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Caviedes-Nozal, Diego; Riis, Nicolai A.B.; Heuchel, Franz M.; Brunskog, Jonas; Gerstoft, Peter; Fernandez-Grande, Efren

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Diego Caviedes-Nozal, Nicolai A. B. Riis, Franz M. Heuchel, Jonas Brunskog, Peter Gerstoft, and Efren Fernandez-Grande

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I. INTRODUCTION

The analysis, reconstruction, and visualization of sound fields over space are fundamental problems in acoustics that appear in many of its applications, such as sound field control. A conventional approach to these problems is to measure the field at multiple locations and project these observations onto a finite, linear combination of spatial basis functions with unknown complex valued coefficients. It is common to choose the basis from known solutions to the Helmholtz equation, such as plane waves, and spherical waves like in the equivalent source method (ESM). Projecting the field onto basis functions by determining the complex coefficients enables estimation of the field at any spatial location.

While conventional linear regression has shown to accurately reconstruct sound fields, the main issue that arises with these methods is to determine the number of basis functions that provide a sufficiently accurate reconstruction of the field. In order to overcome this issue, kernel Ridge regression has been introduced in recent years for sound field reconstruction. This approach consists of the projection of the observations onto a basis of spatial correlation functions known as kernels. Ueno et al. and Ito et al. used the Bessel function kernel that corresponds to the correlation function of the sound field in a reverberation room driven by a pure tone. This kernel (Bessel), which also satisfies the Helmholtz equation, defines the spatial correlation of the field created by an infinite set of propagating plane waves where no truncation is needed. Results showed better reconstruction of a plane wave field than that achieved by means of truncated linear regression on spherical harmonics, or kernel Ridge regression with spatially isotropic radial basis function (RBF).

The promising results achieved with kernel Ridge regression for sound field reconstruction are limited to the aforementioned kernels (Bessel and isotropic RBF) and sound fields (a plane wave), and there are several open questions that need to be addressed. The spatially isotropic kernels used in the literature are suboptimal for the reconstruction of anisotropic fields, such as those produced by a few propagating waves (e.g., the sound field in a room below the Schroeder frequency, or the sound field in urban conditions). In addition, in the current literature in acoustics and audio signal processing, the physical connection between conventional linear regression and kernel regression methods has not been provided. Finally, even though uncertainty quantification using Bayesian formulation has been shown useful in several acoustic applications, it has received little attention in sound field reconstruction and analysis.

In order to tackle these issues, this paper investigates Gaussian process (GP) regression for sound field reconstruction and analysis.

ABSTRACT:
This study examines the use of Gaussian process (GP) regression for sound field reconstruction. GPs enable the reconstruction of a sound field from a limited set of observations based on the use of a covariance function (a kernel) that models the spatial correlation between points in the sound field. Significantly, the approach makes it possible to quantify the uncertainty on the reconstruction in a closed form. In this study, the relation between reconstruction based on GPs and classical reconstruction methods based on linear regression is examined from an acoustical perspective. Several kernels are analyzed for their potential in sound field reconstruction, and a hierarchical Bayesian parameterization is introduced, which enables the construction of a plane wave kernel of variable sparsity. The performance of the kernels is numerically studied and compared to classical reconstruction methods based on linear regression. The results demonstrate the benefits of using GPs in sound field analysis. The hierarchical parameterization shows the overall best performance, adequately reconstructing fundamentally different sound fields. The approach appears to be particularly powerful when prior knowledge of the sound field would not be available.

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Gaussian processes for sound field reconstruction

Diego Caviedes-Nozal,1,a) Nicolai A. B. Riis,2,b) Franz M. Heuchel,1,c) Jonas Brunskog,1,d) Peter Gerstoft,3,e) and Efren Fernandez-Grande1,f)

1Acoustic Technology, Department of Electrical Engineering, Technical University of Denmark, Kongens Lyngby, 2800, Denmark
2Noise Lab, University of California San Diego, La Jolla, California 92093-0238, USA
3Department of Applied Mathematics and Computer Science, Technical University of Denmark, Kongens Lyngby, 2800, Denmark
4Department of Electrical Engineering, Technical University of Denmark, Kongens Lyngby, 2800, Denmark
5Noise Lab, University of California San Diego, La Jolla, California 92093-0238, USA

Electronic mail: dicano@elektro.dtu.dk, ORCID: 0000-0001-6756-3375.
reconstruction over extended regions in space. GP regression is the Bayesian equivalent to kernel Ridge regression, and considers the observations on the sound field as the outcome of a GP which mean and covariance represent the optimal field prediction and the uncertainty of the obtained reconstruction, respectively. GP regression, also known as kriging or probabilistic kernel regression, has been broadly applied in geophysics, image processing, and digital signal processing, including audio source separation, but have received little attention in the acoustic literature.

Using the GP regression framework, we explicitly connect conventional linear regression methods in acoustics with kernel regression. In addition, we study state-of-the-art kernels based on the RBF for the reconstruction of sound fields, where properties such as periodicity and anisotropy are included. An additional contribution of this paper is to derive acoustically informed kernels based on a plane wave expansion for both isotropic and anisotropic fields. We propose a hierarchical Bayesian parameterization over plane waves expansions for adaptation to the spectrum of sparse and non-sparse fields by using a kernel whose degree of sparsity changes depending on the sparsity of the sound field. The reconstruction performance of these acoustically informed kernels is analyzed for fundamental sound fields relevant in acoustics (a plane wave field, a diffuse like field, and the near and far-fields of a point source), and compared to two conventional linear regressions, three state-of-the-art kernels, and sparse Bayesian learning (SBL), on simulated data.

II. FROM LINEAR REGRESSION TO GP

Consider the problem of reconstructing a sound field in space from a limited set of measured data. In the following, the locations \( X = \{x_1, \ldots, x_N\}^T \), \( x \in \mathbb{R}^D \) refer to the spatial locations where the field was measured and \( X_s = \{x_{1s}, \ldots, x_{Ns}\}^T \), \( x_s \in \mathbb{R}^D \) refers to the locations where the field is predicted. The field at the spatial locations \( X \) is denoted by \( f = [f(x_1), \ldots, f(x_N)]^T \), where the function \( f : \mathbb{R}^D \times \mathbb{R} \rightarrow \mathbb{C} \) is unknown. The function \( f \) also depends on the wavenumber \( k = \omega/c \in \mathbb{R} \), where \( \omega \) is the angular frequency and \( c \) the speed of sound. At the predicted locations, the field is \( f_s \in \mathbb{C}^N \).

A. Conventional linear regression

The measured pressure \( p \in \mathbb{C}^N \) at \( N \) locations \( X \) is described by

\[
\mathbf{p} = \mathbf{f} + \mathbf{e},
\]

where \( \mathbf{e} \in \mathbb{C}^N \) is measurement noise, and \( \mathbf{f} \in \mathbb{C}^N \) the unknown sound field. In conventional sound field reconstruction, the unknown function \( f(x) \) is approximated by a linear combination of basis functions, e.g., a superposition of elementary acoustic waves \( \phi_i(x) \) such as plane or spherical waves.

\[
f(x) \simeq \Phi(x)w = \sum_{i=1}^{L} w_i \phi_i(x),
\]

mapping \( x \) to the basis \( \Phi(x) : \mathbb{R}^D \rightarrow \mathbb{C}^L \). Equation (1) is now

\[
\mathbf{p} = \Phi \mathbf{w} + \mathbf{e},
\]

where

\[
\Phi = \begin{bmatrix}
\Phi^T(x_1) \\
\vdots \\
\Phi^T(x_N)
\end{bmatrix} = \begin{bmatrix}
\phi_1(x_1) & \cdots & \phi_L(x_1) \\
\vdots & \ddots & \vdots \\
\phi_1(x_N) & \cdots & \phi_L(x_N)
\end{bmatrix}.
\]

In the acoustics literature, the estimated coefficients \( \mathbf{w} \) are usually the outcome of minimizing a least squares problem regularized by a \( \ell_q \)-norm \( \| \mathbf{w} \|_q \) penalty term over \( \mathbf{w} \) with regularization parameter \( \nu > 0 \),

\[
\mathbf{w} = \arg\min_{\mathbf{w}} \| \mathbf{p} - \Phi \mathbf{w} \|_2^2 + \nu \| \mathbf{w} \|_q^q.
\]

Once the coefficients \( \mathbf{w} \) are estimated through Eq. (5), predictions of the field at the new locations \( \mathbf{X} \), are

\[
f_s = \Phi \mathbf{w},
\]

where \( \Phi \) are the basis functions in Eq. (4) evaluated at \( \mathbf{X} \). Two common regularization schemes in Eq. (5) are the Lasso (q = 1), and Tikhonov or Ridge regression (q = 2). For the latter, Eq. (5) presents a known closed form for \( \mathbf{w} \) in Eq. (5), which can be used for predictions

\[
f_s = \Phi \mathbf{w} = \Phi \mathbf{w}^{\text{opt}} = \Phi \mathbf{w}^{\text{opt}} (\Phi \mathbf{w}^{\text{opt}} + \nu \mathbf{I})^{-1} \mathbf{p}.
\]

These are the conventional approaches to reconstruct a field where measured data is projected onto the basis. In the following, we consider the reconstruction via Bayesian linear regression.

B. Bayesian linear regression with Gaussian priors

In the Bayesian framework, the unknown basis coefficients \( \mathbf{w} \) of Eq. (3) and the measured pressure \( \mathbf{p} \) are considered stochastic variables. The field reconstruction problem is formulated in terms of probability density functions via the Bayes’ theorem,

\[
\pi(\mathbf{w}|\mathbf{p}) = \frac{\pi(\mathbf{p}|\mathbf{w}) \pi(\mathbf{w})}{\pi(\mathbf{p})},
\]

where \( \pi(\mathbf{w}|\mathbf{p}) \) is the posterior density, \( \pi(\mathbf{p}|\mathbf{w}) \) the likelihood density, \( \pi(\mathbf{w}) \) the prior density, and \( \pi(\mathbf{p}) \) is the evidence. Predictions \( \mathbf{f}_s \) at unobserved locations \( \mathbf{X} \), are obtained via the posterior predictive density or marginal likelihood

\[
\pi(\mathbf{f}_s|\mathbf{p}) = \frac{\pi(\mathbf{p}, \mathbf{f}_s)}{\pi(\mathbf{p})} = \int \pi(\mathbf{f}_s|\mathbf{w}) \pi(\mathbf{w}|\mathbf{p}) dw,
\]

where \( \pi(\mathbf{f}_s|\mathbf{w}) \) is the likelihood and \( \pi(\mathbf{w}|\mathbf{p}) \) is the posterior in Eq. (8). For mathematical tractability of Eq. (9), both
w and e are assumed to follow a multivariate zero mean proper (i.e., circularly symmetric) complex Gaussian densities,\textsuperscript{31,34,36} such that
\[
\pi(w) = \mathcal{CN}(0, K_w), \\
\pi(e) = \mathcal{CN}(0, \Sigma),
\]
(10a, 10b)
where \(E[w\bar{w}] = K_w \in \mathbb{C}^{L \times L}\) is the covariance of the basis coefficients, and \(E[ee^H] = \Sigma \in \mathbb{C}^{N \times N}\) is the noise covariance. A thorough explanation of the physical implications of the densities in Eqs. (10a)-(10b) can be found in Refs. 34 and 36.

From Eq. (1), the measured sound field \(f\) will also be described by a complex Gaussian density
\[
\pi(f) = \mathcal{CN}(0, K),
\]
(11)
with mean and covariance
\[
E[f] = \mu_f = \Phi E[w] = 0, \\
E[ff^H] = K = \Phi E[ww^H] \Phi^H = \Phi K_w \Phi^H.
\]
(12a, 12b)
The noise \(e\) in Eq. (10b) is also assumed zero mean proper complex Gaussian with covariance \(\Sigma \in \mathbb{C}^{N \times N}\), which from Eq. (1) leads to the prior density over the measured pressure \(p\),\textsuperscript{35}\n\[
\pi(p) = \mathcal{CN}(0, K + \Sigma).
\]
(13)
Because both \(f\) and \(p\) are Gaussian, the joint density is also Gaussian,\textsuperscript{26}
\[
\pi\left(\begin{bmatrix} p \\ f \end{bmatrix}\right) = \mathcal{CN}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} K + \Sigma & K_c \\ K_c^\dagger & K_c \end{bmatrix}\right),
\]
(14)
where
\[
K_c = E[ff^H] = \Phi K_w \Phi^H, \\
K_{cc} = E[ff^H] = \Phi K_w \Phi^H,
\]
(15a, 15b)
are the assumed covariance of the sound field between measured and predicted locations, and at predicted locations.

From Eq. (14), the posterior predictive density in Eq. (9) needed to make predictions of the field \(f\), given the measured pressure \(p\), is calculated as the conditional Gaussian,\textsuperscript{26,35}
\[
\pi(f|p) = \mathcal{CN}(\mu_{f|p}, K_{f|p}),
\]
(16)
with predictive mean and covariance
\[
\mu_{f|p} = K_c^H (K + \Sigma)^{-1} p, \\
K_{f|p} = K_{cc} - K_c^H (K + \Sigma)^{-1} K_c.
\]
(17a, 17b)
The predictive mean \(\mu_{f|p}\) gives the optimal reconstruction as the value with maximum probability of occurrence in Eq. (16), which in the Gaussian case corresponds to the minimum mean squared error estimate.\textsuperscript{27} The predictive covariance \(K_{f|p}\) represents the uncertainty on the predictions.

Equations (12b), (15a), and (15b) show that given a set of measurements \(p\) and the noise covariance \(\Sigma\), both predictive mean and covariance only depend on the basis \(\Phi\) in Eq. (3), and the covariance \(K_w\) of the unknown coefficients. Interestingly, assuming both \(w\) and \(e\) are independent and identically distributed (i.e., \(K_w = \sigma_w^2 I, \Sigma = \sigma_e^2 I\), respectively), the mean prediction \(\mu_{f|p}\) in Eq. (17a) is
\[
\mu_{f|p} = \Phi \hat{\Phi}^H \left( \Phi \Phi^H + \frac{\sigma_e^2}{\sigma_w^2} I \right)^{-1} p,
\]
(18)
which is equivalent to the \(\ell_2\)-norm regularization in Eq. (7) with \(\nu = \sigma_e^2/\sigma_w^2\).\textsuperscript{20,34} However, the Bayesian formulation allows for uncertainty quantification via the predictive covariance \(K_{f|p}\). Further analysis of both predictive mean and covariance is carried in the following sections.

C. GPs

Sections II A and II B show the relationship between conventional linear regression and Bayesian linear regression with Gaussian priors. In this section we show that GPs are a generalization of Bayesian linear regression with Gaussian priors, where the reconstruction of the sound field can be calculated using Eqs. (17a) and (17b) and a defined spatial covariance function or kernel. All the definitions are based on zero mean proper complex Gaussian variables and processes, as defined in the Appendix.

A GP is a stochastic process such that any finite set of variables has a multivariate Gaussian density function,\textsuperscript{35}
\[
f(x) \sim \mathcal{CGP}(0, \kappa(x, x')),
\]
(19)
where \(\mathcal{CGP}\) indicates Complex GP with mean 0 and spatial correlation function \(\kappa(x, x')\), the kernel. With this definition, the sound field vector \(f\) in Eq. (11) is a set of \(N\) variables that are the outcome of a GP with zero mean and kernel \(\kappa(x, x') = \Phi^H(x) K_w \Phi(x')\) that constructs \(K_c\).

Whereas Eq. (11) in Bayesian linear regression is the probability density of a vector, Eq. (19) is the probability density over functions with a continuous domain.\textsuperscript{35} Each evaluation of Eq. (19) at \(N\) locations \(X\) results in \(N\)-dimensional vectors with (probably) different sound field values, but identical spatial correlation. The predicted sound field at locations \(X\) once the measured pressure \(p\) is available \((f|p)\) in Eq. (16)) is the outcome of a GP such that the predicted sound field \(f\), given the measured pressure \(p\), follows:
\[
f(x)\bigg|p \sim \mathcal{CGP}\left(\mu_{f|p}(x), \kappa_{f|p}(x, x')\right).
\]
(20)
The optimal sound field reconstruction is the posterior mean, which is from Eq. (17a),
\[
\mu_{f|p}(x) = K^H (K + \Sigma)^{-1} p.
\]
(21)
where
\[ \kappa = \kappa(x_1, x) \ldots \kappa(x_N, x)^T \]
is the prior spatial correlation of the sound field, centered at each measured location X and evaluated at any location x.

From Eq. (17b), the posterior covariance is
\[ \kappa_{f; p}(x, x') = \kappa(x, x') - \kappa^H(K + \Sigma)^{-1} \kappa', \]
where \( \kappa' = \kappa(x_1, x') \ldots \kappa(x_N, x')^T \). Equation (23) corresponds to the prior correlation function \( \kappa(x, x') \) in Eq. (19), updated by the correlation between the measured and the predicted locations.

Figure 1 illustrates the reconstruction of a one-dimensional (1D) sound field using the GPs in Eqs. (19) and (20) for a set of \( N = 4 \) noiseless observations of the sound field p. The kernel function is assumed to be that of a reverberant field,\(^{2,37}\)
\[ \kappa(x, x') = \sigma_n^2 \text{sinc}(k||x - x'||), \]
where \( k \) is the wavenumber, and \( \sigma_n \) is the variance in Eq. (18) [see Sec. III.C for the derivation of Eq. (24)]. Figure 1(a) shows the sound fields estimated prior to having any observations (as a blind prediction), calculated by using Eq. (19). The resulting sound fields show a chaotic fluctuating pattern due to the lack of observations p, with zero mean and equal uncertainty all over the reconstruction domain.

Figure 1(b) shows the reconstruction after four observations, where the predictions are made using the posterior GP in Eqs. (20)–(23). The predictions near a measurement p are highly influenced by the measured value of the sound field (showing a mean close to the measured value and little uncertainty), while predictions further away become more uncertain. For reverberant field kernel [a sinc, see Eqs. (24)–(33)] the spatial correlation decays with \( 1/(k||x - x'||) \), thus the mean of the predicted sound field far from the observations tends to zero, and the uncertainty of the prediction increases. Specifically, the spatial correlation between a measured location \( x_n \) and a predicted location \( x \), is, from Eq. (23),
\[ \kappa_{f; p}(x, x_n) = \eta - \frac{\kappa^2(x, x_n)}{\eta + \sigma_n^2}, \]
where \( \eta = \kappa(x, x_n) = \kappa(x, x_n) \). If measured and predicted locations coincide [i.e., \( \kappa(x, x_n) = \eta \)], Eq. (25) is \( \kappa_{f; p}(x, x_n) = \eta(1 - (1 + (\sigma_n^2/\eta))^{-1}) \) and the uncertainty is zero in the absence of noise, as it is shown in Fig. 1.

D. The kernel trick
Covariance matrices are Hermitian positive semidefinite (PSD), and thus the elements of K are the outcome of a bilinear PSD kernel function \( \kappa(x, x') : \mathbb{R}^D \times \mathbb{R}^D \rightarrow \mathbb{C} \) where \( \kappa(x, x') = \kappa^*(x', x) \) and \( \kappa(x, x) \geq 0 \).\(^{38,39}\)

A common way of constructing kernels is through inner products in Hilbert spaces. Given any basis function \( \Psi(x) = [\Psi_1(x) \ldots \Psi_L(x)]^T \) such that \( \Psi(x) : \mathbb{R}^D \rightarrow \mathbb{C}^L \), the kernel \( \kappa(x, x') = \langle \Psi(x), \Psi(x') \rangle ) = \Psi^H(x) \Psi(x') \) is always PSD. Specifically, the covariance function that constructs the covariance K in Eq. (12b) is the inner product of \( \Phi \) over \( K_w \). The representation of inner products with a similarity function \( \kappa \) is the kernel trick, which allows the mapping of X into higher dimensional spaces without the explicit evaluation of \( \Psi \) on every element of X. However, the design of kernel functions is not limited to inner products, and existing kernels can be combined into new valid kernels in multiple ways,\(^{25,39,40}\) such as summation (\( \kappa(x, x') = \sum \kappa_i(x, x') \)) and multiplication (\( \kappa(x, x') = \prod \kappa_i(x, x') \)), to mention a few.

III. KERNEL FUNCTIONS
Kernels are generally classified on behalf of stationarity and isotropy.\(^{38}\) A kernel is stationary if it is translation invariant, i.e., only a function of the difference \( \delta = x - x' \). A kernel is isotropic if it depends on the euclidean distance \( ||\delta|| \). For example, the spatial correlation of a diffuse field is stationary and isotropic, while the spatial correlation of a plane wave field is stationary but anisotropic (as the correlation depends on the direction). In this section, we show some of the conventional stationary kernel functions existing in audio and acoustics literature, and we introduce a kernel based on plane wave expansions that adapts to the spectrum of sparse and non-sparse fields.

A. RBF kernels
In this section, we introduce the widely used RBF in its basic isotropic, anisotropic, and wave-like forms.\(^{27,35,39,40}\) The basic isotropic RBF defines the covariance between two points as
\[ \kappa(x, x) = e^\frac{-1}{2 \rho^2} \exp\left(-\frac{1}{2 \rho^2} ||\delta||^2 \right), \]
where \( \rho \) is the length scale and \( z \) is a scaling factor. The length scale defines the decay rate of the correlation function. The RBF kernel is an example of a PSD kernel not derived from inner products.\(^{31}\)

For anisotropic fields, it seems natural to consider direction dependent covariance. Anisotropic RBF kernels can be
constructed multiplying several isotropic RBF with independent length scales, such that
\[ \kappa(\mathbf{x}, \mathbf{x}') = \alpha^2 \exp \left( -\frac{1}{2} \sum_{l=1}^{L} \frac{|\mathbf{u}_l^T \delta|^2}{\rho_l^2} \right), \] (27)
where \( \mathbf{u}_l \in \mathbb{R}^D \) is the unitary vector defining the \( l \)th direction and \( \rho_l \) is the length scale in that direction.

The kernel in Eq. (27) is not periodic, which might be problematic for describing wave propagation. A periodic version of the RBF kernel is
\[ \kappa(\mathbf{x}, \mathbf{x}') = \alpha^2 \exp \left( -\sum_{l=1}^{L} \frac{1}{2 \rho_l^2} \sin^2 \left( \frac{k ||\mathbf{u}_l^T \delta||}{2} \right) \right), \] (28)
where the covariance repeats every wavelength \( \lambda = 2\pi/k \).

The kernels in Eqs. (26)–(28) are real valued [i.e., \( \kappa(\mathbf{x}, \mathbf{x}') : \mathbb{R}^D \times \mathbb{R}^D \rightarrow \mathbb{R} \)], and thus the correlation between real and imaginary parts of the field is not quantified [\( \kappa_{ii}(\mathbf{x}, \mathbf{x}') = 0 \) in Eq. (A9), see the Appendix]. These kernels are independent of the sign of \( \mathbf{u}_l^T \delta \) and only directions in half space should be considered in order to avoid identifiability problems.

**B. The plane waves kernels**

Plane wave expansions are widely used for sound field reconstruction and direction of arrival estimation. At wavenumber \( k \), the field at any point in space \( \mathbf{x} \) is defined as a plane wave superposition
\[ f(\mathbf{x}) = \sum_{l=1}^{L} w_l e^{-ik_l^T \mathbf{x}}, \] (29)
where \( k_l = k \mathbf{u}_l \) is the wavenumber vector. Given a linear regression model where \( \mathbf{w} = [w_1, \ldots, w_L]^T \) follows a complex Gaussian density with covariance \( \mathbf{K}_w \) as in Eq. (10a), the covariance \( \mathbf{K} \) is calculated as in Eq. (12b). It is common to assume the covariance \( \mathbf{K}_w \) of the coefficients \( \mathbf{w} \) to be diagonal with either shared variance (i.e., \( \mathbf{K}_w = \sigma_w^2 \mathbf{I} \)) or independent variance (i.e., \( \mathbf{K}_w = \text{diag}(\sigma_1^2, \ldots, \sigma_L^2) \)). If the complex coefficients share the variance, the kernel is
\[ \kappa(\mathbf{x}, \mathbf{x}') = \sigma_w^2 \Phi^H(\mathbf{x}) \Phi(\mathbf{x}') = \sigma_w^2 \sum_{l=1}^{L} e^{-jk_l^T \delta}, \] (30)
where \( \delta = \mathbf{x} - \mathbf{x}' \). If independent variance is assumed, the kernel is
\[ \kappa(\mathbf{x}, \mathbf{x}') = \Phi^H(\mathbf{x}) \mathbf{K}_w \Phi(\mathbf{x}') = \sum_{l=1}^{L} \sigma_l^2 e^{-jk_l^T \delta}. \] (31)

The plane waves kernel is complex valued and it includes direction of propagation [\( \kappa_{ii} \neq 0 \) in Eq. (A9), see the Appendix]. The standard deviations \( \sigma_l \) determine the activation of the plane waves in the expansion.

In acoustics, Eq. (31) is applied in SBL, which is effective for problems where promotion of sparsity is beneficial, such as acoustic source localization. Regarding sound field reconstruction, SBL would be suitable for those fields that result from a few propagating waves.

**C. The reverberant field kernel**

A model used to characterize the spatial correlation and coherence in reverberant fields is the superposition of an infinite amount of plane waves with random phases, which corresponds to Eq. (30) in the limit \( L \rightarrow \infty \), such that
\[ \kappa(\mathbf{x}, \mathbf{x}') = \sigma_w^2 \lim_{L \rightarrow \infty} \sum_{l=1}^{L} e^{-jk_l^T \delta}. \] (32)
The series is convergent where the limit kernel is calculated in the full solid angle translating into spherical coordinates
\[ \kappa(\mathbf{x}, \mathbf{x}') = \frac{\sigma_w^2}{4\pi} \int_0^\pi \int_0^{2\pi} e^{-jk||\delta||\cos \theta} \sin \theta \cos \phi d\phi d\theta, \] (33)
where we have used Eq. (10.1.14) from Ref. 43. In the two-dimensional (2D) case, the kernel is the Bessel function of order zero [Eq. (9.1.18) from Ref. 43,
\[ \kappa(\mathbf{x}, \mathbf{x}') = \frac{\sigma_w^2}{2\pi} \int_{-\pi}^{\pi} e^{-jk||\delta||\cos \phi} d\phi = \sigma_w^2 J_0(k||\delta||). \] (34)
These kernels could be considered the acoustic counterpart to the isotropic RBF kernel in Eq. (26).

**D. Variable sparsity plane waves kernel**

So far, the introduced kernels based on plane wave expansions can be divided in two groups: suitable for sound fields that result from the interference of multiple waves [Eqs. (30), (33), and (34)], and suitable for sound fields that result from few propagating waves [Eq. (31)]. In this work, we introduce a hierarchical model over Eq. (31) based on SBL such that the kernel adapts to few and multiple propagating waves.

SBL is a particular case of what is known as automatic relevance determination (ARD). In ARD, the coefficients \( \mathbf{w} \) are zero mean complex Gaussian with diagonal covariance \( \mathbf{K}_w \) as in Eq. (31). The parameters \( \sigma_l \) are assumed as inverse gamma distributed,
\[ \sigma_l \sim \Gamma^{-1}(a, b) = \frac{b^a}{\Gamma(a)} (1/\sigma_l)^{a+1} \exp \left( -b/\sigma_l \right), \] (35)
where \( a > 0 \) is the shape parameter and \( b > 0 \) is the scale parameter of the density function.

Heavy tailed priors as the inverse gamma are suitable to promote sparse solutions by localizing most of the probability density close to zero, implying that \textit{a priori} most of the \( L \)
components are likely to be zero.\(^{45}\) The heavy tail towards large values activates only those components that add significant contribution to the solution. In this way, if \(\sigma_i \rightarrow 0\) in Eq. (31), the Gaussian density function in Eq. (10a) shrinks to a delta function around zero, and \(|w_i| \simeq 0\). If \(\sigma_i > 0\), the Gaussian density broadens and it becomes more likely to obtain coefficients different from zero. SBL in Refs. 30 and 32 is the particular case of ARD where \(\sigma_i\) follows a uniform distribution (i.e., \(a = b = 0\) and all values of \(\sigma_i\) are equally probable).

The proposed model in order to promote both sparse and non-sparse solutions is hierarchical over \(b\)

\[
\sigma_i \sim \Gamma^{-1}(1, b), \quad b \sim \mathcal{N}(\mu_b, \sigma_b) ,
\]

with \(\sigma_i, b > 0\). Figure 2 shows the impact of the hyper-parameterization in the inverse gamma density function. For \(a = 1\), the smaller \(b\) the sparser solutions are promoted as more mass is concentrated towards zero. When \(b\) is large, non-sparsity is promoted.

**IV. KERNEL HYPER-PARAMETER ESTIMATION**

The kernels introduced in Secs. III A–III D depend on hyper-parameters \(\gamma \in \mathbb{R}^P\) that need to be estimated. Specifically, the hyper-parameters for the RBF kernels are the length scale \(\rho (\rho_i)\) in the anisotropic cases, which defines the decay ratio of the spatial correlation with distance and \(z\), which scales the kernel to the data magnitude. For the plane waves kernels, the hyper-parameters are the standard deviations \(\sigma_w\) when the waves share the variance and \((\sigma_1, \ldots, \sigma_L)\) when the waves have independent variance. In the case of variable sparsity plane waves kernel, the parameter \(b\) is also inferred. A typical approach to estimate these hyper-parameters is to maximize the marginal likelihood with respect to the hyper-parameters.\(^{30–32,42}\) However, the marginal likelihood may not be convex with respect to \(\gamma\) and may have local minima. Furthermore, the approach only provides a point estimate of the hyper-parameters. We avoid these issues by sampling the hyper-parameters using Monte Carlo sampling.\(^3\) That is, to sample the joint posterior density

\[
\pi(f|\gamma)p \propto \pi(p|f, \gamma)\pi(f|\gamma)\pi(\gamma) ,
\]

\[\text{with prior and likelihood}
\]

\[
\pi(f|\gamma) = \mathcal{N}(\mathbf{0}, \mathbf{K}(\gamma)) ,
\]

\[\pi(p|f, \gamma) = \mathcal{N}(f, \Sigma).
\]

\(\mathbf{K}(\gamma)\) stands for the covariance in Eq. (11) as a function of the hyper-parameters. The benefit of this approach is two-fold. First, the true density function is sampled with no approximations. Second, samples from the marginal probability density functions of the hyper-parameters are also obtained, which can be used to quantify the uncertainty in the estimates, and can also be used for the sampling of the conditional density in Eq. (16). We use Hamiltonian Monte Carlo sampling-based methods that have been shown to effectively sample this posterior.\(^{46}\)

**V. NUMERICAL RESULTS**

Here, we analyze the performance of GPs for sound field reconstruction. The five kernels in Sec. III are studied: RBF isotropic [Eq. (26)], RBF anisotropic [Eq. (27)], RBF anisotropic periodic [Eq. (28)], radial Bessel [Eq. (34)] and the plane waves kernel in Eq. (31) with the hierarchical hyper-parameterization presented in Eq. (36). The fields to reconstruct are two spatially stationary fields (i.e., a plane propagating wave and a random wave field) and two spatially non-stationary fields (i.e., a point source in the near-field and in the far-field relative to the measurement positions). The fields are reconstructed in a 2D area \((z = 0)\) plane of approximately \(5 \lambda^2 (2.36 \lambda \times 2.1 \lambda)\). The fields are normalized such that the mean magnitude is 1 Pa. The added noise is complex normal, independent, and identically distributed, i.e., \(\Sigma = \sigma_r^2 \mathbf{I}\) with \(\sigma_r = 0.1\). Without loss of generality, Eqs. (17a) and (17b) enable us to consider noise with more complex correlation structures, as long as the noise is Gaussian.

The performance of the kernels is compared to three plane wave regression schemes common in acoustics: two conventional regressions [Lasso \((\ell_1\)-norm) and Tikhonov \((\ell_2\)-norm) as in Eq. (5)], and SBL as applied by Gemba et al.\(^{30,32}\) The optimal regularization parameters \(\nu\) of the conventional regression schemes are calculated via grid search cross-validation for Lasso\(^{47}\) and generalized cross-validation for Tikhonov.\(^{48}\) The search grid consists, in both cases, of 50 evenly spaced values in the range \(\nu \in [10^{-4}, 10^2]\).

Two metrics are defined to assess the performance of the models. The first is the normalized mean squared error (NMSE) between the measured \(p_i\) and predicted \(f_i\) fields, defined as

\[
\text{NMSE} = \frac{1}{N} \sum_{i=1}^{N} \frac{||p_i - f_i||}{||p_i||^2}.
\]

Normalizing each term in the sum separately assures that locations with low and high pressure magnitude contribute
equally to the error estimate. The second measure is the model assurance criteria (MAC),

$$\text{MAC} = \frac{\| f'f_1 \|^2}{(p'p_1)(f'f_1)},$$  \hspace{1cm} (41)

which evaluates the degree of spatial similarity, ranging from 0 (maximally dissimilar) to 1 (identical).

A. Models and priors

RBF kernels were modelled using conventional one level ARD. The prior densities over the parameters of the RBF kernels are

$$\alpha \sim \mathcal{N}(0,1), \quad \rho, \rho_t \sim \Gamma^{-1}(a_\rho, b_\rho),$$  \hspace{1cm} (42)

where $$a_\rho = 5, \ b_\rho = 5.$$

The plane waves kernel in Eq. (31) is modelled with the proposed hierarchical model in Eq. (36) with parameters

$$b = 10^{-b_{\text{we}}}, \ b_{\log} \sim \mathcal{N}(2,1).$$  \hspace{1cm} (43)

Because the values of $$b$$ are rather small, the logarithmic parameterization avoids numerical instability in sampling the probability density $$\pi(b)$$ and subsequently $$\pi(\sigma).$$

The number of directions $$u_i$$ for the anisotropic kernels is $$L = 64$$ equally spaced over half the circle for RBF kernels $$[\angle u_i \in [0, \pi]]$$ and over the entire circle for the plane waves kernel $$[\angle u_i \in [0, 2\pi]].$$

ALGORITHM 1: sound field reconstruction using GPs and Monte Carlo sampling.

input: Measurements $$p$$ at $$N$$ locations $$X$$. Kernel type $$\kappa(x, \cdot)$$. Prior distributions of the kernel hyper-parameters $$\pi(\gamma).
$
output: Sound field reconstruction and uncertainty at $$N$$ locations $$X.$$

1 for $$i = 1$$ to 400 do
2 $$\gamma^i$$ \text{ sample}$$\pi(f, \gamma | p); \hspace{1cm} \text{Eq. (37)}$$
3 $$\kappa^i(x, x') = \kappa(x, x', \gamma^i)$$;
4 for $$j = 1$$ to $$N$$ do
5 $$\mu_{i,j} = \operatorname{mean}(\kappa^i(x \bullet \{j\}, x \bullet \{j\})) ,$$ \hspace{1cm} \text{Eq. (21)}
6 $$\sigma_{i,j} = \operatorname{cov}(\kappa^i(x \bullet \{j\}, x \bullet \{j\})$$; \hspace{1cm} \text{Eq. (23)}
7 end
8 end

B. Hyper-parameter estimation and sound field reconstruction

Algorithm 1 shows the steps to estimate the kernel hyper-parameters and reconstruct a field using GPs together with Monte Carlo sampling. For a given set of observations $$p$$ at $$N$$ locations $$X,$$ 400 samples from the posterior densities of the kernels hyper-parameters are drawn as in Sec. IV. For each posterior sample, a plausible kernel is calculated and predictions are obtained at locations $$X,$$ using Eqs. (21)–(23).

The field reconstruction performance is tested for spatial densities $$N/\lambda^2 = [1, 2, 4, 8],$$ where $$\lambda = 2\pi/k$$ is the wavelength in meters. The spatial sampling is incremental, such that the locations selected at low densities are included at the corresponding higher-density sets. The field is reconstructed in a grid of $$N$$ locations equally distributed over the area with density $$N/\lambda^2 = 140.$$

C. Spatially stationary fields

The reconstruction of two spatially stationary fields is investigated in this section: a plane wave field and a random wave field. These sound fields are found in relevant acoustic scenarios such as the sound field produced by a line array of sources or the sound field in reverberation rooms.

Figure 3 shows the real part of the reconstruction of a plane wave field with random amplitude, phase and direction of propagation. The single plane wave field is anisotropic and periodic. The isotropic RBF poorly reconstructs the plane wave as this kernel is neither anisotropic nor periodic. The uncertainty is high away from the measured locations, and it is not until $$N/\lambda^2 = 8$$ that the mean prediction starts to resemble a plane wave. The anisotropic RBF is able to explain the spatial anisotropy of the plane wave, but it poorly extrapolates along the direction of propagation if a wave front is not sampled. This is observed at the bottom right corner of the RBF anisotropic case for $$N/\lambda^2 = 4$$ (fourth row in Fig. 3), where the kernel wrongly extends the wave front without shifting phase. Better extrapolation is achieved if periodicity is included in the RBF anisotropic kernel, referred to as “RBF periodic” in Fig. 3. The results shown by the RBF kernels illustrate the consequences of choosing a more or less wave-like kernel.

The Bessel kernel shows better reconstruction of the plane wave than its isotropic RBF counterpart due to two main reasons: first, the Bessel kernel oscillates with positive correlation at $$||\delta|| \approx \lambda$$ and negative correlation at $$||\delta|| \approx \lambda/2,$$ better resembling the correlation of a plane wave in the direction of propagation. Second, the kernel decay rate from its central point is defined by the wavenumber $$k,$$ which is a known parameter, contrary to an unknown length scale $$\rho$$ in the RBF, requiring less data to converge into adequate kernel shapes.

As expected, the plane waves kernel gives the best performance, as it corresponds to the kernel of the reconstructed field where the only source of error is the misalignment between the considered directions $$u_i$$ and the actual direction of propagation of the plane wave.

Figure 4 shows the real part of the reconstruction of a random wave field produced by 2000 plane waves with unit magnitude but random phase and direction of propagation. The Bessel kernel corresponds to the kernel of a random wave field and consequently performs best. The RBF kernels show again that a wave-like spatial correlation improves the reconstruction. The plane waves kernel with the proposed hierarchical parameterization shows its versatility with very good reconstruction of the random wave field.
Figure 5 details a 1D zoom-in slice of the 2D reconstruction of the random wave field shown in Fig. 4 using the Bessel, the RBF anisotropic and the plane waves kernels for increasing number of observations $N$. The first three columns show the field reconstruction. The plane waves kernel is able to replicate the reconstruction of the Bessel kernel, whereas the RBF kernel is unable to either interpolate or extrapolate the field away from the observations. In the rightmost column of the figure, the real component of the kernels resulting from the posterior samples are shown for the three models. Both plane waves and RBF anisotropic kernels converge towards the Bessel kernel, which is the closest to the random wave field spatial correlation. The plane waves kernel shows almost identical superposition with the Bessel kernel.

Figures 6(a) and 6(b) show a quantitative summary of the accuracy of the plane wave and the random wave field reconstructions. In order to test the sensitivity of the methods to the locations $X$, the reconstruction in Algorithm 1 is done for 20 independent sets of observations $p$. The relative error [Eq. (40)] and the spatial similarity [Eq. (41)] are calculated for each of the reconstructions. The reconstruction with conventional plane wave regression is also included for comparison, tagged as “Tikhonov,” “Lasso,” and “SBL,” respectively. The markers show the median of the corresponding metrics.

Figures 6(a) and 6(b) show a similar performance between Bessel kernel and conventional regression with Tikhonov regularization, with almost identical results in terms of field reconstruction error and spatial similarity. The field reconstruction with the Bessel kernel is an elegant solution, as it represents in just one parameter $r_w$ a linear expansion on an infinitely dimensional series of plane waves. Conventional regression with Tikhonov regularization is an approximation of the solution provided by the Bessel kernel, as it is constructed with the equivalent truncated series of plane waves [Eq. (30)].

Figures 6(a) and 6(b) show the versatility of the hierarchical plane waves kernel compared to conventional regression and SBL. It shows low reconstruction error and high spatial similarity in the single plane wave reconstruction while improving the performance of both sparse methods (Lasso and SBL) in the random wave field reconstruction, with reconstruction error and spatial similarity closer to that achieved by regression with Tikhonov regularization. This confers potential to the approach when little prior information about the sound field is available.
The anisotropic periodic RBF kernel presents good performance at high spatial sampling densities, outperforming the Bessel kernel in the plane wave case. This indicates that acoustically relevant spatial correlation functions can be constructed directly in the definition of the kernel without necessarily being the solution to the Helmholtz equation.

D. Spatially non-stationary fields

In this section, the reconstruction of the sound field in the near-field and the far-field of a point source is studied. The reconstruction of such sound fields is relevant in applications such as sound field control. The field produced by a point source is spatially non-stationary (i.e., translation variant), as the spatial correlation between two locations \( \langle x, x' \rangle \) depends on their specific spatial coordinates with respect to the point source location \( x_s \), such that

\[
E \left[ f(x) f(x')^H \right] \propto \frac{1}{r^2} e^{-jk(r-r')},
\]

where \( r = ||x - x_s||_2 \) and \( r' = ||x' - x_s||_2 \).
Figure 7(a) shows the reconstruction error and the spatial similarity of the point source field close to the source. When the area to reconstruct is in the near field [Fig. 7(a)], the mismatch between the non-stationary field and the stationary models is high, as none of the kernels investigated in the study can effectively reproduce the inverse distance law. Interestingly, the lowest reconstruction error in this case is achieved by the RBF isotropic kernel. The RBF isotropic kernel does not provide any prediction far from the observations (it estimates on average 0 Pa at locations far from the observations as it was seen in Figs. 3 and 4). Other kernels are very dependent on the observations also far from the measurements, which ends in an incorrect prediction in the case of a very non-stationary field. The proposed hierarchical regularization shows again its capability of reducing the reconstruction error compared to its sparse counterpart SBL when the sound field cannot be represented with a sparse set of propagating waves.

Figure 7(a) shows the reconstruction in the far field. The results are similar to those presented in Fig. 6(a) for a plane wave field, as the inverse distance term is close to constant over the entire observation area and the wave fronts arriving onto the measurement points become locally planar. In this case, those methods that promote sparse solutions, (including the proposed hierarchical model) are the most suitable.

E. Adaptation of the variable sparsity plane waves kernel

The hierarchical plane waves kernel introduced in this paper has shown very good overall performance for the studied cases. The proposed hierarchical model (Sec. III D) enables us to have solutions of variable sparsity, depending on the observations.

Figure 8 shows a summary of the posterior distributions of the sampled parameter $b$ for the fields in Secs. VC and VD, and two extra synthetic fields: a field consisting of five point sources randomly placed around the area with a radial distance of $3 \lambda$ away from the middle point of the reconstruction area, and a field consisting of a point source at $3 \lambda$ from the middle point of the area with six plane waves with a random angle of incidence. The parameter $b$ reacts to the sparsity of the field in terms of a plane wave decomposition. Notice that a “multi point source” field is captured by the model as a less sparse field than a “point source + plane waves” field because each spherical wave needs several plane waves to be adequately reconstructed.

VI. CONCLUSIONS

We proposed and examined the use of GPs for sound field reconstruction. GP regression is shown to have several benefits with respect to conventional reconstruction.
methods: GPs provide uncertainty information about the reconstruction of the field (interpolation and extrapolation) in a closed form. In addition, GPs allow for the reconstruction of fields based on both linear regression and spatial correlation functions by means of the “kernel trick.” Kernels commonly used in other disciplines have been analyzed in the context of sound field reconstruction and compared to kernels based on the Helmholtz equation. Specifically, we propose a hierarchical Bayesian parameterization approach based on the covariance of a plane wave expansion. It adapts to a variable number of sources as well as sparse and non-sparse fields (i.e., fields containing few acoustic waves, or a high number of waves).

Three types of acoustic fields have been tested numerically: a single plane wave, a random wave field (diffuse-like), and a point source field. The performance of the kernels is also compared to a conventional plane wave expansion. The results indicate that kernels that are not a solution to the Helmholtz equation, such as those based on RBF, can properly reconstruct sound field if adapted to include wave-like properties, such as periodicity and directionality. In general, the performance of each kernel is case dependent, showing good performance for a certain type of sound field (e.g., a single plane wave or the far field of a point source), while underperforming in other sound fields (e.g., a random wave field). The proposed hierarchical kernel proves to be versatile, showing the best overall performance. It shows similar reconstruction to SBL when the field is sparse (for both plane wave and point source cases), as well as similar reconstruction accuracy to the diffuse field kernel when the field is far from sparse (e.g., random wave field or reverberant field). This indicates that the method adapts well to variable degrees of sparsity in the field, and confers potential to the approach when little prior information about the sound field is available.

A Python package for sound field reconstruction using GPs has been developed and is available here: https://github.com/d-caviedes/acoustic_gps/.

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![FIG. 7. (Color online) As Fig. 6 for (a) the near-field of a point source and (b) the far-field of a point source.](image-url)
where the complex-valued covariance matrix $\mathbf{R}$ is kept by a simple transformation, is fully characterized by three real-valued kernels, $\kappa_\mathbf{r}(x, x')$, $\kappa_\mathbf{i}(x, x')$, and $\kappa_\mathbf{r}(x, x')$. Complex representation, convenient in acoustics signal processing, is kept by a simple transformation, and the probability density function in Eq. (A1) is rewritten as

$$\pi(\mathbf{f}) = \frac{1}{\pi^N |\mathbf{K}|^{1/2}} \exp \left( -\frac{1}{2} \mathbf{f}^H \mathbf{K}^{-1} \mathbf{f} \right),$$

(A4)

where the complex-valued covariance matrix $\mathbf{K} \in \mathbb{C}^{2N \times 2N}$ is

$$\mathbf{K} = \mathbf{T} \mathbf{R} \mathbf{T}^H = \mathbb{E} \left[ \mathbf{f} \mathbf{f}^H \right] = \begin{bmatrix} \mathbf{K}_\mathbf{r} & \mathbf{K}_\mathbf{i} \\ \mathbf{K}_\mathbf{i}^* & \mathbf{K}_\mathbf{r}^* \end{bmatrix}.$$

(A5)

Here, $\mathbf{K} = \mathbb{E}[\mathbf{f} \mathbf{f}^H ]$ is the covariance and $\mathbf{K} = \mathbb{E}[\mathbf{f} \mathbf{f}^H ]$ is the pseudo-covariance, and only two complex-valued kernels, $\kappa(x, x')$ and $\kappa(x, x')$, are needed to characterize $\mathbf{K}$. Note that even though $\mathbf{K}$ is complex-valued, the resulting output of Eq. (A4) is real-valued. In this way, the function $f$ follows a complex GP such that $f(x) \sim \mathcal{GP}(0, \kappa(x, x'), \tilde{\kappa}(x, x'))$. Both covariance and pseudo-covariance can be expressed in terms of the real-valued bivariate matrices

$$\mathbf{K} = \mathbf{R}_\mathbf{r} + \mathbf{R}_\mathbf{i} + j(\mathbf{R}_\mathbf{i} - \mathbf{R}_\mathbf{r}),$$

(A6a)

$$\tilde{\mathbf{K}} = \mathbf{R}_\mathbf{r} - \mathbf{R}_\mathbf{i} + j(\mathbf{R}_\mathbf{i} + \mathbf{R}_\mathbf{r}),$$

(A6b)

with kernels

$$\kappa(x, x') = \kappa_\mathbf{r}(x, x') + \kappa_\mathbf{i}(x, x')$$

$$+ j(\kappa_\mathbf{r}(x, x') - \kappa_\mathbf{i}(x, x')),$$

(A7a)

$$\tilde{\kappa}(x, x') = \kappa_\mathbf{r}(x, x') - \kappa_\mathbf{i}(x, x')$$

$$+ j(\kappa_\mathbf{r}(x, x') + \kappa_\mathbf{i}(x, x')).$$

(A7b)

When $f$ and $f^H$ are uncorrelated, the pseudo-covariance is zero $\tilde{\mathbf{K}} = \mathbf{0}$ and the random vector $\mathbf{f}$ is said to be proper ($f(x) \sim \mathcal{GP}(0, \kappa(x, x'))$). In this case, $\mathbf{R}_\mathbf{i} = -\mathbf{R}_\mathbf{i}$, and the covariance $\mathbf{K}$ is

$$\mathbf{K} = 2\mathbf{R}_\mathbf{r} - 2j\mathbf{R}_\mathbf{i},$$

(A8)

with kernel

$$\kappa(x, x') = 2\kappa_\mathbf{r}(x, x') - 2\kappa_\mathbf{i}(x, x').$$

(A9)

In the sound field reconstruction literature, it is assumed that the studied cases are proper and only the covariance between $f$ and its complex conjugate $f^H$ is relevant. This assumption implies that $\mathbb{E}[\mathbf{f}^H \mathbf{f}] = \mathbb{E}[\mathbf{f}\mathbf{f}^H]$, $\forall \mathbf{f} \in \mathbb{R}$, and thus $f$ is stationary in the time domain. 49


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