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Published in:
Journal of the Mechanics and Physics of Solids

Link to article, DOI:
10.1016/j.jmps.2020.104286

Publication date:
2021

Document Version
Peer reviewed version

Citation (APA):
Computational rate-independent strain gradient crystal plasticity

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Abstract

Size effects in metal plasticity are widely accepted, and different theoretical approaches to handle the phenomenon are developing in the literature. The present work considers the Fleck and Willis (2009) framework [A mathematical basis for strain-gradient plasticity theory. Part II: tensorial plastic multiplier. J. Mech. Phys. Solids 57;1045-1057], and creates a new gradient enhanced rate-independent crystal plasticity FE-implementation. The study considers both energetic and dissipative gradient hardening and strengthening, and adopts a general form of the gradient enhanced effective slip rate. Monotonic and cyclic shearing of an infinite crystal slab containing a single slip system at an angle of 90° to the loading direction is a first benchmark case. A second case considers combined shear and tension in 2D of a constrained HCP single crystal. The HCP crystal is loaded in its basal plane by a so-called butterfly deformation path that inflicts repeated loading and unloading of the three crystallographic slip systems. Finally, the evolution and interaction of multiple plastic zones are demonstrated by considering a notched tensile sample. A direct comparison to visco-plastic (rate-dependent) simulations confirms that the proposed crystal plasticity framework forms a rate-independent limit for the gradient enhanced Fleck-Willis theory. The model response also reduces to that of conventional crystal plasticity in the limit of zero length parameters.

Keywords: Crystal plasticity, Finite element method, Fleck-Willis theory, Size effects
1. Introduction

Rate-independent crystal plasticity inherits a non-uniqueness problem in determining which (and how many) slip systems will be active for specific loading scenarios. Let $s_i^{(\alpha)}$ be the slip direction, $m_i^{(\alpha)}$ the slip plane normal, and $\gamma^{(\alpha)}$ the increment of crystallographic slip on slip system “$\alpha$”. When employing a Schmid-like description of the macroscopic plastic strain increment, $\dot{\varepsilon}_{ij}^P$, in conventional crystal plasticity, such that

$$\dot{\varepsilon}_{ij}^P = \sum_\alpha \mu_{ij}^{(\alpha)} \dot{\gamma}^{(\alpha)}, \text{ with } \mu_{ij}^{(\alpha)} = \frac{1}{2} \left( s_i^{(\alpha)} m_j^{(\alpha)} + s_j^{(\alpha)} m_i^{(\alpha)} \right),$$

(1)

Taylor (1938) noted that plastic incompressibility requires five linearly independent slip systems. However, the choice of active slip systems becomes non-unique if more than five slip systems simultaneously reach their individual activation stress (critical resolved shear stress). This issue, combined with the fact that the plastic strain increments, and thereby the increment of slip, is based on increments of the stresses, are at the root of the problem in conventional rate-independent crystal plasticity. Pierce et al. (1982) cemented the implication of this issue in their seminal work on non-uniform deformation in single crystals, where the loss of uniqueness was observed in the slip system configuration required to produce the deformation at shear localization. The problem was further underpinned in Pierce et al. (1983), where the same authors demonstrated the capacity of a corresponding rate sensitive (visco-plastic) formulation to extend the parameter window in which a solution exists. In contrast to the rate-independent formulation, the corresponding conventional rate sensitive crystal plasticity formulation lets the slip rate depend on the current stress state - thus, relating increments of slip to a material state variable. This circumvents the problem of non-uniqueness and allows for a unique slip system configuration to be determined for any number of “potentially active” slip systems\(^1\) (Roters et al., 2010, presents an overview of widely used approaches). Adopting (and accepting) rate sensitivity in the material response has been a successful way to generate a number of interesting studies published in the literature. In an early study, Pan and Rice (1983) considered the questionable formation of yield

\(^1\)Note that all slip systems are active in a visco-plastic model, while the current stress state determines the magnitude of the slip rate.
surface vertices. They, too, demonstrated that even a slight rate sensitivity (approaching the rate-independent limit) allows for a slip system configuration to be determined uniquely, which is otherwise non-unique in the rate-independent formulation.

A great effort has been put into formulating (and re-formulating) conventional crystal plasticity in relation to various numerical set-ups in order to allow for co-existing elastic and elastic-plastic domains while circumventing the uniqueness problem (Cuitino and Ortiz, 1992; Borja and Wren, 1993; Anand and Kothari, 1996; Schröder and Miehe, 1997; Schmidt-Baldassari, 2003; Einav, 2012; Forest and Rubin, 2016). To pin down a particular solution (a unique set of active slip systems) in rate-independent plasticity, the published approaches either make use of special (additional) constitutive assumptions or rely on sophisticated numerical algorithms. As discussed by Schmidt-Baldassari (2003), this tends to blur the picture, and the various add-on can, at times, make it hard to distinguish between constitutive behaviour and numerical peculiarities.

The present work takes a new approach to develop a rate-independent crystal plasticity framework, based on known constitutive behaviour and numerics, by taking as a starting point the Fleck and Willis gradient plasticity theory (Fleck and Willis, 2009b). The following work is a first step on the way, and future investigations will cement if the uniqueness-problem has been fully overcome. To this end, the 2D finite element implementation presented in this work allows repeated loading and unloading of multiple slip systems to develop independently. The underlying idea of the proposed modeling approach is that the trial field (discussed in Nielsen and Niordson, 2013, 2014, 2019) for the slip on the individual systems, being determined from the Minimum Principle I, is driven by the current stress state. Once the field shape has been obtained, only the magnitude of the slip is determined from the kinematics in a subsequent solution step where also the increment of the displacement field is determined. That is, the slip field shape is tied to a material state variable, while the actual size of the slip increment for a specific material point is related to the surrounding material deformation and history through the incorporated material length parameter. In the Fleck and Willis gradient crystal plasticity, the Minimum Principle I, that delivers the trial field, states

\[
H = \inf_{\dot{\gamma}^{(a)}_*} \int_V \left( \tau_F [\dot{\gamma}^{(a)}_e] \dot{\gamma}^{(a)}_e + \xi E^{(a)} s^{(a)} \dot{\gamma}^{(a)}_* - s_{ij}^\mu \mu_{ij}^{(a)} \dot{\gamma}^{(a)}_* \right) dV - \int_S \tau^{(a)} \dot{\gamma}^{(a)}_* dS, \tag{2}
\]
where $\tau_F[\gamma_c^{(\alpha)}]$ is the current flow stress on slip system “$\alpha$”, $\xi^{E(\alpha)}$ is the energetic higher order stress with $r^{(\alpha)}$ being the related higher order tractions, $s_i^{(\alpha)}$ is the slip plane direction, $\mu_{ij}^{(\alpha)}$ is the Schmid orientation tensor (see Eq. (1)), and $s_{ij}$ is the Cauchy stress deviator. The trial field for the slip increment on slip system “$\alpha$” is denoted $\dot{\gamma}^{(\alpha)*}$ with its gradient enhanced effective value being $\dot{\gamma}_{e}^{(\alpha)*}$. Throughout, the present work demonstrates that the proposed rate-independent crystal plasticity implementation readily allows for i) co-existence of elastic and active slip regions, with increasing slip activity, ii) multiple slip regions to co-exist and merge, and iii) complete, or temporarily, elastic unloading of plastic zones. In fact, the overall material response and the plastic slip activity is directly comparable to that of a corresponding rate sensitive (visco-plasticity) formulation in the limit of zero rate sensitivity.

The paper is structured as follows: The material model is laid out in Section 2 and the related finite element framework is presented in Section 3. The benchmark cases are outlined in Section 4, while Section 5 presents the results and compares the proposed rate-independent solution procedure directly to its rate sensitive (visco-plastic) counterpart. Some concluding remarks are given in Section 6.

2. Material model: Crystal plastic version of the Fleck-Willis theory

The material model builds upon the work by Niordson and Kysar (2014) which takes as a starting point the work presented in Gurtin et al. (2007); Borg (2007); Fleck and Willis (2009b). The present work is restricted to a small strain formulation and, thus, the total strain rate, $\dot{\varepsilon}_{ij} = \frac{1}{2} (\dot{u}_{i,j} + \dot{u}_{j,i})$, is decomposed into an elastic part, $\dot{\varepsilon}^{E}_{ij}$, and a plastic part, $\dot{\varepsilon}^{P}_{ij}$, such that $\dot{\varepsilon}_{ij} = \dot{\varepsilon}^{E}_{ij} + \dot{\varepsilon}^{P}_{ij}$. Here, the plastic strain rate originates from the crystallographic slip rate, $\dot{\gamma}^{(\alpha)}$, on the individual slip systems denoted by “$\alpha$” (see Eq. (1)). In Cartesian components, the gradient enhanced virtual work principle (Niordson and Kysar, 2014) can be written as

$$
\int_V \sigma_{ij} \delta \dot{\varepsilon}_{ij} + \sum_{\alpha} (q^{(\alpha)} - \tau^{(\alpha)}) \delta \dot{\gamma}^{(\alpha)} + \sum_{\alpha} \xi^{(\alpha)} s_{i}^{(\alpha)} \delta \dot{\gamma}^{(\alpha)} \delta_{i} dV = \int_S T_i \delta \dot{u}_i + \sum_{\alpha} r^{(\alpha)} \delta \dot{\gamma}^{(\alpha)} dS
$$

(3)
from which the governing equilibrium equations (the strong form) can be identified as

\[ q^{(a)} - \tau^{(a)} - \xi^{(a)}_{ij} s_i^{(a)} = 0 \]  
\[ \text{(Higher order equilibrium)} \quad (4) \]

\[ \sigma_{ij,j} = 0 \]  
\[ \text{(Conventional equilibrium).} \quad (5) \]

Here, \( \sigma_{ij} \) is the symmetric Cauchy stress tensor, \( \tau^{(a)} = \sigma_{ij} \mu^{(a)}_{ij} \) are the Schmid stress on the individual slip systems, while \( q^{(a)} \) and \( \xi^{(a)} \) are the slip system related micro-stress and higher order stress, respectively. The right-hand side of Eq. (3) includes both conventional tractions, \( T_i = \sigma_{ij} n_j \), and higher order tractions, \( r^{(a)} = \xi^{(a)} s_i^{(a)} n_i \), with \( n_i \) being the outward unit normal to the surface, \( S \), that bounds the volume, \( V \). The difference between the Schmid stress, \( \tau^{(a)} \), and the micro-stress, \( q^{(a)} \), follows from a non-homogeneous higher order stress field as its gradient enters the higher order equilibrium (see Eq. (4)). This is built into the theory in order to mimic the effect of movement and/or storage of geometrically necessary dislocations (GNDs). The higher order stress, here, decomposes into an energetic part, \( \xi^{E(a)} \), and a dissipative part, \( \xi^{D(a)} \), such that \( \xi^{(a)} = \xi^{E(a)} + \xi^{D(a)} \). In contrast, the micro-stress, \( q^{(a)} \), is assumed to have a dissipative part only, \( q^{(a)} = q^{D(a)} \). Assuming a quadratic free energy, such that

\[ \Psi = \frac{1}{2} (\varepsilon_{ij} - \varepsilon_{ij}^P) L_{ijkl} (\varepsilon_{kl} - \varepsilon_{kl}^P) + \sum_{\alpha} \frac{1}{2} G \left( L_E \right)^2 \left( s_i^{(a)} \gamma_{ij}^{(a)} \right)^2 \]  
\[ \text{(6)} \]

the energetic part of the higher order stress can then be derived as \( \xi^{E(a)} = \partial \Psi / \partial (\gamma_{ij}^{(a)} s_i^{(a)}) = G \left( L_E \right)^2 s_i^{(a)} \gamma_{ij}^{(a)} \). The conventional stesses are given by \( \sigma_{ij} = \partial \Psi / \partial \varepsilon_{ij} = L_{ijkl} (\varepsilon_{kl} - \varepsilon_{kl}^P) \), with \( L_{ijkl} \) being the isotropic elastic stiffness tensor. The shear modulus is denoted \( G \) and the energetic length parameter is \( L_E \). By employing a generalized version of the gradient enhanced stress and strain, such that

\[ \tau^{(a)}_{\mu/\mu-1} = \left( q^{(a)} \right)^{\mu/\mu-1} + \left( L_D^{-1} \xi^{D(a)} \right)^{\mu/\mu-1} \]  
\[ \text{(7)} \]

and

\[ \dot{\gamma}_{\mu}^{C} = \left( \left( \dot{\gamma}^{(a)} \right)^2 \right)^{\mu/2} + \left( L_D^{\gamma_{i}^{(a)} s_i^{(a)}} \right)^{\mu/2} \]  
\[ \text{(8)} \]

the dissipative quantities can be derived as

\[ q^{D(a)} = \mu_A \cdot \frac{\tau^{(a)}_{\mu/\mu-1}}{\dot{\gamma}_{\mu}^{C}} \gamma_{\mu}^{(a)}, \text{ with } \mu_A = \left( \frac{\left| \dot{\gamma}_{\mu}^{(a)} \right|}{\gamma_{\mu}^{(a)}} \right)^{\mu-2} \]  
\[ \text{(9)} \]
\[ \xi^{D(\alpha)} = \mu_B \cdot \left( \frac{\tau_F^{(\alpha)}}{\gamma_e} \right)^2 \dot{\gamma}_i \hat{s}_i^{(\alpha)} \], with \[ \mu_B = \left( \frac{|L_D \gamma_{i}^{(\alpha)} \hat{s}_i^{(\alpha)}|}{\gamma_e \dot{\gamma}_i^{(\alpha)}} \right)^{\mu-2} \] (10)

where the gradient of the slip, \( \gamma^{(\alpha)} \), on slip system “\( \alpha \)” is denoted \( \gamma_{i}^{(\alpha)} \), and \( L_D \) is identified as the dissipative length parameter. The gradient enhanced reference stress on the individual slip system is denoted \( \tau_{C}^{(\alpha)} \). It is worth to notice that for \( \mu = 2 \), Eqs. (7)-(10) reduced to the quadratic form treated by Niordson and Kysar (2014) (although in a visco-plastic version). Bardella (2007, 2010) further discusses the motivation and consequences of different constitutive assumptions related to both the energetic and dissipative contributions (quadratic and non-quadratic).

To complete the theoretical framework, two minimum principles can subsequently be formulated such that: Minimum principle I governs the evolution of the slip rate field, and Minimum principle II governs the displacement field (and the size of the slip increment for the rate-independent formulation). Minimum principle I and II were first derived by Fleck and Willis (2009b) for isotropic materials under a small strain assumption, and the two minimum principles have recently been developed in a corresponding finite strain setting in Nielsen and Niordson (2019). The principles are given below for both a rate-independent and a visco-plastic formulation of crystal plasticity.

2.1. Rate-independent formulation

By taking the strong form of the higher order equilibrium in Eq. (4) as a starting point, Niordson and Kysar (2014) stated the minimum principle I for the Fleck-Willis theory in a crystal visco-plastic setting. The Minimum principle I for the corresponding rate-independent version can be written as

\[ H = \inf_{\dot{\gamma}^{(\alpha)}_*} \int_V \left( \tau_F^{(\alpha)} \dot{\gamma}_i^{(\alpha)} \dot{\gamma}_i^{(\alpha)} + \xi^{E(\alpha)} \dot{\gamma}_i^{(\alpha)} \dot{\gamma}_i^{(\alpha)} - s_{ij} \mu_{ij}^{(\alpha)} \dot{\gamma}_i^{(\alpha)} \right) dV - \int_S \tau_F^{(\alpha)} \dot{\gamma}_i^{(\alpha)} dS \] (11)

where \( \tau_F^{[\gamma_e^{(\alpha)}]} \) is the current flow stress on slip system “\( \alpha \)” that follows the hardening relation outlined in Section 2.3. Based on the current stress field, the slip rate is determined from Eq. (11) to within a plastic multiplier, such that \( \dot{\gamma}^{(\alpha)} = \lambda^{(\alpha)} \dot{\gamma}^{(\alpha)}_* \). This is achieved by rendering stationarity of Eq. (11) whereby the following variational
statement can be discretized by the finite element method

\[
\int_V q^{D(\alpha)} \delta \dot{\gamma}^{(\alpha)\ast} + \xi^{D(\alpha)} s_i^{(\alpha)} \delta \dot{\gamma}_{i, \alpha} \, dV = \\
\int_V s_{ij} \mu_{ij}^{(\alpha)} \delta \dot{\gamma}^{(\alpha)\ast} - \xi^{E(\alpha)} s_i^{(\alpha)} \delta \dot{\gamma}_{i, \alpha} \, dV + \int_S r^{(\alpha)} \delta \dot{\gamma}^{(\alpha)\ast} \, dS. \tag{12}
\]

It is worth to notice that, the slip field is obtained independently of the stress increment within the rate-independent formulation and thus Eq. (12) delivers only a trial field (scaled to unity). Moreover, the fact that Eq. (12) only depends on the total stresses, and not their increments, allows decoupling the search for a slip fields on the individual slip systems. The actual size of the slip field is obtained from Minimum principle II in a subsequent step together with the displacement increment. The Minimum principle II can be written as

\[
J[\dot{\gamma}(\alpha)] = \frac{1}{2} \int_V \mathcal{L}_{ijkl} \left( \dot{\varepsilon}_{ij} - \sum_\alpha \mu_{ij}^{(\alpha)} \dot{\gamma}_{(\alpha)} \right) \left( \dot{\varepsilon}_{kl} - \sum_\alpha \mu_{kl}^{(\alpha)} \dot{\gamma}_{(\alpha)} \right) + \sum_\alpha h[\gamma^{(\alpha)}] \left( \dot{\gamma}_{(\alpha)} \right)^2 dV \\
- \int_S \dot{T}_{i} \dot{\gamma}_{i} + \hat{t}_{ij} \sum_\alpha \mu_{ij}^{(\alpha)} \dot{\gamma}_{(\alpha)} \, dS \tag{13}
\]

The strain hardening is here denoted \( h[\gamma^{(\alpha)}] \), which can include both self-hardening and latent hardening. However, only self-hardening are considered in this work and thus \( h[\gamma^{(\alpha)}] = d\tau F/\dot{\gamma}^{(\alpha)} \).

2.2. Visco-plastic formulation

Following Niordson and Kysar (2014), the minimum principle I for a visco-plastic single crystal material can be derived as

\[
H = \inf_{\dot{\gamma}(\alpha)} \int_V \left( \Phi[\gamma^{(\alpha)}_e, \dot{\gamma}^{(\alpha)}_e] + \xi^{E(\alpha)} s_i^{(\alpha)} \dot{\gamma}_{i}^{\ast} - s_{ij} \mu_{ij}^{(\alpha)} \dot{\gamma}_{(\alpha)}^{\ast} \right) dV - \int_S r^{(\alpha)} \dot{\gamma}^{(\alpha)\ast} \, dS \tag{14}
\]

where the visco-plastic potential is given by \( \Phi[\gamma^{(\alpha)}_e, \dot{\gamma}^{(\alpha)}_e] = \tau_C \frac{\dot{\gamma}^0}{m+1} \left( \frac{\dot{\gamma}^0}{\gamma^0} \right)^{m+1} \) such that \( \tau_C[\gamma^0, \dot{\gamma}^0] = \tau_F[\gamma^0] (\dot{\gamma}^0/\dot{\gamma}^0)^m \). The slip rate hardening exponent is denoted \( m \) and the reference slip rate is \( \dot{\gamma}^0 \). Rendering stationarity of Eq. (14) yields an expression similar to Eq. (12), but with the stresses now related to the increments of slip through the visco-plastic relation for the reference stress \( \tau_C[\gamma^0, \dot{\gamma}^0] \). Thus, in the visco-plastic version, the minimum principle I delivers the actual slip rate field, \( \dot{\gamma}^{(\alpha)} \), rather than only a trial
field. The corresponding displacement rate is determined from minimum principle II that can be written as

\[ J[\dot{u}_i] = \frac{1}{2} \int_V \mathcal{L}_{ijkl} \left( \dot{\varepsilon}_{ij} - \sum_\alpha \mu_{ij}^{(\alpha)} \dot{\gamma}^{(\alpha)} \right) \left( \dot{\varepsilon}_{kl} - \sum_\alpha \mu_{kl}^{(\alpha)} \dot{\gamma}^{(\alpha)} \right) \, dV - \int_S \dot{T}_i \dot{u}_i \, dS \quad (15) \]

It is worth to notice that latent hardening is cumbersome to treat in the visco-plastic setup since the individual slip systems will interact through the slip rate. Thus, the variational statement in Eq. (12) will have to be fulfilled in a nested loop to ensure simultaneous solution on all slip systems (contrasting the rate-independent model). The displacement field is determined in a subsequent step once a solution that obeys Eq. (14) if found for all slip system.

2.3. Strain hardening relation

Only self-hardening is considered in this work. The individual slip systems follow a linear hardening relation in both the rate-independent and the visco-plastic model, such that \( \tau_F[\gamma_e^{(\alpha)}] = \tau_{\gamma_Y}^{(\alpha)} + h^{(\alpha)} \gamma_e^{(\alpha)} \). Here, \( \tau_{\gamma_Y}^{(\alpha)} \) is the initial yield stress and \( h^{(\alpha)} \) is the slip hardening on slip system “\( \alpha \)”. Approximately zero strain hardening is dealt with throughout and the hardening modulus is set to \( h^{(\alpha)} = 10^{-6} E \) in 1D calculations and \( h^{(\alpha)} = 10^{-3} E \) in the 2D calculations, where \( E \) is Young’s modulus. It is worth mentioning that the simplified hardening relation is chosen for clarity of the work while the implementation of either of the models put no limitation to the choice of the hardening relation.

3. Rate-independent crystal plastic finite element framework

Niordson and Hutchinson (2011) demonstrated in an early study that the visco-plastic model lends itself nicely to numerical implementation, whereas untraditional numerics are required to treat the corresponding rate-independent model. The following finite element implementation and solution procedure are inspired by the numerical framework for isotropic materials laid out in Nielsen and Niordson (2013, 2014, 2019) and a similar two-step solution procedure is adopted in the following. The current formulation focuses on small strains and small deformations\(^2\). A conventional finite element

\(^2\)A corresponding finite strain version should take as a starting point the present work combined with the work by Lynggard et al. (2019); Nielsen and Niordson (2019)
interpolation is used, such that

\[ \dot{u}_i = \sum_{n=1}^{N_{II}} N_i^{(n)} \dot{U}^{(n)} \quad \text{and} \quad \dot{\gamma}^{(\alpha)} = \sum_{n=1}^{N_I} M^{(n)} \dot{\gamma}^{(\alpha)(n)} \],

where the displacement field is discretized by quadratic shape functions, \( N_i^{(n)} \), whereas linear shape functions, \( M^{(n)} \), are employed to describe the slip rate field. Here, \( U^{(n)} \) and \( \dot{\gamma}^{(\alpha)(n)} \) are the quantities of the two fields at the nodes of the finite element mesh. The number of degrees of freedom for the element displacement field and the slip rate field are denoted \( N_{II} \) and \( N_I \), respectively. By discretizing Eqs. (12)-(13), the two step solution procedure progresses as follows for the rate-independent crystal plastic formulation (see also Fig. 1 for an illustration):

**Step 1.** Consider a deformation state where the plastic strain history are known such that the stresses can be evaluated through the elastic relationship. The slip rate field associated with the individual slip systems can then be evaluated by requiring stationarity of Minimum principle I (see Eq. (12)). By adopting the generalized version of the gradient enhanced slip rate (see Eq. (8)) and the associated stress quantities (see Eqs. (9)-(10)), the discretized version of Eq. (12) for the rate-independent formulation can be written as

\[
\int_{V_e} \frac{\tau^{(\alpha)}}{\dot{\gamma}^{(\alpha)(m)}} \left( \mu_A M^{(n)} - \mu_B L^2 D s_i^{(\alpha)} s_j^{(\alpha)} M^{(n)}_{i,j} \right) dV_e \cdot \dot{\gamma}^{(\alpha)(n)} = \int_{V_e} \left( s_{ij} \mu_i^{(\alpha)} M^{(n)} - \xi E^{(\alpha)} s_i^{(\alpha)} M_{i} \right) dV_e + \int_{S_e} \tau^{(\alpha)} M^{(n)} dS_e
\]

which conveys to the corresponding system for the visco-plastic model by replacing the flow stress, \( \tau_F \), with visco-plastic relation for the effective stress, \( \tau_C \). The non-linear system of equations in Eq. (17) closely resembles the system obtained in Nielsen and Niordson (2013, 2014, 2019) for an isotropic solid and, thus, the same iterative solution procedure is employed. The trial slip field for the individual slip systems can be obtained separately in the rate-independent model as their interactions are dealt with in the subsequent Step 2 (see Fig. 1). In contrast, the individual treatments are only the case in the visco-plastic model when omitting latent hardening.

Despite continuously normalizing the trial field to unity (such that \( \dot{\gamma}^{(\alpha)(n)}/|\dot{\gamma}^{(\alpha)(n)}|_{\text{max}} \)) while iterating on the solution to Eq. (17), the field can take values different from
zero even in (inactive slip) regions which must be governed by elasticity. Thus, all regions of noticeable slip activity cannot be declared active at this stage in the solution procedure. Instead, the concept of “potentially active” slip regions is introduced and a search for these are conducted by image analysis before entering into Step 2. The search progresses as follows: classical connected-component-labelling is used to determine which elements belongs to regions of noticeable slip activity (typically using \( \min(|\dot{\gamma}^{(\alpha)}|) > 10^{-3} - 10^{-4} \) as threshold\(^3\)), denoted “potentially active”, and a multiplier is only assigned to the slip system if it furthermore obeys \( \Phi^{(\alpha)}_{H(k)} > 0 \), with \( \Phi^{(\alpha)}_{H(k)} \) being the value of the Minimum Principle I integrated within that specific “potentially active” slip region.

\[
\Phi^{(\alpha)}_{H(k)} = - \left( \int_{V^{pl(k)}} \left( \tau_F \gamma_e^{(\alpha)} \dot{\gamma}^{(\alpha)} + \xi \gamma^{(\alpha)} \dot{\gamma}^{(\alpha)} - s_{ij} \mu_{ij}^{(\alpha)} \dot{\gamma}^{(\alpha)} \right) dV^{pl(k)} \right) \geq 0 \quad (18)
\]

Minimum Principle I, in this way, constitutes a non-local yield criterion (see also Danas et al., 2012). The total number of “potentially active” slip zones is denoted “\( K \)”. It is, however, worth to notice that, in general, there can exist a number of multipliers within a given region of the material domain as the plastic multiplier is tied to a specific slip system. Whether or not the individual slip system can be regarded as active is determined through the plastic multiplier obtained from Minimum principle II in the subsequent Step 2. Bear in mind that the intermediate image analysis of the trial field is conducted independently for the individually slip systems and is only required in the rate-independent model (see also Fig. 1).

**Step 2.** Based on the slip rate (trial) field obtained in Step 1, the subsequent Step 2 determines the incremental displacement field. The presentation, here, focuses on the rate-independent formulation as this second step also determines the plastic multipliers assigned to the individual slip system. Rendering stationarity of Minimum Principle II (Eq. (13)) and partitioning the material domain into “potentially active” (elastic-plastic) and “inactive” (elastic) regions, such that \( V = V^{pl} + V^{el} \) (bounded by

---

\(^3\)The threshold value must be small compared to the convergence criterion related to the iterative solution for (17)
the surface $S = S^{pl} + S^{el}$, yields
\[
\sum_{k=1}^{K} \int_{V^{pl}(k)} L_{ijkl} \left( \dot{\varepsilon}_{ij} - \sum_{\alpha} \mu_{ij}^{(\alpha)} \lambda_{(k)}^{(\alpha)} \dot{\gamma}_{(k)}^{(\alpha)*} \right) \left( \delta \dot{\varepsilon}_{kl} - \sum_{\alpha} \mu_{kl}^{(\alpha)} \delta \lambda_{(k)}^{(\alpha)} \dot{\gamma}_{(k)}^{(\alpha)*} \right) + \sum_{\alpha} h[\gamma_{e}^{(\alpha)}] \lambda_{(k)}^{(\alpha)} (\gamma_{e}^{(\alpha)*})^2 \delta \lambda_{(k)}^{(\alpha)} \right] dV^{pl}(k) + \int_{V^{el}} L_{ijkl} \dot{\varepsilon}_{ij} \dot{\varepsilon}_{kl} dV^{el} = \int_{S} \dot{T}_{i} \delta u_{i} dS + \sum_{k=1}^{K} \int_{S^{pl}} \dot{t}_{ij} \sum_{\alpha} \mu_{ij}^{(\alpha)} \delta \lambda_{(k)}^{(\alpha)} \dot{\gamma}_{(k)}^{(\alpha)*} dS^{pl}(k)
\]

which holds for any kinematically admissible variation of the displacements field, $\delta u_{i}$, and any admissible variation in the plastic multipliers, $\delta \lambda_{(k)}^{(\alpha)}$. A finite element discretization of Eq. (19) then gives two discrete systems of equations (similar to isotropic gradient plasticity, see Nielsen and Niordson, 2013, 2014), given by
\[
\int_{V_{e}} L_{ijkl} B_{ij}^{(n)} B_{kl}^{(m)} dV_{e} \cdot \dot{U}_{(m)} - \sum_{\alpha} \int_{V_{e}} L_{ijkl} B_{ij}^{(n)} \mu_{kl}^{(\alpha)} \dot{\gamma}_{(k)}^{(\alpha)*} dV_{e} \cdot \lambda_{(k)}^{(\alpha)} = \int_{S_{e}} \dot{T}_{i} N_{i}^{(n)} dS_{e}
\]

The finite element volume is denoted $V_{e}$ and the finite element surface is $S_{e}$, while $B_{ij}^{(n)} = (N_{ij}^{(n)} + N_{ji}^{(n)})/2$. The plastic multipliers are denoted $\lambda_{(k)}^{(\alpha)}$, with “$\alpha$” referring to the current slip system displaying noticeable slip activity within the “potential active” slip region with number “$k$” ($k = 1, 2, \ldots K$). Once the system of equations in Eq. (21) is solved, then the requirement $\lambda_{(k)}^{(\alpha)} > 0$ is considered (see Fleck and Willis, 2009b). In the case $\lambda_{(k)}^{(\alpha)} < 0$, the slip system is termed “inactive” (elastic) in that particular region and the multiplier is removed from the system of equations. The system of equations is hereafter solved again. The repeated solving of Eq. (21) is continued until each slip system satisfies $\Phi_{H_{(k)}}^{(\alpha)} > 0$ and $\lambda_{(k)}^{(\alpha)} > 0$ for all “potentially active” slip region, after which the slip systems are declared active and the incremental solution is updated.

3.1. Visualizing the 2-step solution procedure for the rate-independent model

Imagine deforming the material until reaching the stress field in Fig. 1 (on the left). The subsequent increment then enables visualizing the 2-step solution procedure as
well as the strategy for handling several slip systems with (potentially) multiple active slip regions (see Fig. 1). Starting from the left in Fig. 1, the known stress field, $\sigma_{ij}^{(n-1)}$, from the previous increment acts as driving force for determining the trial slip field, $\dot{\gamma}^{(a)*}$, (solving Eq. (17)). The trial fields thereby ties to a material state variable (and not on the increments hereof). The equation solving for a trial field solution for the individual slip systems decouples and enables individual identification of potentially active slip regions through image analysis. A raster image of the trial field solution is first prepared based on the irregular mesh used for the FE calculations, whereafter connected component labeling identifies the potentially active slip regions. In Fig. 1, the image analysis identifies two slip regions for slip system one and three, respectively, and four regions for slip system two. Once identified, the non-local yield criterion from Eq. (18) determines if a plastic multiplier must be assigned to the individual slip regions. For the iteration depicted in Fig. 1, only the slip regions associated with slip system one and three have $\Phi^{(1)}_{H(k)}>0$ for $k=1,2$, while $\Phi^{(2)}_{H(k)}<0$ for $k=1,2,3,4$. In Step 2, the Minimum principle II is enriched by the multipliers and solved (and resolved) through Eq. (19) until $\lambda^{(a)}_{(k)} \geq 0$ for all “$k$” and “$a$”. The obtained plastic multiplies allow for updating the slip field, plastic strain, and stress field, whereafter the next load increment can begin.

Unique to the propose solution procedure for the rate-independent model is that drift from equilibrium typically shows in the non-local yield criteria, either by $\Phi^{(a)}_{H(k)}$ drifting from zero such that $\Phi^{(a)}_{H(k)} \gg 0$, or by temporarily unloading of individual slip regions. Typically, the issue is resolved by dividing the deformation into additional increments. To enable efficient finite element modeling, future work should focus on i) improving the solution procedure for the trial field (e.g. solving Eq. (17) efficiently), and ii) improve the staggered 2-step approach either to ensure equilibrium or by a one-step correction with the residual from the previous load increment.

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4Basically, determining what is foreground and background in the “image” corresponding to $|\dot{\gamma}^{(a)*}|$, with background determined by $|\dot{\gamma}^{(a)*}| < 10^{-3} - 10^{-4}$. 
4. Benchmark cases: loading, unloading, and slip zone interactions

The new gradient enhanced rate-independent crystal plasticity framework has been implemented in both a 1D and 2D finite element model allowing for a demonstration of the procedure. The focus is on the three benchmark cases summarized below and, unless anything else is stated, the material properties collected in Table 1 are used.

4.1. Shearing of a constrained infinite slab modelled in 1D

The first benchmark case considers shear loading of an infinite slab of a homogeneous material constrained between two rigid platens (see Fig. 2). A single slip system is modeled with an orientation such that the slip direction is along the $x_2$-axis ($\theta^{(\alpha)} = 90^\circ$). This crystal configuration is highly artificial but it serves to demonstrate the material response in a wide parameter space and to illustrate the key features readily accounted for by the proposed modeling procedure. Micro-hard boundaries that prohibit the penetration of dislocations (zero slip) are enforced at the top ($x_2 = H$) and bottom ($x_2 = 0$) interface of the domain, while shear loading is prescribed by the deformation history, $\Delta(t)$, at the top boundary (see Fig. 2). Both monotonic and cyclic shearing are prescribed to demonstrate the model’s capacity to handle changes in the yielding condition (loading versus unloading). For comparison to the new rate-independent model, the imposed deformation rate is $\dot{\Delta}/(H\dot{\gamma}_0) = 1$ in the corresponding visco-plastic model. The setup maps out the influence of the individual model parameters and serves as a baseline for comparison to existing model results in the literature.

4.2. Combined shear and tension modelled in 2D

Following a butterfly deformation path, combined shear and tension is enforced onto a constrained infinite slab of material resembling an HCP single crystal. The setup composes the second benchmark case. The crystal orientation is such that the deformation occurs in the basal plane of the HCP crystal inflicting plane strain deformation. Thus, three crystallographic slip systems are included in the analyses, as illustrated in Fig. 3(a). The deformation is prescribed through $\Delta_1(t)$ and $\Delta_2(t)$ according to the butterfly deformation path in Fig. 3(b) where the arrows indicated the deformation direction. Micro-hard boundaries that prohibit the penetration of dislocations (zero slip) are enforced at the top ($x_2 = H$) and bottom ($x_2 = 0$) interface of
the domain modeled in 2D plane strain. For comparison to the new rate-independent model, the imposed deformation rates are \( \Delta_1/(H\dot{\gamma}_0) = 1 \) and \( \Delta_2/(H\dot{\gamma}_0) = 1 \) in the corresponding visco-plastic model.

4.3. Tensile loading of a notched sample

The third benchmark case focuses on a notched sample subject to 2D plane strain tension. The half domain height is \( H \), the half domain width is \( W \), and the notch radius is \( R \), with \( H/W = 3 \) and \( R/W = 0.5 \) (see Fig. 4). A far-field tensile load is applied by prescribing the displacement at the top and bottom boundaries (\( \dot{u}_2 = \pm \Delta \) at \( x_2 = \pm H \)), while the domain is constrained to contract along the same boundaries (\( \dot{u}_1 = 0 \) at \( x_2 = \pm H \)). Moreover, micro-hard boundaries are enforced at the top and bottom (\( \dot{\gamma}^{(\alpha)} = 0 \) at \( x_2 = \pm H \)). The material resembles an HCP single crystal oriented such that plane strain deformation takes place in the basal plane (see Fig. 4(c)). For comparison to the new rate-independent model, the imposed deformation rate is \( \dot{\Delta}/(H\dot{\gamma}_0) = 1 \) in the corresponding visco-plastic model.

5. Results

5.1. Material response and slip activity in a constrained layer subject to shear

Shearing deformation of a homogeneous slab (one 90° slip system) yields a single slip region that spans the entire material domain. The appearance and evolution of slip, however, change dramatically with the length scale. The material response and related slip are presented in Figs. 5-7 for monotonic shearing and in Fig. 8 for cyclic shearing.

The overall load-deflection curves in Fig. 5(a) display an increase in the apparent yield stress when increasing the dissipative length parameter, and a further increase is obtained when \( \mu < 2 \). In contrast, the apparent yield stress decreases for \( \mu > 2 \). The \( \mu \)-parameter, however, only affects the strengthening and only when \( L_D > 0 \), while the subsequent hardening of the material is unaffected (see also the discussion in Nellmann et al., 2017). Furthermore, Fig. 5(a) shows that the energetic length parameter, \( L_E \), gives rise to additional hardening of the material and that the material response only depends on \( \mu \)-parameter when \( L_D > 0 \) (in line with Eqs. (7)-(8)). Thus, the
\(\mu\)-parameter only has a small effect on the effective hardening but the \(\mu\)-parameter re-distributes the slip as shown in Fig. 5(b) due to a local increase in hardening. The dependence on \(\mu\) and \(L_E\) is further demonstrated in Fig. 6 showing the effective hardening \(\frac{H_{eff}}{G} = 3/(G/G_t - 1)\), with \(G\) being the shear modulus and \(G_t\) being the tangent shear modulus extracted from the last part of the overall stress-strain response (adopted from Niordson and Legarth, 2010).

The interplay between the \(\mu\)-parameter and the dissipative length parameter on the shear response if mapped out in Fig. 7. Here, showing the apparent yield stress as a function of dissipative length parameter in Fig. 7(a) and as a function of the \(\mu\)-parameter in Fig. 7(b). A non-linear relationship between the apparent yield stress and the dissipative length parameter exists for small \(L_D/H\) values. In contrast, the dependence on the dissipative length parameter is almost linear for \(L_D/H \gtrapprox 0.3\). Niordson and Legarth (2010) reports the same trend for isotropic materials. Figure 7(a) further shows that the dependence on \(L_D\) is little affected by the \(\mu\)-parameter. However, the changes in the distribution of slip across the material domain are dramatic when diverging from a quadratic formulation (\(\mu \neq 2\)). For the case with \(L_E = 0\), Fig. 5(c) shows the effect on the distribution of slip at maximum deformation for various combinations of \(L_D\) and \(\mu\)-parameter\(^5\).

5.2. Cyclic loading and unloading of a single slip system

Repeated elastic-plastic loading and unloading is readily accounted for in the new rate-independent modeling procedure by evaluating the non-local yield criteria (Eq. (18)) in each increment. This key feature is demonstrated by considering cyclic shearing of a constrained homogeneous material slab with a single slip system located at 90° (see Fig. 2). Two shearing cases are modeled where cyclic shear takes place i) around zero mean strain (see Fig. 8a), and ii) at a non-zero mean strain (see Fig. 8b). Figure 8 shows the load-deflection curves at various length scales including an approximate conventional response \((L_D/H = 1/50\) and \(L_E = 0\)). The cyclic response curves show that

\(^5\)This weak coupling between the distribution of slip and the apparent yield stress might help matching the gradient (crystal and isotropic) plasticity theory by Fleck and Willis (2009b) to experimental findings (see also El-Naaman et al., 2016).
i) the material transitions to an elastic stage when unloading occurs, and hereafter follows an elastic response until subsequent yielding. ii) Subsequent yielding occurs either at a stress level corresponding to $|\Sigma_{12}| = |\Sigma_{12}^{\text{max}}|$ (isotropic hardening for $L_D > 0$), or at a stress level $|\Sigma_{12}| = |\Sigma_{12}^{\text{max}} - 2\tau_Y|$ (kinematic hardening for $L_E > 0$). Here, $\Sigma_{12}^{\text{max}}$ is the highest value of $\Sigma_{12}$ obtained throughout the deformation history. iii) the cyclic hysteresis is a closed loop due to the perfectly plastic material response. The findings are consistent for all combinations of the length parameters (energetic and dissipative) and loading scenarios. Figure 8 focuses on the quadratic formulation ($\mu = 2$) since additional calculations (not presented here) for the cyclic loading has shown little effect of the $\mu$-parameter. However, convergence is harder to achieve in Step 1 of the procedure when $\mu = 3$ (see Eq. (17)) and more increments were required to circumvent this issue (elaborated in Section 6).

Comparing the rate-independent model results to that of a corresponding visco-plastic model ($m = 0.01$) yields a nearly perfect match between curves (see Fig. 8). The match was intended by Fleck and Willis (2009a,b) and important, bearing in mind that the visco-plastic versus the rate-independent model implementations are very different.

5.3. Combined shear and tension modeled in 2D under plane strain conditions

Forest and Rubin (2016) considers a so-called biaxial butterfly test to demonstrate a corresponding rate-independent crystal plasticity formulation (without gradient effects). Figure 3 illustrates a related test setup, where a constrained infinite material slab is subject to combined shear and tension according to the deformation path in Fig. 3(b). The material resembles an HCP single crystal, and the domain is modeled within a 2D plane strain finite element framework (see Section 4.2 for details).

Figure 9 shows the stress response, corresponding to the butterfly deformation path, obtained with the new rate-independent model for two distinct length scales and zero conventional strain hardening ($h^{(0)}(\alpha) = 10^{-6}E$). Figure 9(a) shows the stress path for an approximate conventional material ($L_D/H = 1/50$), and Fig. 9(b) shows the effect of including gradient strengthening at a small scale ($L_D/H = 1/2$). The solution by a corresponding visco-plastic model is shown for comparison, and nearly coinciding stress responses are obtained. The new rate-independent model, thereby, readily i) activates
independent slip on the three individual slip systems, ii) accounts for multiple slip systems acting together and even in opposite directions (negative versus positive slip), and iii) takes care of the changes in the yielding conditions as the complex butterfly loading progresses. The strategy for handling multiple slip systems is described in Section 3 and visualized in Fig. 1.

5.4. Load-deflection response and slip in a notched tensile sample

The 2D finite element implementation of the new rate-independent modeling procedure is demonstrated on a notched tensile sample considering the irregular mesh in Fig. 4(b). The material resembles an HCP single crystal orientated such that plane strain deformation takes place in its basal plane (see Fig. 4(c)). Figure 10(a) shows the load-deflection response for four different length scales using a quadratic formulation ($\mu = 2$) and a nearly perfectly plastic material ($N = 10^{-3}E$) along with the total slip field at $\Delta/(H\varepsilon_Y) = 4$ (see Fig. 10(b-e)). A direct comparison between the rate-independent model and the corresponding visco-plastic model yields nearly coinciding results. The load-deflection response shows a gradual transition from initial yielding until the load-carrying capacity reaches the plateau set by the zero conventional strain hardening. The gradual transition reflects a development in the slip activity, which starts at the notch on both sides of the sample, whereafter the active slip regions evolve and connect the two notches. Figure 11 shows the evolution in the active slip region, the slip on system one ($\alpha = 1$) (see Fig. 4), and the total slip given by $\gamma_{\text{tot}} = \sum_\alpha |\gamma^{(\alpha)}|$ for $L_D/H = 0.2$. Three deformation stages are considered. As the tensile load increases, two independent slip regions becomes active (one at each notch, see Fig. 11(a)), for both slip system one ($\alpha = 1$) and slip system three ($\alpha = 3$), whereas slip system two ($\alpha = 2$) remains inactive. The fact that the slip on system one is positive and the slip on system three is negative (not shown here) is readily accounted for and requires no special attention. Upon further loading, the active slip regions merge and evolve, as depicted in Fig. 11(b-e). The merging of active slip regions requires a shift from two (or more) plastic multipliers, $\lambda^{(\alpha)}_k$, to one multiplier, and this is readily accomplished by the image analysis adopted to identify slip regions. Only two active slip regions (belonging to $\alpha = 1$ and 3) exist in the final deformation state, and the total slip has intensified between the notched in bands along the direction of the crystallographic
6. Concluding remarks

The proposed solution procedure, suitable for finite element modeling, extends the gradient enhanced crystal visco-plasticity model by Niordson and Kysar (2014) to rate-independent crystal plasticity. The work takes as a starting point the size-dependent plasticity theory by Fleck and Willis (2009b), and finds great inspiration in the modeling techniques for corresponding isotropic materials (Nielsen and Niordson, 2012, 2014, 2019). The new rate-independent crystal plasticity framework readily accounts for i) the evolution of slip in individual active slip regions and how the region expands into the surrounding elastic material (that gradually becomes plastic). ii) Co-existing and merging of active slip regions and unequal development of slip in individual slip regions. iii) Repeated loading and unloading of slip systems. Moreover, the proposed method readily accounts for iv) co-existence of slip systems with either identical or different orientations (see also in Nielsen, 2019), and v) no special action is required to distinguish between positive and negative slip on a given slip system.

The proposed solution method forms a rate-independent limit for the corresponding visco-plastic (rate sensitive) model by Niordson and Kysar (2014). Thus, the model handles the abrupt activation of slip systems when yielding occurs rather than the smooth transition modeled by the visco-plastic version. Furthermore, conventional rate-independent crystal plasticity acts as the limit case for the proposed framework when $L_D \rightarrow 0$ and $L_E = 0$. The limit case is, however, hard to reach in these types of implementations (also discussed in Nielsen and Niordson, 2012, 2019). Similar to the visco-plastic version, the new rate-independent crystal plasticity model suffers from convergence issues for the trial slip field when $L_D \rightarrow 0$ and $N \rightarrow 0$, as well as for increasing $L_E$ (see also Niordson and Legarth, 2010).

7. Acknowledgements

The work is financially supported partly by the Danish Council for Independent Research in the project “Advanced Damage Models with InTrinsic Size Effects”, grant DFF-7017-00121, and partly by VILLUM FOUNDATION EXPERIMENT in the project
“Micron-Scale Crashworthiness”, grant no. 00028205. Professor Christian F. Niordson is acknowledged for fruitful discussions.


Tables

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Figures

Figure 1: Schematic of the staggered procedure used to identify the slip regions in a 2D setup. The (known) stress field, $\sigma_{ij}$, from the previous increment feeds to minimum principle I which delivers a unit trial field for the slip rate for each slip system. Image analysis on each field is conducted to identify potentially active slip regions whereafter the non-local yield criterion is evaluated. A plastic multiplier is enforced when $\Phi_{H(k)}^{(\alpha)} > 0$, whereafter minimum principle II is solved and field variables are updated when all $\lambda_{(k)}^{(\alpha)} > 0$. 

Solve Eq. (17)  \hspace{1cm} \text{Image analysis}  \hspace{1cm} \text{Solve Eqs. (20)-(21)}  \hspace{1cm} \text{Updates field variables}
Figure 2: a) Schematic of a material slab constrained between rigid platens subject to shear deformation (prescribed by $\Delta(t)$ along the $x_1$-direction). Constrained plastic flow is enforced at the interface between the slab and the platens ($\gamma^{(\alpha)} = 0$ at $x_2 = [0, H]$). b) Definition of slip system “$\alpha$” and its orientation given by $\theta^{(\alpha)}$. The slip direction is $s^{(\alpha)}_i$ and the slip normal is $m^{(\alpha)}_i$.

Figure 3: Schematic of an infinite 2D domain, representing an HCP crystal in its basal plane, constrained between rigid platens and subject to a combined tension and shear (prescribed $\Delta_1(t)$ along the $x_1$-direction and $\Delta_2(t)$ along the $x_2$-direction). Constrained plastic flow is enforced at the interface between the slab and platens ($\gamma^{(\alpha)} = 0$ at $x_2 = [0, H]$). b) A butterfly deformation path is testing loading and unloading of the systems (suggested by Forest and Rubin, 2016). The arrows indicates the loading direction.
Figure 4: a) Schematic of a notch plane strain tension sample of height $2H$, width $2W$, and notch radius $R$. The sample has $H/W = 3$ and $R/W = 0.5$. b) Finite element mesh with the deformation prescribed at top and bottom ($x_2 = \pm \Delta$ at $x_2 = \pm H$) where also the plastic slip is constrained ($\gamma^{(\alpha)} = 0$ at $x_2 = \pm H$). c) The Slip system configuration resembling an HCP single crystal oriented such that the deformation takes place in the basal plane (plane strain condition).
Figure 5: The effect of the energetic and dissipative length parameters shown by a) the stress-strain response comparing the results from the proposed rate-independent model to that of a corresponding visco-plastic formulation (homogeneous material slab of height $H$), and the evolution of slip across the material slab for b) $L_D/H = 1/2$ and c) $L_E/H = 0$ ($\gamma_Y = \sigma_Y/\sqrt{3}$, $\gamma_Y = 2\tau_Y(1 + \nu)/E$ and $h^{(\alpha)} = 10^{-6}E$).
Figure 6: The effect of the energetic length parameter on the effective hardening ($H_{\text{eff}}/G = 3/(G/G_1 - 1)$) for fixed $\mu = [1.5, 2, 3]$ and two values of the dissipative length parameter ($L_D/H = [1/50, 1/2]$).
Figure 7: Strengthening effect of the dissipative length parameter (with $L_E = 0$) shown by a) the apparent yield stress as function of the dissipative length parameter for fixed $\mu = [1.5, 2, 3]$, and b) the apparent yield stress as function of $\mu$ for three values of the dissipative length parameter, $L_D/H = [1/50, 1/4, 1/2]$. 
Figure 8: Cyclic shearing for a) large amplitude and zero mean strain and b) small amplitude and prescribed mean strain, showing results for conventional plasticity as well as the effect of the dissipative and energetic length parameter ($\tau_Y = \sigma_Y / \sqrt{3}$, $\gamma_Y = 2\tau_Y (1 + \nu)/E$ and $h^{(\alpha)} = 10^{-6}E$).
Figure 9: Cyclic stress response corresponding the prescribed butterfly deformation path. Here, showing the solution from both the visco-plastic model and the new rate-independent model. Two distinct length scale are considered representing a structure a) on the conventional scale with $L_D/H = 1/50$, and b) on the micron scale with $L_D/H = 1/2$. Throughout, a quadratic formulation is used ($\mu = 2$), $L_E = 0$, and $h^{(\alpha)} = 10^{-6} E$. 
Figure 10: (a) Load-deflection response for the notched sample for increasing dissipative length parameter showing coinciding curves for the visco-plastic model and the new rate-independent implementation. Total slip field $\gamma_{\text{tot}}/\gamma_Y = \sum_\alpha |\gamma^{(\alpha)}|/\gamma_Y$, where $\gamma_Y = \sigma_Y/(E/(2(1+\nu)))$, showing results for four length parameters b) $L_D/W = 0.1$, c) $L_D/W = 0.2$, d) $L_D/W = 0.4$, and e) $L_D/W = 0.8$ at $\Delta/(H\varepsilon_Y) = 4$. Throughout, a quadratic formulation is used ($\mu = 2$), $L_E = 0$, and $h^{(\alpha)} = 10^{-3} E$. 
Figure 11: Notched sample at three stages of deformation a,d,g) $\Delta/(H\gamma_Y) = 2$, b,e,h) $\Delta/(H\gamma_Y) = 3$, and c,f,i) $\Delta/(H\gamma_Y) = 4$. The results show a-c) The active slip regions detected by the image analysis for slip system $\alpha = 1$, d-f) The slip, $\gamma^{(1)}/\gamma_Y$, for $\alpha = 1$, and g-i) The total slip $\gamma_{\text{tot}}/\gamma_Y = \sum_\alpha |\gamma^{(\alpha)}|/\gamma_Y$. Throughout, $\gamma_Y = \sigma_Y/(E/(2(1+\nu)))$, $h^{(\alpha)} = 10^{-3}E$, and $L_D/W = 0.2$. 

$\gamma_{\text{tot}}/\gamma_Y$