A at time $t = 0$. This idea has previously been used in the experimental investigations of Baldock et al. [37] and Johannessen & Swan [38] (which we will refer to as 'JS' throughout this

Figure 9: The formation of the L2 pattern over time. The figure shows the surface elevation at time (a) $t = 37T$, (b) $t = 39T$, (c) $t = 41T$, (d) $t = 43T$, (e) $t = 45T$ and (f) $t = 47T$ when using the parameters $N_x = 64$, $N_y = 16$, $N_s = 8$, $\epsilon = 10^{-9}$, $\alpha = 0.5$ and $\Delta t = T/50$. The amplitude of the initial perturbation is $\epsilon = 10^{-4}$.

Figure 10: The surface elevation at $y = \pi/k_q$ at some point in time during the formation of the L2 pattern showing the definition of the quantities λ_1 , λ_2 , h_{11} , h_{12} , h_{21} and h_{22} .

Figure 11: The ratios λ_1/λ_2 , h_{11}/h_{12} , h_{11}/h_{21} and h_{21}/h_{22} as a function of time for different values of ε . The figures have been produced using the parameters $N_x = 64$, $N_y = 16$, $N_s = 8$, $\epsilon = 10^{-9}$, $\alpha = 0.5$ and $\Delta t = T/50$. The legend applies to all figures.

	Present method												Su [33]
t/T	36	37	38	39	40	41	42	43	44	45	46	47	
λ_2/λ_1	1.05	1.06	1.06	1.08	1.09	1.12	1.14	1.16	1.19	1.21	1.21	1.20	1.28
h_{11}/h_{12}	1.03	1.04	1.04	1.05	1.06	1.07	1.09	1.11	1.14	1.18	1.25	1.34	1.10
h_{11}/h_{21}	1.10	1.13	1.15	1.19	1.23	1.29	1.36	1.45	1.57	1.71	1.80	1.80	1.66
h_{21}/h_{22}	0.97	0.96	0.96	0.95	0.93	0.92	0.90	0.87	0.84	0.79	0.73	0.69	0.88
$\max \partial_x \eta$	0.58	0.60	0.62	0.64	0.68	0.72	0.78	0.86	0.95	1.03	0.94	0.64	0.65, 1.02

Table 4: The values of the wavelength and wave height ratios as a function of time computed with the parameters $N_x = 64$, $N_y = 16$, $N_s = 8$, $\epsilon = 10^{-9}$, $\alpha = 0.5$, $\Delta t = T/50$ and $\varepsilon = 10^{-4}$ together with the values of Su [33].

Figure 12: The time evolution of $|\hat{\eta}_{6,2}|$ as a function of time when using the parameters $\varepsilon = 10^{-4}$, $N_x = 64$, $N_y = 16$, $N_s = 8$, $\epsilon = 10^{-9}$, $\alpha = 0.5$ and $\Delta t = T/50$. The dashed line is given by $\exp(0.0316\sqrt{gk}t)$ and corresponds to the evolution predicted by McLean's analysis.

section) as well as in the numerical studies of Bateman et al. [39] and Fuhrman & Madsen [40]. While the mentioned authors all studied a wide variety of focusing events, we will here confine ourselves to simulating the case denoted as B78s4 by JS. For this case $A = 78$ mm while the frequency, ω , and angle of propagation, θ (measured counter-clockwise from the x -axis), of the initial wave field are distributed according to the distribution $\mu(\omega, \theta) = \chi(\omega)\Gamma(\theta)$, where

$$\chi(\omega) \propto \begin{cases} \omega^{-2} & \text{if } \omega_{\min} \leq \omega \leq \omega_{\max} \\ 0 & \text{otherwise} \end{cases} \quad (30)$$

and

$$\Gamma(\theta) \propto \begin{cases} \cos\left(\frac{\theta}{2}\right)^4 & \text{if } \theta_{\min} \leq \theta \leq \theta_{\max} \\ 0 & \text{otherwise.} \end{cases} \quad (31)$$

In the above equations $\omega_{\min} = 92\pi/64$ rad/s, $\omega_{\max} = 212\pi/64$ rad/s, $\theta_{\min} = -\pi/4$ rad and $\theta_{\max} = \pi/4$ rad. The natural starting point for our numerical method is a wave number spectrum rather than a frequency-direction spectrum, and for that reason we convert $\mu(\omega, \theta)$ to the corresponding wave number spectrum, $\rho(\mathbf{k})$, where $\mathbf{k}$ is the vector of length κ and angle θ relative to the x -axis. To do the conversion, we assume that $\rho(\mathbf{k}) = \psi(\kappa)\Gamma(\theta)$ and that ω and $\mathbf{k}$ are related through the linear dispersion relation $\omega(\mathbf{k})^2 = g\kappa \tanh(\kappa h)$. Using the fact that for all values of κ_1 , κ_2 , θ_1 and θ_2 the energy contained in the wave components with wave vectors in the set $\{\mathbf{k} | \kappa_1 \leq \kappa \leq \kappa_2 \text{ and } \theta_1 \leq \theta \leq \theta_2\}$ is the same as the energy contained in the wave components with frequencies and propagation angles in the set $\{(\omega, \theta) | \omega(\kappa_1) \leq \omega \leq \omega(\kappa_2) \text{ and } \theta_1 \leq \theta \leq \theta_2\}$, it can be shown that $\chi(\omega)$ and $\psi(\kappa)$ are connected through the relation (see for example Chapter 2 in the book by Young [28])

$$\psi(\kappa) = \frac{1}{\kappa} \frac{d\omega}{d\kappa} \chi(\omega(\kappa)) \quad (32)$$

Figure 13: A zoom-in on the surface elevation at the focus time computed using the parameters $L_x = L_y = 25.6$ m, $h = 1.2$ m, $A = 78$ mm, $t_0 = -20$ s, $\Delta t = 0.01$ s, $N_x = N_y = 128$, $N_s = 14$, $b = 0.12$ m, $\epsilon = 10^{-9}$ and $\alpha = 0.7$.

implying that the total wave number spectrum is given by

$$\rho(\mathbf{k}) \propto \begin{cases} \frac{1}{\kappa \omega(\kappa)^2} \frac{d\omega}{d\kappa} \cos\left(\frac{\theta}{2}\right)^4 & \text{if } \kappa_{\min} \leq \kappa \leq \kappa_{\max} \text{ and } \theta_{\min} \leq \theta \leq \theta_{\max} \\ 0 & \text{otherwise.} \end{cases} \quad (33)$$

From this distribution we choose the Fourier coefficients of η at time $t = 0$ such that they are all positive numbers and add up to A . From η at time $t = 0$ we obtain η at time $t = t_0$ by using linear theory, and we obtain Φ_s at t_0 utilizing the relation between the Fourier coefficients of Φ_s and η from linear theory.

Employing this procedure for initializing η and Φ_s at $t_0 = -20$ s, we have simulated the B78s4 case of JS using the parameters $L_x = L_y = 25.6$ m, $h = 1.2$ m, $A = 78$ mm, $\Delta t = 0.01$ s, $N_x = N_y = 128$, $N_s = 14$, $b = 0.12$ m (roughly corresponding to reduction of the domain size by 90%), $\epsilon = 10^{-9}$ and $\alpha = 0.7$. In our simulations the focus occurs at the time $t_f = 0.15$ s (relative to $t_f = 0$ predicted by linear theory) and the surface elevation at that time is shown in figure 13. Moreover, our results for the time evolution of the surface elevation at the focus point as well as the vertical profile of the velocity in the x -direction, $u = \partial_x \Phi$, at the focus point and time are shown in figure 14 together with the experimental results of JS. Although the numerical results cannot be expected to match the experimental results of JS perfectly, since the initial condition of the simulation necessarily differs from that used in the experimental wave tank, the overall agreement is very good indeed. To quantify the deviation, we here note that our result for the maximum surface elevation is about 1.8% smaller than that of JS. Our results for the velocity profile are also in good agreement with the results of JS, and we note that the present numerical method avoids the small oscillations in the computed velocity profile near the seabed of the Boussinesq-type model used by Fuhrman & Madsen [40].

Simulation of a Gaussian hump initially at rest

As a final test of the method, we use it to simulate the time evolution of a Gaussian hump which is initially at rest. Testing the quality of the solution obviously requires a reference solution, and to the best of our knowledge, there is no (numerically) exact solution to the nonlinear case. In the limit of small height-to-width-ratio of the hump, the time evolution is, however, going to be predicted by the linear equations obtained from (1) by linearizing around the rest state, and this gives us the opportunity to test the numerical method. Aside from testing the method in this limit, we also present results for a nonlinear case with the aim of providing a reference solution for future tests.

To carry out the simulation we let $L_x = L_y = L$ where $L = 1$ m and denote the initial height and characteristic width of the hump by H and σ , respectively. We take the initial surface elevation at the (n_x, n_y) th grid point to be

$$\eta(x_{n_x}, y_{n_y}, 0) = H \exp\left(-\frac{(x_{n_x} - L/2)^2 + (y_{n_y} - L/2)^2}{\sigma^2}\right) \quad (34)$$

and we take the initial surface potential, $\Phi_s(x, y, 0)$, to be a constant over the domain. It is clear that the initial condition for η is not periodic in x and y , but we note here, that it becomes an exponentially accurate approximation to a periodic function in the limit where σ/L is small. Denoting the (n_x, n_y) th Fourier coefficient of $\eta(x, y, 0)$ by $\hat{\eta}_{n_x, n_y}$

Figure 14: (a): The surface elevation at the focus point as a function of time relative to the focusing to the time of focus t_f . (b): The vertical profile of the velocity in the x -direction at the focus point and time. Our results have been obtained using the parameters $L_x = L_y = 25.6$ m, $h = 1.2$ m, $A = 78$ mm, $t_0 = -20$ s, $\Delta t = 0.01$ s, $N_x = N_y = 128$, $N_s = 14$, $b = 0.12$ m, $\epsilon = 10^{-9}$ and $\alpha = 0.7$ and are shown as the blue line in both figures. The red circles are the results of Johannessen & Swan [38].

it follows that the linearization of (1) predicts the surface elevation at time t to be given by

$$\eta^{(L)}(x, y, t) = \sum_{n_x=-N_x}^{N_x-1} \sum_{n_y=-N_y}^{N_y-1} \left(\hat{\eta}_{n_x, n_y} \cos(\omega_{n_x, n_y} t) \right) \exp \left(i(\mathbf{k}_{n_x, n_y} \cdot \mathbf{r}) \right), \quad (35)$$

where $\omega_{n_x, n_y}^2 = g \kappa_{n_x, n_y} \tanh(\kappa_{n_x, n_y} h)$ which is the solution we test against in the limit of small H/σ . Based on the order of magnitude estimates given by Lautrup [41] (see Section 24 in his book) we measure time in units of $\sigma/\sqrt{gh}$ and we choose the time step as

$$\Delta t = C \frac{\sigma}{\sqrt{gh}} \quad (36)$$

with $C \leq 1$ a dimensionless number. We take $b = H$ which in all cases will give a reduction in domain size by about 94% or more and we perform the simulations in this section without any artificial damping.

7.1 Convergence to the result of linear theory in the limit $H/\sigma \rightarrow 0$

To test the convergence of our numerical solution to (35) we consider a Gaussian hump with $h/L = 1$ and $\sigma/L = 0.02$ and use the computational parameters $N_x = N_y = 16$, $N_s = 6$, $C = 1/10$ and $\epsilon = 10^{-14}$ to integrate the system in time until $t_f = 25\sigma/\sqrt{gh}$. At this time we compute the deviation between the simulated result and the result of linear theory as

$$\text{Deviation}[\eta] = \sqrt{\frac{\sum_{n_x=0}^{2N_x-1} \sum_{n_y=0}^{2N_y-1} \left| \eta(x_{n_x}, y_{n_y}, t_f) - \eta^{(L)}(x_{n_x}, y_{n_y}, t_f) \right|^2}{\sum_{n_x=0}^{2N_x-1} \sum_{n_y=0}^{2N_y-1} \left| \eta^{(L)}(x_{n_x}, y_{n_y}, t_f) \right|^2}}. \quad (37)$$

To steer clear of any possible misconceptions, we stress that $\text{Deviation}[\eta]$ does not directly measure the error of the numerical simulations since we do not compare to the exact solution but only to the solution of the linearized problem. Our results for $\text{Deviation}[\eta]$ as a function of H/σ are given in figure 15 and it is clear to see that the deviation increases linearly with H/σ as long as $H/\sigma \geq 10^{-6}$. The fact that the deviation decreases with H/σ for $H/\sigma < 10^{-6}$ we ascribe to rounding errors, for when computing the absolute deviation (i.e. (37) with a denominator of 1), we have found that the deviation stagnates for $H/\sigma < 10^{-6}$. As such we are confident that our model works as intended.

Figure 15: Deviation[η] as a function of H/σ for the case $h/L = 1$ and $\sigma/L = 0.02$. The computational parameters used to produce the figure are $N_x = N_y = 16$, $N_s = 6$, $C = 1/10$ and $\epsilon = 10^{-14}$.

Figure 16: The surface elevation at $(x, y) = (L/2, L/2)$ as a function of time when using the parameters $h/L = 1$, $\sigma/L = 0.02$, $H/L = 2^{-6}$, $N_y = N_x$, $N_s = 14$, $C = 0.1$ and $\epsilon = 10^{-14}$.

7.2 A nonlinear benchmark result

From the results shown in figure 15 it can be seen that the prediction of linear theory is off by roughly 20% at $t = t_f$ for $H/L = 2^{-6}$ (corresponding to $H/\sigma = 0.78$) when $h/L = 1$ and $\sigma/L = 0.02$. In an attempt to find the exact, nonlinear behavior of this case, we will in this section study its convergence as a function of the horizontal resolution using the computational parameters $N_s = 14$, $C = 1/10$ and $\epsilon = 10^{-14}$. The quantity, which we use to measure the convergence, is the surface elevation at the domain center, $(x_c, y_c) = (L/2, L/2)$.

The surface elevation at (x_c, y_c) is shown as a function of time in figure 16 for different values of $N_x = N_y$. As expected, the surface elevation oscillates up and down in time and it is seen that the curves come closer together as N_x increases. To quantify the convergence, the values of $\eta(L/2, L/2, t)$ at the times $t\sqrt{gh}/\sigma = 10, 20, 30, 40$ and 50 are given in table 5 as a function of $N_x = N_y$. For a fixed time, the surface elevation converges to some value although the rate of convergence is clearly larger for small t than for large t . One might of course wonder why so fine resolution is needed in order to obtain accurate results for this case. For that reason we show a zoom-in on the non-zero part of the free surface elevation at the times $t\sqrt{gh}/\sigma = 10, 20, 30, 40$ and 50 in figure 17 which illustrates the fact that $\eta(x, y, t)$ locally exhibits large spatial variations and thereby explains the need for fine resolution.

8 Computational cost

Having shown that the present numerical method is capable of accurately simulating a number of nonlinear wave phenomena, we use this section to discuss its computational cost. We first derive the expected scaling of the computational cost in terms of the parameters N_x , N_y and N_s , before we report on the actual CPU time per time step required to run the simulations. To put these results into perspective we mention here that the numerical method has been programmed in Matlab R2019a and executed on a HP Elitebook laptop with 16 GB RAM and an Intel i5-8250U 1.6 GHz processor.

N_x	$t\sqrt{gh}/\sigma$				
	10	20	30	40	50
16	0.182547	-0.538534	0.022585	0.346630	-0.062052
32	0.163578	-0.574976	0.230362	0.370330	-0.138438
48	0.147206	-0.538029	0.188467	0.350112	-0.145491
64	0.149869	-0.544301	0.190157	0.335021	-0.150387
96	0.149139	-0.543174	0.195174	0.336819	-0.143987
128	0.149069	-0.543270	0.195443	0.336397	-0.140960
192	0.149062	-0.543283	0.195385	0.336422	-0.139693
256	0.149062	-0.543283	0.195379	0.336397	-0.139814

Table 5: The values of $\eta(L/2, L/2, t)/H$ for different values of N_x and t when using the parameters $h/L = 1$, $\sigma/L = 0.02$, $H/L = 2^{-6}$, $N_y = N_x$, $N_s = 14$, $C = 0.1$ and $\epsilon = 10^{-14}$. The bold digits indicate those matching the result obtained with $N_x = 256$.

8.1 Expected computational cost

In order to advance η and Φ_s a single time step, the method in short performs three tasks: It solves the transformed Laplace equation (15), computes w_s using (23) and finally evaluates the right hand sides of (1c) and (1d). Of these, the computational cost of the first task overwhelms the cost of the latter two, and the computational cost associated with a single time step can thus be analyzed by considering the computational cost of solving (15).

Now, when using the preconditioning strategy outlined in Section 3.2.3 we have found that the number of GMRES iterations needed to solve (15) for all practical purposes is bounded by a constant, say, 40. As such, the computational complexity of the method must be the same as the computational complexity of a single iteration, which we recall consists of applying the preconditioner and evaluating the right hand side of (15). The cost associated with applying the preconditioner is dominated by the inversion of the matrices $\mathcal{L}$ and $\mathcal{U}$, whose sparsity pattern is shown in Figure 1. Since these matrices both have N_s bands of non-zero entries with a width of one, it is possible to invert them using e.g. Gaussian elimination in at most $O(N_x N_y N_s^2)$ operations. The cost associated with evaluating the right hand side of (15) is dominated by the computation of the derivatives of F , because the pointwise multiplication of the derivatives with their coefficients has a cost which scales linearly with N_x , N_y and N_z . The computation of x -derivatives requires $O(N_y N_s)$ fast Fourier transforms of vectors of length $O(N_x)$, meaning that the total computational complexity becomes $O(N_x \log(N_x) N_y N_s)$. Using a similar argument one may show that the computation of y -derivatives requires $O(N_x N_y \log(N_y) N_s)$ operations in total. To compute the s -derivatives of F , $O(N_x N_y)$ matrix vector products with complexity $O(N_s^2)$ are needed, since the differentiation matrices in the s -direction are dense. From this it follows that the computational cost associated with the computation of the s -derivatives is $O(N_x N_y N_s^2)$. In total we therefore conclude that a single iterations requires $O(N_x \log(N_x) N_y N_s + N_x N_y \log(N_y) N_s + N_x N_y N_s^2)$ operations.

Denoting the total number of grid points as $N = 4N_x N_y (N_s + 1)$ we deduce the following from the above estimation: If N_y (or alternatively N_x) and N_s are held constant while N_x (or alternatively N_y) is, the computational cost of the method should be proportional to $N \log(N)$. If, on the other hand N_x and N_y are held constant while N_s is varied, the computational cost of the method should be proportional to N^2 . Finally, if only N_s is held constant while N_x and N_y are both varied such that we always have $N_x = N_y$, the computational cost is expected to be proportional to $N \log(N)$.

8.2 Actual computational cost

To illustrate the actual computational cost of the numerical method we consider the cost associated with the simulation of the nonlinear benchmark result for the Gaussian hump initially at rest described in Section 7.2. To that end we again use the parameters $h/L = 1$, $\sigma/L = 0.02$, $H/L = 2^{-6}$, $N_y = N_x$, $C = 0.1$ and $\epsilon = 10^{-14}$ in connection with no artificial damping, and consider the CPU time per time step when keeping N_s fixed while varying $N_x = N_y$ and when keeping N_x fixed while varying N_s . The results of this computation are shown together with the expected computational cost in Figure 18, from which it may be seen that the actual cost matches the expected cost very well. We note, however, that only the first of the two cases is actually relevant for applications, since it is never necessary to choose large values of N_s due to the artificial boundary condition limiting the vertical extent of the fluid domain.

It is of course interesting to compare the computational scaling of the present method to that of other methods. Engsig-Karup et al. [4] found the computational effort of their method to scale linearly with N which as such appears more efficient than the present method. In that connection one should, however, keep in mind that the present method

Figure 17: Zoom-ins on the free surface elevation at the times (a) $t\sqrt{gh}/\sigma = 0$, (b) 10, (c) 20, (d) 30, (e) 40 and (f) 50 when using the parameters $h/L = 1$, $\sigma/L = 0.02$, $H/L = 2^{-6}$, $N_x = N_y = 256$, $N_s = 14$, $C = 0.1$ and $\epsilon = 10^{-14}$. The horizontal scales are the same for all figures while the vertical scale is different for each figure.

is expected to be more accurate than that of Engsig-Karup et al. when using the same number of degrees of freedom. The reason for this is, of course, that the pseudospectral discretization used in this work gives an exponential decrease of the error with the number of degrees of freedom while the error in the finite difference method of Engsig-Karup et al. decreases with the number of degrees of freedom to a fixed, negative power. Ducroz et al. [16] found their implementation of the HOS method to have a computational effort proportional to $N \log(N)$, but since the vertical dimension is not resolved in the HOS method, their N corresponds to $4N_x N_y$ in our notation. In conclusion, the HOS method therefore seems substantially more efficient than the present method. Whether this performance gap can be closed (for example by executing the present method in parallel on a graphics processing unit as done by e.g. Engsig-Karup et al. [17]) is not known at the moment and remains to be clarified through further research.

9 Conclusions

We have described and tested a pseudospectral method for the simulation of nonlinear water waves in three dimensions. The method discretizes the spatial part of the time dependent boundary conditions at the free surface using the Fourier collocation method and the temporal part using the classical fourth order Runge-Kutta method. Moreover, the solution of the Laplace problem is based on the introduction of an artificial boundary condition and a discretization which uses the Fourier collocation method in the horizontal directions and the Legendre collocation method in the vertical direction. In order to solve the Laplace problem efficiently we have used the GMRES method preconditioned from the left by an ordered and sparse matrix which arises from the Laplace equation when setting the free surface elevation to zero.

The method was first tested on steady nonlinear wave trains and we have shown that the method is capable of computing the vertical surface velocity with a relative error as low as 10^{-12} for practically all water depth and relative wave heights. In addition, it was shown that if the vertical velocity was to be computed with an error less than some specified tolerance, only about half the degrees of freedom were needed in this method compared to the Galerkin method of Klahn et al. [10] when $kh = 2\pi$. As such, using the artificial boundary condition does in fact enable substantial computational savings when the water depth is comparable in magnitude to the wavelength. By testing how well the method can compute the vertical profile of the horizontal velocity, we found it to be much more accurate than the methods of Madsen et al. [27] and Engsig-Karup et al. [9]. The stability of the time integration was shown

Figure 18: The CPU time per time step when simulating the nonlinear benchmark case for the Gaussian hump as a function of the total number of grid points $N = 4N_x N_y (N_s + 1)$ when varying (a) N_x with $N_s = 10$ and (b) N_s with $N_x = 64$ together with the expected scaling. The simulations have been performed with the parameters $h/L = 1$, $\epsilon/L = 0.02$, $H/L = 2^{-6}$, $N_y = N_x$, $C = 0.1$ and $\epsilon = 10^{-14}$ and no artificial damping has been used.

to be dependent on the spatial resolution, the relative water depth and the relative wave height, and (like seemingly all other nonlinear water wave models) it was found that steady waves of large relative wave height could not be integrated in time for more than a few periods unless artificial damping was introduced. When introduced, however, the artificial damping enabled stable time integration of waves with a relative height close to the limiting steepness in deep and intermediately deep water. For shallow water, the maximum relative amplitude that could be integrated in time in a stable and accurate way was somewhat lower.

The second test of the method consisted of simulating the L2 crescent wave pattern which arises due to the three-dimensional instability of steep steady nonlinear wave trains in deep water. Here we found a good qualitative agreement with experimental results and demonstrated the quantitative significance of the magnitude of the perturbation amplitude. Moreover, we found a reasonably good quantitative agreement between our simulations and experimentally obtained results, as well as between the simulated growth rate and that predicted by the analysis of McLean [34].

As a third test, we assessed the method's ability to simulate a nonlinear focusing event. We compared our results for the time series of the surface elevation at the focus point and the vertical profile of the velocity in the x -direction at the focus point and time to the results obtained experimentally by Johannessen & Swan [38]. Overall, a very good agreement was found.

In the fourth and final test, the method was tested on a Gaussian hump initially at rest. Due to the lack of reference solutions we compared the results to those predicted by linear theory for small height-to-width ratio and found that the method produced the correct answer in that limit. To enable future comparisons, we presented a convergence study for a nonlinear case poorly described by linear theory.

Following the test of accuracy and stability of the method, we discussed the computational cost associated with using the method. Using a rather rough estimate, we showed that $O(N_x \log(N_x) N_y \log(N_y) N_s^2)$ operations are required in order to advance the solution one time step, and when comparing this estimate to the actual CPU time per time step when simulating the nonlinear benchmark result for the Gaussian hump initially at rest, a very good match was found both when varying the horizontal and the vertical resolution.

As a last remark, we note that, despite the fact that the lowest point of the surface elevation varies significantly with time for the three last test cases, all simulations were carried out with a fixed vertical position of the artificial boundary condition. The position of the boundary condition has been such that, in all but the shallow water and intermediately deep water cases for the steady nonlinear wave trains, the domain size required to solve a given problem has in general been reduced by around 90% or more.

Declaration of Interests

The authors report no conflict of interest.

Data availability statement

Data have not been shared.

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