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Gradient-based prestress and size optimization for the design of cable domes

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Abstract
Cable domes are cable-strut structural systems widely used as large span roofs of arenas, stadiums, and open spaces for their lightweight forms and aesthetic impact. We first recall the matrix theory for the static and kinematic analysis of pin-jointed cable-strut structural assemblies. This theory is used to identify potential states of self-equilibrated prestress for a given structure, and to evaluate their stiffening effect on the structure internal mechanisms. Then, a problem for simultaneous optimization of prestress and size of nonlinear cable-strut structural systems is formulated. Constraints on internal forces and displacements are imposed while the structural weight is minimized. The resulting optimization problem is solved with a gradient-based approach based on sequential linear programming. The gradients of the aggregated constraint functions are consistently calculated with adjoint sensitivity analyses. The optimization approach is applied to the design of a simple illustrative structure and of a realistic cable dome. The results show that the proposed approach can identify optimized designs with modest computational efforts, and with significant savings in terms of structural weight compared to initial design guesses.

Keywords: cable structures; spatial structures; structural optimization; prestress; adjoint sensitivity analysis

1. Introduction
Cable structures are lightweight, versatile structures with a strong aesthetic and architectural impact. They have been widely used as large span roofs for arenas, stadiums, and open squares. Broadly speaking, cable structures can be divided into: i) pure tensile structures made of two sets of cables - one set supports the structure while the second stabilizes the structure; ii) tensegrity structures made of cables and struts (Motro, 2003) - they are made of a discontinuous set of compressed elements (struts) inside a continuous set of tensioned elements (cables). They are internally self-equilibrated with suitable stable prestress distributions; iii) hybrid tensile structures made of cable and struts - they are designed based on the engineers’ intuition and share many concepts of the first two typologies. Cable domes are hybrid tensile structures that are widely used in practice (Guo & Zhou, 2016). The cable dome structural concept was first proposed by Geiger and employed for the roof of the Olympic Gymnastics Hall and the Fencing Hall for the Korean Olympics in Seoul (Geiger et al., 1986). A new cable dome concept was then proposed in occasion of the Atlanta Olympics of 1996 for the Georgia Dome with the so-called Levy form (Levy, 1994; Terry, 1994). Yuan et al. (2007) proposed three new cable dome forms: the Kiewitt form, the Hybrid form, and the Bird nest form. Other well-known cable domes are the Redbird Arena in Illinois (USA), the Suncoast Dome in Florida (USA), the La Plata Stadium in Argentina, and the Tao-Yuan County Arena in Taiwan.

Cable domes are made of cable and strut struc-
tural elements. In general, cable domes can have internal mechanisms that jeopardize their serviceability and load bearing capacity in practical applications, because the equilibrium is achieved only for large displacements. However, through a state of self-equilibrated initial prestress, in some cases it may be possible to stiffen the internal mechanisms, and turn an unserviceable cable dome structure into a serviceable one.

This paper deals with the optimization-based design of the prestress and size of the structural elements of cable domes. In previous work, several authors focused on the optimization of cable-strut structural assemblies. Kawaguchi et al. (1999) first discussed the shape optimization of cable domes. In their work, they maximize the stiffness of a realistic elliptical dome with geometric design variables. They also investigate the dependence of the optimized results upon the shape design variables. Yuan & Dong (2003) discuss a method for the identification and optimization of a new equilibrium state of integral feasible prestress for cable domes. The optimization of tensegrity structures is considered by Masic et al. (2006). They propose a procedure to identify and design new tensegrity structures through a gradient-based optimization-based approach. Yuan et al. (2007) discuss the optimization of the prestress of the outer hoop of a Kiewitt dome. The optimization of the prestress of a Kiewitt dome is also discussed by Chen et al. (2015), where the strain energy is minimized. A multi-objective optimization approach is presented by Ohsaki et al. (2008) for the simultaneous maximization of the lowest eigenvalue of the tangent stiffness matrix and minimization of the compliance for a set of specified external loads. The design variables are the combination coefficients of the self-equilibrated states of prestress, and the optimization is performed with a gradient-based approach. Chen et al. (2012) propose a form finding approach for tensegrity structures that relies on the maximization of the first eigenvalue of the tangent stiffness matrix with an ant colony algorithm. Lee & Lee (2014) use a genetic algorithm to optimize the prestress of cable-strut structures to maximize the first natural frequency. The optimization of the prestress and size of cable domes using a genetic algorithm is discussed by Albertin et al. (2012) and Quagliaroli et al. (2015). A genetic algorithm is used also by Ashwear et al. (2016) to maximize the first natural frequency and the frequency gap of the first two modes of modular tensegrity structures. Zhang & Feng (2017) optimize the prestress of cable-strut structures by maximizing the minimum eigenvalue of the geometric stiffness matrix with constraints on prestress variance and structural stability. A comparative study using meta-heuristic algorithms for the optimization of single layer domes is presented in (Kaveh & Talatahari, 2010). The optimization of domes is discussed also in (Kaveh & Dehkordi, 2003) where neural networks are trained for the analysis, design and prediction of the displacements of domes using back propagation and radial basis functions networks. In several articles the Enhanced Colliding Bodies Optimization (ECBO) method is used for the topology optimization of different types of domes (Kaveh & Rezaei, 2015, 2016). The topology optimization of domes is also discussed by Kaveh & Talatahari (2011) where a meta-heuristic algorithm, known as the Charged System Search (CSS) is used. A hybrid algorithm based on a vibrating particles system (VPS) algorithm, multi-design variable configuration (Multi-DVC) cascade optimization, and an upper bound strategy (UBS) for optimizing large-scale dome truss structures is presented in (Kaveh & Ilchi Ghazaan, 2018). Recent work focused on the optimization of dome structures with frequency constraints based on Enhanced Colliding Bodies optimization (Kaveh & Ghazaan, 2016), and chaos-based firefly algorithms (Kaveh & Javadi, 2019). The optimization with a meta-heuristic algorithm of large-scale domes is discussed also in (Kaveh et al., 2018). The multi-objective optimization for the prestress design of cable-strut structures is discussed in recent work by Ma et al. (2019). Four optimization objectives are considered: the minimization of structural weight; the maximization of the minimum eigenvalue of the stiffness matrix; the minimization of the prestress variance of the cables; and the minimization of the maximum eigenvalue of the error sensitivity matrix. For optimization a genetic algorithm is used. In more recent work, Chen et al. (2020) use a particle swarm algorithm to identify feasible prestress
modes of prestressed cable-strut structures with multiple self-stress states. They formulate the problem as a multi-objective optimization problem, and convert it into a single objective problem utilizing the weight coefficient method.

Most of the work above mentioned concerns the optimization of cable-strut structural assemblies with gradient-free optimization algorithms. Very few contributions rely on first-order information for optimization. Both gradient-free and gradient-based methods are effective solution techniques for structural optimization problems. Gradient-free methods typically require higher computational cost and time compared to gradient-based methods because they rely on heuristic search techniques. Gradient-based methods on the other hand, require the calculation of the gradients of the functions involved, which in some cases may be a complicated task. To the best of the author’s knowledge, gradient-based optimization of the prestress and size of cable domes has been discussed previously only by Yuan & Dong (2002). They propose a two-level optimization approach where the optimization of the prestress and elements’ size is done separately and sequentially.

Thus, this paper presents a novel approach for the simultaneous gradient-based optimization of the prestress and size of the structural members of cable domes. The matrix theory for the static and kinematic analysis of pin-jointed cable-strut structural assemblies is first recalled. This theory is used to identify possible states of self-equilibrated prestress and to evaluate their stiffening effect on the internal mechanisms of the structures considered. Once a suitable state of prestress that stabilizes the internal mechanisms and verifies the requirement of cables in tension and struts in compression is found, the prestress magnitude and the structural elements’ size are simultaneously optimized by the proposed approach. The structural weight is minimized, and constraints are imposed on the structure nodal displacements, the maximum tension in the cables, and the maximum compression in the struts under different loading conditions. The optimization is solved efficiently with a gradient-based algorithm based on Sequential Linear Programming (SLP). This algorithm relies on first order information, hence the calculation of the gradients of the nonlinear displacement and force constraints requires particular care. To this end, adjoint sensitivity analyses are developed and their details are discussed thoroughly in the paper. The computational cost of the adjoint sensitivity analyses is reduced by aggregating each set of constraints (i.e., displacement constraints, force constraints) separately into a single constraint on the maximum value. The optimization approach is first applied to the design of a simple structure made of two cables with three hinges aligned. This example is used to present and discuss visually the capabilities of the proposed optimization approach. Then, promising results related to the optimization of the prestress and size of a realistic Geiger dome are presented. The numerical results reveal that the weight of the optimized Geiger dome is reduced up to approximately 80% with respect to the initial design guess. Moreover, the optimization analysis requires only few minutes of computational time on a standard personal computer, making the proposed approach suitable for practical engineering applications.

The reminder of the article is organized as follows: in Sec. 2 we briefly recall the matrix analysis of spatial pin-jointed structures and we provide the details of the nonlinear structural analysis adopted; in Sec. 3 we discuss the formulation of the optimization problem considered, with details on the objective and constraint functions; the sensitivity analysis for the calculation of the constraint functions’ gradients is discussed thoroughly in Sec. 4, where important details related to the implementation of the SLP optimization algorithm are also provided; numerical results are discussed in Sec. 5 and final conclusions are drawn in Sec. 6.

2. Analysis of cable-strut assemblies

In the following section we first provide important details on the matrix analysis of pin-jointed spatial structures used to study the static and kinematic properties of cable domes. Next, the details of the nonlinear structural analysis adopted in this work are provided.
2.1. Matrix analysis of pin-jointed structures

We briefly review the matrix analysis of spatial pin-jointed structures. More details can be found in (Quagliaroli et al., 2015). In the context of matrix analysis of pin-jointed structures, major contributions are due to Pellegrino and Calladine (Calladine, 1982; Pellegrino & Calladine, 1986; Pellegrino, 1990; Calladine & Pellegrino, 1991, 1992). The essential results of their approach based on the Singular Value Decomposition (SVD) of the equilibrium matrix are here recalled for the sake of clarity and completeness of the discussion. These results are then used in the optimization approach discussed later in the paper.

In the matrix analysis of spatial structures the assumptions are: the structural members are connected by pin-joints; the connectivity of the elements is known; self-weight of the structural elements is neglected by pin-joints; the connectivity of the elements are subjected only to axial forces, either in tension or compression. If we write the equilibrium equations in compact form isolating the free degrees of freedom we have:

\[ \mathbf{A} t = \mathbf{f}_{ext} \quad (1) \]

where \( \mathbf{A} \) is the equilibrium matrix, \( t \) is the vector with the axial forces in the elements, and \( \mathbf{f}_{ext} \) is the vector of nodal forces applied in correspondence of the free degrees of freedom. Similarly, we have that:

\[ \mathbf{B} \mathbf{q} = \mathbf{e} \quad (2) \]

where \( \mathbf{B} \) is the compatibility matrix; \( \mathbf{q} \) is the vector of nodal displacements; and \( \mathbf{e} \) is the vector of the elements’ elongations. Because \( \mathbf{B} = \mathbf{A}^T \), the SVD of the equilibrium matrix reveals information on both the static and kinematic properties of the structure considered, as described in (Pellegrino, 1993).

In particular, from the SVD of the equilibrium matrix \( \mathbf{A} \) we obtain information on: 1) the basis vectors \( \mathbf{t}_i \) that define the space of self-equilibrated prestress states, i.e. \( \mathbf{A} \mathbf{t}_i = \mathbf{0} \) for \( i = 1, \ldots, N_t \); 2) the basis vectors \( \mathbf{q}_i \) that define the space of inextensible internal mechanisms, i.e. \( \mathbf{B} \mathbf{q}_i = \mathbf{0} \) for \( i = 1, \ldots, N_q \); 3) the rank \( r_A \) of the equilibrium matrix \( \mathbf{A} \).

In general, structures with internal mechanisms are avoided in practice because their equilibrium is reached only for large displacements. However, there is the possibility that by prescribing a state of self-equilibrated prestress to a pin-jointed structure with internal mechanisms, the structure is able to carry the loads in a serviceable manner. This happens thanks to a first-order geometrical stiffness provided by the prestress that stabilizes the internal mechanisms. A generic self-equilibrated prestress state can be obtained as a linear combination of the basis vectors \( \mathbf{t}_i \) obtained form the SVD of \( \mathbf{A} \):

\[ \mathbf{t}_\alpha = \alpha_1 \mathbf{t}_1 + \cdots + \alpha_{N_t} \mathbf{t}_{N_t} = \mathbf{M}_t \mathbf{\alpha}, \quad \mathbf{\alpha} \in \mathbb{R}^{N_t} \quad (3) \]

where the columns of the matrix \( \mathbf{M}_t \) are the vectors \( \mathbf{t}_1, \ldots, \mathbf{t}_{N_t} \). Similarly, a generic internal mechanism can be obtained as linear combination of the basis vectors \( \mathbf{q}_i \) obtained form the SVD of \( \mathbf{A} \) as well:

\[ \mathbf{q}_\beta = \beta_1 \mathbf{q}_1 + \cdots + \beta_{N_q} \mathbf{q}_{N_q} = \mathbf{M}_q \mathbf{\beta}, \quad \mathbf{\beta} \in \mathbb{R}^{N_q} \quad (4) \]

where the columns of the matrix \( \mathbf{M}_q \) are the vectors \( \mathbf{q}_1, \ldots, \mathbf{q}_{N_q} \). To check whether the prestress \( \mathbf{t}_\alpha \) stabilizes the internal mechanisms of a given pin-jointed structure, we prescribe the prestress \( \mathbf{t}_\alpha \) to the structure. We then apply a perturbation to the structural configuration by imposing a generic inextensible mechanism \( \mathbf{q}_\beta \) on the structure, as described in (Quagliaroli et al., 2015). If the total work is positive, the activation of the mechanism \( \mathbf{q}_\beta \) requires a positive amount of energy, and hence the configuration of the structure with the prestress \( \mathbf{t}_\alpha \) is stable.

The total work can be expressed as a sum of elastic and geometric work contributions as follows:

\[ \mathcal{L} = \mathcal{L}_E + \mathcal{L}_G = \frac{1}{2} \mathbf{q}_\beta^T \mathbf{K}_E \mathbf{q}_\beta + \frac{1}{2} \mathbf{q}_\beta^T \mathbf{K}_G \mathbf{q}_\beta \quad (5) \]

where \( \mathbf{K}_E \) is the elastic stiffness matrix and \( \mathbf{K}_G \) is the geometric stiffness matrix. As discussed in (Quagliaroli et al., 2015), because \( \mathbf{q}_\beta \) is a mechanism that does not result in any elongation in the structural members \( \mathbf{A}^T \mathbf{q}_\beta = \mathbf{0} \) the elastic work is zero. Hence, the total work depends only on the geometric stiffness:

\[ \mathcal{L} = \mathcal{L}_G = \frac{1}{2} \mathbf{q}_\beta^T \mathbf{K}_G \mathbf{q}_\beta = \frac{1}{2} \mathbf{\beta}^T \mathbf{M}_G^T \mathbf{K}_G \mathbf{M}_G \mathbf{\beta} = \frac{1}{2} \mathbf{\beta}^T \mathbf{A} \mathbf{\beta} > 0, \quad \forall \mathbf{\beta} \in \mathbb{R}^m \quad (6) \]
The structural configuration is stable if the total work \( L \) is positive, or in other words if the eigenvalues of \( \Lambda \) are positive, with \( \Lambda = M_q^T K_q M_q \). It should be noted that in general a dome can remain stable even if some struts are subjected to tension and/or certain cables become slack. However, for safety reasons during the optimization process only the combination vectors \( \alpha \) that ensure that all cables are in tension and all struts are in compression are accepted. A technique used to ensure these requirements is discussed in (Yuan et al., 2007). Once a combination vector \( \alpha \) has been defined, the stability of the structural configuration can be checked as defined in Eq. (6). In the design approach discussed herein, the entries of the vector \( \alpha \) are part of the variables of the optimization problem.

2.2. Geometrically nonlinear structural analysis

![Figure 1: Global coordinate system \((x, y, z)\), local coordinate system \((p, q, r)\), and local displacements \((u, v, w)\) of the cable/strut element](image)

In the following, the geometrically nonlinear cable/strut element model considered for the structural analysis of cable domes is presented. It is based on the model described in (Broughton & Ndumbaro, 1994). In particular, the vector of internal forces of a cable/strut element \( \varepsilon \) in global nodal coordinates is:

\[
f_{\alpha} = T^T \mathbf{H}(u)^T \left( t_{\alpha,e} + \frac{E A}{L_0} H(u) \right)
\]

where \( t_{\alpha,e} \) is the \( e \)-th component of the prestress vector \( t_\alpha \), and \( u = [u, v, w]^T \) is the displacement vector of the element in local coordinates \((p, q, r)\) (see Fig. 1). It is defined as \( u = T q \), in which \( q \) is the vector of the element nodal displacements in global coordinates \((x, y, z)\) and \( T \) is a matrix that transforms the element nodal displacements from the global to the local coordinate system. Moreover, in Eq. (7): \( L_0 \) is the original element length; \( E A \) is the axial stiffness; \( H(u) \) relates the element elongation \( \varepsilon \), and the element displacements in local coordinates \( u \), i.e.:

\[
\varepsilon = H(u) = \sqrt{(L_0 + u)^2 + v^2 + w^2} - L_0;
\]

the row vector \( \mathbf{H}(u) \) relates the element elongation \( \varepsilon \) and the element local displacements \( u \) incrementally, i.e.:

\[
\delta \varepsilon = \mathbf{H}(u) \delta u = \left[ \frac{L_0 + u}{L_0 + \varepsilon} \frac{v}{L_0 + \varepsilon} \frac{w}{L_0 + \varepsilon} \right] \delta u.
\]

If we write the relation between the element internal forces and nodal displacements in global coordinates incrementally we obtain:

\[
\delta f_{\alpha} = T^T (\dot{\mathbf{H}}^T E A \dot{\mathbf{H}} + D) T \delta q
\]

where the expression

\[
\ddot{K}^e = T^T (\dot{\mathbf{H}}^T E A \dot{\mathbf{H}} + D) T
\]

defines the element incremental tangent stiffness matrix. In Eq. (11) the matrix \( D \) is defined as follows:

\[
D = \begin{bmatrix}
\frac{t}{(L_0 + \varepsilon)^3} & -\frac{tw}{(L_0 + \varepsilon)^3} & -\frac{tw}{(L_0 + \varepsilon)^3} \\
-\frac{tw}{(L_0 + \varepsilon)^3} & \frac{t((L_0 + u)^2 + v^2)}{(L_0 + \varepsilon)^3} & -\frac{tvw}{(L_0 + \varepsilon)^3} \\
-\frac{tw}{(L_0 + \varepsilon)^3} & -\frac{tvw}{(L_0 + \varepsilon)^3} & \frac{t((L_0 + u)^2 + v^2)}{(L_0 + \varepsilon)^3}
\end{bmatrix}
\]

where \( t \) is the total axial force in each cable/strut element due to the prestress and external loads (i.e., total tension or compression).

The nonlinear equilibrium equations are solved in terms of nodal displacements using a Newton–Raphson iterative scheme. The global system of equilibrium equations takes the form:

\[
r(q, t_\alpha, x) = f_{\alpha} - f_{\alpha \varepsilon}(x) = 0
\]
where $f_{int}$ is a vector with dimensions $[N_d \times 1]$ collecting the resultants of the elements’ internal forces defined in Eq. (7), i.e. $f_{int} = \sum_{e=1}^{N_e} f_{int}^e$, where $N_d$ is the number of free degrees of freedom, $N_e$ is the number of structural elements, and $\sum$ here represents the assembly operation. $f_{ext}$ is a vector with dimensions $[N_d \times 1]$ with the values of the external loads applied on the structural joints (including the self-weight of the structure).

3. Optimization problem formulation

In the optimization problem discussed herein, both the prestress magnitude ($\alpha$) and the diameters of the structural elements ($\phi$) of cable domes are simultaneously optimized. We consider a basis made of $N_t$ vectors describing the prestress vector space. These vectors are precomputed as described in Sec. 2.1. Hence, a generic self-equilibrated state of prestress given to the structure is a combination of these vectors. We also consider $N_s$ structural elements subdivided into $N_{gr}$ size-groups, i.e. elements with the same diameter size. The elements are grouped based on symmetry of the structure or by engineering judgment. As a consequence, in the optimization problem there are $N = N_t + N_{gr}$ variables, namely the pre-stress variables $\alpha_i$ for $i = 1, \ldots, N_t$, and the diameter variables $\phi_j$ for $j = 1, \ldots, N_{gr}$. In the optimization problem formulation, we consider normalized optimization variables $x_{\alpha,i} \in [0, 1]$ and $x_{\phi,j} \in [0, 1]$, such that: $\alpha_i = x_{\alpha,i} \bar{\alpha}$ for $i = 1, \ldots, N_t$, and $\phi_j = x_{\phi,j} \bar{\phi}$ for $j = 1, \ldots, N_{gr}$. The coefficient $\bar{\alpha}$ defines the maximum available value of prestress, and the coefficient $\bar{\phi}$ defines the maximum available diameter. These transformations of variables have beneficial effects from a numerical point of view, because they define a design domain that is handled more easily by optimization algorithms. For convenience, the vectors with the design variables $x_\alpha$ and $x_\phi$ are collected in the vector $x$. In the following, we provide more details regarding the functions involved in the optimization problem formulation. In particular, we discuss the formulation of the objective function minimized, and the formulation of the displacement and force constraints.

3.1. Objective function

The objective function minimized in the optimization problem is the structural weight:

$$f(x) = \sum_{i=1}^{N_e} \sum_{j=1}^{N_{gr}} \pi \left( \frac{c_{ij} x_{\phi,j} \bar{\phi}}{4} \right)^2 \bar{\rho}_s$$

(14)

where $c_{ij}$ is equal to 1 if the $i$-th element belongs to the $j$-th size-group and it is equal to 0 otherwise, $\bar{\rho}_s$ is the initial length of each element, and $\rho_s$ is the material density of the structural elements.

3.2. Displacement constraint

We impose a constraint on the vertical displacements in the z direction of the structural joints. To reduce the number of constraints, and hence the computational cost required for the calculation of the gradients of the constraints in the sensitivity analysis, we aggregate the displacement constraints into a single constraint on the maximum (i.e. largest) displacement in the $x$ direction. Thus, the aggregated displacement constraint is defined as follows:

$$g_1(q, x) = \max \left( \frac{I_q \cdot q(x)}{q_{z,allow}} \right) \leq 1$$

(15)

where $I_q$ is a diagonal matrix with ones on the diagonal only in correspondence of the $z$ degrees of freedom, $q$ is the vector of displacements, and $q_{z,allow}$ is the maximum displacement allowed, which is predefined. Eq. (15) defines a non-differentiable constraint. We redefine the constraints of Eq. (15) by approximating the max function:

$$g_1(q, x) = \frac{1}{q_{z,allow}} \frac{1^T \mathcal{D}(I_q \cdot q(x))^{p+1} 1}{1^T \mathcal{D}(I_q \cdot q(x))^{p} 1} \leq 1$$

(16)

where $1$ is a unity vector with dimensions $[N_d \times 1]$, and $\mathcal{D}(\cdot)$ is an operator that transforms a vector into a diagonal matrix with the vector entries on the diagonal. For increasing values of $p$, Eq. (16) approximates with increasing level of accuracy Eq. (15). The effect of the parameter $p$ is analyzed in Appendix A. The advantage is that this reformulation is differentiable and allows for the calculation of the analytical gradient of the displacement constraint function $g_1$. 


3.3. Prestress force constraint

We consider also constraints on the axial forces in the cable and strut elements due to the initial prestress only. For each cable $i$ we consider the following force constraint based on the material strength:

$$
\bar{t}_{ca,i} = \frac{t_{ca,i}(x)}{\sigma_{adm} A_i(x)} \leq 1, \quad \text{for } i = 1, \ldots, N_{cable}
$$

where $t_{ca,i}$ is the tension force in the cable $i$ due to the prestress $t_a$ only, $\sigma_{adm}$ is the maximum admissible stress in the cables, $A_i$ is the cross-section area of the cable $i$, $\gamma_{ca}$ is a safety factor, and $N_{cable}$ is the number of cable elements. For each strut $j$ we consider the following force constraint on the strut instability:

$$
\bar{t}_{sa,j} = \frac{t_{sa,j}(x)}{-\pi^2 E I_j(x)} \leq 1, \quad \text{for } j = 1, \ldots, N_{strut}
$$

where $t_{sa,j}$ is the compression force in the strut $j$ due to the prestress $t_a$ only, $E$ is the Young's modulus, $I_j$ is the cross-section inertia, $L_{0,j}$ is the initial strut length, $\gamma_{sa}$ is a safety factor, and $N_{strut}$ is the number of strut elements. Eq. (17) and Eq. (18) generate $N_{cable} + N_{strut} = N_e$ force constraints. We can group the constraints on the elements’ axial forces due to the initial prestress in a single vector $\bar{t}_a$:

$$
\bar{t}_a = [\bar{t}_{ca,1} \ldots \bar{t}_{ca,N_{cable}} \bar{t}_{sa,1} \ldots \bar{t}_{sa,N_{strut}}]^T
$$

and aggregate the constraints of Eq. (17) and Eq. (18) into a single constraint as follows:

$$
g_2(\bar{t}_a, x) = \max_i(\bar{t}_{ca,i}(x)) \leq 1, \quad \text{for } i = 1, \ldots, N_e
$$

As for the displacement constraint, here too we approximate the constraint of Eq. (20) with a differentiable formulation:

$$
g_2(\bar{t}_a, x) = \frac{1^T D(\bar{t}_a(x)) p + 1}{1^T D(\bar{t}_a(x)) p} \leq 1
$$

where as in Eq. (16), for increasing values of $p$ Eq. (21) approximates with increasing level of accuracy Eq. (20).

3.4. Total force constraint

Lastly, we introduce additional constraints on the axial forces in the cable and strut elements due to the initial prestress, the structural self-weight, and the external loads applied to the structural joints. For each cable $i$ we consider the following force constraint based on the material strength:

$$
\bar{t}_{c,i} = \frac{t_{c,i}(x)}{\sigma_{adm} A_i(x)} \leq 1, \quad \text{for } i = 1, \ldots, N_{cable}
$$

where $t_{c,i}$ is the total force in the cable $i$, and $\gamma_c$ is a safety factor. For each strut $j$ we consider the following force constraint on the strut instability:

$$
\bar{t}_{s,j} = \frac{t_{s,j}(x)}{-\pi^2 E I_j(x)} \leq 1, \quad \text{for } j = 1, \ldots, N_{strut}
$$

where $t_{s,j}$ is the total force in the strut $j$, and $\gamma_s$ is a safety factor. Eq. (22) and Eq. (23) generate $N_{cable} + N_{strut} = N_e$ total force constraints. We group the total force constraints in a single vector $\bar{t}$:

$$
\bar{t} = [\bar{t}_{c,1} \ldots \bar{t}_{c,N_{cable}} \bar{t}_{s,1} \ldots \bar{t}_{s,N_{strut}}]^T
$$

and aggregate the constraints of Eq. (22) and Eq. (23) into a single constraint as follows:

$$
g_3(\bar{t}, x) = \max_i(\bar{t}_i(x)) \leq 1, \quad \text{for } i = 1, \ldots, N_e
$$

As for the previous displacement and prestress constraints, here too we approximate the constraint of Eq. (25) with a differentiable formulation:

$$
g_3(\bar{t}, x) = \frac{1^T D(\bar{t}(x)) p + 1}{1^T D(\bar{t}(x)) p} \leq 1
$$

where, as already explained for $g_1$ and $g_2$, for increasing values of $p$ Eq. (26) approximates with increasing level of accuracy Eq. (25).

In principle, we could introduce additional bounds for the total forces in the elements. These could be defined as $t_{c,i} > 0$ and $t_{s,j} < 0$ if the $i$-th element is a cable and the $j$-th element is a strut. Nevertheless, through initial numerical experiments it was observed that these constraints were not needed because the requirement of cables in tension and struts in compression was already enforced by the prestress given to the structural elements (for which $t_{c,i} > 0$ and $t_{s,j} < 0$) and the effect of the vertical loads.
3.5. Final optimization problem formulation

The final optimization problem at hand for the pre-stress and size design of cable-strut structural assemblies is stated as follows:

\[
\begin{align*}
\text{minimize:} & \quad f(x) \\
\text{subject to:} & \quad g_1(q, x) \leq 1 \\
& \quad g_2(f, x) \leq 1 \\
& \quad g_3(t, x) \leq 1 \\
& \quad 0 < x_i \leq 1, \quad \text{for } i = 1, \ldots, N \\
\text{with:} & \quad r(q, t, x, x) = 0
\end{align*}
\]

where \( N \) is the number of variables; \( x \) is the vector of the design variables; \( f, g_1, g_2, \) and \( g_3 \) are the objective and constraint functions previously defined; and \( r(q, t, x, x) \) is the residual vector of the equilibrium equations defined in Eq. (13). The optimization problem of Eq. (27) is a nonlinear and nonconvex optimization problem. The problem is solved with a Sequential Linear Programming (SLP) algorithm implemented by the author. For more information and details regarding SLP and nonlinear constrained optimization, the interested reader is referred to the relevant literature (Nocedal & Wright, 2006; Christensen & Klarbring, 2008). The SLP algorithm used in this work is an iterative optimization process, where in every iteration the problem is linearized and solved locally. Thus, the gradients of the objective function (\( \nabla f \)) and of the constraints (\( \nabla g_1, \nabla g_2, \nabla g_3 \)) are evaluated in every optimization iteration. The gradient \( \nabla f \) is straightforward because the objective function is defined explicitly in terms of the optimization problem variables. Similarly, also the gradient \( \nabla g_2 \) can be calculated directly. The constraints’ gradients \( \nabla g_1, \nabla g_2, \nabla g_3 \), instead, require additional adjoint sensitivity analyses for their calculation. The details of the sensitivity analysis and other computational considerations related to the optimization algorithm implementation are provided in the following section.

4. Sensitivity analysis and computational considerations

In the following section, the details of the sensitivity analyses for the calculation of the gradients of the constraint functions are provided (Michaleris et al., 1994). The objective function gradient is straightforward because it is formulated explicitly in terms of the design variables. Hence, it does not require particular care. The section is concluded with practical computational considerations for the implementation of the SLP algorithm.

4.1. Sensitivity analysis

The gradient of the constraint function \( g_1 \) requires and adjoint sensitivity analysis and it is calculated by first defining the augmented function \( \hat{g}_1 \):

\[
\hat{g}_1(q, x) = g_1(q, x) + \lambda^T(I_{\text{int}} - I_{\text{ext}})
\]

Eq. (28) defines the augmented function \( \hat{g}_1(q, x) \), obtained by adding a zero term to \( g_1(q, x) \). At equilibrium \( I_{\text{int}} - I_{\text{ext}} = 0 \) (Eq. (13)), and thus \( \hat{g}_1(q, x) = g_1(q, x) \). The gradient of \( \hat{g}_1 \) is then calculated as follows:

\[
\nabla \hat{g}_1 = \frac{\partial \hat{g}_1}{\partial q} + \frac{\partial \hat{g}_1}{\partial x} + \left( \frac{\partial f_{\text{int}}}{\partial q} + \frac{\partial f_{\text{int}}}{\partial x} - \frac{df_{\text{ext}}}{dx} \right) \lambda_1
\]

(29)

where \( \frac{\partial g_1}{\partial x} = 0 \). To avoid the calculation of the implicit derivative of the displacement vector with respect to the design variables (i.e., \( \frac{\partial d_0}{\partial x} \)), in Eq. (29) we collect the terms multiplied by \( \frac{\partial d_0}{\partial q} \) and equate them to zero:

\[
K \lambda_1 = -\frac{\partial g_1}{\partial q}
\]

(30)

where we have used the definition of Eq. (11), which implies that \( K = \frac{\partial f_{\text{int}}}{\partial q} \). Moreover, in Eq. (30) the vector \( \frac{\partial d_0}{\partial q} \) is defined as follows:

\[
\frac{\partial g_1}{\partial q} = \frac{1}{\text{den}_{g_1}} \left( \text{den}_{g_1} (p + 1) D(I, q(x))^p I_1 \right)
\]

(31)

\[
- \text{num}_{g_1} p D(I, q(x))^{p-1} I_1
\]

(32)

with

\[
\text{num}_{g_1} = I^T D(I, q(x))^{p+1} I_1, \quad \text{den}_{g_1} = I^T D(I, q(x))^p I_1
\]
The adjoint sensitivity analysis necessary for the calculation of $\nabla g_1$ consists in the solution of the linear system given in Eq. (30). Once the adjoint variables $\lambda_1$ have been calculated, and given that $\hat{g}_1(x) = g_1(x)$, we can finally calculate the displacement constraint gradient:

$$\nabla g_1 = \left( \frac{\partial f_{int}}{\partial x} - \frac{\partial f_{ext}}{\partial x} \right) \lambda_1$$

(33)

At the element level, in Eq. (33) the derivative $\frac{\partial f_{int}}{\partial x}$ for an element $e$ with respect to the $i$-th prestress basis vector is:

$$\frac{\partial f_{int}}{\partial x_{\alpha,i}} = T^T \bar{H}(u)^T \bar{t}_{i,e}$$

(34)

Similarly, the derivative with respect to the diameter size variable for an element $e$ that belongs to the $j$-th diameter size-group is calculated as follows:

$$\frac{\partial f_{int}}{\partial x_{\phi,j}} = T^T \bar{H}(u)^T \left( \frac{E H(u)}{2} \frac{L_0}{L_0} \phi_j \right) \bar{\phi}$$

(35)

The derivative $\frac{\partial f_{ext}}{\partial x}$ in Eq. (33) at the element level is:

$$\frac{d f_{ext,i}}{d x_{\alpha,i}} = \frac{d f_{ext,y}}{d x_{\alpha,i}} = \frac{d f_{ext,z}}{d x_{\alpha,i}} = 0$$

$$\frac{d f_{ext,i}}{d x_{\phi,i}} = \frac{d f_{ext,y}}{d x_{\phi,i}} = \frac{d f_{ext,z}}{d x_{\phi,i}} = 0$$

(36)

where the expressions of Eq. (36) apply to both ends of each structural element $e$. Moreover, it is assumed that the element $e$ belongs to the $i$-th diameter size-group, and $N_{L,i}$ is the number of loaded joints on which the total structural weight is applied as external force in the $z$ direction.

The gradient of the constraint function $g_2(x)$ does not require an adjoint sensitivity analysis because it is defined explicitly in terms of the design variables. In particular, the gradient is first defined as:

$$\nabla g_2(x) = \frac{d \bar{t}_{\alpha}}{d x} \frac{\partial g_2}{\partial \bar{t}_{\alpha}} + \frac{\partial g_2}{\partial x}$$

(37)

where $\frac{\partial g_2}{\partial x} = 0$ and:

$$\frac{\partial g_2}{\partial \bar{t}_{\alpha}} = \frac{1}{\text{den}_g^2} \left( \frac{\text{den}_g^2}{(p+1)} D(\bar{t}_{\alpha}(x))^p \right) \left( \frac{\text{num}_g^2}{p D(\bar{t}_{\alpha}(x))^{p-1}} \right) \mathbf{1}$$

(38)

with

$$\text{num}_g^2 = 1^T D(\bar{t}_{\alpha}(x))^{p+1} \mathbf{1}, \text{den}_g^2 = 1^T D(\bar{t}_{\alpha}(x))^p \mathbf{1}$$

(39)

In Eq. (37), the derivative $\frac{\partial f_{int}}{\partial x}$ for an element $i$ that belongs to the geometric size-group $j$ is defined as follows:

$$\frac{d \tilde{f}_{i,j}}{d x_{\alpha,k}} = \frac{1}{\gamma_{\alpha,i} \gamma_{\phi,j}} \tilde{t}_{k,i}$$

$$\frac{d \tilde{f}_{i,j}}{d x_{\phi,j}} = \frac{1}{\gamma_{\phi,j} \gamma_{\phi,j}} (\frac{E}{L_0} \bar{\phi})$$

(40)

if $i$ is a cable, or

$$\frac{d \tilde{f}_{i,j}}{d x_{\alpha,k}} = -\frac{1}{\gamma_{\alpha,i} \gamma_{\phi,j}} \tilde{t}_{k,i}$$

$$\frac{d \tilde{f}_{i,j}}{d x_{\phi,j}} = -\frac{1}{\gamma_{\phi,j} \gamma_{\phi,j}} (\frac{E}{L_0} \bar{\phi})$$

(41)

if $i$ is a strut, and where $\tilde{t}_{k,i}$ is the $i$-th component of the $k$-th vector of the prestress vector basis.

Lastly, the gradient of the constraint function $g_3$ is calculated with an adjoint sensitivity analysis. We first define the augmented function $\tilde{g}_3$:

$$\tilde{g}_3(x) = g_3(x) + \lambda_3^T (f_{int} - f_{ext})$$

(42)

We then differentiate the augmented function as follows:

$$\nabla \tilde{g}_3(x) = \frac{d q}{d x} \frac{\partial \tilde{t}}{\partial \tilde{t}} \frac{\partial g_3}{\partial \tilde{t}} + \frac{\partial q}{d \bar{t}} \frac{\partial g_3}{\partial \bar{t}} + \frac{\partial g_3}{\partial x}$$

(43)
where \( \frac{\partial g_3}{\partial x} = 0 \) and:

\[
\frac{\partial g_3}{\partial t} = \frac{1}{\text{den}_{g_3}} \left( \text{den}_{g_3} (p+1) D(t(x))^p \mathbf{1} - \text{num}_{g_3} p D(t(x))^{p-1} \mathbf{1} \right)
\]

(44)

with

\[
\text{num}_{g_3} = 1^T D(t(x))^{p+1} \mathbf{1}, \quad \text{den}_{g_3} = 1^T D(t(x))^{p} \mathbf{1}
\]

(45)

Based on Eq. (22) and (23), \( \frac{\partial t}{\partial x} \) is defined as follows:

\[
\frac{\partial t}{\partial x} = D \left( \frac{1}{t_{allow}} \right)
\]

(46)

where

\[
t_{allow,i} = \frac{\sigma_{adm} A_i}{\gamma_c} \text{ if element } i \text{ is a cable}
\]

(47)

\[
t_{allow,i} = -\frac{\pi^2 E I_i}{\gamma_s L_0,i} \text{ if element } i \text{ is a strut}
\]

(48)

In Eq. (43), the \( i \)-th component of the derivative \( \frac{\partial t}{\partial x} \), which is associated to the \( i \)-th structural element, is:

\[
\frac{\partial t_i}{\partial q} = \frac{E A_i}{L_{0,i}} \tilde{H}(u) \mathbf{T}
\]

(49)

Moreover, in Eq. (43), the components of the derivative \( \frac{\partial t}{\partial x} \) associated to the \( i \)-th element are defined as follows:

\[
\frac{\partial t_i}{\partial x_{a,k}} = \frac{1}{\sigma_{adm} A_i} \tilde{t}_{k,i} \tilde{\alpha}, \\
\frac{\partial t_i}{\partial x_{\phi,j}} = \frac{t_i}{\sigma_{adm} \phi} \frac{d(A_i^{-1})}{d\phi_j} \phi + \frac{1}{\sigma_{adm} A_i} E H(u) \frac{dA_i}{d\phi_j} \phi = \frac{\pi \phi_j}{2}
\]

(50)

if element \( i \) is a cable, or

\[
\frac{\partial t_i}{\partial x_{a,k}} = -\frac{2 E T_k}{\gamma_c I_0} \tilde{t}_{k,i} \tilde{\alpha}, \\
\frac{\partial t_i}{\partial x_{\phi,j}} = -\frac{E I_i}{\gamma_s L_0,i} \frac{d(I_i^{-1})}{d\phi_j} \phi + \frac{1}{\sigma_{adm} A_i} E H(u) \frac{dA_i}{d\phi_j} \phi
\]

(51)

if element \( i \) is a strut, for \( i = 1, \ldots, N_e \), \( j = 1, \ldots, N_{gr} \), and \( k = 1, \ldots, N_t \). In Eq. (49) and (48) \( \frac{d(A_i^{-1})}{d\phi_j} \) and \( \frac{d(I_i^{-1})}{d\phi_j} \) are defined in Eq. (40) and (41). Also in this case, as for the gradient of \( g_1 \), we collect the terms multiplied by the unknown implicit derivatives \( \frac{d\lambda_3}{dt} \) in Eq. (43) and equate them to zero. As a result of this procedure, we obtain the following adjoint system of equations:

\[
K \lambda_3 = -\frac{\partial t}{\partial q} \frac{\partial t}{\partial x} \frac{\partial g_3}{\partial t} - \lambda_3 \left( \frac{\partial f_{int}}{\partial x} - \frac{\partial f_{ext}}{\partial x} \right)
\]

(52)

where we used the equivalence \( \tilde{g}_3(x) \equiv g_3(x) \), and the results of Eq. (34), (35) and (36).

4.2. Computational considerations

To successfully adopt existing algorithms for the nonlinear optimization problem at hand defined in Eq. (27), practical and conservative measures are included in the implementation of the optimization approach proposed here. These include the management of the linearized constraints; a continuation scheme for the control of the values attained by certain parameters; and convergence criteria.

4.2.1. Managing the linear approximations of the constraints

We solve the optimization problem stated in Eq. (27) with a modified SLP approach inspired by the cutting planes method (Kelley, 1960), and implemented by the author. In every iteration of standard SLP a linear subproblem is solved. In the algorithm used herein, the subproblems grow in dimension because in each iteration a new linearized local approximation of each of the aggregated constraints (Eq. (16), (21), (26)) is added to the set of constraints considered. Because the problem at hand is in principle nonconvex, it may happen that a constraint is active even though the current solution strictly falls into
the feasible domain. In other words, it may happen that a linearized constraint cuts the feasible domain directing the algorithm towards a very conservative solution. In the SLP algorithm developed and used in this work, these undesired constraints are disregarded and removed from the set of linearized constraints considered in the subsequent optimization iterations.

4.2.2. Continuation scheme

The optimization problem stated in Eq. (27) includes several highly nonlinear components, namely, the differential equivalents of the max functions in the aggregated constraints. Therefore, difficulties to converge smoothly towards a good optimized solution are expected. A common approach for promoting a smooth convergence of the optimization process is to gradually increase the parameters that control the degree of nonlinearity. This applies to the parameter $p$ in Eq. (16), (21), and (26), which is increased during the optimization analysis from a minimum to a maximum value by given steps: $p = [p_{\text{min}} : \Delta p : p_{\text{max}}]$. Furthermore, a conservative move limit strategy is applied in the solution of the subproblems, meaning that in each optimization iteration $i$ the updates of $x$ are searched in a close neighborhood of the solution corresponding to the previous iteration $i-1$: $x_{i-1} - m_i \leq x_i \leq x_{i-1} + m_i$. Specific details regarding the values of these parameters are given in the numerical examples of Section 5.

4.2.3. Convergence criteria

The methodology is assumed to have reached the final solution in a $i$-th optimization iteration after a maximum of $i_{\text{max}}$ iterations, or once we have that $\Delta x < \delta$, where $\Delta x = \|x_i - x_{i-1}\|$. The values of $i_{\text{max}}$ and $\delta$ are given in Section 5.

5. Numerical applications

In this section, we discuss the results obtained in two numerical applications. The first concerns the prestress and size optimization of a simple structure made of two cables with three aligned hinges. This first illustrative example is used to discuss the optimization approach and the results also through graphic illustrations. Next, we discuss the prestress and size optimization of a Geiger dome. This represents a more realistic structure that shows the potential of the optimization approach discussed herein.

The nonlinear structural analysis and sensitivity analysis codes have been implemented by the author in Python 3.7. Also the SLP optimization algorithm has been implemented in Python 3.7 by the author, using the Linear Programming solver of Gurobi Optimizer 9.0.2 (Gurobi Optimization LLC, 2020). All numerical simulations have been performed on a PC with 16 Gb of RAM and a Intel i7 processor at 2.11 GHz running Microsoft Windows 10.

Lastly, in the following examples we consider cables and struts made of steel and with circular cross-sections, with a Young’s modulus $E = 210 \text{ GPa}$ and a density $\rho_{\text{steel}} = 7860 \text{ kg/m}^3$.

5.1. An illustrative example

A schematic representation of the first structure considered is shown in Fig. 2. It is made of two pin-jointed prestressed cables aligned and hinged to the ground. An external load $F_{\text{ext}} = -7.5 \text{ kN}$ is applied to the internal hinge. The initial length of the cables is $L_0 = 1 \text{ m}$. If we perform the matrix analysis described in Sec. 2.1 on this structure, we learn that the structure considered has one mechanism and one state of prestress capable of stabilizing the mechanism (i.e., $N_t = 1$, $N_q = 1$). The mechanism allows for a vertical displacement of point B. However, through a state of prestress $t_0$, the cables gain a first order geometric stiffness that can resist the external load and that stabilizes the internal mechanism: $dF_{\text{ext}} = 2\frac{L_0}{L_0^3}dz + 2\frac{EA}{L_0^3}dz^3$, where $dz$ is an infinitesimal vertical displacement of the internal hinge B. The prestress vector basis is made of one vector with equal unit entries $t = [1.0 \ 1.0]^T$. A positive sign is assigned to the vector $t$ entries to ensure that both cables are...
in tension. Thus, for this structure we consider two
design variables: one prestress amplification factor
\( \alpha = x_\alpha \Delta \alpha \) such that \( t_\alpha = \alpha \bar{t} \); and one size-group for
the cables defined by the diameter \( \phi = x_\phi \bar{\phi} \) (i.e. both
cables have the same size). The fact that we consider
only two variables facilitates the visualization of the
optimization problem design domain, which is shown
in Fig. 3.

In the optimization analysis \( x_\alpha \) and \( x_\phi \) are both
initialized to one, \( \bar{\alpha} = 5.5 \cdot 10^2 \text{kN} \), and \( \bar{\phi} = 50 \text{ mm} \). These values are selected in order to start
from a feasible initial design. For the numerical
simulations we consider the following settings: the
maximum allowed vertical displacement in Eq. (16)
is \( q_{z,\text{allow}} = 2 L_0 / 250 = -8 \text{ mm} \); the admissible
stress is \( \sigma_{\text{adm}} = 430 \text{ MPa} \); the safety coefficients
are \( \gamma_{c\alpha} = \gamma_{s\alpha} = 1.25 \), and \( \gamma_c = \gamma_s = 1.0 \). The coeffi-
cient \( \rho \) in Eq. (16), (21), (26) is initialized to 1 and
kept constant in this example because there is only
one vertical displacement constrained. The moving
limit described in Sec. 4.2.2 for the prestress variable
\( x_\alpha \) is set to \( m_{l\alpha} = 0.005 \), and for the diameter vari-
able \( x_\phi \) is set to \( m_{l\phi} = 0.01 \). For convergence, the
maximum number of optimization iterations allowed
is \( i_{\text{max}} = 500 \), and the relative norm of the variables’
update needs to be less or equal to \( \delta = 10^{-4} \) (see Sec.
4.2.3).

The optimization analysis converged in 29 itera-
tions after 0.6 sec, and the final solution was \( x^{\text{opt}} = [x_\phi^{\text{opt}} \quad x_\alpha^{\text{opt}}]^T = [0.838 \quad 0.860]^T \). Fig. 3 shows a
portion of the design domain, with the contour of the
objective function and of the three constraints. In the
figure the blue circles represents intermediate designs
attained during the optimization analysis. The final
optimized design is labeled with a star. It can be
observed that in correspondence of the final solution
two constraints are active, namely the displacement
constraint \( g_1 \) and the force constraint due to the pre-
stress only \( g_2 \). This can also be observed in Fig. 4,
that shows the evolution of the values of the objective
function and of the three constraints during the
optimization iterations. From Fig. 4 it is possible
also to observe that through the optimization process
the structural weight is reduced approximately of the 30%.
The final structural weight of the opti-
mized structure is 21.7 kg.
5.2. **Optimization of a Geiger dome**

We discuss now the results obtained from the optimization of a Geiger dome without inner hoop. In particular, Fig. 5 shows a perspective view of the structure considered. The top and side views of the structure are shown in Fig. 6 and Fig. 7, respectively. The structure is composed of 121 structural elements of which 25 are struts and 96 cable elements.

![Figure 5: View of the Geiger dome optimized in Sec. 5.2](image)

There are 62 joints, of which 12 are fixed and connected to the ground. For this structure, the matrix analysis of Sec. 2.1 reveals that the prestress vector basis of the structure is made of one prestress vector $\tilde{t}$ that stabilizes all of the 30 internal mechanisms of the structure. The prestress vector is listed in Table 1, and its entries satisfy the requirement of cables in tension and struts in compression. Thus, this prestress vector is prescribed to the structure and its amplification factor $\alpha = x_{\alpha} \bar{\alpha}$ is optimized. The prestress prescribed for the structure is then $t_{\alpha} = \alpha \tilde{t}$. Due to the high degree of symmetry of the structure, the structural elements are divided into 11 size-groups (i.e., groups of elements with the same properties). The grouping is done based on the topology of the structure as shown in Fig. 8. The diameters assigned to each size-group are also design variables of the optimization problem: $\phi_i = x_{\phi,i} \bar{\phi}$ for $i = 1, ..., 11$. Hence, in this example the optimization problem consists of

![Figure 6: Top view of the Geiger dome optimized in Sec. 5.2](image)

![Figure 7: Side view of the Geiger dome optimized in Sec. 5.2](image)

<table>
<thead>
<tr>
<th>Size-group</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prestress</td>
<td>-0.389</td>
<td>0.391</td>
<td>0.232</td>
<td>-0.096</td>
<td>0.627</td>
<td>0.374</td>
</tr>
<tr>
<td>Size-group</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>Prestress</td>
<td>-0.196</td>
<td>1.000</td>
<td>0.483</td>
<td>0.696</td>
<td>0.851</td>
<td></td>
</tr>
</tbody>
</table>

![Table 1: Prestress vector $\tilde{t}$ of the Geiger dome optimized in Sec. 5.2. The size-groups are shown in Fig. 8](image)

![Figure 8: Size-group division of the structural elements of the Geiger dome optimized in Sec. 5.2](image)
12 design variables.

In the optimization analysis $\bar{\alpha} = 30 \text{ MN}$, and $\bar{\phi} = 300 \text{ mm}$. These values are chosen such that the resulting design domain has feasible solutions. If needed, different values could be selected depending on the requirements of the problem. The variables are initialized such that $x_{\alpha} = 0.5$, and $x_{\phi, i} = 1$ for $i = 1, \ldots, 11$. Moreover, the maximum allowed vertical displacement in Eq. (16) is $D/250 = 60 \text{ m}/250$ where $D$ is the diameter of the Geiger dome considered. Thus, $q_{z,\text{allow}} = -240 \text{ mm}$. A distributed external load of $5 \text{ kN/m}^2$ is imposed on the external surface of the structure. This load could represent a snow load, for example. The load is applied in terms of equivalent concentrated forces on the top joints of the structure. Additionally, also the structural self-weight is considered. The moving limit described in Sec. 4.2.2 for the prestress variable $x_{\alpha}$ is set to $m_{\alpha} = 5 \cdot 10^{-4}$, and for the diameter variables $x_{\phi, i}$ is set to $m_{\phi} = 5 \cdot 10^{-3}$. Moreover, the coefficient $p$ in Eq. (16), (21), (26) is initialized to 100 and increased by steps of 100 every optimization iteration up to a maximum value of 1000. All of the remaining parameters are set as in the previous example of Sec. 5.1.

The optimization analysis terminated after 500 iterations with a computational time of approximately 3 min and 8 sec. The final values of the optimized design variables are listed in Table 2. The evolution of the objective and constraint functions during the optimization analysis is shown in Fig. 9. In correspondence of the final optimized solution, the constraints values are $g_1 = 0.559$, $g_2 = 0.924$ and $g_3 = 0.969$. Moreover, during the optimization analysis the initial structural weight is reduced approximately of 80%. The final structural weight is 121.9 t.

6. Final considerations

A novel optimization approach for the simultaneous design of the prestress and size of the structural elements of cable domes is presented. The matrix theory for the analysis of pin-jointed cable-strut structural assemblies is used to identify possible states of self-equilibrated prestress. The stiffening effect of the prestress on the internal mechanisms of the structures considered is then evaluated. Once a suitable state of prestress is found, its magnitude and the structural elements’ sizes are simultaneously optimized by the proposed approach. In particular, in the optimization process the structural weight is minimized, and constraints are imposed on the nodal displacements, on the tension in the cables, and on the compression in the struts for different loading conditions. The optimization problem at hand is efficiently solved with a sequential linear programming approach. To reduce the computational cost required by the sensitivity analyses, each set of displacement and force constraints is aggregated into a single differentiable constraint on the maximum value. The constraints’ gradients are consistently calculated with adjoint sensitivity analyses, whose details are provided in the paper.

The optimization approach discussed herein is first applied to the design of a simple 2-D structure made of two pin-jointed aligned cables. This structure has an internal mechanism that makes it unserviceable from a practical engineering point of view. However, in the example it is shown that it is possible to identify a prestress state that stabilizes the internal mechanism turning the structure into a serviceable one. The prestress and size of the cables are then optimized with the proposed approach. This simple example allows also for a graphical visualization of the optimization problem at hand. It is thus possible to observe that proposed approach in this case converges towards the global optimal solution. Moreover, the optimized design has a weight reduced of 30
Table 2: Final optimized values of the design variables in the example of Sec. 5.2

<table>
<thead>
<tr>
<th>Variable</th>
<th>$x_\alpha$</th>
<th>$x_{\phi,1}$</th>
<th>$x_{\phi,2}$</th>
<th>$x_{\phi,3}$</th>
<th>$x_{\phi,4}$</th>
<th>$x_{\phi,5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimized value</td>
<td>0.250</td>
<td>0.436</td>
<td>0.363</td>
<td>0.278</td>
<td>0.365</td>
<td>0.460</td>
</tr>
<tr>
<td>Optimized design</td>
<td>7.5 MN</td>
<td>130.8 mm</td>
<td>108.9 mm</td>
<td>83.4 mm</td>
<td>109.5 mm</td>
<td>138.0 mm</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variable</th>
<th>$x_{\phi,6}$</th>
<th>$x_{\phi,7}$</th>
<th>$x_{\phi,8}$</th>
<th>$x_{\phi,9}$</th>
<th>$x_{\phi,10}$</th>
<th>$x_{\phi,11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimized value</td>
<td>0.354</td>
<td>0.563</td>
<td>0.585</td>
<td>0.442</td>
<td>0.486</td>
<td>0.587</td>
</tr>
<tr>
<td>Optimized design</td>
<td>106.2 mm</td>
<td>168.9 mm</td>
<td>175.5 mm</td>
<td>132.6 mm</td>
<td>145.8 mm</td>
<td>176.1 mm</td>
</tr>
</tbody>
</table>

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Appendix A. Approximation of the max function

We analyze the effect of the parameter $p$ in the max function approximation used in Eq. (16), Eq. (21), and Eq. (26). We consider the vector $\mathbf{v}$ with entries ranging from 1 to 10 spaced with steps of 1, i.e. $\mathbf{v} = [1, 2, ..., 10]^T$. The maximum value of $\mathbf{v}$ is: $v_{\text{max}} = \max(\mathbf{v}) = 10$. We approximate the max function with:

$$\tilde{v}_{\text{max}} = \frac{1^T D(\mathbf{v})^p + 1}{1^T D(\mathbf{v})^p + 1} \quad (A.1)$$

with $\tilde{v}_{\text{max}} \approx v_{\text{max}}$. We define the error of the max function approximation as: $\text{error} = (v_{\text{max}} - \tilde{v}_{\text{max}})/v_{\text{max}}$. Fig. A.10 shows the relative error between the true maximum value and the approximated maximum value of $\mathbf{v}$. For $p = 1$ we have a 30% error, and for $p = 50$ we have a 0.06% error. If we increase the resolution of $\mathbf{v}$ with entries ranging from 9.9 to 10 and spaced with steps of 0.002 (i.e. $\mathbf{v} = [9.9, 9.902, ..., 9.998, 10]^T$), then the error is: 0.5% for $p = 1$; 0.46% for $p = 50$; 0.41% for $p = 100$; 0.09% for $p = 1000$. Hence, it is recommended to increase the value of $p$ up to 1000 in order to reach a good approximation during the optimization analysis.
References


