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Abstract/Resumé

Abstract

In this thesis, we study pseudo-differential operators on a real-analytic manifold, which is either compact Riemannian, or a Lie group with a bi-invariant metric. Our aim is to obtain algebras of such operators, acting on real-analytic functions, but preserving a tube domain into which the functions extend holomorphically. The tube domain, contained in a complexification, is known as a "Grauert tube". We show that all the operators commuting with the Laplacian have this property, and in so doing, we make use of the Poisson transform introduced by Stenzel [56]. The transform is derived from a special case of a claim by Boutet de Monvel in [3], which was proved only recently by Stenzel [55] and Zelditch [65] in different ways. We demonstrate that the same would be true of many other real-analytic operators, if Boutet de Monvel's claim holds in general, and briefly discuss approaches to it. Finally, in the setting of operators on a Lie group carrying a bi-invariant metric, without using the transform, we obtain a non-trivial algebra with this property. This algebra is determined by a subspace of the global matrix-valued symbols, which was introduced by Ruzhansky, Turunen and Wirth [49].

Resumé

I denne afhandling undersøger vi pseudo-differentialoperatorer på reel-analytiske mangfoldigheder, som enten er kompakt Riemmanske, eller Lie grupper med en bi-invariant metrik. Vores mål er at finde algebraer af sådanne operatorer som virker på reel-analytiske funktioner, men også bevarer et "rør" i en kompleksificering af mangfoldigheden som funktionerne kan udvides ind i. Dette rør kaldes et "Grauert rør". Vi viser at alle operatorer som kommuterer med Laplace operatoren har denne egenskab, og til dette anvender vi Poisson transformationen, introduceret af Stenzel i [56]. Denne transformation er udledt af et specielt tilfælde af en påstand af Boutet de Monvel [3], som kun for nylig er blevet bevist af Stenzel [55] og Zelditch [65] på forskellige måder. Vi demonstrerer at det samme ville være sandt for mange andre reel-analytiske operatorer, hvis Boutet de Monvel's påstand holder generelt. Til sidst, i tilfældet af operatorer på en Lie gruppe med bi-invariant metrik, så finder vi en ikke-triviel algebra med egenskaben, uden at bruge Poisson transformationen. Denne algebra er bestemt af et underrum af de globale matrix-symboler, som blev introduceret af Ruzhansky, Turunen og Wirth i [49].

Contents

Symbols and Abbreviations

Symbol Description

ON	Orthonormal.
ONB	Orthonormal basis.
DCT	Dominated convergence theorem.
MCT	Monotone convergence theorem.
FTT	Fubini-Tonelli theorem.
RLL	Riemann-Lebesgue lemma.
FID	Fourier integral distribution(s).
FIO	Fourier integral operator(s).
PHG	Poly-homogeneous.
w.r.t.	with respect to.
$\mathbb{N}$	Natural numbers.
$\mathbb{N}_0$	Natural numbers with zero.
$\mathbb{Z}$	Integers.
$\mathbb{R}$	Real numbers.
$\mathbb{C}$	Complex numbers.
$\mathbb{S}^n$	The n-sphere.
$\mathbb{T}^n$	The n-torus.
$B(x, r)$	Ball of radius r about $x \in \mathbb{R}^n$.
1_K	Characteristic function for K.
$C^\infty(\mathbb{R}^n)$	Smooth functions on $\mathbb{R}^n$.
$C_0^\infty(\mathbb{R}^n)$	As above, with compact support.
$\mathcal{S}(\mathbb{R}^n)$	The Schwartz functions on $\mathbb{R}^n$.
$\mathcal{D}'(\mathbb{R}^n)$	Distributions on $\mathbb{R}^n$.
$\mathcal{E}'(\mathbb{R}^n)$	As above, with compact support.
$\mathcal{S}'(\mathbb{R}^n)$	Tempered distributions on $\mathbb{R}^n$.
$L^2(\mathbb{R}^n)$	Standard L^2-functions on $\mathbb{R}^n$.
$H^s(\mathbb{R}^n)$	Order $s \in \mathbb{R}$ Sobolev space on $\mathbb{R}^n$.
$[\cdot, \cdot]$	The commutator bracket.
$\lvert\alpha\rvert$	Size of a multi-index $\alpha \in \mathbb{N}_0^n$.
∂_x^α	The composition $\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.
x^α	The product $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.
$\mathrm{ad}_{\partial_{x_j}} u$	The commutator $[\partial_{x_j}, u]$.
$\mathrm{ad}_{x_j} u$	The commutator $[x_j, u]$.
$\mathrm{ad}_{\partial_x}^\alpha u$	$\mathrm{ad}_{\partial_{x_1}}^{\alpha_1} \circ \cdots \circ \mathrm{ad}_{\partial_{x_n}}^{\alpha_n} u$.
$\mathrm{ad}_x^\alpha u$	$\mathrm{ad}_{x_1}^{\alpha_1} \circ \cdots \circ \mathrm{ad}_{x_n}^{\alpha_n} u$.
$\wedge$	Wedge product of exterior forms.
$\otimes$	Tensor product (various).

M	Typically a smooth manifold.
$d\Phi_x$	Differential of a map Φ at x.
Ω_M^1	1-density bundle over M.
$\Omega_M^{1/2}$	$\frac{1}{2}$-density bundle over M.
G	Typically a Lie group.
$\mathfrak{g}$	Left-invariant vector fields on G. (Or the Lie algebra of G.)
L, R	The regular representations of G. Left L and Right R.
ad	Adjoint representation of $\mathfrak{g}$.
Ad	Adjoint representation of G.
$\widehat{G}$	The unitary dual of G.
$\mathrm{Mat}(\widehat{G})$	Matrix-valued sequences on $\widehat{G}$.
$\mathcal{S}'(\widehat{G})$	Tempered sequences on $\widehat{G}$.
Tr	Hilbert-Schmidt trace functional. (Ordinary trace on matrices.)
Δ_G	A Laplacian on the Lie group G. (Relative to a bi-invariant metric.)
$\Delta_{G_\mathbb{C}}$	Induced Laplacian on $G_\mathbb{C}$.
$(\Delta_G)_\mathbb{C}$	Holomorphic "lift" of Δ_G to $G_\mathbb{C}$.
d_ξ	Dimension of ξ in $[\xi] \in \widehat{G}$.
λ_ξ	Eigenvalue of $[\xi] \in \widehat{G}$ w.r.t. Δ_G.
$\langle\xi\rangle$	The function $(1 + \lvert\xi\rvert^2)^{\frac{1}{2}}$ on $\mathbb{R}^n$. The function $(1 + \lvert\lambda_\xi\rvert^2)^{\frac{1}{2}}$ on G.
$M_\mathbb{C}$	Bruhat-Whitney complexification of a real-analytic manifold M.
$G_\mathbb{C}$	Universal group complexification of G (real-analytic structure).
M_ϵ	The Grauert tube of radius $\epsilon > 0$ of $M_\mathbb{C}$ when M is compact.
G_ϵ	The Grauert tube of radius $\epsilon > 0$ of $G_\mathbb{C}$ when G is compact.
$\mathcal{P}_\epsilon$	The Poisson transform on M_ϵ. Fixed tube with radius $\epsilon > 0$.
$\mathcal{C}_t$	Segal-Bargmann transform on $G_\mathbb{C}$. Evaluated at fixed time $t > 0$.

N	Typically a Kähler manifold.
J	An almost complex structure.
Θ	A Jacobian factor associated to Φ in the decomposition of $G_{\mathbb{C}}$.
$\Omega^{p,q}(N)$	Differential forms of type (p,q).
$T^*_{p,q}N$	(p,q) cotangent tensor bundle.
$T^{p,q}N$	(p,q) tangent tensor bundle.
$\mathcal{O}(N)$	Holomorphic functions on N.
$C(M)$	Continuous functions on M.
$C^m(M)$	The C^m-functions on M.
$C^\omega(M)$	Real-analytic functions on M.
$C^\infty(M)$	Smooth functions on M.
$C_0^\infty(M)$	As above, with compact support.
$\mathcal{D}'(M)$	Distributions on M.
$\mathcal{E}'(M)$	As above, with compact support.
$L^2(M)$	Square-integrable functions on M. (Fixed smooth positive 1-density.)
$H^s(M)$	Order $s \in \mathbb{R}$ Sobolev space on M.
$H^s_K(M)$	As above, but supported in K.
$\mathcal{O}^s(\partial N)$	A space of order $s \in \mathbb{R}$ traces.
supp	The support.
supp_0	As above, in the zero-section.
$\langle u, \varphi \rangle$	Action of a distribution u on φ.
$u \otimes v$	Tensor product of distributions.
Φ^*	Pullback by Φ.
Φ_*	Pushforward by Φ.
$K(P)$	Schwartz kernel of P.
P^*	Adjoint (formal, Hilbert) of P.
$\mathrm{dom}(P)$	The domain of P.
$\sigma(P)$	The spectrum of P.
I	Identity operator.
$d\xi$	Integration w.r.t. measure on $\mathbb{R}^n$. (Standard Lebesgue measure.)
$đ\xi$	$(2\pi)^{-n} d\xi$.
$d\mu$	Integration w.r.t. measure μ.
$\mathcal{F}$	Fourier transformation.
$\mathcal{F}_G$	Fourier transformation on G.

Δ	Laplacian (various).
Δ_g	Laplacian (relative to metric g).
$\mathrm{U}(H)$	Unitary operators $H \to H$.
$\mathrm{U}(m)$	Dimension m unitary group.
$\mathfrak{u}(m)$	The Lie algebra of $\mathrm{U}(m)$.
S^d	Order $d \in \mathbb{R}$ local symbols on $\mathbb{R}^n$. Order $d \in \mathbb{R}$ matrix-symbols on G. (The type is always $(1,0)$).
$\mathrm{Op}(p)$	Operator associated to a symbol p. (On a compact Lie group, or $\mathbb{R}^n$.)
$HL^2(N)$	Holomorphic L^2 -functions.
$HH^s(M_\epsilon)$	Holomorphic Sobolev spaces.
$B(X,Y)$	Bounded operators $X \to Y$.
$F(X,Y)$	Fredholm operators $X \to Y$.
$\mathrm{Mat}(m,\mathbb{C})$	$m \times m$ matrices.
$K \subset\subset U$	Means "K compact in U".
WF	Wavefront set.
sing supp	The singular support.
cone supp	The conic support.
ess supp	The essential support.
$\mathrm{Diff}(M)$	Differential operators on M.
$\mathrm{Diff}^d(M)$	As above, of order $d \in \mathbb{N}$.
$\Psi(M)$	Pseudo-differential operators.
$\Psi^d(M)$	As above, of order $d \in \mathbb{R}$.
$\Psi^d_{\mathrm{phg}}(M)$	As above, but also PHG.
$C^\infty(X, \Omega^{\frac{1}{2}})$	Smooth $\frac{1}{2}$-density sections.
$C_0^\infty(X, \Omega^{\frac{1}{2}})$	Smooth $\frac{1}{2}$-density sections.
$\mathcal{D}(X, \Omega^{\frac{1}{2}})$	$\frac{1}{2}$-density distributions.
$\mathcal{D}_\Gamma(X, \Omega^{\frac{1}{2}})$	As above, with WF in Γ.
$\mathcal{E}_\Gamma(X, \Omega^{\frac{1}{2}})$	And with compact support.
$L^2(X, \Omega^{\frac{1}{2}})$	The L^2 $\frac{1}{2}$-density sections.
$I(X,\Lambda)$	FID associated to Λ.
$I^d(X,\Lambda)$	As above, of order $d \in \mathbb{R}$.
$I^d_{\mathrm{phg}}(X,\Lambda)$	As above, but also PHG.
$I(X,Y;C)$	FIO associated to C.
$I^d(X,Y;C)$	As above, of order $d \in \mathbb{R}$.
$I^d_{\mathrm{phg}}(X,Y;C)$	As above, but also PHG.

1

Introduction

In the engineering sciences, one sometimes finds mathematical problems of a deeper kind. These problems can not be approached by the basic tools of calculus, matrices or the like. Instead, stronger mathematical tools must be invoked, often just to gain understanding, before a numerical scheme, or an algorithm, can be fashioned to approximate a solution. The study of inverse problems, and their regularization, is a prototypical example of this. Here, an operator-theoretic analysis of the inverse problem precedes algorithm synthesis, and the scheme that is ultimately implemented is just a finite-dimensional approximation of something that really takes place in infinite-dimensional (Banach) spaces of functions. Another source of such problems are numerical schemes for solving boundary problems, where convergence and stability are the main issues to be addressed by a deeper analysis. Many advanced schemes do not have well-understood convergence or stability properties, but rely on benchmarking and rules of thumb instead.

The actual starting point for this thesis was the so-called interior source point methods. This is an umbrella term for a class of methods applicable to scattering-type problems, and includes direct acoustic and electromagnetic (analytic elliptic) scattering problems. They are also applicable to inverse scattering, via reconstruction of scatterer near-field, as a step in the Kirsch-Kress decomposition technique [7]. These methods are fairly new, and are quite efficient, but have a notable flaw that limits their applicability in general. Their stability and convergence depend on a piece information about the actual solution: A region outside of the problem domain into which the true solution extends analytically. The sources must be placed "near" true singularities [63, 9, 37].

Inspired by these recent advances, we ask a simple, and essentially classical, question. Suppose that we are given a Dirichlet boundary value problem for an elliptic operator, where the operator, boundary and boundary data are all understood to be real-analytic. How far outside the problem domain can solutions to the Dirichlet problem be extended? The topic of analyticity of solutions to PDE problems goes back by more than a century, but it is still being actively researched. See the survey by Khavinson and Lundberg [38]. Of course, at this point, our question is rather vague, so we illustrate with an example: Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply-connected domain with real-analytic boundary $\partial\Omega$. Suppose that $g_0 \in C^\omega(\partial\Omega)$ extends to a real-analytic function on a neighbourhood of $\overline{\Omega}$. We seek a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = g_0. \end{cases}$$

It is then well-known [38] that there always exists a unique solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, and this u is real-analytic in Ω, extending analytically to an open subset containing $\overline{\Omega}$. Concretely, in the metric of $\mathbb{R}^2$, how big is this neighbourhood?

It turns out that our question is extremely difficult to answer, even for very simple Ω. So far, most concrete results require Ω to be an ellipse [38]. This is not very satisfactory. But then again, why should we expect the solution to reveal its full domain of existence? Coming from an engineering point of view, maybe we could ask a more humble question. Is there a way to get an *estimate* of a collar neighbourhood of $\partial\Omega$ into which u extends? A crude approach would be to follow the constants in estimates of all derivatives of u, which yield a convergence radius for a power series expansion around each point on $\partial\Omega$. However, $\partial\Omega$ is not a half-plane, so we must "flatten" it locally using real-analytic charts, and thus we are also forced to supply information about all the derivatives of this chart. Needless to say, not only does the level of complexity in doing all these estimates explode, but the final convergence radii may still be uselessly small.

Recently, a seemingly promising alternative was explored by Karamehmedović [37]. The idea is to solve a Cauchy problem near $\partial\Omega$ instead. Let $g_1 \in C^\omega(\partial\Omega)$ be some function. We seek a domain $U \subset \mathbb{R}^2$, containing $\partial\Omega$, and a solution $\widetilde{u} \in C^\omega(U)$ to

$$\begin{cases} \Delta\widetilde{u} = 0 & \text{in} \quad U, \\ \widetilde{u}|_{\partial\Omega} = g_0, \\ \partial_\nu\widetilde{u}|_{\partial\Omega} = g_1, \end{cases}$$

where ν is the interior unit normal to $\partial\Omega$, and $\partial_\nu|_{\partial\Omega}$ is the normal derivative operator. Now, by the Cauchy-Kovalevskaya theorem [38], the above problem is uniquely solvable, and we must conclude that $\widetilde{u}$ coincides with u on U if the boundary data is $g_1 = \partial_\nu u|_{\partial\Omega}$. But if we knew g_1, then U could be "enlarged" by using the so-called Zerner's theorem. This approach gives more freedom, but we need to know something about g_1.

To explain how it works, we take a closer look at the Cauchy-Kovalevskaya theorem. Let $D(0,R)$ be the open disc of radius $R > 0$ centred at 0 in $\mathbb{C}$. If $\delta > 0$, we put

$$\Omega_{R,\delta} = D(0,R)^{n-1} \times D(0,R\delta),$$

where $n \in \mathbb{N}$ is the dimension in the following version of the theorem.

Theorem 1.0.1 (Cauchy-Kovalevskaya [31])**.** *Let $d \in \mathbb{N}$ and put $\beta = (0,\cdots,0,d) \in \mathbb{N}_0^n$. Suppose $\{a_\alpha\}_{|\alpha|\le d}$ and f are bounded holomorphic functions on $\Omega_{R,\delta}$ with $a_\beta = 1$, and*

$$2(2^n e)^d \sum_{\alpha\neq\beta} R^{d-|\alpha|}\delta^{d-\alpha_n|}|a_\alpha(z)| \le 1 \quad \text{for all} \quad z \in \Omega_{R,\delta}.$$

Then there is a unique holomorphic solution u in $\Omega_{\frac{R}{2},\delta}$ to

$$\begin{cases} \sum_{|\alpha|\le m} a_\alpha \partial_z^\alpha u = f & \text{in} \quad \Omega_{\frac{R}{2},\delta}, \\ \partial_{z_n}^j u|_{z_n=0} = 0 & \text{for} \quad j < d, \end{cases}$$

where $u \mapsto \partial_{z_n}^j u|_{z_n=0}$ is the jth normal derivative with respect to the $z_n = 0$ hyperplane.

The above has a reformulation where $z_n = 0$ is replaced by an analytic hypersurface. It is obtained by simply applying the above after a biholomorphic change of coordinates, and then returning by the same transformation.

Let P be a complex-analytic differential operator of order $d \in \mathbb{N}$ on an open $U \subset \mathbb{C}^n$, and let $S \subset U$ be a complex hypersurface (co-dimension one complex submanifold) in U. The operator P has the form

$$P = \sum_{|\alpha| \le d} a_\alpha (-i\partial_z)^\alpha,$$

where $\{a_\alpha\}_{|\alpha|\le d}$ are now holomorphic functions on U, but which need not be bounded. Associated to P there is the principal symbol p defined by

$$p(z,\zeta) = \sum_{|\alpha|=d} a_\alpha(z)\zeta^\alpha \quad \text{for all} \quad (z,\zeta) \in U \times \mathbb{C}^n.$$

Instances of the following results can be found in the standard text by Hörmander [31]. The corollary is just Cauchy-Kovalevskaya at S, the other is known as Zerner's theorem. In both, f is a holomorphic function on U, and $\{g_j\}_{j=0}^{d-1}$ are holomorphic functions on S.

Corollary 1.0.1. *Let $\Phi : U \to \mathbb{C}^n$ be a biholomorphism with $\Phi(S)$ given by $z_n = 0$. Suppose, if $e_n = (0, \cdots, 0, 1) \in \mathbb{C}^n$, that*

$$p(z, (d\Phi_z)^t e_n) \neq 0 \quad \text{for all} \quad z \in S.$$

Then there is an open $U_0 \supset S$ in U and a unique holomorphic u on U_0 solving

$$\begin{cases} Pu = f & \text{in} \quad U_0, \\ \rho_j u = g_j & \text{for} \quad j < d, \end{cases}$$

where $\rho_j u = \partial_{z_n}^j (u \circ \Phi^{-1}) \circ \Phi|_S$ is the holomorphic trace on S with respect to Φ.

Now if S is a real C^1 hypersurface, it is locally the zero set of a real C^1 function ψ: Near any $z_0 \in S$, there exists such a ψ, with $d\psi_{z_0} \neq 0$, and $\psi(z) = 0$ if and only if $z \in S$. In that case, we say that $z_0 \in S$ is non-characteristic w.r.t. P if

$$p(z_0, \partial\psi_{z_0}) \neq 0,$$

and S is non-characteristic for P if it is so at every point.

Theorem 1.0.2 (Zerner [65])**.** *Let $U_0 \subset U$ be open, and u a holomorphic solution to*

$$Pu = f \quad \text{in} \quad U_0.$$

Suppose that $z_0 \in \partial U_0 \cap U$ and that ∂U_0 is C^1 and non-characteristic at this z_0 w.r.t. P. Then u continues analytically on to a neighbourhood of z_0 inside U.

Thus, if we have a continuum of such neighbourhoods, u will "bleed" through them, but may still be obstructed by the possible occurrence of some non-characteristic points. Of course, the larger the initial U_0, the easier it is to expand by using Zerner's theorem. We want to apply this locally near a complex surface $(\partial\Omega)_\mathbb{C} \subset \mathbb{C}^2$ with $\partial\Omega = (\partial\Omega)_\mathbb{C} \cap \mathbb{R}^2$, but to do so, we need to know whether g_0 and g_1 even extend holomorphically to $(\partial\Omega)_\mathbb{C}$, and, once we have that, we can get a U_0 containing a piece of $(\partial\Omega)_\mathbb{C}$.

The idea of Karamehmedović [37] was to estimate how far $\partial_\nu u|_{\partial\Omega}$ extends into $(\partial\Omega)_{\mathbb{C}}$. If g_0 extends holomorphically to some domain in $(\partial\Omega)_{\mathbb{C}}$, what can we say about $\partial_\nu u|_{\partial\Omega}$? The answer should be hidden in the Dirichlet-to-Neumann (DN) map

$$\Lambda : g_0 \mapsto \partial_\nu u|_{\partial\Omega},$$

which is a pseudo-differential operator. In the above case, $\Lambda \in \Psi^1(\partial\Omega)$. See Grubb [15]. It can be formed from the constituent parts of another operator, the Calderón projector. This is a matrix of pseudo-differential operators, which, in turn, is formed from the PDE, and the Cauchy trace operator (more generally, boundary differential operators) on $\partial\Omega$. In his paper [37] Karamehmedović obtained some explicit local mapping properties of Λ, and managed to answer the question locally wherever $\partial\Omega$ is a piece of a straight line. Combined with Zerner's theorem, this gives a way to estimate the domain of existence, but without determining u. The result is explicit, and depends only on knowledge of g_0, and how far g_0 extends holomorphically into $(\partial\Omega)_{\mathbb{C}}$ near the linear pieces.

It was basically done by showing that the Calderón projectors have analytic symbols. This type of pseudo-differential symbol was first introduced by Boutet de Monvel in [2], and [37] essentially reuses these symbols, but introduces some unnecessary technicalities. The technicalities appear to be caused by Boutet de Monvel's failure to write out details, and, especially, details surrounding the convergence of some "contour deformed" integrals. Unfortunately, [37] does not cite subsequent improvements to [2] by Trèves [61] either. So to correct this state of affairs, we will now improve and simplify the first part of [37], which is just filling out details from the paper [2], and is not the subject of this thesis. However, it was the starting point. Therefore, we consider it important to include it here, and it will lead us to the questions we actually seek to answer.

In the following, $B(x, r)$ denotes the open ball with center at $x \in \mathbb{R}^n$ and radius $r > 0$, and $\mathrm{Op}(p)$ is the (unique) pseudo-differential operator associated to a symbol p on $\mathbb{R}^n$. If $f : U \to \mathbb{C}$ is the restriction of a holomorphic function $\widetilde{f} : U_{\mathbb{C}} \to \mathbb{C}$ to $U = U_{\mathbb{C}} \cap \mathbb{R}^n$, where $U_{\mathbb{C}} \subset \mathbb{C}^n$ is open, we say that f extends holomorphically to $U_{\mathbb{C}}$.

Theorem 1.0.3. *Fix $R > 0$ and $\epsilon > 0$, and let $p \in S^d(\mathbb{R}^n \times \mathbb{R}^n)$ be a symbol with $d \in \mathbb{R}$. Assume $p|_{B(0,r_0)\times\mathbb{R}^n}$ extends holomorphically into $(B(0, r_0) + iB(0, \delta_0)) \times W_\epsilon$, where*

$$W_\epsilon = \{\zeta \in \mathbb{C}^n \mid |\mathrm{Im}\,\zeta| < \epsilon|\mathrm{Re}\,\zeta|\} \cap \{\zeta \in \mathbb{C}^n \mid |\mathrm{Re}\,\zeta| > R\},$$

and satisfies

$$\sup_{(x,\zeta)\in K\times W_\epsilon} \langle \mathrm{Re}\,\zeta\rangle^{-d} |p(x, \zeta)| < \infty \quad \text{for any} \quad K \subset\subset B(0, r_0) + iB(0, \delta_0)$$

Let $u \in C_0^\infty(\mathbb{R}^n)$. Suppose that $u|_{B(0,r)}$ extends holomorphically into $B(0, r) + iB(0, \delta)$. Choose $r > r' > 0$ and $\delta \geq \delta' > 0$ so that

$$\frac{\delta'}{r - r'} < \epsilon.$$

Then $\mathrm{Op}(p)u|_{B(0,\min\{r',r_0\})}$ extends similarly into $B(0, \min\{r', r_0\}) + iB(0, \min\{\delta', \delta_0\})$.

Proof. A deformation of $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{C}^n \times \mathbb{C}^n$ allows us to continue $\mathrm{Op}(p)u$ explicitly. It is done by dividing the oscillatory integral into two parts, and applying Stokes' theorem.

Take $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ with $\chi_2(\xi) = 1$ for $\xi \in \overline{B(0,2R)}$ but $\chi_2(\xi) < 1$ for $\xi \notin \overline{B(0,2R)}$. Furthermore, let $\chi_1 \in C_0^\infty(B(0,r))$ be a cutoff such that $\chi_1(y) = 1$ whenever $y \in B(0,r'')$, where $r > r'' > r' > 0$ and $\delta \geq \delta' > 0$ are chosen so that

$$\frac{\delta'}{r - r'} < \frac{\delta'}{r'' - r'} < \epsilon.$$

Now let $\sigma : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^n \times \mathbb{C}^n$ be defined by

$$(t,y,\xi) \mapsto (w,\zeta) = \Big(y - it\delta'\chi_1(y)(1-\chi_2(\xi))\frac{\xi}{|\xi|}, \xi - it\frac{\delta'(1-\chi_1(y))}{r''-r'}(1-\chi_2(\xi))|\xi|\frac{y}{|y|}\Big),$$

and put

$$\mathcal{C}(t) = \sigma(\{t\} \times \mathbb{R}^n \times \mathbb{R}^n) \quad \text{for all} \quad t \in [0,1].$$

The above ensures that pullbacks by σ, or $\sigma(t,\cdot,\cdot)$ at $t \in [0,1]$, yield convergent integrals. Under the σ deformation, if $\chi_2(\xi) = 0$ and $|\mathrm{Re}\,(x)| < r'$, we get

$$\begin{aligned}
\mathrm{Re}\,(i(x-w)\cdot\zeta) &= -\xi \cdot \Big(\mathrm{Im}\,(x) + t\delta'\chi_1(y)\frac{\xi}{|\xi|}\Big) + t\frac{\delta'(1-\chi_1(y))}{r''-r'}|\xi|\frac{y}{|y|}\cdot(\mathrm{Re}\,(x) - y) \\
&\leq -|\xi|\Big(\frac{\xi}{|\xi|}\cdot \mathrm{Im}\,(x) + t\delta'\chi_1(y) + t\frac{\delta'(1-\chi_1(y))}{r''-r'}\Big(|y| - |\mathrm{Re}\,(x)|\Big)\Big) \\
&\leq -|\xi|\Big(-|\mathrm{Im}\,(x)| + t\delta'\chi_1(y) + t\frac{\delta'(1-\chi_1(y))}{r''-r'}\Big(|y| - |\mathrm{Re}\,(x)|\Big)\Big) \\
&\leq -|\xi|\Big(t\delta' - |\mathrm{Im}\,(x)|\Big).
\end{aligned}$$

Take $x \in B(0,r') + itB(0,\delta')$, and fix $\rho > 2R$ and $1 \geq t_2 > t_1 \geq 0$. Put

$$Q(\rho) = (t_1,t_2) \times \mathbb{R}^n \times (B(0,\rho) \setminus \overline{B(0,2R)}),$$

and note then that σ is injective on $Q(\rho)$, and

$$\sigma(\overline{Q(\rho)}) \subset \mathbb{C}^n \times W_\epsilon \quad \text{for all} \quad \rho > 2R.$$

In the following, we use notation $dw = dw_1 \wedge \cdots \wedge dw_n$ and $đ\zeta = (2\pi)^{-n} d\zeta_1 \wedge \cdots \wedge d\zeta_n$. Define for $(w,\zeta) \in \mathbb{C}^n \times W_\epsilon$ the $2n$-form

$$\mu_x = G(w,\zeta)\,dw \wedge đ\zeta = e^{i\zeta\cdot(x-w)}p(x,\zeta)u(w)\,dw \wedge đ\zeta,$$

where $\sigma^*\mu_x$ is smooth and compactly supported in $\overline{Q(\rho)}$, and

$$d\mu_x = \sum_{j=1}^n \partial_{\overline{w}_j}\left[e^{i\zeta\cdot(x-w)}p(x,\zeta)u(w)\right] d\overline{w}_j \wedge dw \wedge đ\zeta.$$

Then $\sigma^* d\mu_x$ vanishes on $Q(\rho)$ by holomorphy in $y \in B(0,r)$ and reality in w if $y \notin B(0,r)$.

Using the above, we can now, without any convergence issues, apply Stoke's theorem. Thus, by Stokes' theorem for manifolds with corners applied to $\overline{Q(\rho)}$, we get

$$0 = \int_{Q(\rho)} \sigma^* d\mu_x = \int_{Q(\rho)} d(\sigma^* \mu_x) = \int_{\partial Q(\rho)} \sigma^* \mu_x.$$

Also, by the above estimate, there is some $C > 0$ such that

$$|(G \circ \sigma)(t, y, \xi) \det d_{(y,\xi)}\sigma(t, y, \xi)| \leq Ce^{-|\xi|(t\delta' - \mathrm{Im}\,(x))} \langle \xi \rangle^d 1_{\mathrm{supp}(u)}(y),$$

which guarantees existence of $\int_{\mathcal{C}(t)} \mu_x$ when $t > 0$. If $t = 0$, it is meaningful if $p \in S^{-\infty}$, but x must then have zero imaginary part. The aim is to show equivalence with $t = 1$. Let $\sigma_\rho : [t_1, t_2] \times \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{C}^n \times \mathbb{C}^n$ be defined by

$$(t, y, \omega) \mapsto \Big(y - it\delta' \chi(y)\omega, \rho\Big[\omega - it\frac{\delta'(1 - \chi(y))}{r'' - r'} \frac{y}{|y|}\Big]\Big).$$

Again, by the estimate, if $x \in B(0, r')$, we get $C' > 0$ such that

$$|(G \circ \sigma_\rho)(t, y, \omega) \det(d\sigma_\rho)(t, y, \omega)| \leq C' e^{-\rho t\delta'} \langle \rho \rangle^d 1_{\mathrm{supp}(u)}(y),$$

and since $\sigma(t, y, \xi) = (y, \xi)$ for $\xi \in \overline{B(0, 2R)}$, $\sigma^*\mu_x$ vanishes on $(t_1, t_2) \times \mathbb{R}^n \times \partial B(0, 2R)$. Therefore, combining integrals of opposite orientation, and using the DCT, we get

$$\begin{aligned}\int_{\mathcal{C}(t_2)} \mu_x - \int_{\mathcal{C}(t_1)} \mu_x &= \lim_{\rho\to\infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \int_{\xi \in \partial B(0,\rho)} (\sigma^*\mu_x)(t, y, \xi) \\ &= \lim_{\rho\to\infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \Big[\int_{\omega\in\mathbb{S}^{n-1}} [(G \circ \sigma_\rho) \det(d\sigma_\rho)](t, y, \omega) \,\mathrm{vol}_{\mathbb{S}^{n-1}}(\omega)\Big]\, dy\, dt,\end{aligned}$$

where the integrand is compactly supported in y, and is bounded as above for all $\rho > R$. It follows that the limit is zero, and we conclude that $\int_{\mathcal{C}(t_2)} \mu_x = \int_{\mathcal{C}(t_1)} \mu_x$ if $x \in B(0, r')$. Pick $t_0 \in (0, 1)$ so that $\mathcal{C}(t_0) \subset \mathbb{C}^n \times W_{\frac{1}{2}}$. Using the DCT, we then get

$$\begin{aligned}\mathrm{Op}(p)u(x) &= \lim_{\lambda\to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)} [e^{-\lambda^2 \xi\cdot\xi} p(x, \xi)] u(y)\, dy\, đ\xi \\ &= \lim_{\lambda\to 0} \int_{\mathcal{C}(t_0)} e^{i\zeta\cdot(x-w)} [e^{-\lambda^2\zeta\cdot\zeta} p(x, \zeta)] u(w)\, dw \wedge đ\zeta \\ &= \int_{\mathcal{C}(t_0)} e^{i\zeta\cdot(x-w)} p(x, \zeta) u(w)\, dw \wedge đ\zeta \\ &= \int_{\mathcal{C}(1)} e^{i\zeta\cdot(x-w)} p(x, \zeta) u(w)\, dw \wedge đ\zeta,\end{aligned}$$

which makes sense, because if $\lambda \in \mathbb{R}$, we have

$$|e^{-\lambda^2\zeta\cdot\zeta}| \leq e^{-\lambda^2(|\mathrm{Re}\,\zeta|^2 - |\mathrm{Im}\,\zeta|^2)} \leq e^{-\frac{1}{2}\lambda^2|\mathrm{Re}\,\zeta|^2} \quad \text{if} \quad |\mathrm{Im}\,\zeta| < \frac{1}{2}|\mathrm{Re}\,\zeta|.$$

But now the last integral extends holomorphically in x to the right open set. □

Note that for those $y \notin B(0,r)$ the function u in μ_x may fail to extend holomorphically. But this is not an issue, as the contour deformation then only happens in the ζ-variable.

Corollary 1.0.2. *Let* $u \in \mathcal{E}'(\mathbb{R}^n)$ *and* $u|_{B(0,r)}$ *extend, as before, into* $B(0,r) + iB(0,\delta)$*. Then* $\mathrm{Op}(p)u|_{B(0,\min\{r',r_0\})}$ *extends similarly into* $B(0,\min\{r',r_0\}) + iB(0,\min\{\delta',\delta_0\})$*.*

Proof. Take $\chi \in C_0^\infty(B(0,r))$ such that $\chi(y) = 1$ for $y \in \mathrm{supp}(\chi_1)$. Then χu is smooth. Let $\sigma_\xi : [0,1] \times \mathbb{R}^n \to \mathbb{C}^n$ be defined by

$$(t,\xi) \mapsto \zeta = \xi - it\frac{\delta'(1-\chi_1(y))}{r''-r'}(1-\chi_2(\xi))|\xi|\frac{y}{|y|},$$

and put $\mathcal{C}_\xi(1) = \sigma_\xi(\{1\} \times \mathbb{R}^n)$. As before, if $\chi_2(\xi) = 1$ and $|\mathrm{Re}\,(x)| < r'$, we have

$$\mathrm{Re}\,(i(x-y)\cdot\zeta) \leq -|\xi|(t\delta'(1-\chi_1(y)) - \mathrm{Im}\,(x)).$$

Taking $\varphi \in C_0^\infty(B(0,r'))$, we get

$$\begin{aligned}\langle \mathrm{Op}(p)u, \varphi\rangle &= \langle \mathrm{Op}(p)(\chi u), \varphi\rangle + \langle \mathrm{Op}(p)((1-\chi)u), \varphi\rangle \\ &= \langle \mathrm{Op}(p)(\chi u), \varphi\rangle + \int_{\mathbb{R}^n} \Big\langle u(y), K(x,y)\Big\rangle \varphi(x)\,dx,\end{aligned}$$

where the kernel $K : B(0,r') \times \mathbb{R}^n \to \mathbb{C}$ is smooth, rapidly decaying in y uniformly in x. It has the form

$$\begin{aligned}K(x,y) &= \lim_{\lambda\to 0}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}[e^{-\lambda^2\xi\cdot\xi}(1-\chi)(y)p(x,\xi)]\,đ\xi \\ &= \int_{\mathcal{C}_\xi(1)} e^{i(x-y)\cdot\zeta}(1-\chi)(y)p(x,\zeta)\,đ\zeta,\end{aligned}$$

which extends holomorphically in x for each y to the stated open set. □

Note the obstruction that $\epsilon > 0$ places on extensions near analytic singularities of u. This is embodied in the condition

$$\frac{\delta'}{r-r'} < \epsilon.$$

Corollary 1.0.3. *Suppose that* $\epsilon > 0$ *can be made arbitrarily large (independent of* R*). Then* $\mathrm{Op}(p)u|_{B(0,\min\{r,r_0\})}$ *extends, as before, into* $B(0,\min\{r,r_0\}) + iB(0,\min\{\delta,\delta_0\})$*.*

Among other things, the above theorem removes the topology on the symbols in [37], and reduces the situation to symbols defined by Boutet de Monvel [2] and Trèves [61]. However, the whole approach has a massive weakness not addressed by Karamehmedović. There is no easy way to construct parametrices with symbols in the *same* analytic class, and thus no easy way to generalize the approach taken in [37] via the Calderón projectors. It only works in [37] because the geometry of the setup is extremely simple - a half-plane. Trèves overcame this by replacing the analytic symbols with pseudo-analytic amplitudes, which have weaker conditions imposed on them - analyticity is replaced by an estimate. These amplitudes are equivalent to analytic amplitudes up to an exponential error term, where the decay rate constrains the domain of extension. But this constant is not explicit, and there seems to be no way to determine or estimate it.

The consequence is that the results above, and also those in [37], are mostly useless. Only in the case of constant coefficients and a planar boundary are they of any real use. The parametrices constructed by Trèves are not helpful for the above mentioned reasons. Although, in principle, it should be possible to follow the estimates obtained by Trèves, and get a lower bound on the convergence radii preserved by the various operators there. But the usefulness of this approach is questionable at best, because of the difficulty level. In that case, estimating derivatives of u would probably be easier.

All these difficulties prompted us to search the recent literature for a better approach, and, perhaps, a modern, well-developed, and tangible framework for asking our questions. It turns out that there is one. In $(\partial\Omega)_{\mathbb{C}}$ there are special domains, called Grauert tubes, and these provide an invariant way to measure how far an extension "reaches" into $(\partial\Omega)_{\mathbb{C}}$. These domains $(\partial\Omega)_\epsilon$ are parametrized by $\epsilon > 0$ up to a possibly finite maximum $\epsilon_{\max} > 0$. On each $(\partial\Omega)_\epsilon$ there are natural Hilbert spaces of holomorphic functions, $HH^s((\partial\Omega)_\epsilon)$, which mirror $H^s(\partial\Omega)$, the Sobolev spaces, for each $s \in \mathbb{R}$. Our question then crystallizes: If $\mathcal{R}_\epsilon : \mathcal{O}((\partial\Omega)_\epsilon) \to C^\omega(\partial\Omega) : f \mapsto f|_{\partial\Omega}$ is the operator restricting functions onto $\partial\Omega$, then we ask for an $\epsilon > 0$ and a $\widetilde{\Lambda}$ such that the following diagram commutes:

$$\begin{array}{ccc} HH^s((\partial\Omega)_\epsilon) & \xrightarrow{\widetilde{\Lambda}} & HH^{s-1}((\partial\Omega)_\epsilon) \\ \downarrow{\scriptstyle \mathcal{R}_\epsilon} & & \downarrow{\scriptstyle \mathcal{R}_\epsilon} \\ H^s(\partial\Omega) & \xrightarrow{\Lambda} & H^{s-1}(\partial\Omega) \end{array}$$

In particular, what we are asking here is; if g_0 extends to $(\partial\Omega)_\epsilon$ is the same true of Λg_0? This is just an abstract generalization of the question raised by Karamehmedović in [37]. However, Λ is a special pseudo-differential operator associated to the Dirichlet problem, and it is unknown if there are non-trivial operators satisfying a diagram like this at all. Instead of considering the DN map, we simply ask if such non-trivial operators even exist. Are there many? Do they form an algebra? Does it contain non-identity elliptic elements? If so, are there parametrices for these elliptic elements that also belong to this algebra? Can we somehow characterize them by their symbols? None of this concerns the DN map, but the hope is that there are enough such operators that the DN map is one of them. Another question is the size of ϵ. Can we put a lower bound on ϵ where any of this holds? This is the most important (and most difficult) problem.

In light of recent research by Stenzel, Zelditch and others, there are a few answers. Our questions are connected to the Poisson transform $\mathcal{P}_\epsilon$, introduced by Stenzel in [56], which is based on an old theorem/conjecture by Boutet de Monvel [3] about propagators. Only very recently did proofs (by Zelditch [65] and Stenzel [55]) of a special case appear. In fact, we will find that the algebra generated by the Laplacian has the right properties, but the size of ϵ is constrained by the unknown existence radius of the Poisson transform. The full conjecture in [3] remains unproven. If it were true, the Laplacian is not special, and there are very many such operators preserving holomorphically extendible functions. More precisely, any elliptic real-analytic differential operator P would generate domains, like $(\partial\Omega)_\epsilon$, via the holomorphic extension of the Hamiltonian flow of its principal symbol, and operators commuting with P would satisfy a commutative diagram like the one above, where the holomorphic functions now live on these domains instead.

Our questions are global (on the whole manifold, not just in a chart) from the outset. There are two reasons for this. The first is just compatibility with the recent literature, but the second is that we seek characterizations in terms of simpler intrinsic properties, like the principal symbol, or algebraic relations, that can be verified or manipulated easily. In this way, we hope to obtain results on the existence and plurality of such operators, and ways to construct them. But it should be said that there are few explicit examples, because pseudo-differential operators are mainly a tool for proving abstract theorems, and their defining formulas are very forbidding when it comes to explicit calculations. The textbooks by Hörmander [33, 34] and Trèves [61, 60] also lack examples.

However, there is an easy way to see that such global operators should exist on $\mathbb{R}^n$. The idea is to use exact quantization, which is available because $\mathbb{R}^n$ has a group structure. Let $\mathbb{R}^n_\epsilon = \{z \in \mathbb{C}^n \mid |\mathrm{Im}\,(z)| < \epsilon\}$ be the tube neighbourhood of radius $\epsilon > 0$ about $\mathbb{R}^n$. Suppose $p \in S^d(\mathbb{R}^n \times \mathbb{R}^n)$ extends holomorphically in the first variable to $\mathbb{R}^n_\epsilon$, and

$$\sup_{(z,\xi)\in K\times\mathbb{R}^n} \langle\xi\rangle^d |p(z,\xi)| < \infty \quad \text{for any} \quad K \subset\subset \mathbb{R}^n_\epsilon,$$

and that $u \in \mathcal{S}(\mathbb{R}^n)$ extends to $\mathbb{R}^n_\epsilon$ in such a way that

$$\max_{|\alpha|\le N} \sup_{z\in\mathbb{R}^n_\epsilon} \langle\mathrm{Re}\,(z)\rangle^N |\partial_z^\alpha u(z)| < \infty \quad \text{for all} \quad N \in \mathbb{N}_0.$$

In that case, if $z \in \mathbb{R}^n_\epsilon$, we can write

$$\begin{aligned}\mathrm{Op}(p)u(z) &= \lim_{\lambda\to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(z-y)\cdot\xi}[e^{-\lambda^2\xi\cdot\xi} p(z,\xi)]u(y)\,dy\,đ\xi \\ &= \int_{\mathbb{R}^n} e^{i\mathrm{Re}\,(z)\cdot\xi} p(z,\xi)\Big[\int_{\mathbb{R}^n} e^{-iy\cdot\xi} u(y + i\,\mathrm{Im}\,(z))\,dy\Big]\,đ\xi,\end{aligned}$$

where $y \mapsto y + i\mathrm{Im}\,(z)$ is justified by an analytic continuation argument in the first line. Using the DCT, we see that we can take Wirtinger derivatives through the last integrals, and therefore, $\mathrm{Op}(p)u$ has the sort of property we seek; it extends holomorphically to $\mathbb{R}^n_\epsilon$. Of course, it fails completely if $u \in C_0^\infty(\mathbb{R}^n)$. There is no way the above can be localized. This leads us to look for such pseudo-differential operators on a compact Lie group G, where, like on $\mathbb{R}^n$, a global quantization is available. It turns out to be modestly fruitful, and we will find a non-trivial subalgebra of $\Psi(G)$ with all the properties that we desire. The next step would be to see if some of this transfers to homogeneous spaces.

Without doubt, the mathematics involved in these questions is difficult to understand, and requires a lot of background. Especially in basic real, complex and functional analysis. We assume basic familiarity with local distributions and pseudo-differential operators. The book by Shubin [51] is sufficient. Beyond this, differential topology and geometry, and some basic knowledge of Lie groups, symplectic and Kähler geometry, is necessary. This can be acquired by reading basic parts of Lee [41], Warner [40] and Moroianu [44]. Elementary representation theory will appear later in the analysis on compact Lie groups. A good reference for this would be Folland [13]. The theory used for the Poisson transform, and the Boutet de Monvel theorem, can be found in the treatises by Hörmander [33, 34]. See also the articles by Hörmander [27] and Hörmander and Duistermaat [30].

Finally, a few words about conventions, notation, and basic definitions (already used). See also Symbols and Abbreviations. By a manifold we always mean *without* boundary, and when a manifold has a boundary, we will explicitly call it a manifold-with-boundary. The manifold topology is always required to be second-countable (then it is paracompact). A submanifold is understood to be embedded, that is, it carries the subspace topology, and an atlas of slice charts. An immersion is said to give rise to an immersed submanifold. Let $n, N, k \in \mathbb{N}$ and $d \in \mathbb{R}$. Let $V \subset \mathbb{R}^n$ be open, and $\alpha \in \mathbb{N}_0^N$ and $\beta \in \mathbb{N}_0^n$ multi-indices. The Hörmander amplitudes $S^d(V \times \mathbb{R}^N)$ consist of $a \in C^\infty(V \times \mathbb{R}^N)$ such that

$$\sup_{(x,\theta)\in V\times\mathbb{R}^N} \langle\theta\rangle^{|\alpha|-d} |\partial_x^\beta \partial_\theta^\alpha a(x,\theta)| < \infty,$$

and these form a separating family of semi-norms, making $S^d(V \times \mathbb{R}^N)$ a Frechet space. The PHG amplitudes $a \in S^d_{\mathrm{phg}}(V \times \mathbb{R}^N)$ are Hörmander amplitudes with

$$\sup_{(x,\theta)\in V\times\mathbb{R}^N} \langle\theta\rangle^{|\alpha|-d-k} \Big|\partial_x^\beta \partial_\xi^\alpha \Big(a(x,\theta) - \sum_{j=0}^{k-1} \chi(\theta) a_{d-j}(x,\theta)\Big)\Big| < \infty,$$

where $a_{d-j} \in C^\infty(V \times (\mathbb{R}^N \setminus 0))$ are functions satisfying

$$a_{d-j}(x, t\theta) = t^{d-j} a_{d-j}(x,\theta) \quad \text{for all} \quad (x,\theta) \in V \times \mathbb{R}^N \quad \text{and} \quad t > 0,$$

and $\chi \in C^\infty(\mathbb{R}^n)$ is any cutoff such that

$$\chi(\theta) = \begin{cases} 1 & \text{if} \quad |\theta| \geq 1, \\ 0 & \text{if} \quad |\theta| \leq \frac{1}{2}. \end{cases}$$

The functions a_{d-j} are not symbols, but they are uniquely determined by a if it is PHG. In the Euclidean setting, if $p \in S^d(V \times \mathbb{R}^n)$, we write

$$\mathrm{Op}(p)u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi) \mathcal{F}u(\xi)\, đ\xi \quad \text{for any} \quad x \in \mathbb{R}^n,$$

where we use the notation $đ\xi = (2\pi)^{-n} d\xi$ for the scaled Lebesgue measure.

Author comments: This research monograph documents a lot of background theory, and, in the final chapter, my poor attempts to approach the questions in the introduction. Many threads in the text do not lead anywhere, and are regurgitations of known theory. I made attempts to reach out to established researchers, but was met with little interest. In the course of writing, I made contributions to two articles not related to this research, but they are not included here. One article is now published [6], the other is on arxiv [36]. To be clear, only the very last chapter contains a few sparse original ideas.

2

Functions and Distributions

In this section we construct and define the most important spaces that will be used later. Most important of all are the distributions, and their classification in terms of regularity, which is captured by Sobolev spaces of arbitrary real order, both positive and negative. Other spaces will appear as the text proceeds, but those presented here are ubiquitous. Also, we introduce the wavefront set, which plays a central role in microlocal analysis, and show how it is changed by some operations on distributions.

Why do so many different spaces occur in the theory of PDE, and what is their use? The different subspaces of the mother space of distributions exist to classify regularity. This could be in terms of differentiability, growth, decay and/or strength of singularities. They can often be equipped with more structure than the mother space of distributions, which is much too large to be given anything beyond variations of the weak*-topology. On the other hand, the space of C_0^∞-functions is much too small for many applications. The path to a rich analysis is through the spaces in between - the middle way, as it were. By far, the most important example of such spaces "in between" are the Sobolev spaces, which are Hilbert spaces, and make available the powerful tools of functional analysis. Applications include existence, uniqueness and regularity for many boundary problems, and spectral analysis, eigenvalue asymptotics, and the index theory of elliptic operators. We will mainly be concerned with the L^2-Sobolev spaces as defined by e.g. Shubin [51], and assume prior familiarity with pseudo-differential operators on just smooth functions. The first five sections of the first chapter of Shubin's book [51] would be enough.

A useful notion of distributions requires an identification for the ordinary functions. On $\mathbb{R}^n$ it is provided by the "Du Bois-Reymond lemma". It appears, for example, in [15]. But on a smooth manifold X, we need to integrate globally to get a similar identification. Of course, if X were orientable, we could just integrate with respect to a volume n-form, but there is no reason to assume this. We can use the 1 and $\frac{1}{2}$-density bundles instead, which we denote (customarily suppressing the subscript), respectively, by

$$\Omega_X^1 \quad \text{and} \quad \Omega_X^{\frac{1}{2}}$$

These are smooth complex line bundles over X, described in [41] under different notation, and it is shown in [41] that any smooth 1-density section can be integrated invariantly. The smooth sections of these bundles are denoted, respectively, by

$$C^\infty(X, \Omega^1) \quad \text{and} \quad C^\infty(X, \Omega^{\frac{1}{2}}),$$

and, in the PDE literature, the various spaces are often defined using $\frac{1}{2}$-density sections.

Let X and Y be two smooth (real) manifolds, not necessarily compact or connected. The density bundles are trivial [41], so we fix a smooth positive 1-density $\omega_0 \in C^\infty(X, \Omega^1)$, and we can form a measure locally equal to a smooth function times Lebesgue measure. To this end, fix a partition of unity $\{\chi_j\}_{j\in\mathbb{N}}$ subordinate to charts $\kappa_j : U_j \to \mathbb{R}^n$ of X, where $\{U_j\}_{j\in\mathbb{N}}$ is a locally finite cover of X, $\operatorname{supp}\chi_j \subset U_j$, and each U_j is precompact. Declare $S \subset X$ to be measurable in X if $\kappa_j(U_j \cap S)$ is a Borel set in $\mathbb{R}^n$ for any $j \in \mathbb{N}$. The collection of these form the Borel σ-algebra as generated by the manifold topology, and integration of any $\omega \in C^\infty(X, \Omega^1)$ is performed by

$$\int_S \omega = \sum_{j\in\mathbb{N}} \int_{\kappa_j(U_j\cap S)} (\kappa_j^{-1})^*(\chi_j \omega).$$

This leads to a positive Borel measure μ_0, defined on such $S \subset X$, by setting

$$\mu_0(S) = \int_S \omega_0,$$

and integration of Borel measurable $f : X \to \mathbb{C}$ is then performed by

$$\int_S f\, d\mu_0 = \int_S f\omega_0 = \sum_{j\in\mathbb{N}} \int_{\kappa_j(U_j\cap S)} (\kappa_j^{-1})^*(\chi_j f\omega_0).$$

which follows by taking an approximating simple monotone sequence and using the MCT. Starting this way allows us to circumvent issues with defining L^2-spaces of sections later. One checks that these definitions are all independent of the charts and partition of unity, and measures induced by any such 1-densities all have the same null-sets.

Now that we have a measure fixed by the 1-density ω_0, we can define L^2-spaces on X. Construct compact subsets $\{K_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ such that

$$K_{i,j} \subset \kappa_j(U_j) \quad \text{and} \quad K_{i,j} \subset K^\circ_{i+1,j} \quad \text{for each} \quad (i,j) \in \mathbb{N}^2,$$

which exhaust the charts in the sense that we have $\cup_{i\in\mathbb{N}} K_{i,j} = \kappa_j(U_j)$ for each chart κ_j.

Definition 2.0.1. *Let $L^2(X, \omega_0)$ consist of a.e. equal measurable $u : X \to \mathbb{C}$ such that*

$$||u||_{L^2(X,\omega_0)} = \Big(\int_X |u|^2\omega_0\Big)^{\frac{1}{2}} < \infty.$$

Also, we let $L^2_{\text{loc}}(X, \omega_0)$ consist of those $[u]$ with $[1_K u] \in L^2(X, \omega_0)$ for every $K \subset\subset X$. This latter space is given the countable, separating family semi-norms

$$L^2_{\text{loc}}(X, \omega_0) \to [0, \infty) : [u] \mapsto \Big(\int_{K_{i,j}} |u|^2\omega_0\Big)^{\frac{1}{2}}.$$

In the usual way, $L^2(X, \omega_0)$ becomes a Banach space, and $L^2_{\text{loc}}(X, \omega_0)$ a Frechet space. Of course, $L^2(X, \omega_0)$ is also a Hilbert space with the natural inner product

$$L^2(X, \omega_0) \times L^2(X, \omega_0) \to \mathbb{C} : (u, v) \mapsto (u, v)_{L^2(X,\omega_0)} = \int_X u\overline{v}\, \omega_0.$$

The spaces $C^\infty(X)$, $C^\infty(X,\Omega^1)$ and $C^\infty(X,\Omega^{\frac{1}{2}})$ are all isomorphic once ω_0 is fixed. But let us equip $C^\infty(X,\Omega^{\frac{1}{2}})$ with the appropriate topology. Put

$$S_k = \cup_{j\le k}\kappa_j^{-1}(\cup_{i\le k}K_{i,j}) \quad \text{for each} \quad k\in\mathbb{N},$$

and let $\tau_j : \Omega^{\frac{1}{2}}|_{U_j} \to \mathbb{C}$ be the vector trivialization that is associated to κ_j for each $j\in\mathbb{N}$. These induce local trivializations of sections

$$\Phi_j : C^\infty(U_j,\Omega^{\frac{1}{2}}) \to C^\infty(\kappa_j(U_j)) : u \mapsto \Phi_j(u) = \tau_j \circ u \circ \kappa_j^{-1}.$$

Definition 2.0.2. *Give $C^\infty(X,\Omega^{\frac{1}{2}})$ the countable, separating family semi-norms*

$$C^\infty(X,\Omega^{\frac{1}{2}}) \to [0,\infty) : u \mapsto \max_{|\gamma|\le k}\sup_{x\in K_{i,j}} |\partial^\gamma\Phi_j(u|_{U_j})(x)|.$$

Recall that uniform convergence of the derivatives implies differentiability of the limit. It follows that $C^\infty(X,\Omega^{\frac{1}{2}})$ is a Frechet space when given the above semi-norms.

Definition 2.0.3. *The spaces of compactly supported functions are topologized as follows: Denote by a subscript $K \subset\subset X$ those u, or $[u]$ with representative u, so that* $\mathrm{supp}(u)\subset K$.

1. *Give $C^\infty_{S_k}(X,\Omega^{\frac{1}{2}})$ the subspace topology to get continuous inclusions*

$$C^\infty_{S_1}(X,\Omega^{\frac{1}{2}}) \hookrightarrow C^\infty_{S_2}(X,\Omega^{\frac{1}{2}}) \hookrightarrow \cdots \hookrightarrow C^\infty_0(X,\Omega^{\frac{1}{2}}),$$

and give the inductive limit Frechet (LF) topology to

$$C^\infty_0(X,\Omega^{\frac{1}{2}}) = \bigcup_{k=1}^\infty C^\infty_{S_k}(X,\Omega^{\frac{1}{2}}).$$

2. *Give $L^2_{S_k}(X,\omega_0)$ the subspace topology to get continuous inclusions*

$$L^2_{S_1}(X,\omega_0) \hookrightarrow L^2_{S_2}(X,\omega_0) \hookrightarrow \cdots \hookrightarrow L^2_{\mathrm{comp}}(X,\omega_0),$$

and give the inductive limit Frechet (LF) topology to

$$L^2_{\mathrm{comp}}(X,\omega_0), = \bigcup_{k=1}^\infty L^2_{S_k}(X,\omega_0).$$

The topologies are independent of the choice of charts and the exhaustion $\{S_k\}_{k=1}^\infty$, Let $L^2_{\mathrm{loc}}(X,\Omega^{\frac{1}{p}})$ consist of a.e. equal sections of the form $u\omega_0^{1/2}$ for some $[u]\in L^2(X,\omega_0)$. This leads to a well-defined linear bijective correspondence

$$L^2_{\mathrm{loc}}(X,\omega_0) \to L^2_{\mathrm{loc}}(X,\Omega^{\frac{1}{2}}) : [u] \mapsto [u\omega_0^{\frac{1}{2}}],$$

which defines some remaining spaces via the diagram

$$\begin{array}{ccccccc}
 & & L^2_{\mathrm{comp}}(X,\omega_0) & \hookrightarrow & L^2(X,\omega_0) & \hookrightarrow & L^2_{\mathrm{loc}}(X,\omega_0) \\
 & & \updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong \\
C^\infty_0(X,\Omega^{\frac{1}{2}}) & \hookrightarrow & L^2_{\mathrm{comp}}(X,\Omega^{\frac{1}{2}}) & \hookrightarrow & L^2(X,\Omega^{\frac{1}{2}}) & \hookrightarrow & L^2_{\mathrm{loc}}(X,\Omega^{\frac{1}{2}})
\end{array}$$

Note that elements of $L^p_{\text{comp}}(X,\omega_0) \subset L^p_{\text{loc}}(X,\omega_0)$ are independent of the choice of ω_0. The same is true of $L^p(X,\omega_0)$ when X is compact, hence for the spaces of L^2-sections. There is a canonical pairing, $\langle\cdot,\cdot\rangle$, of the L^2_{loc} and L^2_{comp} sections of $\Omega^{\frac{1}{2}}$. It is given by

$$L^2_{\text{loc}}(X,\Omega^{\frac{1}{2}}) \times L^2_{\text{comp}}(X,\Omega^{\frac{1}{2}}) : (u_1,u_2) \mapsto \langle u_1,u_2\rangle = \int_X u_1u_2.$$

Definition 2.0.4. *Give C^∞ and C_0^∞ the locally convex topologies in the above definitions. The spaces of distributions are the topological duals (themselves given the weak*-topology):*

$$\mathcal{E}'(X,\Omega^{\frac{1}{2}}) = (C^\infty(X,\Omega^{\frac{1}{2}}))',$$
$$\mathcal{D}'(X,\Omega^{\frac{1}{2}}) = (C_0^\infty(X,\Omega^{\frac{1}{2}}))',$$

and with continuous identification $\iota : L^2_{\text{loc}}(X,\Omega^{\frac{1}{2}}) \hookrightarrow \mathcal{D}'(X,\Omega^{\frac{1}{s}})$ given by

$$L^2_{\text{loc}}(X,\Omega^{\frac{1}{2}}) \times C_0^\infty(X,\Omega^{\frac{1}{2}}) : (f,\varphi) \mapsto \langle\iota(f),\varphi\rangle = \int_X f\varphi.$$

Injectivity of the identification follows from the Du Bois-Reymond lemma chart-wise. All the local facts about distributions on open subsets of $\mathbb{R}^n$ extend to the global setting.

$$\begin{array}{ccc} C_0^\infty(X,\Omega^{\frac{1}{2}}) & \hookrightarrow & C^\infty(X,\Omega^{\frac{1}{2}}) \\ \iota\downarrow & & \iota\downarrow \\ \mathcal{E}'(X,\Omega^{\frac{1}{2}}) & \hookrightarrow & \mathcal{D}'(X,\Omega^{\frac{1}{2}}) \end{array}$$

Proposition 2.0.1. *The space $C_0^\infty(X,\Omega^{\frac{1}{2}})$ is weak*-dense in $\mathcal{D}'(X,\Omega^{\frac{1}{2}})$.*

The (local) notion of tensor product of distributions extends to the manifold setting. This is recorded in the following theorem.

Theorem 2.0.1. *Given $u \in \mathcal{D}'(X,\Omega^{\frac{1}{2}})$ and $v \in \mathcal{D}'(Y,\Omega^{\frac{1}{2}})$, then*

$$\big\langle u(x),\langle v(y),\chi(x,y)\rangle\big\rangle = \big\langle v(y),\langle u(x),\chi(x,y)\rangle\big\rangle \quad \text{for any} \quad \chi \in C_0^\infty(X\times Y,\Omega^{\frac{1}{2}}).$$

The two equivalent actions on χ define a distribution $u\otimes v \in \mathcal{D}'(X\times Y,\Omega^{\frac{1}{2}})$.

Finally, this leads us to the central theorem in the analysis of PDE, the kernel theorem. It applies to the broad class of continuous linear operators of the form

$$A : C_0^\infty(Y,\Omega^{\frac{1}{2}}) \to \mathcal{D}'(X,\Omega^{\frac{1}{2}}).$$

Theorem 2.0.2. *There is a unique distribution $K(A) \in \mathcal{D}'(X\times Y,\Omega^{\frac{1}{2}})$ such that*

$$\langle A\psi,\varphi\rangle = \langle K(A),\varphi\otimes\psi\rangle \quad \text{for any} \quad (\varphi,\psi) \in C_0^\infty(X,\Omega^{\frac{1}{2}}) \times C_0^\infty(Y,\Omega^{\frac{1}{2}}).$$

This distribution uniquely defines A.

If $\operatorname{supp} K(A) \to X : (x,y) \mapsto x$ and $\operatorname{supp} K(A) \to Y : (x,y) \mapsto y$ are proper maps, then both A and $K(A)$ are said to be properly supported.

2.1 The Wavefront Set

The simplest question one could ask about a distribution is where it fails to be smooth, and this leads to the (in the local case, well-known) notion of (smooth) singular support. In the global setting, if X is a smooth manifold of $\dim(X) = n$, it is defined analogously. That is, if $u \in \mathcal{D}'(X, \Omega^{\frac{1}{2}})$, we write $x \notin \operatorname{sing\,supp}(u)$ if and only if

$$u|_U \in C^\infty(U, \Omega^{\frac{1}{2}}) \quad \text{for some open} \quad U \subset X \quad \text{with} \quad x \in U,$$

where "$\in$" is understood in the sense of the identification established in previous section. The study of singularities, using the cotangent bundle, is known as microlocal analysis, and the tools have applications to the study of the operators themselves, via the kernel. Microlocal analysis revolves around the wavefront set (also called the singular spectrum). It is a set that encodes location and "direction" in which a distribution fails to be regular. There are several types. For example, measuring the failure to be C^∞, C^ω or H^s regular. An extensive treatment is given by Hörmander [31, 32] of types "between" C^∞ and C^ω. We consider only C^∞ regularity.

But now let $X \subset \mathbb{R}^n$ be open, and let $u \in \mathcal{D}'(X)$ be an arbitrary distribution in X. Henceforth, $\pi_x : X \times (\mathbb{R}^n \setminus \{0\}) \to X : (x, \xi) \mapsto x$ is the projection onto the first factor. We define $\mathrm{WF}(u)$ for such X and u first, and then globalize later.

Definition 2.1.1. *Let $(x_0, \xi_0) \in X \times \mathbb{R}^n$. Put $(x_0, \xi_0) \notin \mathrm{WF}(u)$ if the following holds: There is an open cone $\xi_0 \in \Gamma \subset \mathbb{R}^n \setminus \{0\}$, and $\chi \in C_0^\infty(X)$ with $\chi(x_0) \neq 0$ so that*

$$\sup_{\xi \in \Gamma} \langle \xi \rangle^k |\mathcal{F}(\chi u)(\xi)| < \infty \quad \text{for each} \quad k \in \mathbb{N}.$$

Otherwise $(x_0, \xi_0) \in \mathrm{WF}(u)$ when $\xi_0 \neq 0$.

Example 2.1.1. Put $x = (x', x_n)$. Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be the unit step along x_n at $x_n = 0$. In this case, we have

$$\operatorname{sing\,supp}(u) = \{x \in \mathbb{R}^n \mid x_n = 0\} \neq \emptyset,$$

and

$$\mathcal{F}(\chi u)(\xi) = \int_0^\infty e^{-ix_n\xi_n} \mathcal{F}_{x'}\chi(\xi', x_n)\, dx_n \quad \text{for all} \quad \xi \in \mathbb{R}^n,$$

where we get rapid decay near $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ as long as $\xi_0' \neq 0$. This shows that

$$\mathrm{WF}(u) = \{(x, \xi) \mid x_n = 0, \xi' = 0\}.$$

A basic fact about $\mathrm{WF}(u)$ is that it generalizes $\operatorname{sing\,supp}(u)$ and respects restrictions. This is provable within the scope of a few pages.

Proposition 2.1.1. *If $U \subset X$ is open, then*

$$\pi_x \mathrm{WF}(u) = \operatorname{sing\,supp}(u) \quad \text{and} \quad \mathrm{WF}(u|_U) = \mathrm{WF}(u) \cap \pi_x^{-1}(U).$$

Proof. Suppose first that $u \in \mathcal{E}'(\mathbb{R}^n)$ and that the estimates hold with χu replaced by u. That is, there is an open cone Γ containing $\xi_0 \in (\mathbb{R}^n \setminus \{0\})$ such that

$$\sup_{\xi\in\Gamma} \langle\xi\rangle^k |\mathcal{F}u(\xi)| < \infty \quad \text{for each} \quad k \in \mathbb{N}.$$

Now $\mathcal{F}u$ is entire with at most degree $N \in \mathbb{N}_0$ polynomial growth in the real directions. This means that there is a constant $c > 0$ such that

$$\sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{-N} |\mathcal{F}u(\xi)| \le c.$$

Choose $0 < \epsilon < 1$ so small that there exists a conic neighbourhood Γ_ϵ of ξ_0 in $\mathbb{R}^n \setminus \{0\}$, which has the property that

$$\xi - \eta \in \Gamma \quad \text{if} \quad |\eta| \le \epsilon|\xi| \quad \text{and} \quad \xi \in \Gamma_\epsilon.$$

The point here is that φu satisfies similar estimates, but with $\xi \in \Gamma_\epsilon$, when $\varphi \in C_0^\infty(X)$. Observe that since $|\xi - \eta| \ge (1-\epsilon)|\xi|$ when $|\eta| \le \epsilon|\xi|$, we get

$$\int_{B(0,\epsilon|\xi|)} |\mathcal{F}\varphi(\eta)||\mathcal{F}u(\xi-\eta)|\, đ\eta \le c_k\langle\xi\rangle^{-k}||\mathcal{F}\varphi||_{L^1(\mathbb{R}^n)},$$

and because $|\xi - \eta| \le (\epsilon^{-1}+1)|\eta|$ when $|\eta| \ge \epsilon|\xi|$, we can estimate

$$\begin{aligned}
|\mathcal{F}(\varphi u)(\xi)| &= \frac{1}{(2\pi)^n}|(\mathcal{F}\varphi * \mathcal{F}u)(\xi)| \\
&\le c_k\langle\xi\rangle^{-k}||\mathcal{F}\varphi||_{L^1(\mathbb{R}^n)} + \int_{\mathbb{R}^n\setminus B(0,\epsilon|\xi|)} |\mathcal{F}\varphi(\eta)||\mathcal{F}u(\xi-\eta)|\, d\eta \\
&\le c_k\langle\xi\rangle^{-k}||\mathcal{F}\varphi||_{L^1(\mathbb{R}^n)} + c\int_{\mathbb{R}^n\setminus B(0,\epsilon|\xi|)} |\mathcal{F}\varphi(\eta)|\langle\xi-\eta\rangle^N\langle\epsilon\xi\rangle^{-k}\langle\eta\rangle^k\, d\eta \\
&\le C_k\langle\xi\rangle^{-k}\Big[||\mathcal{F}\varphi||_{L^1(\mathbb{R}^n)} + \int_{\mathbb{R}^n\setminus B(0,\epsilon|\xi|)} |\mathcal{F}\varphi(\eta)|\langle\eta\rangle^{N+k}\, d\eta\Big].
\end{aligned}$$

This shows that

$$\sup_{\xi\in\Gamma_\epsilon} \langle\xi\rangle^k|\mathcal{F}(\varphi u)(\xi)| \le C_k\int_{\mathbb{R}^n}\langle\eta\rangle^{N+k}|\mathcal{F}\varphi(\eta)|\, d\eta.$$

As a consequence, we may multiply χu by a smooth cutoff with support contained in U, and still get similar estimates for $\xi \in \Gamma_\epsilon$. Hence WF respects restrictions.

Now for the other identity. The "$\subset$" inclusion is trivial. So assume $x_0 \notin \pi_x\mathrm{WF}(u)$. Cover $\mathbb{S}^{n-1}$ by open cones $\{\Gamma_j\}_{j=1}^M$ with associated $\chi_j \in C_0^\infty(X)$ such that

$$\sup_{\xi\in\Gamma_j}\langle\xi\rangle^k|\mathcal{F}(\chi_j u)(\xi)| < \infty \quad \text{for each} \quad k \in \mathbb{N}.$$

In this case, given any $\xi_0 \in \Gamma_j$, there is some Γ_ϵ containing ξ_0 such that

$$\sup_{\xi\in\Gamma_\epsilon}\langle\xi\rangle^k\Big|\mathcal{F}\Big(\prod_{r=1}^M\chi_r u\Big)(\xi)\Big| \le C_k\int_{\mathbb{R}^n}\langle\eta\rangle^{N_j+k}\Big|\mathcal{F}\Big(\prod_{r\neq j}^M\chi_r\Big)(\eta)\Big|\, d\eta,$$

and by compactness of $\mathbb{S}^{n-1}$, we get rapid decay. Thus $x_0 \notin \operatorname{sing\,supp}(u)$. □

Turning to distributions on a smooth manifold X, there are many ways to define WF. One is to use pseudo-differential operators, another to test against waves on the manifold. The latter approach is simpler. Let now $u \in \mathcal{D}'(X, \Omega^{\frac{1}{2}})$.

Definition 2.1.2. *Let $(x_0, \xi_0) \in T^*X \setminus 0$. Put $(x_0, \xi_0) \notin \mathrm{WF}(u)$ if the following holds: Given any $p \in \mathbb{N}$ and any real $f \in C^\infty(X \times \mathbb{R}^p)$, then two conditions hold:*

1. *There is some $a_0 \in A$ with $d_x f(x_0, a_0) = \xi_0$.*
2. *There is an open $U_0 \times A_0 \ni (x_0, a_0)$ so that if $\varphi \in C_0^\infty(U_0, \Omega^{\frac{1}{2}})$, we have*

$$\sup_{a \in A_0} |\langle u(x), e^{-i\tau f(x,a)} \varphi(x) \rangle| = O_{\tau \to \infty}(\tau^{-N}) \quad \text{for any} \quad N \in \mathbb{N}.$$

Otherwise $(x_0, \xi_0) \in \mathrm{WF}(u)$ when $\xi_0 \neq 0$.

If X is an open set in $\mathbb{R}^n$, we verify that this is consistent with the earlier definition. Only local computations are required, and we may disregard $\Omega^{\frac{1}{2}}$.

Proof. Suppose that $(x_0, \xi_0) \notin \mathrm{WF}(u)$. So there is an open cone Γ about ξ_0 in $\mathbb{R}^n \setminus \{0\}$, and a cutoff $\chi \in C_0^\infty(X)$ equal to 1 in a neighbourhood $U \subset X$ of x_0, such that

$$\sup_{\xi \in \Gamma} \langle \xi \rangle^k |\mathcal{F}(\chi u)(\xi)| < \infty \quad \text{for each} \quad k \in \mathbb{N}.$$

This can be achieved, for example, by the arguments in the proof of Proposition 2.1.1. Take $\varphi \in C_0^\infty(U_0)$ with $U_0 \subset U$ open, and write

$$\begin{aligned} \langle u(x), e^{-i\tau f(x,a)} \varphi(x) \rangle &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \tau f(x,a))} \varphi(x) \mathcal{F}(\chi u)(\xi)\, dx\, đ\xi \\ &= \tau^n \int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} e^{i\tau(x \cdot \xi - f(x,a))} \varphi(x)\, dx \Big] \mathcal{F}(\chi u)(\tau \xi)\, đ\xi. \end{aligned}$$

Since $d_x f(x_0, a_0) = \xi_0$, we can choose U_0 and $A_0 \subset \mathbb{R}^p$ such that

$$|d_x f(x,a) - \xi| \geq \epsilon > 0 \quad \text{for all} \quad (x,a) \in U_0 \times A_0 \quad \text{and} \quad \xi \notin \Gamma.$$

In this situation, we define the operator

$$L = \frac{\xi - d_x f(x,a)}{|\xi - d_x f(x,a)|^2} \cdot d_x,$$

and integrate by parts $k \in \mathbb{N}$ times to get the estimates

$$\Big| \int_{\mathbb{R}^n} e^{i\tau(x \cdot \xi - f(x,a))} \varphi(x)\, dx \Big| \leq \tau^{-k} \int_{\mathbb{R}^n} |(L^t)^k \varphi(x)|\, dx \leq C_k \tau^{-k} \langle \xi \rangle^{-k}.$$

Combining this with rapid decay of $\mathcal{F}(\chi u)$ in Γ, and the Paley-Wiener-Schwartz theorem, then we obtain the right asymptotic. The other direction is trivial. □

Therefore, $\mathrm{WF}(u)$ is a closed cone in $T^*X \setminus 0$, with the same rules as in the local case. That is, if $\pi : T^*X \to X$ is the bundle projection, and $U \subset X$ is open, then

$$\pi(\mathrm{WF}(u)) = \operatorname{sing\,supp}(u) \quad \text{and} \quad \mathrm{WF}(u|_U) = \mathrm{WF}(u) \cap \pi^{-1}(U).$$

2.1.1 Operations on Distributions

Many operations on ordinary functions have extensions to the setting of distributions, both locally and globally, provided that the wavefront sets of the arguments are aligned. The most important such operations are pullbacks and pushforwards by a smooth map, which, for example, can be used to obtain the kernel of a composition of two operators. This is used by Hörmander and Duistermaat [30] to describe propagation of singularities. Let X, Y and Z be three smooth manifolds. If $u \in \mathcal{D}'(X, \Omega^{\frac{1}{2}})$, we put

$$\operatorname{supp}_0(u) = \{(x, 0) \in T^*X \mid x \in \operatorname{supp}(u)\}.$$

Proposition 2.1.2. *If $u \in \mathcal{D}'(X, \Omega^{\frac{1}{2}})$ and $v \in \mathcal{D}'(Y, \Omega^{\frac{1}{2}})$, then*

$$\mathrm{WF}(u \otimes v) \subset (\mathrm{WF}(u) \times \mathrm{WF}(v)) \cup (\mathrm{WF}(u) \times \operatorname{supp}_0(v)) \cup (\operatorname{supp}_0(u) \times \mathrm{WF}(v)).$$

Proof. Use the Paley-Wiener-Schwartz theorem and the definition of $\mathrm{WF}(u \otimes v)$. □

Definition 2.1.3. *If $\Gamma \subset T^*X \setminus 0$ is a non-empty closed cone, we define*

$$\mathcal{D}'_\Gamma(X, \Omega^{\frac{1}{2}}) = \left\{ u \in \mathcal{D}'(X, \Omega^{\frac{1}{2}}) \,\middle|\, \mathrm{WF}(u) \subset \Gamma \right\}.$$

(It is easily checked that this is a subspace.) It is equipped with additional semi-norms. Let $f \in C^\infty(X \times \mathbb{R}^p)$ be real, $\varphi \in C_0^\infty(X, \Omega^{\frac{1}{2}})$, and $A \subset \mathbb{R}^p$ compact, such that

$$(x, d_x f(x, a)) \notin \Gamma \quad \text{when} \quad (x, a) \in \operatorname{supp} \varphi \times A.$$

Then for each $N \in \mathbb{N}$, we also equip $\mathcal{D}'_\Gamma(X)$ with the semi-norm

$$\mathcal{D}'_\Gamma(X, \Omega^{\frac{1}{2}}) \to [0, \infty) : u \mapsto \sup_{(\tau, a) \in [1, \infty) \times A} \tau^N \left| \left\langle u(x), e^{-i\tau f(x,a)} \varphi(x) \right\rangle \right|,$$

which is the smallest constant in the τ^N-asymptotics for this test wave.

These spaces make precise the idea of wavefront sets aligned relative to a closed cone. The following can be found in Duistermaat [10] or Hörmander [31].

Proposition 2.1.3. *The space $C_0^\infty(X, \Omega^{\frac{1}{2}})$ is sequentially dense in $\mathcal{D}'_\Gamma(X, \Omega^{\frac{1}{2}})$.*

Theorem 2.1.1. *Let $\Phi : Y \to X$ be smooth, and $\Gamma \subset T^*X \setminus 0$ a non-empty closed cone. Suppose that $N_\Phi^* \cap \Gamma = \emptyset$, where N_Φ^* is the set of $(x, \xi) \in T^*X \setminus 0$ such that*

$$x = \Phi(y) \quad \text{and} \quad d\Phi_y^t \xi = 0 \quad \text{for some} \quad y \in X.$$

Then Φ^* extends to $\Phi^* : \mathcal{D}'_\Gamma(X, \Omega^{\frac{1}{2}}) \to \mathcal{D}'_{\Phi^*\Gamma}(Y, \Omega^{\frac{1}{2}})$, where $\Phi^*\Gamma$ is the closed cone

$$\Phi^*\Gamma = \left\{ (y, d\Phi_y^t \xi) \in T^*Y \setminus 0 \,\middle|\, (\Phi(y), \xi) \in \Gamma \right\}.$$

This extension is unique, sequentially continuous, and has the property that

$$\operatorname{supp}(\Phi^* u) \subset \Phi^{-1}(\operatorname{supp} u) \quad \text{for any} \quad u \in \mathcal{D}'_\Gamma(X, \Omega^{\frac{1}{2}}).$$

The transpose of the pullback Φ^* is the pushforward Φ_*. It is more easily understood. This map also has an extension. If $v \in \mathcal{D}'(Y, \Omega^{\frac{1}{2}})$, it is just given by

$$\langle (\Phi_* v)(x), \varphi(x) \rangle = \langle v(y), (\Phi^* \varphi)(y) \rangle \quad \text{for any} \quad \varphi \in C_0^\infty(X, \Omega^{\frac{1}{2}}).$$

Proposition 2.1.4. *Let $\Phi : Y \to X$ again be a smooth map, as in the above theorem. The pushforward $\Phi_* v$ is defined for $v \in \mathcal{D}'(Y, \Omega^{\frac{1}{2}})$ if $\Phi : \operatorname{supp}(v) \to X$ is a proper map. This extension is sequentially continuous $v \mapsto \Phi_* v \in \mathcal{D}'(X, \Omega^{\frac{1}{2}})$, and*

$$\mathrm{WF}(\Phi_* v) \subset \{\, (\Phi(y), \xi) \in T^*X \setminus 0 \,|\, (y, (d\Phi_y)^t \xi) \in \mathrm{WF}(v) \,\}.$$

Proof. Take $(x_0, \xi_0) \in T^*X \setminus 0$, and let $f \in C^\infty(X \times \mathbb{R}^p)$ be real with $d_x f(x_0, a_0) = \xi_0$. Observe that if $\varphi \in C_0^\infty(X, \Omega^{\frac{1}{2}})$ and $\tau > 0$, then

$$\langle (\Phi_* v)(x), e^{-i\tau f(x,a)} \varphi(x) \rangle = \langle v(y), e^{-i\tau f(\Phi(y),a)} (\Phi^* \varphi)(y) \rangle,$$

where we have

$$d_x f(\Phi(y_0), a_0) \circ d\Phi_{y_0} = (d\Phi_{y_0})^t \xi_0 \quad \text{if} \quad x_0 = \Phi(y_0).$$

It is clear that Φ_* is sequentially continuous for such distributions. □

Definition 2.1.4. *Let $A : C_0^\infty(Y, \Omega^{\frac{1}{2}}) \to \mathcal{D}'(X, \Omega^{\frac{1}{2}})$ be a continuous and linear operator. The wavefront relation of A is the set*

$$\mathrm{WF}'(A) = \Big\{ ((x,\xi),(y,\eta)) \in T^*X \setminus 0 \times T^*Y \setminus 0 \,\Big|\, ((x,y),(\xi,-\eta)) \in \mathrm{WF}(K(A)) \Big\}.$$

Furthermore, we write

$$\begin{aligned} \mathrm{WF}'_X(A) &= \{(x,\xi) \in T^*X \setminus 0 \,|\, \exists y \in Y : ((x,y),(\xi,0)) \in \mathrm{WF}(K(A))\}, \\ \mathrm{WF}'_Y(A) &= \{(y,\eta) \in T^*Y \setminus 0 \,|\, \exists x \in X : ((x,y),(0,-\eta)) \in \mathrm{WF}(K(A))\}. \end{aligned}$$

Theorem 2.1.2. *Momentarily, we denote*

$$\begin{aligned} &\pi_K : X \times Y \times Z \to X \times Z : (x,y,z) \mapsto (x,z), \\ &\Delta_K : X \times Y \times Z \to X \times Y \times Y \times Z : (x,y,z) \mapsto (x,y,y,z). \end{aligned}$$

Consider two continuous and linear operators

$$A : C_0^\infty(Y, \Omega^{\frac{1}{2}}) \to \mathcal{D}'(X, \Omega^{\frac{1}{2}}) \quad \text{and} \quad B : C_0^\infty(Z, \Omega^{\frac{1}{2}}) \to \mathcal{D}'(Y, \Omega^{\frac{1}{2}}).$$

If A and B are properly supported, and $\mathrm{WF}'_Y(A) \cap \mathrm{WF}'_Y(B) = \emptyset$, then

$$K(A \circ B) = (\pi_K)_* (\Delta_K)^* (K(A) \otimes K(B)).$$

This is the kernel of $A \circ B$, when $A \circ B$ is defined. It has the following properties:

1. $\operatorname{supp} K(A \circ B) \subset \operatorname{supp} K(A) \circ \operatorname{supp} K(B)$.
2. $\mathrm{WF}'(A \circ B) \subset \mathrm{WF}'(A) \circ \mathrm{WF}'(B) \cup (\mathrm{WF}'_X(A) \times 0_{T^*Z}) \cup (0_{T^*X} \times \mathrm{WF}'_Z(B))$.

In particular, if Γ is a closed cone in $T^*Y \setminus 0$ with $\mathrm{WF}'_Y(A) \cap \Gamma = \emptyset$, then

$$\mathrm{WF}'(Au) \subset \mathrm{WF}'(A) \circ \mathrm{WF}(u) \cup \mathrm{WF}'_X(A) \quad \text{if} \quad u \in \mathcal{E}'_\Gamma(Y, \Omega^{\frac{1}{2}}).$$

3

Operators on Manifolds with More Structure

The structure of a compact manifold M influences the structure of $\Psi(M)$ defined on it. If the underlying manifold M carries a Riemannian metric (or just an affine connection), then there is an isomorphism $S^d(T^*M)/S^{-\infty}(T^*M) \to \Psi^d(M)/\Psi^{-\infty}(M)$ for each $d \in \mathbb{R}$. This stronger symbol map is obtained by Sharafutdinov in [50] via the exponential map. The situation is even better for Lie groups, where, like $\mathbb{R}^n$, there is an exact quantization, but the symbols on the cotangent bundle must be replaced with matrix-valued functions, which live on the group and its equivalence classes of irreducible unitary representations. Let $\epsilon > 0$. Consider $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, which is embedded within

$$\mathbb{T}^n_\epsilon = (\mathbb{R}_\epsilon/2\pi\mathbb{Z})^n,$$

where $\mathbb{R}_\epsilon$ is the strip $\{z \in \mathbb{C} \,|\, |\mathrm{Im}\,(z)| < \epsilon\}$ in $\mathbb{C}$, with $2\pi\mathbb{Z} \subset \mathbb{R}$ acting on the real part. The irreducible unitary representations of $\mathbb{T}^n$ are one-dimensional, parametrized by $\mathbb{Z}^n$, and are for each $k \in \mathbb{Z}^n$ just given by the complex exponential

$$e_k : \mathbb{T}^n \to \mathbb{C} : x \mapsto e^{ik\cdot x}$$

If $A \in \Psi^d(\mathbb{T}^n)$, we define a "symbol" $p(x,k) = e_k(x)^*(Ae_k)(x)$ for all $(x,k) \in \mathbb{T}^n \times \mathbb{Z}^n$. By a Fourier expansion of $u \in C^\infty(\mathbb{T}^n)$, which converges in $C^\infty(\mathbb{T}^n)$, we have

$$Au(x) = \sum_{k\in\mathbb{Z}} e^{ik\cdot x} p(x,k) \int_{\mathbb{T}^n} u(y) e^{-ik\cdot y}\, dy \quad \text{for all} \quad x \in \mathbb{T}^n.$$

This represents A point-wise in terms of p. Let us try to put a global condition on p. Suppose $x \mapsto p(x,k)$ is real-analytic, extending holomorphically to $z \in \mathbb{T}^n_\epsilon$ for each $k \in \mathbb{Z}$, and that p also satisfies

$$\sup_{z\in\mathbb{T}^n_\epsilon} \langle k\rangle^{-d} |p(z,k)| < \infty.$$

In that case, if $u \in C^\omega(\mathbb{T}^n)$ extends to $\mathbb{T}^n_\epsilon$, then

$$\begin{aligned} Au(z) &= \sum_{k\in\mathbb{Z}^n} e^{ik\cdot z} p(z,k) \int_{\mathbb{T}^n} e^{-ik\cdot y} u(y)\, dy \\ &= \sum_{k\in\mathbb{Z}^n} e^{ik\cdot \mathrm{Re}\,(z)} p(z,k) \int_{\mathbb{T}^n} e^{-ik\cdot y} u(y + i\,\mathrm{Im}\,(z))\, dy. \end{aligned}$$

It follows that Au extends holomorphically to $\mathbb{T}^n_\epsilon$ if u does. This is the property we seek. Of course, we have to address whether there are non-trivial p satisfying the requirements. In light of this observation, we are inspired to look for algebras of operators on Lie groups, and their homogeneous spaces, as a step towards a better understanding of the situation.

3.1 $\Psi(M)$ for a Compact Riemannian Manifold M

To get a good start, we begin by recalling the basic theory of pseudo-differential operators. We assume M is a compact, smooth Riemannian manifold with metric g from the outset, and we fix (arbitrarily) a smooth positive 1-density ω_0 on M to make things much simpler. In that case, $\Omega_M^{1/2}$ is just removed from the notation. Put $n = \dim(M)$.

Definition 3.1.1. *Let $P : C^\infty(M) \to C^\infty(M)$ be any continuous and linear operator. Then we write $P \in \Psi^d(M)$ for $d \in \mathbb{R}$ if the following holds:*

1. $\operatorname{sing\,supp} K(P) \subset \{(x,x) \in M \times M \mid x \in M\}$.
2. *Given any chart $\kappa : U \to \mathbb{R}^n$ of M, and $\phi, \psi \in C_0^\infty(U)$, we have*

$$(\kappa^{-1})^*(\phi P \psi)\kappa^* = \operatorname{Op}(p) \quad \text{for some} \quad p \in S^d(\kappa(U) \times \mathbb{R}^n).$$

Also, we write $P \in \Psi^d_{\mathrm{phg}}(M)$ if the above holds with $p \in S^d_{\mathrm{phg}}(\kappa(U) \times \mathbb{R}^n)$ instead.

Definition 3.1.2. *Let $d \in \mathbb{R}$. Define $S^d(T^*M)$ by pulling back $S^d(\mathbb{R}^n \times \mathbb{R}^n)$ to T^*M. That is, write $p \in S^d(T^*M)$ if the following holds:*

1. $p \in C^\infty(T^*M)$
2. *Given any chart $\kappa : U \to \mathbb{R}^n$ of M, we have*

$$(d\kappa^t)^* p \in S^d(\kappa(U) \times \mathbb{R}^n).$$

The local pseudo-differential operators and symbols are preserved by diffeomorphisms. So the above definitions are meaningful. See Shubin [51] or Hörmander [33].

To illustrate what is meant, consider a symbol $p \in S^d(V \times \mathbb{R}^n)$ for some open $V \subset \mathbb{R}^n$. Then, if $\Phi : V \to V'$ is a diffeomorphism onto an open set $V' \subset \mathbb{R}^n$, we have

$$(d\Phi^t)^* p \in S^d(V' \times \mathbb{R}^n),$$

and so $S^d(T^*M)$ is well-defined, because the transition maps of T^*M are of this form. Likewise, suppose that $\operatorname{Op}(p) \in \operatorname{Op} S^d(V \times \mathbb{R}^n)$ is a properly supported operator on V. It acts on $u \in C_0^\infty(V)$ pointwise in $x \in V$ in the usual way by

$$\operatorname{Op}(p)u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi) \mathcal{F}u(\xi)\, đ\xi.$$

But the coordinate-transformed operator is

$$(\Phi^{-1})^* \operatorname{Op}(p) \Phi^* = \operatorname{Op}(p_\Phi),$$

where $\operatorname{Op}(p_\Phi)$ is properly supported on V', p_Φ is related to p by

$$p_\Phi - (d\Phi^t)^* p \in S^{d-1}(V' \times \mathbb{R}^n),$$

and p_Φ vanishes outside a compact subset of V' in the first variable if p does so for V. The same is true if S^d and S^{d-1} are replaced by the spaces S^d_{phg} and S^{d-1}_{phg}, respectively. This justifies the definition (and is a special case of general results about FIO [27]).

Pseudo-differential operators are closed under compositions and formal L^2-adjoints. This is an elementary fact. See e.g. Shubin [51].

Theorem 3.1.1 (Shubin [51]). *Let $d, d_1, d_2 \in \mathbb{R}$ be arbitrary. The following holds:*

1. *If $A \in \Psi^{d_1}(M)$ and $B \in \Psi^{d_2}(M)$, then $AB \in \Psi^{d_1+d_2}(M)$.*
2. *If $A \in \Psi^d(M)$, then A has a formal L^2-adjoint $A^* \in \Psi^d(M)$.*

Then $\Psi(M) = \cup_{d\in\mathbb{R}}\Psi^d(M)$ forms a $*$-algebra under composition and formal L^2-adjoints. The classical operators form a $*$-subalgebra:

1. *If $A \in \Psi^{d_1}_{\mathrm{phg}}(M)$ and $B \in \Psi^{d_2}_{\mathrm{phg}}(M)$, then $AB \in \Psi^{d_1+d_2}_{\mathrm{phg}}(M)$.*
2. *If $A \in \Psi^d_{\mathrm{phg}}(M)$, then A has a formal L^2-adjoint $A^* \in \Psi^d_{\mathrm{phg}}(M)$.*

Attached to each space there is the principal symbol map

$$\sigma_d : \Psi^d(M)/\Psi^{d-1}(M) \to S^d(T^*M)/S^{d-1}(T^*M),$$

which takes any equivalence class $[P]$ of operators to some class $[p] = \sigma_d([P])$ of symbols. It is the unique map with the following two properties:

1. *σ_d is a linear isomorphism.*
2. *Given any chart $\kappa : U \to \mathbb{R}^n$ of M, and $\phi, \psi \in C_0^\infty(U)$, we have*

$$(\kappa^{-1})^*(\phi P\psi)\kappa^* - (\phi\circ\kappa^{-1})\mathrm{Op}((d\kappa^t)^*p)(\psi\circ\kappa^{-1}) \in \mathrm{Op}\, S^{d-1}(\kappa(U)\times\mathbb{R}^n).$$

It also has the following $*$-isomorphism properties:

1. *If $A \in \Psi^{d_1}(M)$ and $B \in \Psi^{d_2}$, then $\sigma_{d_1+d_2}([AB]) = \sigma_{d_1}([A])\sigma_{d_2}([B])$.*
2. *If $A \in \Psi^d(M)$, then $\sigma_d([A^*]) = \overline{\sigma_d([A])}$.*

Finally, if $P \in \Psi^d_{\mathrm{phg}}(M)$, there is a unique function $p_d \in C^\infty(T^*M \setminus 0)$ such that

1. *$p_d(x, t\xi) = t^d p_d(x,\xi)$ for any $(x,\xi) \in T^*M \setminus 0$ and $t > 0$.*
2. *$(1-\chi)p_d \in \sigma_d([P])$ if $\chi \in C_0^\infty(B^*M)$ equals 1 near the zero section.*

The unique p_d obtained from a $P \in \Psi^d_{\mathrm{phg}}(M)$ is called the classical principal symbol. Any operator $P \in \Psi(M)$ dualizes to act on $u \in \mathcal{D}'(M)$ by setting

$$\langle Pu, \varphi\rangle = \langle u, \overline{P^*\overline{\varphi}}\rangle \quad \text{for all} \quad \varphi \in C^\infty(M),$$

which is compatible with the natural identification $L^1(M) \hookrightarrow \mathcal{D}'(M)$ when $u \in C^\infty(M)$. In this way, A extends by duality to a weak$*$-continuous operator

$$P : \mathcal{D}'(M) \to \mathcal{D}'(M),$$

and restrictions of this extension are called (bounded or unbounded) realizations of P. Also, P^* denotes both the formal L^2-adjoint on $C^\infty(M)$ (which is not a Hilbert adjoint), but also Hilbert adjoints when P is realized as an operator between two Hilbert spaces. This abuse is often seen in the literature. We use it with clarification, if necessary.

An important notion when dealing with symbols/operators is that of asymptotic sums. The main result here is encapsulated in the following theorem.

Theorem 3.1.2. *Let $d_j \to -\infty$ as $j \to \infty$. The following holds:*

1. *If $p_j \in S^{d_j}(T^*M)$, then there is a $p \in S^{\max_{j\geq 0} d_j}(T^*M)$ such that*

$$p - \sum_{j=0}^{k-1} p_j \in S^{\max_{j\geq k} d_j}(T^*M) \quad \text{for each} \quad k \in \mathbb{N}.$$

2. *If $P_j \in \Psi^{d_j}(M)$, then there is a $P \in \Psi^{\max_{j\geq 0} d_j}(M)$ such that*

$$P - \sum_{j=0}^{k-1} P_j \in \Psi^{\max_{j\geq k} d_j}(M) \quad \text{for each} \quad k \in \mathbb{N}.$$

A sequence of operators $\{P_j\}_{j=0}^{\infty}$ in the situation above is said to be asymptotic to P. In that case, we write $P \sim \sum_{j=0}^{\infty} P_j$. Similarly for symbols $\{p_j\}_{j=0}^{\infty}$, we write $p \sim \sum_{j=0}^{\infty} p_j$.

Definition 3.1.3. *An $A \in \Psi^d(M)$ is elliptic if there is a $B \in \Psi^{-d}(M)$ such that*

$$AB - I \in \Psi^{-\infty}(M) \quad \text{and} \quad BA - I \in \Psi^{-\infty}(M).$$

Theorem 3.1.3 (Shubin [51])**.** *Let $d \in \mathbb{R}$. The following are equivalent:*

1. *$A \in \Psi^d(M)$ is elliptic.*
2. *$\sigma_d([A])$ is invertible.*

Proof. Applying the isomorphism σ_d to the definition of ellipticity, we see that (1) $\Rightarrow$ (2). To see (2) $\Rightarrow$ (1), take the inverse $[b_0]$ to $\sigma_d(A)$, and write $[B_0] = \sigma_{-d}([b_0])$.

Left: Put $R = I - B_0A$. Define $B_j = R^jB_0$ for $j \in \mathbb{N}_0$ and $\Psi^{-d}(G) \ni B \sim \sum_{j=0}^{\infty} B_j$. Then for any $N \in \mathbb{N}$, we have

$$\begin{aligned}\Psi^{-N}(M) \ni \Big(B - \sum_{j=0}^{N-1} B_j\Big)A - R^N \\ = BA - \sum_{j=1}^{N-1} R^j(I-R) - R^N = BA - I.\end{aligned}$$

Right: Put $R = I - AB_0$. Define $B_j = B_0R^j$ for $j \in \mathbb{N}_0$ and $\Psi^{-d}(G) \ni B \sim \sum_{j=0}^{\infty} B_j$. Then for any $N \in \mathbb{N}$, we have

$$\begin{aligned}\Psi^{-N}(M) \ni A\Big(B - \sum_{j=0}^{N-1} B_j\Big) - R^N \\ = AB - \sum_{j=0}^{N-1} (I-R)R^j - R^N = AB - I.\end{aligned}$$

In either case, we have used Theorem 3.1.2. ◻

The operators $\Psi^0(M)$ turn out to have bounded realizations on $L^2(M)$ back to itself. This is the content of the following theorem:

Theorem 3.1.4 (Shubin [51]). *Let $P \in \Psi^0(M)$ with any principal symbol $p \in S^0(T^*M)$.*

1. *If P is restricted to $L^2(M)$, it realizes a bounded linear operator*

$$P|_{L^2(M)} : L^2(M) \to L^2(M).$$

2. *If P is realized as above, then it is compact if*

$$\lim_{k\to\infty} \sup_{|\xi|\geq k} \Big[\sup_{x\in M} |p(x,\xi)| \Big] = 0.$$

At this point, we make a slight jump in the theory to significantly reduce complexity. It is convenient to simply "download" the following:

Theorem 3.1.5 (See e.g. Shubin [51]). *Write Δ for the Laplacian with respect to g. There is a one-parameter group $\{(I-\Delta)^{\frac{s}{2}}\}_{s\in\mathbb{R}} \subset \Psi(M)$ of invertible operators:*

1. *$(I-\Delta)^{\frac{s}{2}} \in \Psi^s(M)$ is formally self-adjoint for each $s \in \mathbb{R}$.*
2. *If $s \in 2\mathbb{Z}$, then $(I-\Delta)^{\frac{s}{2}}$ coincides with the corresponding power of $I-\Delta$.*

Definition 3.1.4. *Define for any $s \in \mathbb{R}$ the Sobolev space*

$$H^s(M) = \Big\{ u \in \mathcal{D}'(M) \,\Big|\, (I-\Delta)^{\frac{s}{2}} u \in L^2(M) \Big\},$$

and let it be equipped with the inner product

$$(u,v)_{H^s(M)} = ((I-\Delta)^{\frac{s}{2}} u, (I-\Delta)^{\frac{s}{2}} v)_{L^2(M)} \quad \text{for any} \quad u,v \in H^s(M).$$

Here the definition makes it clear that all the Sobolev spaces must be Hilbert spaces. The operator $(I-\Delta)^{\frac{s}{2}} : H^s(M) \to L^2(M)$ is automatically an isometry.

Theorem 3.1.6 (Shubin [51]). *Let $s \in \mathbb{R}$ and $m \in \mathbb{N}_0$. The following holds:*

1. *The canonical pairing extends to a separately continuous dual pairing*

$$H^s(M) \times H^{-s}(M) \to \mathbb{C} : (u,v) \mapsto \int_M [(I-\Delta)^{\frac{s}{2}} u]\, [(I-\Delta)^{-\frac{s}{2}} v]\, \omega_0.$$

2. *If $s' > s$, the inclusion $H^{s'}(M) \hookrightarrow H^s(M)$ is well-defined and compact.*
3. *If $s > \frac{n}{2} + m$, then $\iota^{-1} : H^s(M) \to C^m(M)$ is well-defined and compact.*

The H^s-spaces on M are related to those on $\mathbb{R}^n$ (defined using the Fourier transform). Given any chart $\kappa : U \to \mathbb{R}^n$ of M, we have a linear homeomorphism

$$H^s_K(M) \to H^s_{\kappa(K)}(\mathbb{R}^n) : u \mapsto (\kappa^{-1})^* u \quad \text{if} \quad K \subset\subset U,$$

where $H^s_K(M) = \{u \in H^s(M) \,|\, \mathrm{supp}(u) \subset K\}$ is closed.

In Theorem 3.1.6 above, the third point is known as the Sobolev embedding theorem. It states that $s > \frac{n}{2} + m$ Sobolev regularity forces identification with $C^m(M)$ functions. Combining this with the structure theorem for $\mathcal{E}'(\mathbb{R}^n)$, we have that

$$\cap_{s\in\mathbb{R}} H^s(M) = C^\infty(M) \quad \text{and} \quad \cup_{s\in\mathbb{R}} H^s(M) = \mathcal{D}'(M),$$

which we can use to characterize $\Psi^{-\infty}(M)$ in terms of how these operators act on $\mathcal{D}'(M)$. Of course, $\Psi^{-\infty}(M)$ is the set of operators with smooth kernel, by Definition 3.1.1.

Theorem 3.1.7. *The following are equivalent:*

1. $P \in \Psi^{-\infty}(M)$.
2. $P : C^\infty(M) \to C^\infty(M)$ *is linear and continuous, and extends to*

$$P|_{H^s(M)} \in B(H^s(M), C^\infty(M)) \quad \text{for every} \quad s \in \mathbb{R}.$$

Proof. We begin by showing (1) $\Rightarrow$ (2). If $P \in \Psi^{-\infty}(M)$, then $P^* \in \Psi^{-\infty}(M)$ as well. Therefore, taking $u \in H^s(M)$ and testing Pu against $\varphi \in C^\infty(M)$, we see that

$$\begin{aligned}\langle Pu, \varphi\rangle &= \Big\langle u(y), \overline{\int_M K(P^*)(y,x)\overline{\varphi(x)}\omega_0(x)}\Big\rangle \\ &= \int_M \Big\langle u(y), \overline{K(P^*)(y,x)}\Big\rangle \varphi(x)\,\omega_0(x),\end{aligned}$$

where the integrals converge as Riemann sums in $C^\infty(M)$, so the interchange is allowed. Hence Pu is smooth. So if $D \in \mathrm{Diff}^m(M)$ for some $m \in \mathbb{N}_0$, we can estimate

$$\begin{aligned}\sup_{x\in M} |D(Pu)(x)|^2 &= \sup_{x\in M} \Big|\Big\langle u(y), D_x\overline{K(P^*)(y,x)}\Big\rangle\Big|^2 \\ &\leq \sup_{x\in M} \Big[\int_M |(I-\Delta_y)^{-\frac{s}{2}} D_x\overline{K(P^*)(y,x)}|^2\,\omega_0(y)\Big] ||u||^2_{H^s(M)},\end{aligned}$$

where we have brought forth $(I - \Delta_y)^{\frac{s}{2}}$ onto u, and used the Cauchy-Schwarz inequality. To prove (2) $\Rightarrow$ (1), note that $\delta_y \in H^{-\frac{n}{2}-1}(M)$ if δ_y is the unit point measure at $y \in M$. Therefore, $M \to C^\infty(M) : y \mapsto P\delta_y$ is well-defined. If $x \in M$ is fixed, we see that

$$D_y[P\delta_y(x)] = (PD^*\delta_y)(x),$$

which follows from $y \to \delta_y$ being in $C^m(M, H^{-\frac{n}{2}-1-m}(M))$ and the hypothesis on P. Hence $M \times M \to \mathbb{C} : (x, y) \mapsto P\delta_y(x)$ is in $C^\infty(M \times M)$, and we can write

$$\begin{aligned}\langle P\psi, \varphi\rangle &= \Big\langle \overline{P^*\overline{\varphi(y)}}, \psi(y)\Big\rangle \\ &= \Big\langle\Big\langle \delta_y(x), \overline{P^*\overline{\varphi(x)}}\Big\rangle, \psi(y)\Big\rangle \\ &= \Big\langle\Big\langle P\delta_y(x), \varphi(x)\Big\rangle, \psi(y)\Big\rangle = \langle P\delta_y(x), (\varphi\otimes\psi)(x,y)\rangle,\end{aligned}$$

which means that $K(P)(x, y) = P\delta_y(x)$, so $K(P)$ is smooth. □

3.1.1 Spectral Properties

When realized on the Sobolev spaces, the operators $\Psi(M)$ possess a rich spectral theory. The elliptic self-adjoint operators of strictly positive order are especially well-understood. Operators of this type have only countable point spectrum, accumulating only at infinity, and the eigenspaces are finite dimensional. See also Shubin [51].

Theorem 3.1.8. *Let $d, s \in \mathbb{R}$, and let $P \in \Psi^d(M)$ with formal L^2-adjoint $P^* \in \Psi^d(M)$. Then P realizes a bounded linear operator*

$$P|_{H^s(M)} : H^s(M) \to H^{s-d}(M).$$

If P is also elliptic, it also has the following properties:

1. $P|_{H^s(M)} \in F(H^s(M), H^{s-d}(M))$.
2. $\ker P|_{H^s(M)} = \ker P \subset C^\infty(M)$.
3. $\operatorname{ind} P|_{H^s(M)} = \dim\ker P - \dim\ker P^*$.

Proof. Observe that

$$(I-\Delta)^{\frac{s-d}{2}} P (I-\Delta)^{-\frac{s}{2}} \in \Psi^0(M).$$

Therefore, by L^2-boundedness, if $u \in H^s(M)$, we have

$$||Pu||_{H^{s-d}(M)} \leq \left|\left|\left[(I-\Delta)^{\frac{s-d}{2}} P (I-\Delta)^{-\frac{s}{2}}\right]\right|\right|_{B(L^2(M))} ||u||_{H^s(M)}.$$

Next, let P be elliptic with $Q \in \Psi^{-d}(M)$ so that $PQ - I$ and $QP - I$ are in $\Psi^{-\infty}(M)$. But these are bounded into $C^\infty(M)$, which is compactly embedded in any Sobolev space. Therefore $P|_{H^s(M)}$ is Fredholm, because Q can be realized as the operator

$$Q|_{H^{s-d}(M)} : H^{s-d}(M) \to H^s(M),$$

which is an almost inverse, since the residuals are compact by Theorems 3.1.7 and 3.1.6. To see that $\ker P|_{H^s(M)} \subset C^\infty(M)$, take $u \in H^s(M)$ with $Pu = 0$, and write

$$u = (I - QP)u \in C^\infty(M).$$

Finally, we compute the Hilbert adjoint operator of $P|_{H^s(M)}$ expressed via P^* explicitly. The expression will imply the formula for $\operatorname{ind} P|_{H^s(M)}$ in terms of the formal adjoint. Taking $u, v \in C^\infty(M)$, we have

$$\begin{aligned}
(Pu, v)_{H^{\frac{s-d}{2}}(M)} &= ((I-\Delta)^{\frac{s-d}{2}} Pu, (I-\Delta)^{\frac{s-d}{2}} v)_{L^2(M)} \\
&= ((I-\Delta)^{\frac{s}{2}} u, (I-\Delta)^{-\frac{s}{2}} P^* (I-\Delta)^{s-d} v)_{L^2(M)} \\
&= \left(u, \left[(I-\Delta)^{-s} P^* (I-\Delta)^{s-d}\right] v\right)_{H^s(M)},
\end{aligned}$$

and so, by density, the Hilbert adjoint is

$$(P|_{H^s(M)})^* = (I-\Delta)^{-s} P^* (I-\Delta)^{s-d}|_{H^s(M)},$$

which has kernel in $C^\infty(M)$ of the same dimension as $\ker(P^*)$. □

The notion of a spectrum makes sense for unbounded operators on a Hilbert space. Here we are mainly concerned with $H = L^2(M)$ as the model space.

Definition 3.1.5. *Let $T : \mathrm{dom}(T) \subset H \to H$ be an unbounded linear operator on H. Define for $\lambda \in \mathbb{C}$ the translation*

$$T_\lambda = T - \lambda I : \mathrm{dom}(T) \to H.$$

The spectrum $\sigma(T)$ are those $\lambda \in \mathbb{C}$ that fail to satisfy one of the following:

1. *T_λ is injective.*
2. *T_λ has dense range.*
3. *T_λ has bounded inverse.*

Our aim is now to understand the spectrum of unbounded realizations of $P \in \Psi(M)$. If $d > 0$ and $P \in \Psi^d(M)$, we realize P on $L^2(M)$ as

$$P : H^d(M) \subset L^2(M) \to L^2(M).$$

In fact, if P is also elliptic, this realization is closed, and is both minimal and maximal. To see this, take $u \in L^2(M)$ and a sequence $(u_k)_{k=1}^{\infty}$ in $H^d(M)$ such that

$$u_k \to u \quad \text{and} \quad Pu_k \to f \quad \text{in} \quad L^2(M) \quad \text{as} \quad k \to \infty,$$

and note that

$$Pu_k \to Pu \quad \text{and} \quad Pu_k \to f \quad \text{in} \quad \mathcal{D}'(M) \quad \text{as} \quad k \to \infty.$$

Therefore $Pu = f \in L^2(M)$. Because P is assumed elliptic, $u \in H^d(M)$. So it is closed. In this case, $P - \lambda I$ always has closed range if $\lambda \notin \sigma(P)$, hence is surjective.

Lemma 3.1.1. *If P is elliptic as above, then*

$$\ker(P^* - \overline{\lambda} I) = \ker(P - \lambda I) = \{0\} \quad \text{if and only if} \quad \lambda \notin \sigma(P).$$

Proof. Observe that $\lambda \notin \sigma(P)$ is equivalent to $P - \lambda I : H^d(M) \to L^2(M)$ being bijective, because $(P - \lambda I)^{-1} : L^2(M) \to L^2(M)$ is then bounded by the closed graph theorem. However, since $d > 0$, we also have

$$P - \lambda I \in F(H^d(M), L^2(M)) \quad \text{for all} \quad \lambda \in \mathbb{C},$$

and bijectivity is equivalent to the kernel and co-kernel being zero. □

Lemma 3.1.2. *Let $d \in \mathbb{R}$. If $A \in \Psi^d(M)$ is elliptic and invertible, then $A^{-1} \in \Psi^{-d}(M)$.*

Proof. Choose some parametrix $B \in \Psi^{-d}(M)$, and then write $A^{-1} = B - A^{-1}(AB - I)$. Now A^{-1} is continuous by the open mapping theorem (for Frechet spaces), and

$$A^{-1}(AB - I)|_{H^s(M)} \in B(H^s(M), C^\infty(M)) \quad \text{for any} \quad s \in \mathbb{R},$$

and so A^{-1} differs from B by a remainder in $\Psi^{-\infty}(M)$. □

Now we can show the main properties of elliptic self-adjoint operators of positive order. The existence of an L^2 ONB of eigenfunctions will be especially important later.

Theorem 3.1.9. *Let $d > 0$, and let $A \in \Psi^d(M)$ be both formally self-adjoint and elliptic. Then A realizes a (true) self-adjoint unbounded linear operator*

$$A : H^d(M) \subset L^2(M) \to L^2(M).$$

Furthermore, A has eigenfunctions $\{\phi_k\}_{k=0}^{\infty}$ with the following properties:

1. *The corresponding eigenvalues $\{\lambda\}_{k=0}^{\infty}$ are real, and $|\lambda_k| \to \infty$ as $k \to \infty$.*
2. *The system $\{\phi_k\}_{k=0}^{\infty} \subset C^\infty(M)$ is an ONB for $L^2(M)$.*
3. *The spectrum $\sigma(A)$ coincides with $\{\lambda_k\}_{k=0}^{\infty}$.*

Proof. Using that $A = A^*$, we have

$$||(A - \lambda I)u||_{L^2(M)}^2 \geq |\mathrm{Im}\,(\lambda)|^2 ||u||_{L^2(M)}^2 \quad \text{for all} \quad u \in C^\infty(M),$$

and by Lemma 3.1.1, $\sigma(A) \subset \mathbb{R}$, because the above implies

$$\ker(A - \lambda I) = \ker(A^* - \overline{\lambda} I) = \{0\} \quad \text{if} \quad \mathrm{Im}\,(\lambda) \neq 0.$$

Similarly, we have that

$$\ker(A - \lambda I) \neq \{0\} \quad \text{if} \quad \lambda \in \sigma(A) \subset \mathbb{R}.$$

But $\sigma(A) \neq \mathbb{R}$. Otherwise, $\{\ker(A - \lambda I)\}_{\lambda \in \mathbb{R}}$ are non-zero mutually orthogonal spaces, which would imply that $L^2(M)$ admits an uncountable set of mutually orthogonal vectors. However, this is impossible, because $L^2(M)$ is separable. So there exists a $\lambda_0 \in \mathbb{R} \setminus \sigma(A)$. In that case, since $A - \lambda_0 I \in \Psi^d(M)$ is elliptic, it maps $C^\infty(M)$ bijectively onto itself. Then Lemma 3.1.2 implies that

$$(A - \lambda_0 I)^{-1} \in \Psi^{-d}(M) \quad \text{and} \quad ((A - \lambda_0 I)^{-1})^* = (A - \lambda_0 I)^{-1},$$

which extends by continuity to a compact self-adjoint operator on $L^2(M)$ since $d > 0$. Now we can apply the spectral theorem to

$$R_{\lambda_0} = (A - \lambda_0 I)^{-1}|_{L^2(M)} : L^2(M) \to L^2(M),$$

and obtain an ONB for $L^2(M)$ of eigenfunctions $\{\phi_k\}_{k=0}^{\infty}$ with eigenvalues $\{\mu_k\}_{k=0}^{\infty} \subset \mathbb{R}$. There is no zero eigenvalue, because $\ker(A - \lambda_0 I) = \{0\}$, but they tend to 0 as $k \to \infty$. Note that $\phi_k \in C^\infty(M)$, because A is elliptic of order $d > 0$, and

$$A\phi_k = (\lambda_0 + \mu_k^{-1})\phi_k,$$

which also means that ϕ_k is an eigenfunction of A with the eigenvalue $\lambda_k = \lambda_0 + \mu_k^{-1}$. But then $|\lambda_k| \to \infty$ as $k \to \infty$, and we have $\{\lambda_k\}_{k=0}^{\infty} = \sigma(A)$ since $\{\phi_k\}_{k=0}^{\infty}$ is complete. The unbounded realization is self-adjoint, because

$$A = R_{\lambda_0}^{-1} + \lambda_0 I.$$

□

Without the self-adjointness condition, the situation becomes a bit more complicated. But it is still well-understood.

Corollary 3.1.1. *If A is not self-adjoint, then either $\sigma(A) = \mathbb{C}$ or $\sigma(A)$ is discrete.*

Proof. Suppose that $\sigma(A) \neq \mathbb{C}$. Pick $\lambda_0 \in \mathbb{C} \setminus \sigma(A)$, and write

$$A - \lambda_0 I : H^d(M) \subset L^2(M) \to L^2(M),$$

which is linear and unbounded on $L^2(M)$, but has compact inverse defined on all $L^2(M)$. Then, if $\lambda \in \mathbb{C} \setminus \{\lambda_0\}$, we can write

$$A - \lambda I = -(\lambda - \lambda_0)\Big[(A - \lambda_0 I)^{-1} - (\lambda - \lambda_0)^{-1} I\Big](A - \lambda_0 I),$$

and this implies

$$\lambda \in \sigma(A) \quad \text{if and only if} \quad \lambda \neq \lambda_0 \quad \text{and} \quad (\lambda - \lambda_0)^{-1} \in \sigma((A - \lambda_0 I)^{-1}).$$

But $\sigma((A - \lambda_0 I)^{-1})$ has only 0 as an accumulation point, so $\sigma(A)$ has none. □

The eigenvalues of certain $P \in \mathrm{Diff}^d(M)$ with $d \in \mathbb{N}$ have well-known asymptotics. Associated to such P is a unique $p_d \in \sigma_d([P])$, polynomial in each $\xi \in T_x^*M$ of degree d. It coincides with the classical principal symbol. Let P be formally self-adjoint, and

$$p_d(x, \xi) > 0 \quad \text{if} \quad (x, \xi) \in T^*M \setminus 0.$$

In this case, P is elliptic by Theorem 3.1.3, and turns out to be semi-bounded from below. This, in turn, means that the eigenvalues are bounded from below.

Theorem 3.1.10 (Shubin [51]). *If P and p_d are as above, there is $C > 0$ such that*

$$(Pu, u) \geq -C(u, u) \quad \text{for all} \quad u \in C^\infty(M).$$

Denote by $N_P(\lambda)$ the number of eigenvalues, counted with multiplicity, below $\lambda \in \mathbb{R}$. It turns out that $N_P(\lambda)$ grows asymptotically like $\lambda^{\frac{n}{d}}$, to first order determined by p_d. The formulations below can be found in Shubin [51].

Theorem 3.1.11 (Shubin [51]). *If P and p_d are as above, then*

$$N_P(\lambda) - \frac{1}{(2\pi)^n}\Big[\int_M \Big[\int_{S_x^*M} \frac{p_d(x,\xi)^{-\frac{n}{d}}}{n} \mathrm{vol}_{S_x^*M}(\xi)\Big]\,\omega_0(x)\Big]\lambda^{\frac{n}{d}} = O_{\lambda\to\infty}(\langle\lambda\rangle^{\frac{n-1}{d}})$$

This result is also known as Weyl's (global) asymptotic law for the eigenvalues of P. It will be useful later, when studying operators $\Psi(G)$, where G is a compact Lie group. The dimensions of the irreducible representations of G are controlled by it.

Corollary 3.1.2 (Shubin [51]).

$$N_\Delta(\lambda) - \frac{2}{n!}\lambda^{\frac{n}{2}} = O_{\lambda\to\infty}(\langle\lambda\rangle^{\frac{n-1}{2}})$$

Later, the Poisson transform will draw upon two basic ideas from functional calculus. It will be necessary to take powers and exponentials of elliptic $A \in \Psi^d_{\mathrm{phg}}(M)$ with $d > 0$. The technique requires that $\sigma(A) \neq \mathbb{C}$ is contained inside a sector in $\mathbb{C}$ with vertex at 0. By a closed sector Λ in $\mathbb{C}$, we mean a set of the form

$$\Lambda = \cup_{\theta \in [\theta_1, \theta_2]} e^{i\theta}\mathbb{R} \quad \text{for some} \quad \theta_1, \theta_2 \in [0, 2\pi] \quad \text{with} \quad \theta_1 \leq \theta_2,$$

and for $\epsilon > 0$ we introduce the associated set

$$\Lambda_\epsilon = \Lambda \cap \{\lambda \in \mathbb{C} \,|\, |\lambda| \geq \epsilon\},$$

which is Λ with the open ϵ-disc at 0 removed.

Let a be the classical principal symbol of A, non-zero on $T^*M \setminus 0$ since A is elliptic. The spectral requirement of A is implied by the following condition on a:

Definition 3.1.6. *The operator A is said to be parameter-elliptic w.r.t. Λ if*

$$a(x, \xi) - \lambda \neq 0 \quad \text{for all} \quad (x, \xi) \in T^*M \setminus 0 \quad \text{when} \quad \lambda \in \Lambda.$$

Theorem 3.1.12 (Shubin [51]). *Suppose that the above A is parameter-elliptic w.r.t. Λ. Then the following holds:*

1. *There is an $\epsilon > 0$ such that $\sigma(A) \subset \mathbb{C} \setminus \Lambda_\epsilon$, and*

$$(A - \lambda I)^{-1} \in \Psi^{-d}_{\mathrm{phg}}(M) \quad \text{for all} \quad \lambda \in \Lambda_\epsilon.$$

2. *Given any $s \in \mathbb{R}$ and $l \in [0, d]$, we have the resolvent estimate*

$$\sup_{\lambda \in \Lambda_\epsilon} |\lambda|^{1-\frac{l}{d}} ||(A - \lambda I)^{-1}||_{B(H^s(M), H^{s+l}(M))} < \infty.$$

These estimates allow us to create new operators out of A via a functional calculus. It requires functions holomorphic on a certain domain in the resolvent set.

Theorem 3.1.13. *Let $A \in \Psi^d_{\mathrm{phg}}(M)$, the sector Λ and $\epsilon > 0$ be as in Theorem 3.1.12. Suppose that Λ' is a closed sector in $\mathbb{C}$ with $R > 0$ such that the following holds:*

1. *Λ'_R contains $\sigma(A) \setminus \{0\}$ in its interior.*
2. *Λ_ϵ contains $\Gamma = \partial\Lambda'_R$ except for some disc about the origin.*

Let $f : \Lambda'_R \to \mathbb{C}$ be holomorphic, continuous up to $\overline{\Lambda'_R}$, and

$$\sup_{\lambda \in \Lambda'_R} |\lambda|^{-r} |f(\lambda)| < \infty \quad \text{for some} \quad r < 0.$$

Then "the function f of A", defined below, is a well-defined continuous linear operator. Converging in $C^\infty(M)$, it is given by the Riemann integral

$$f(A) : C^\infty(M) \to C^\infty(M) : u \mapsto \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I - A)^{-1} u \, d\lambda,$$

where Γ is viewed as a counter-clockwise contour about $\sigma(A) \setminus \{0\}$.

Proof. Using Theorem 3.1.12, if $k \in \mathbb{N}$ we get some $C_k > 0$ such that

$$\int_\Gamma |f(\lambda)|\, ||(\lambda I - A)^{-1}u||_{H^k(M)}\, |d\lambda| \leq C_k\Big(\int_\Gamma \frac{1}{|\lambda|^{1-r}}\, |d\lambda|\Big)||u||_{H^k(M)},$$

and so the Riemann integral converges absolutely in the norm of $H^k(M)$ for any $k \in \mathbb{N}$. By the Sobolev embedding theorem, it converges in $C^\infty(M)$, and $f(A)$ is continuous. □

Even if f satisfies the required estimate with $r \geq 0$ growth, $f(A)$ can still be defined. Let $f_{-k}(\lambda) = f(\lambda)\lambda^{-k}$ for all $\lambda \in \Lambda'_R$, with $k \in \mathbb{N}$ chosen (arbitrarily) so that $r < k$. Then f_{-k} is decaying of order $r - k$, and we can define $f(A)$ by

$$f(A) = f_{-k}(A)A^k.$$

To see that it makes sense, observe that

$$\begin{aligned} f_{-k}(A)Au &= \frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda)}{\lambda^k}(\lambda I - A)^{-1}Au\, d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda)}{\lambda^{k-1}}(\lambda I - A)^{-1}u - \frac{f(\lambda)}{\lambda^k}Iu\, d\lambda = f_{-k+1}(A)u, \end{aligned}$$

where the last integral term vanishes, because the integrand is holomorphic inside Λ'_R. Therefore, if $k > k' \in \mathbb{N}$, we have

$$f_{-k}(A)A^k = f_{-k}(A)A^{k-k'}A^{k'} = f_{-k+k-k'}A^{k'} = f_{-k'}(A)A^{k'},$$

which shows that it is independent of the choice of k.

In case $A \in \Psi^d_{\text{phg}}(M)$ is also formally self-adjoint, $f(A)$ can be expressed in a basis. Let $\sigma(A) = \{\lambda_k\}_{k=0}^\infty \subset \mathbb{R}$ have the corresponding ON eigenbasis $\{\phi_k\}_{k=0}^\infty$ for $L^2(M)$.

Corollary 3.1.3. *If A is also formally self-adjoint, then*

$$f(A)u = \sum_{k=0}^\infty f(\lambda_k)(u, \phi_k)\phi_k,$$

where the sum converges in the topology of $C^\infty(M)$.

Proof. Using the continuity on $C^\infty(M)$, we compute

$$\begin{aligned} f(A)u &= \frac{1}{2\pi i}\int_\Gamma f(\lambda)\lambda^{-k}(\lambda I - A)^{-1}A^k\Big[\sum_{k=0}^\infty (u, \phi_k)\phi_k\Big]\, d\lambda \\ &= \sum_{k=0}^\infty (u, \phi_k)\Big[\frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda)}{\lambda - \lambda_k}\, d\lambda\Big]\phi_k \\ &= \sum_{k=0}^\infty f(\lambda_k)(u, \phi_k)\phi_k. \end{aligned}$$

□

As an example, let $A \in \Psi^d_{\mathrm{phg}}(M)$ be formally self-adjoint with no negative eigenvalues. Let A be parameter-elliptic w.r.t. a sector with angles $[\pi - \theta_0, \pi + \theta_0]$ for some $\theta_0 \in (0, \pi)$. In that case, f can be a power of any $z \in \mathbb{C}$, with logarithmic branch cut along $(-\infty, 0]$, and the contour of integration $\Gamma = \partial \Lambda'_R$ a keyhole opening out along the branch cut, where the hole is so small that it encircles no other eigenvalue of A than possibly $0 \in \mathbb{C}$. That is, take $R > 0$ so small that

$$\Lambda'_R = \cup_{\theta \in [\pi - \theta_0, \pi + \theta_0]} e^{i\theta}[R, \infty) \supset \sigma(A) \setminus \{0\}.$$

Then, for any $k \in \mathbb{N}_0$ with $\mathrm{Re}\,(z) - k < 0$, we can unambiguously define

$$f_z(A) : C^\infty(M) \to C^\infty(M) : u \mapsto \frac{1}{2\pi i} \int_\Gamma \lambda^{z-k}(\lambda I - A)^{-1} A^k u \, d\lambda.$$

A straightforward computation shows that $\{f_z(A)\}_{z \in \mathbb{C}}$ is a group under composition. Take $w \in \mathbb{C}$ and $k' \in \mathbb{N}_0$ with $\mathrm{Re}\,(w) - k' < 0$ and form $f_w(A)$ with another contour Γ'. Using the FTT and Cauchy integral theorem, we calculate

$$\begin{aligned} f_z(A) f_w(A) u &= \frac{1}{(2\pi i)^2} \int_\Gamma \lambda^{z-k}(\lambda I - A)^{-1} A^k \Big[\int_{\Gamma'} \mu^{w-k'}(\mu I - A)^{-1} A^{k'} u \, d\mu \Big] \, d\lambda \\ &= \frac{1}{(2\pi i)^2} \int_\Gamma \int_{\Gamma'} \lambda^{z-k} \mu^{w-k'} \Big(\frac{(\lambda I - A)^{-1} - (\mu I - A)^{-1}}{\mu - \lambda} \Big) A^{k+k'} u \, d\mu \, d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma'} \Big[\frac{1}{2\pi i} \int_\Gamma \frac{\lambda^{z-k}}{\lambda - \mu} \, d\lambda \Big] \mu^{w-k'} (\mu I - A)^{-1} A^{k+k'} u \, d\mu \\ &= \frac{1}{2\pi i} \int_{\Gamma'} \mu^{w+z-k'-k} (\mu I - A)^{-1} A^{k+k'} u \, d\mu, \end{aligned}$$

where Γ' is a slightly bigger keyhole contour about Γ, and we have used that

$$\frac{1}{(2\pi i)^2} \int_\Gamma \lambda^{z-k} \Big[\int_{\Gamma'} \mu^{w-k'} \frac{1}{\mu - \lambda} \, d\mu \Big] (\lambda I - A)^{-1} A^{k+k'} u \, d\lambda = 0.$$

This shows that

$$f_z(A) f_w(A) u = f_{w+z}(A) u,$$

and provided that z is not zero or a negative integer, we have

$$\begin{aligned} f_z(A) u &= \sum_{k=0}^\infty (u, \phi_k) \Big[\frac{1}{2\pi i} \int_\Gamma \frac{\lambda^{z-k}}{\lambda - \lambda_k} \, d\lambda \Big] \lambda^k \phi_k \\ &= \sum_{k=0}^\infty \lambda_k^z (u, \phi_k) \phi_k. \end{aligned}$$

However, if A is invertible, we also have

$$f_z(A) = A^z \quad \text{if} \quad z \in \mathbb{Z}.$$

It is therefore customary to write A^z for $f_z(A)$ for all $z \in \mathbb{C}$, even if A is not invertible.

Many operators can be made out of A in this manner from particular choices of f. Later, we will need some built out of A^z with $\operatorname{Re}(z) > 0$, where A may not be invertible, but is parameter elliptic w.r.t. to a sector with angles $[\theta_0, 2\pi - \theta_0]$ for some $\theta_0 \in (0, \frac{1}{2}\pi)$, and is of course still assumed to be formally self-adjoint with no negative eigenvalues. Namely, the family $\{e^{-tA^z}\}_{t\in(0,\infty)}$ of continuous linear operators

$$e^{-tA^z} : C^\infty(M) \to C^\infty(M) : u \mapsto e^{-tA^z} u,$$

which are defined explicitly by

$$e^{-tA^z} u = \frac{1}{2\pi i} \int_{\partial\Lambda'_R} e^{-t\lambda^z} (\lambda I - A)^{-1} u \, d\lambda + \frac{1}{2\pi i} \int_{R\mathbb{S}^1} (\lambda I - A)^{-1} u \, d\lambda,$$

where $R\mathbb{S}^1$ is oriented counter-clockwise, with $R > 0$ chosen so that

$$\Lambda'_R = \cup_{\theta\in[-\theta_0,\theta_0]} e^{i\theta}[R, \infty) \supset \sigma(A) \setminus \{0\},$$

and $\partial\Lambda'_R$ is oriented counter-clockwise relative to $\sigma(A) \setminus \{0\}$, lying in the right half-plane. This last term is necessary in order to have

$$e^{-tA^z} u = \sum_{k=0}^{\infty} (u, \phi_k) \frac{1}{2\pi i} \Big[\int_{\partial\Lambda_R} \frac{e^{-t\lambda^z}}{\lambda - \lambda_k} \, d\lambda + \int_{R\mathbb{S}^1} \frac{1}{\lambda - \lambda_k} \, d\lambda \Big] \phi_k = \sum_{k=0}^{\infty} e^{-t\lambda_k^z} (u, \phi_k) \phi_k,$$

which implies that the limit $t \to 0+$ exists in $C^\infty(M)$ and equals u.

It turns out that the operators A^z are actually always pseudo-differential operators. Their symbols can also be calculated explicitly in terms of a. See Shubin [51].

Theorem 3.1.14 (Shubin [51]. Some statements are contained in the exercises therein). *Let $A \in \Psi^d_{\mathrm{phg}}(M)$ with $d > 0$ be, as above, an elliptic and formally self-adjoint operator. Additionally, let a be its classical principal symbol, and assume that*

$$\sigma(A) \subset [0, \infty).$$

Let $\theta_0 \in (0, \pi)$ be fixed. The following holds:

1. *If A is parameter-elliptic w.r.t. $\cup_{\theta\in[\pi-\theta_0,\pi+\theta_0]} e^{i\theta}[0, \infty)$, then*

$$A^z \in \Psi^{\operatorname{Re}(z)d}_{\mathrm{phg}}(M) \quad \text{for any} \quad z \in \mathbb{C}.$$

2. *$(\mathbb{C}, +) \to \Psi(M) : z \mapsto A^z$ is a group $*$-homomorphism ($A^{\overline{z}} = (A^z)^*$).*
3. *The classical principal symbol of A^z is precisely a^z.*

Let $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$ and $\theta_0 \in (0, \frac{1}{2}\pi)$ be fixed. The following also holds:

1. *If A is parameter-elliptic w.r.t. $\cup_{\theta\in[\theta_0,2\pi-\theta_0]} e^{i\theta}[0, \infty)$, then*

$$e^{-tA^z} \in \Psi^{-\infty}(M) \quad \text{for any} \quad t \in (0, \infty).$$

2. *$([0, \infty), +) \to \Psi(M) : t \mapsto e^{-tA^z}$ is a semi-group homomorphism.*

The one-parameter group in Theorem 3.1.5 can be taken to be the powers of $I - \Delta$. This may seem circular, but it can be reached by another definition of the Sobolev spaces.

3.1.2 Beal's Theorem

This subsection is dedicated to Beal's theorem, the commutator characterisation of $\Psi(M)$. It characterizes $\Psi(M)$ fully in terms of boundedness of commutators on Sobolev spaces, and allows us to determine if an operator belongs to $\Psi(M)$ without local computations. This theorem can be found, for example, in the book by Ruzhansky and Turunen [48], and is useful for identifying pseudo-differential operators on homogeneous spaces.

Lemma 3.1.3. *Let $A : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$ be any continuous and linear operator. Suppose that $A = \psi A \phi$ for some $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$. Put $e_\xi : \mathbb{R}^n \to \mathbb{C} : x \mapsto e^{ix\cdot\xi}$ for $\xi \in \mathbb{R}^n$. Then $A = \mathrm{Op}(a)$ for a unique $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with the following properties:*

1. *$a(x,\xi) = (e_{-\xi} A e_\xi)(x)$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$.*
2. *There is some $k \in \mathbb{N}_0$ such that*

$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n} \langle\xi\rangle^{-k} |a(x,\xi)| < \infty.$$

Proof. Continuity of A gives some $k \in \mathbb{N}_0$ and $C, C' > 0$ such that

$$\sup_{x\in\mathbb{R}^n} |(e_{-\xi} A e_\xi)(x)| \leq C \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} |\psi(x)\partial_x^\alpha(\phi(x)e^{ix\cdot\xi})| \leq C'\langle\xi\rangle^k.$$

If $u \in C_0^\infty(\mathbb{R}^n)$, then the Riemann sums of $\phi u = \int_{\mathbb{R}^n} \phi e_\xi \mathcal{F}u(\xi)\, đ\xi$ converge in $C_0^\infty(\mathbb{R}^n)$, and we can use continuity to take A through the integral to get the representation. $\square$

Theorem 3.1.15. *Let $d \in \mathbb{R}$, and let $P : C^\infty(M) \to C^\infty(M)$ be continuous and linear. The following are equivalent:*

1. *$P \in \Psi^d(M)$.*
2. *If $s \in \mathbb{R}$ and $\{D_j\}_{j=0}^\infty \subset \mathrm{Diff}^1(M)$ are arbitrary, then*

$$\begin{cases} P_0 = P \in B(H^s(M), H^{s-d}(M)) \\ P_k = [P_{k-1}, D_{k-1}] \in B(H^s(M), H^{s-d+\sum_{j=0}^{k-1}(1-\deg(D_j))}(M)) \quad \text{for each} \quad k \in \mathbb{N}. \end{cases}$$

Proof. By the symbolic calculus and mapping properties of $\Psi(M)$, we obtain (1) $\Rightarrow$ (2). The converse is the interesting part. Note that if $\phi, \psi \in C^\infty(M)$, we have

$$[\psi P \phi, D] = \psi[P, D]\phi + \psi P[\phi, D] + [\psi, D]P\phi \quad \text{if} \quad D \in \mathrm{Diff}^1(M).$$

In particular, if ϕ and ψ have disjoint supports, then

$$\psi P \phi = \psi[P, \chi]\phi = \psi[[P, \chi], \chi]\phi = \cdots,$$

where $\chi \in C^\infty(M)$ equals 1 on a neighbourhood of $\mathrm{supp}(\phi)$ but $\mathrm{supp}(\chi) \cap \mathrm{supp}(\psi) = \emptyset$. Thus, if (2) holds, we have that $\psi P \phi$ is bounded from any Sobolev space into $C^\infty(M)$, and so, in this case, $\psi P \phi$ has smooth kernel by the characterization in Theorem 3.1.7. Given any chart $\kappa : U \to \mathbb{R}^n$ of M, and $\phi, \psi \in C_0^\infty(U)$, it remains to show that

$$A = (\kappa^{-1})^*(\phi P \psi)\kappa^* = \mathrm{Op}(a) \quad \text{for some} \quad a \in S^d(\kappa(U) \times \mathbb{R}^n).$$

To do this, we need to show that a of A from Lemma 3.1.3 belongs to $S^d(\kappa(U)\times\mathbb{R}^n)$. By the symbol $a\in C^\infty(\kappa(U)\times\mathbb{R}^n)$, we mean the function

$$a(x,\xi)=e^{-ix\cdot\xi}A_x(e^{ix\cdot\xi})\quad\text{for all}\quad(x,\xi)\in\kappa(U)\times\mathbb{R}^n.$$

If $\{C_j\}_{j=0}^\infty\subset\text{Diff}^1(\mathbb{R}^n)$ is arbitrary, put $D_j=\chi\kappa^*C_j(\kappa^{-1})^*$, and

$$\begin{cases}A_0=A,\\ A_k=[A_{k-1},C_{k-1}]\quad\text{for each}\quad k\in\mathbb{N},\end{cases}$$

where $\chi\in C_0^\infty(U)$ is now a cutoff equal to 1 on a neighbourhood of $\text{supp}(\phi)\cup\text{supp}(\psi)$. Note that the hypothesis (2) holds even if we start with $P_0=\phi P\psi$, regardless of ϕ and ψ. This can be seen by using (2) and applying induction to the above commutator identity. Then, with $P_0=\psi P\phi$, we have $\kappa^*A_k(\kappa^{-1})^*=P_k$, because

$$\kappa^*A_k(\kappa^{-1})^*=[\kappa^*A_{k-1}(\kappa^{-1})^*,D_{k-1}],$$

and so A_k is bounded between Sobolev spaces on $\mathbb{R}^n$ of the same order as P_k is on M. Using the above, if $\mathbb{N}\ni m>\frac{n}{2}$ and $\alpha,\beta\in\mathbb{N}_0^n$, we get a $C>0$ such that

$$\begin{aligned}|\partial_x^\beta\partial_\xi^\alpha a(x,\xi)|^2&\le\Big[\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}e^{-ix\cdot\eta}\partial_x^\beta\partial_\xi^\alpha a(x,\xi)\,dx\Big|\,d\eta\Big]^2\\&\le\Big(\int_{\mathbb{R}^n}\langle\eta\rangle^{-2m}\,d\eta\Big)\int_{\mathbb{R}^n}\langle\eta\rangle^{2m}\Big|\int_{\mathbb{R}^n}e^{-ix\cdot\eta}\partial_x^\beta\partial_\xi^\alpha a(x,\xi)\,dx\Big|^2\,d\eta\\&\le C\sum_{|\gamma|\le m}\int_{\mathbb{R}^n}|\partial_x^{\beta+\gamma}\partial_\xi^\alpha a(x,\xi)|^2\,dx\\&=C\sum_{|\gamma|\le m}\int_{\mathbb{R}^n}|(\text{ad}_{\partial_x}^{\beta+\gamma}\text{ad}_{-x}^\alpha A)(\chi e_\xi)(x)|^2\,d\eta\\&\le C\Big[\sum_{|\gamma|\le m}||\text{ad}_{\partial_x}^{\beta+\gamma}\text{ad}_{-x}^\alpha A||^2_{B(H^{d-|\alpha|}(\mathbb{R}^n),L^2(\mathbb{R}^n))}\Big]||\chi e_\xi||^2_{H^{d-|\alpha|}(\mathbb{R}^n)},\end{aligned}$$

where $e_\xi(x)=e^{ix\cdot\xi}$, the bracketed term is finite, and

$$\begin{aligned}||\chi e_\xi||^2_{H^{d-|\alpha|}(\mathbb{R}^n)}&=\int_{\mathbb{R}^n}\langle\eta\rangle^{2(d-|\alpha|)}|\mathcal{F}(\chi e_\xi)(\eta)|^2\,d\eta\\&=\int_{\mathbb{R}^n}\langle\eta+\xi\rangle^{2(d-|\alpha|)}|\mathcal{F}\chi(\eta)|^2\,d\eta\\&\le 2^{|d-|\alpha||}\Big[\int_{\mathbb{R}^n}\langle\eta\rangle^{2|d-|\alpha||}|\mathcal{F}\chi(\eta)|^2\,d\eta\Big]\langle\xi\rangle^{2(d-|\alpha|)}.\end{aligned}$$

Note that we here have

$$\langle\eta+\xi\rangle^{2(d-|\alpha|)}\le 2^{|d-|\alpha||}\langle\xi\rangle^{2(d-|\alpha|)}\langle\eta\rangle^{2|d-|\alpha||},$$

and because $\mathcal{F}\chi$ is a Schwartz function, the last bracketed term above is certainly finite. Thus (2) $\Rightarrow$ (1) as well. □

3.2 $\Psi(G)$ for a Compact Lie Group G

To start everything off, we need some tools from harmonic analysis on topological groups. Only the essential tools needed to study $\Psi(G)$ when G is a Lie group are presented here. See Folland [13], Faraut [12] or Ruzhansky and Turunen [48] for more details.

Let G be a compact Hausdorff topological group with neutral/identity element $e \in G$. Then G and its closed subgroups each have a unique bi-invariant probability measure, which is a Radon measure on the Borel sigma algebra of the group; the Haar measure. On G we denote the Haar measure by dx, and on a closed subgroup $H \subset G$ we write dh. Also, we always give the coset (homogeneous) space G/H the natural quotient topology. The usual Lebesgue spaces $L^s(G)$ are defined for $s \geq 1$ with respect to this Haar measure, and $L^1(G)$ is given the usual Banach $*$-algebra structure.

Theorem 3.2.1 (Folland [13])**.** *The following holds:*

1. *Any irreducible unitary representation of G must be finite-dimensional.*
2. *Any unitary representation of G is a direct sum of irreducible ones.*

The set of all unitary equivalence classes of irreducible unitary representations ξ is $\widehat{G}$. It is customary to let $d_\xi = \dim(\xi) < \infty$ denote the dimension of ξ, and write

$$\xi_{ij} : G \to \mathbb{C} : x \mapsto (\xi(x)e_j, e_i)_{L^2(G)},$$

where $\{e_j\}_{j=1}^{d_\xi}$ is some arbitrary fixed orthonormal basis of the representation space of ξ. These belong to $C(G) \subset L^2(G)$ due to continuity of ξ in the strong operator topology.

The elements of $\widehat{G}$ are the building blocks for most global analysis on a Lie group. Basically, this is because they lead to a sort of "Fourier expansion" of $L^2(G)$ functions, which generalizes the elementary Fourier series of periodic functions on $\mathbb{R}$.

Theorem 3.2.2 (Peter-Weyl and Schur orthogonality theorems. See either [13] or [12])**.** *Given any irreducible unitary representation ξ of G, define*

$$\mathcal{E}_\xi = \operatorname{span}\{\sqrt{d_\xi}\,\xi_{ij}\}_{i,j=1}^{d_\xi},$$

where ξ_{ij} are the matrix elements relative to an ONB of the representation space of ξ. Then the following holds:

1. *$\mathcal{E}_\xi$ depends only on the class $[\xi]$.*
2. *If $[\xi] \neq [\eta]$ are elements of $\widehat{G}$, then $\mathcal{E}_\xi$ and $\mathcal{E}_\eta$ are orthogonal in $L^2(G)$.*
3. *$\{\sqrt{d_\xi}\,\xi_{ij}\}_{i,j=1}^{d_\xi}$ is an orthonormal basis for $\mathcal{E}_\xi$ in $L^2(G)$ for any $[\xi] \in \widehat{G}$.*

The set $\operatorname{span} \cup_{[\xi]\in\widehat{G}} \mathcal{E}_\xi$ *is a dense $*$-subalgebra of $C(G)$, and*

$$L^2(G) = \bigoplus_{[\xi]\in\widehat{G}} \operatorname{span}\{\sqrt{d_\xi}\,\xi_{ij}\}_{i,j=1}^{d_\xi}.$$

A slightly more general version is provided by Folland [13], with two different proofs. Let us collect the immediate consequences:

Corollary 3.2.1 (Folland [13]). *The following are equivalent:*

1. *G is a Lie group.*
2. *G has a faithful finite-dimensional representation.*

The decomposition of $L^2(G)$ can be rephrased in terms of the Peter-Weyl expansion. Given $f \in L^1(G)$, we shall (with some abuse) write

$$\mathcal{F}_G f(\xi) = \int_G f(x)\xi(x)^* \, dx \quad \text{for all} \quad [\xi] \in \widehat{G}.$$

Corollary 3.2.2. *If $f \in L^2(G)$, then*

$$f = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi \mathcal{F}_G f(\xi)] \quad \text{in} \quad L^2(G).$$

In particular, we obtain a Parseval identity from the orthonormality of $\{\sqrt{d_\xi}\xi_{ij}\}_{i,j=1}^{d_\xi}$. Given $f, g \in L^2(G)$, we compute directly

$$\begin{aligned}(f,g)_{L^2(G)} &= \sum_{[\xi]\in\widehat{G}} \sum_{[\eta]\in\widehat{G}} d_\xi d_\eta \int_G \mathrm{Tr}(\xi(x)\mathcal{F}_G f(\xi))\overline{\mathrm{Tr}(\eta(x)\mathcal{F}_G g(\xi))}\, dx \\ &= \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\mathcal{F}_G g(\xi)^* \mathcal{F}_G f(\xi)),\end{aligned}$$

where the Hilbert-Schmidt inner product appears in the last expression.

This suggests that we define Lebesgue spaces $L^s(\widehat{G})$ of "L^s matrix sequences" on $\widehat{G}$, where the individual matrices are measured using the Hilbert-Schmidt (the trace) norm. Write $\mathrm{Mat}(\widehat{G})$ for those functions $a : \widehat{G} \to \cup_{[\xi]\in\widehat{G}} \mathrm{Mat}(d_\xi, \mathbb{C})$ such that $a([\xi]) \in \mathrm{Mat}(d_\xi, \mathbb{C})$, and write $a(\xi)$. We assume always that a representative ξ has been fixed in each class $[\xi]$. The appropriate Banach spaces (with the obvious norms) are

$$\begin{aligned} L^s(\widehat{G}) &= \Big\{ a \in \mathrm{Mat}(\widehat{G}) \,\Big|\, \Big(\sum_{[\xi]\in\widehat{G}} d_\xi^{2-\frac{s}{2}} \mathrm{Tr}(a(\xi)^* a(\xi))^{\frac{s}{2}} \Big)^{\frac{1}{s}} < \infty \Big\}, \\ L^\infty(\widehat{G}) &= \Big\{ a \in \mathrm{Mat}(\widehat{G}) \,\Big|\, \sup_{[\xi]\in\widehat{G}} d_\xi^{-\frac{1}{2}} \sqrt{\mathrm{Tr}(a(\xi)^* a(\xi))} < \infty \Big\}, \end{aligned}$$

which for $s = 2$ is equipped with

$$(a,b)_{L^2(\widehat{G})} = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(b(\xi)^* a(\xi)) \quad \text{for any} \quad a, b \in L^2(\widehat{G}),$$

and this also makes $L^2(\widehat{G})$ into a Hilbert space.

Note that while $L^2(G) \subset L^1(G)$, the same is not true in general for the spaces on $\widehat{G}$. Given $a \in L^2(\widehat{G})$, we shall (meaningfully, by orthonormality) write

$$\mathcal{F}_G^{-1} a = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi a(\xi)] \quad \text{in} \quad L^2(G),$$

where $\mathcal{F}_G^{-1} a$ is also well-defined uniformly convergent for $a \in L^1(\widehat{G})$, hence in $C(G)$ then. To see this, take $a \in L^1(\widehat{G})$ and write

$$\begin{aligned} ||\mathcal{F}_G^{-1} a||_{C(G)} &\leq \sum_{[\xi]\in\widehat{G}} d_\xi |\mathrm{Tr}[\xi a(\xi)]| \\ &= \sum_{[\xi]\in\widehat{G}} d_\xi^{\frac{3}{2}} \sqrt{\mathrm{Tr}(a(\xi)^* a(\xi))} = ||a||_{L^1(\widehat{G})}, \end{aligned}$$

where we use that $d_\xi = \mathrm{Tr}(\xi^*\xi)$ and

$$|\mathrm{Tr}[\xi a(\xi)]|^2 \leq \mathrm{Tr}(\xi^*\xi)\mathrm{Tr}(a(\xi)^* a(\xi)),$$

and it is then automatic that $\mathcal{F}_G^{-1} : L^1(\widehat{G}) \to C(G)$ is well-defined and also continuous. Sometimes we will drop the subscript G from $\mathcal{F}_G$ and $\mathcal{F}_G^{-1}$.

Theorem 3.2.3. *The transforms $\mathcal{F}_G$ and $\mathcal{F}_G^{-1}$ are unitary and inverses on the L^2-spaces.*

$$\mathcal{F}_G : L^2(G) \to L^2(\widehat{G}) \quad \text{and} \quad \mathcal{F}_G^{-1} : L^2(\widehat{G}) \to L^2(G).$$

Furthermore, given any $f, g \in L^1(G)$, the following holds:

1. $\mathcal{F}_G(f * g) = \mathcal{F}_G g \mathcal{F}_G f$.
2. $\mathcal{F}_G(f^*) = (\mathcal{F}_G f)^*$.

Proof. The above Parseval identity and Theorem 3.2.2 combine to the first statement, which is purely a consequence of the way that $L^2(\widehat{G})$ and the transform $\mathcal{F}_G$ are defined. Using the FTT and left invariance, we compute

$$\begin{aligned} \mathcal{F}_G(f * g) &= \int_G \Big[\int_G f(y) g(y^{-1}x)\, dy\Big] \eta(x)^*\, dx \\ &= \int_G \int_G f(y) g(y^{-1}x) \eta(x)^*\, dx\, dy \\ &= \int_G \int_G g(x) f(y) \eta(yx)^*\, dx\, dy = \mathcal{F}_G g \mathcal{F}_G f, \end{aligned}$$

and likewise

$$\begin{aligned} \mathcal{F}_G(f^*) &= \int_G \overline{f(x^{-1})} \eta(x)^*\, dx \\ &= \int_G \overline{f(x)} \eta(x)\, dx = (\mathcal{F}_G f)^*. \end{aligned}$$

□

There exists a canonical continuous and linear projection $\Pi_{G/H} : C(G) \to C(G/H)$. Let $\pi : G \to G/H$ be the natural quotient. Then $\pi^* : C(G/H) \to C(G)$ is an isometry. This π^* will make the word "projection" meaningful.

Definition 3.2.1. *If $f \in C(G)$, then we put*

$$\Pi_{G/H} f(xH) = \int_H f(xh)\, dh \quad \text{for all} \quad xH \in G/H.$$

Clearly, $\Pi_{G/H} f \in C(G/H)$, by left-invariance, since G/H has the quotient topology. Also, $\Pi_{G/H}$ is the projection map onto $C(G/H)$, viewed as a closed subspace of $C(G)$. More precisely, we have

1. $||\Pi_{G/H} f||_{C(G/H)} \leq ||f||_{C(G)}$ for any $f \in C(G)$.
2. $\Pi_{G/H}(g \circ \pi) = g$ for any $g \in C(G/H)$.

Theorem 3.2.4 (Folland [13])**.** *Let the space G/H have the natural quotient topology. Then there is a unique G-invariant positive Radon measure μ on G/H such that*

$$\int_G f(x)\, dx = \int_{G/H} \int_H f(xh)\, dh\, d\mu(xH) \quad \text{for any} \quad f \in C(G).$$

Proof. Because $\Pi_{G/H}$ is surjective, the formula completely determines μ, so it is unique. The functional $C(G/H) \to \mathbb{C} : g \mapsto \int_G (g \circ \pi)(x)\, dx$ is linear, G-invariant and bounded. Since it is clearly also positive, the Riesz representation theorem provides the desired μ. To see that it has the right property, note that

$$\begin{aligned} 0 &= \int_G \int_H (f - \Pi_{G/H} f \circ \pi)(xh)\, dh\, dx \\ &= \int_H \Big[\int_G (f - \Pi_{G/H} f \circ \pi)(xh)\, dx \Big]\, dh = \int_G (f - \Pi_{G/H} f \circ \pi)(x)\, dx, \end{aligned}$$

and put $g = \Pi_{G/H} f$ to get

$$\int_G f(x)\, dx = \int_G (\Pi_{G/H} f \circ \pi)(x)\, dx = \int_{G/H} \Pi_{G/H} f(xH)\, d\mu(xH).$$

□

Corollary 3.2.3. *If $s < \infty$, then*

$$||\Pi_{G/H} f||_{L^s(G/H,\mu)} \leq ||f||_{L^s(G)} \quad \text{for any} \quad f \in C(G).$$

Proof. Using the Hölder inequality with $\frac{1}{s} + \frac{1}{t} = 1$ when $1 < s < \infty$, we compute

$$\begin{aligned} \int_{G/H} \Big| \int_H f(xh)\, dh \Big|^s d\mu(xH) &\leq \int_{G/H} \Big[\int_H |f(xh)|^s\, dh \Big] \Big[\int_H 1^t\, dh \Big]^{\frac{s}{t}} d\mu(xH) \\ &= \int_{G/H} \int_H |f(xh)|^s\, dh\, d\mu(xH), \end{aligned}$$

and the case $s = 1$ is similar. □

Corollary 3.2.4. *If $s < \infty$, then*

$$||g||_{L^s(G/H,\mu)} = ||g \circ \pi||_{L^s(G)} \quad \text{for any} \quad g \in C(G/H).$$

Proof. Applying the above Theorem 3.2.4 to $|g|^s$, we get

$$\int_{G/H} |g(xH)|^s \, d\mu(xH) = \int_G (|g|^s \circ \pi)(x) \, dx.$$

□

Henceforth suppressing μ, the space $L^s(G/H)$ is isometrically embedded into $L^s(G)$, and we may view it as a closed subspace. Thus we can make sense of $\mathcal{F}_G$ on $L^1(G/H)$. In particular, if $g \in L^1(G/H)$, we have

$$\mathcal{F}_G(g \circ \pi)(\xi) = \int_{G/H} g(xH) \Big[\int_H \xi(xh)^* \, dh \Big] \, d\mu(xH) \quad \text{for all} \quad [\xi] \in \widehat{G},$$

and this might motivate us to define $\{[\xi] \in \widehat{G} \mid \int_H \xi(h) \, dh \neq 0\}$ as the "dual" of G/H. However, we stop here. See Connolly [8] for more in this direction.

Finally, we record a few easily proved properties of $\mathcal{F}_G$ and the Peter-Weyl expansion. Write L_x and R_x for the usual left and right regular representations at $x \in G$, respectively.

$$L : G \to \mathrm{U}(L^2(G)) \quad \text{and} \quad R : G \to \mathrm{U}(L^2(G)).$$

Proposition 3.2.1. *Let $[\xi] \in \widehat{G}$. The following statements hold:*

1. Let $j \in \{1, \cdots, d_\xi\}$. Then the subspace $\mathrm{span}\{\xi_{ij}\}_{i=1}^{d_\xi}$ is invariant under L, and restricted to this space, L is equivalent to $\overline{\xi}$ under

$$(c_1, \cdots, c_{d_\xi}) \mapsto \sum_{j=1}^{d_\xi} c_i \xi_{ij}.$$

2. Let $i \in \{1, \cdots, d_\xi\}$. Then the subspace $\mathrm{span}\{\xi_{ij}\}_{j=1}^{d_\xi}$ is invariant under R, and restricted to this space, R is equivalent to ξ under

$$(c_1, \cdots, c_{d_\xi}) \mapsto \sum_{i=1}^{d_\xi} c_j \xi_{ij}.$$

Furthermore, given any $f \in L^2(G)$ with $x \in G$ fixed, the following holds:

1. $\mathcal{F}_G(L_x f)(\xi) = \mathcal{F}_G f(\xi)\xi(x^{-1})$.

2. $\mathcal{F}_G(R_x f)(\xi) = \xi(x)\mathcal{F}_G f(\xi)$.

Finally, $\mathcal{F}_G$ and $\mathcal{F}_G^{-1}$ are bounded on the L^1-spaces in the following way:

1. $||\mathcal{F}_G f||_{L^\infty(\widehat{G})} \leq ||f||_{L^1(G)}$ for all $f \in L^1(G)$.

2. $||\mathcal{F}_G^{-1} a||_{L^\infty(G)} \leq ||a||_{L^1(\widehat{G})}$ for all $a \in L^1(\widehat{G})$.

Henceforth G is a compact (not necessarily connected) Lie group with Lie algebra $\mathfrak{g}$. It is always given a bi-invariant metric induced from an Ad invariant inner product on $\mathfrak{g}$. Let $\Delta_G = \Delta$ be the associated Laplacian (quadratic Casimir), and put $n = \dim(G)$.

Theorem 3.2.5 (Faraut [12]). *The following holds:*

1. *Any eigenspace of* $-\Delta$ *is of the form* $\operatorname{span}\{\sqrt{d_\xi}\,\xi_{ij}\}_{i,j=1}^{d_\xi}$ *for some* $[\xi] \in \widehat{G}$.
2. *The eigenvalues* $\{\lambda_\xi\}_{[\xi]\in\widehat{G}}$ *are non-negative, countable, increasing to* ∞.

It follows immediately that $\xi_{ij} \in C^\infty(G)$ for each $[\xi] \in \widehat{G}$ by the ellipticity of $-\Delta$. Also, since $(I-\Delta)^{\frac{s}{2}}$ has kernel in $L^2(G\times G)$ for $s < -\frac{n}{2}$, it is Hilbert-Schmidt, and

$$\sum_{[\xi]\in\widehat{G}} d_\xi^2 \langle\xi\rangle^{2s} < \infty \quad \text{when} \quad s < -\frac{n}{2},$$

where the eigenvalues of $(I-\Delta)^{\frac{1}{2}}$ are $\langle\xi\rangle = (1+\lambda_\xi^2)^{\frac{1}{2}}$. They serve to count the classes. In fact, the dimensions are controlled by this function.

Proposition 3.2.2.

$$d_\xi = O(\langle\xi\rangle^{\frac{n}{2}}).$$

Proof. The dimension of the eigenspace of $(1-\Delta)^{\frac{1}{2}}$ corresponding to $\langle\xi\rangle$ is precisely d_ξ^2, and the statement follows from Corollary 3.1.2, the Weyl asymptotics

$$\sum_{\langle\xi\rangle\leq\lambda} d_\xi^2 - C\lambda^n = O_{\lambda\to\infty}(\lambda^{n-1}),$$

where $C > 0$ is an intrinsic constant. □

Analogous to the space of Schwartz functions on $\mathbb{R}^n$, we define a sequence space $\mathcal{S}(\widehat{G})$. Let us first take $d \in \mathbb{R}$, $k, l \in \mathbb{Z}$ and define auxiliary weighted L^2 spaces

$$\mathcal{S}^d(\widehat{G}) = \Big\{a \in \operatorname{Mat}(\widehat{G}) \,\Big|\, \sum_{[\xi]\in\widehat{G}} d_\xi \langle\xi\rangle^{2d} \operatorname{Tr}(a(\xi)^* a(\xi)) < \infty\Big\},$$

which are equipped with the inner products

$$(a,b)_{\mathcal{S}^d(\widehat{G})} = \sum_{[\xi]\in\widehat{G}} d_\xi \langle\xi\rangle^{2d} \operatorname{Tr}(b(\xi)^* a(\xi)) \quad \text{for any} \quad a, b \in \mathcal{S}^d(\widehat{G}),$$

and give $\mathcal{S}(G) = \cap_{k\in\mathbb{Z}} \mathcal{S}^k(\widehat{G})$ the Frechet topology generated by the collection of norms. Note that $\mathcal{S}(G)$ is sequentially dense in every $\mathcal{S}^k(G)$ by truncating the sequences in $\mathcal{S}^k(G)$. Let us equip each space with the pairing

$$\mathcal{S}^{-k}(G) \times \mathcal{S}^k(G) \to \mathbb{C} : (a,b) \mapsto \langle a,b\rangle = \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}(a(\xi)b(\xi)).$$

The Riesz representation theorem and denseness of $\mathcal{S}(G)$ implies $\mathcal{S}'(G) \cong \cup_{k\in\mathbb{Z}} \mathcal{S}^k(\widehat{G})$. This is understood in the above pairing, which extends to $\mathcal{S}'(\widehat{G}) \times \mathcal{S}(\widehat{G})$.

Now we will show that $\mathcal{S}(G)$ is the right space for $\mathcal{F}_G$ when it is restricted to $C^\infty(G)$. It mirrors the Schwartz functions in view of the following mapping properties:

Proposition 3.2.3. *The transforms $\mathcal{F}_G$ and $\mathcal{F}_G^{-1}$ restrict to linear homeomorphisms:*

$$\mathcal{F}_G : C^\infty(G) \to \mathcal{S}(\widehat{G}) \quad \text{and} \quad \mathcal{F}_G^{-1} : \mathcal{S}(\widehat{G}) \to C^\infty(G).$$

Proof. Observe that if $N \in \mathbb{N}_0$, $f \in C^\infty(G)$ and $[\xi] \in \widehat{G}$, then

$$\begin{aligned}\langle\xi\rangle^N \mathcal{F}_G f(\xi) &= \int_G f(x)(I-\Delta)^{\frac{N}{2}}\xi(x)^*\,dx \\ &= \int_G (I-\Delta)^{\frac{N}{2}} f(x)\xi(x)^*\,dx = \mathcal{F}_G(I-\Delta)^{\frac{N}{2}} f(\xi),\end{aligned}$$

and combining with the Parseval identity, we get

$$\sum_{[\xi]\in\widehat{G}} d_\xi \langle\xi\rangle^{2N} \mathrm{Tr}(\mathcal{F}_G f(\xi)^* \mathcal{F}_G f(\xi)) = ||(I-\Delta)^{\frac{N}{2}} f||^2_{L^2(G)}.$$

But any semi-norm of $C^\infty(G)$ is controlled by a Sobolev norm of some fixed order $N \in \mathbb{N}_0$, and compactness ensures the L^2-norm of $C(G)$ functions is bounded by the sup-norm. The equality implies that they restrict to isometries

$$\mathcal{F}_G : H^N(G) \to \mathcal{S}^N(\widehat{G}) \quad \text{and} \quad \mathcal{F}_G^{-1} : \mathcal{S}^N(\widehat{G}) \to H^N(G).$$

Therefore the statement follows immediately. □

Corollary 3.2.5. *The Peter-Weyl expansion of $f \in C^\infty(G)$ is convergent in $C^\infty(G)$.*

Proof. It follows from the above that the expansion must converge in every Sobolev norm. The Sobolev embedding theorem then implies that it converges in $C^\infty(G)$. □

The space $\mathcal{S}'(\widehat{G})$ is naturallly equipped with the weak$*$ topology in the above pairing. However, it is also possible to equip this space with a natural inductive limit topology, which comes from the sequence of embeddings $\cdots \hookrightarrow \mathcal{S}^k(\widehat{G}) \hookrightarrow \mathcal{S}^{k-1}(\widehat{G}) \hookrightarrow \cdots \hookrightarrow \mathcal{S}'(\widehat{G})$.

Proposition 3.2.4. *If $a \in \mathcal{S}^k(\widehat{G})$ and $\mathcal{S}^l(\widehat{G})$, then $ab \in \mathcal{S}^{k+l}(\widehat{G})$, and*

$$||ab||_{\mathcal{S}^{k+l}(\widehat{G})} \leq ||a||_{\mathcal{S}^k(\widehat{G})} ||b||_{\mathcal{S}^l(\widehat{G})}$$

Proof. Using that $d_\xi \geq 1$, we apply Cauchy-Schwarz to estimate

$$\begin{aligned}||ab||_{\mathcal{S}^{k+l}(\widehat{G})} &\leq \sum_{[\xi]\in\widehat{G}} d_\xi \langle\xi\rangle^{k+l} \sqrt{\mathrm{Tr}(a(\xi)^*a(\xi))}\sqrt{\mathrm{Tr}(b(\xi)^*b(\xi))} \\ &\leq \Big[\sum_{[\xi]\in\widehat{G}} d_\xi \langle\xi\rangle^{2k} \mathrm{Tr}(a(\xi)^*a(\xi))\Big]^{\frac{1}{2}} \Big[\sum_{[\xi]\in\widehat{G}} d_\xi \langle\xi\rangle^{2l} \mathrm{Tr}(b(\xi)^*b(\xi))\Big]^{\frac{1}{2}}.\end{aligned}$$

□

Corollary 3.2.6. *The pointwise matrix-product of elements in $\mathcal{S}'(\widehat{G})$ is again in $\mathcal{S}'(\widehat{G})$.*

This means that we have a well-defined product $\mathcal{S}'(\widehat{G}) \times \mathcal{S}'(\widehat{G}) \to \mathcal{S}'(\widehat{G}) : (a,b) \mapsto ab$. In fact, it has the following pleasant property:

Corollary 3.2.7. *The bilinear product is separately continuous in the weak∗-topology.*

It remains to extend $\mathcal{F}_G$ and $\mathcal{F}_G^{-1}$ to the distributions $\mathcal{D}'(G)$ and $\mathcal{S}'(\widehat{G})$, respectively. The right dualization is obtained by observing how they interact with the dual pairings. Note that if $f \in C^\infty(G)$ and $a \in \mathcal{S}(\widehat{G})$, then

$$\begin{aligned}\langle \mathcal{F}_G f(\xi), a(\xi)\rangle &= \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}\Big(\Big[\int_G f(x)\xi(x)^*\,dx\Big]a(\xi)\Big)\\ &= \int_G f(x)\Big[\sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\xi(x^{-1})a(\xi))\Big]\,dx = \langle f(x), \mathcal{F}_G^{-1}a(x^{-1})\rangle.\end{aligned}$$

This motivates the next definition. Note that the transforms are still mutual inverses, and are continuous in the weak∗-topologies on $\mathcal{D}'(G)$ and $\mathcal{S}'(G)$.

Definition 3.2.2. *Overloading notation, we extend $\mathcal{F}_G$ and $\mathcal{F}_G^{-1}$ by duality as follows:*

1. *Define $\mathcal{F}_G : \mathcal{D}'(G) \to \mathcal{S}'(\widehat{G}) : f \mapsto \mathcal{F}_G f$ by*

$$\langle \mathcal{F}_G f(\xi), a(\xi)\rangle = \langle f(x), \mathcal{F}_G^{-1}a(x^{-1})\rangle \quad \text{for all} \quad a \in \mathcal{S}(\widehat{G}).$$

2. *Define $\mathcal{F}_G^{-1} : \mathcal{S}'(\widehat{G}) \to \mathcal{D}'(G) : a \mapsto \mathcal{F}_G^{-1}a$ by*

$$\langle \mathcal{F}_G^{-1}a(x), f(x^{-1})\rangle = \langle a(\xi), \mathcal{F}_G f(\xi)\rangle \quad \text{for all} \quad f \in C^\infty(G).$$

Many identities valid for the smooth functions on G can be extended to distributions. Let us demonstrate how separate continuity is used to do this.

Proposition 3.2.5.

$$\mathcal{F}_G(u * v) = \mathcal{F}_G v \mathcal{F}_G u \quad \text{for any} \quad u, v \in \mathcal{D}'(G).$$

Proof. Convolution $\mathcal{D}'(G) \times \mathcal{D}'(G) \to \mathcal{D}'(G) : (u,v) \mapsto u * v$ is separately continuous. Take $\{u_j\}_{j=1}^\infty$ and $\{v_k\}_{k=1}^\infty$ in $C^\infty(G)$ so that

$$u_j \to u \quad \text{and} \quad v_j \to v \quad \text{in} \quad \mathcal{D}'(G) \quad \text{as} \quad j \to \infty.$$

Then, we can write

$$\begin{aligned}\mathcal{F}_G(u * v) &= \lim_{k\to\infty} \mathcal{F}_G(u * v_k)\\ &= \lim_{k\to\infty}\lim_{j\to\infty} \mathcal{F}_G(u_j * v_k)\\ &= \lim_{k\to\infty}\lim_{j\to\infty} \mathcal{F}_G v_k \mathcal{F}_G u_j\\ &= \lim_{k\to\infty} \mathcal{F}_G v_k \mathcal{F} u = \mathcal{F}_G v \mathcal{F}_G u.\end{aligned}$$

□

3.2.1 The Exact Operator Calculus

By Corollary 3.2.1, there exists a faithful representation $\rho : G \to \mathrm{U}(m)$ for some $m \in \mathbb{N}$. Now ρ, being continuous, is automatically smooth. It is a closed map by the compactness. It follows that ρ is an embedding onto the closed subgroup $\rho(G)$ in the relative topology, which is a Lie subgroup of $\mathrm{U}(m)$. Therefore G identifies with a closed subgroup of $\mathrm{U}(m)$. Because $\mathrm{U}(m) \cong \mathrm{O}(2m) \cap \mathrm{GL}(m, \mathbb{C}) \subset \mathrm{GL}(2m, \mathbb{R})$ by representing $\mathbb{C}$ with real matrices, we may assume that $\rho : G \to \mathrm{O}(2m)$. Let us choose a complement to $d\rho(\mathfrak{g})$ in $\mathfrak{gl}(2m, \mathbb{R})$. Then, by the inverse function theorem, there is an open $V \subset \mathrm{GL}(2m, \mathbb{R})$ with $\rho(G) \subset V$, and a neighbourhood U of 0 in the complement, giving a diffeomorphism

$$G \times U \to V : (g, Y) \mapsto \rho(g) \exp(Y),$$

which in turn gives a smooth "projection" map

$$\varrho : V \to G : \rho(g) \exp(Y) \mapsto g,$$

and this ϱ absorbs G from the left, that is, $x\varrho(y) = \varrho(\rho(x)y)$ when $x \in G$ and $y \in V$. Choose a cutoff $\chi \in C_0^\infty(V)$ equal to 1 on a $\rho(G)$-invariant neighbourhood of $\rho(G)$ in V. Note that $\chi(\rho(x)y) = \chi(\rho(x)) = 1$ for $x \in G$ and y in this neighbourhood.

Proposition 3.2.6. *Any $u \in C^\infty(G)$ has a global Taylor series expansion in $N \in \mathbb{N}$, with the sums running over $\alpha \in \mathbb{N}_0^{m \times m}$, given by*

$$u(x) = \sum_{|\alpha|<N} \frac{1}{\alpha!} d_\alpha(x^{-1}) \partial^\alpha [\chi(u \circ \varrho)](I) + \sum_{|\alpha|=N} \frac{N}{\alpha!} d_\alpha(x^{-1}) R_\alpha(x) \quad \text{for all} \quad x \in G.$$

where $\chi \in C_0^\infty(V)$ is the above cutoff for $\rho(G)$, and

$$R_\alpha(x) = \int_0^1 (1-t)^{N-1} \partial^\alpha [\chi(u \circ \varrho)](I + t(\rho(x) - I))\, dt.$$

Proof. Taylor expand $\chi(u \circ \varrho)$ at I evaluated in $\rho(x) \in \rho(G) \hookrightarrow \mathrm{GL}(2m, \mathbb{R})$ to get

$$u(x) = \sum_{|\alpha|<N} \frac{1}{\alpha!} (\rho(x) - I)^\alpha \partial_x^\alpha [\chi(u \circ \varrho)](I) + \sum_{|\alpha|=N} \frac{N}{\alpha!} (\rho(x) - I)^\alpha R_\alpha(x),$$

where $d_\alpha(x^{-1}) = (\rho(x) - I)^\alpha$. □

It will be convenient to define two families of operators δ_ξ^α and δ_x^α associated to d_α.

Definition 3.2.3. *Let δ_x^α be differential operators acting on $u \in C^\infty(G)$ by*

$$\delta_x^\alpha u(x) = \partial_y^\alpha \left[\chi(u \circ \varrho)(\rho(x)y)\right]\Big|_{y=I} \quad \text{for all} \quad x \in G,$$

and define "difference operators" δ_ξ^α acting on $\mathcal{S}'(\widehat{G})$ by

$$\delta_\xi^\alpha = \mathcal{F} d_\alpha \mathcal{F}^{-1}.$$

Before developing the symbolic calculus using the above global Taylor series expansion, we need to be able to move δ_ξ^α through integrals when the argument depends on a variable. This is due the remainder term in the expansion, where there is an integral over $t \in [0,1]$. It is possible to do this because of the following lemma:

Lemma 3.2.1. *If $a \in \mathcal{S}'(\widehat{G})$, then*

$$\delta_\eta^\alpha a(\eta) = \sum_{[\xi]\in\widehat{G}} d_\xi \int_G d_\alpha(y) \mathrm{Tr}(\xi(y)a(\xi))\eta(y)^*\, dy \quad \text{for all} \quad [\eta] \in \widehat{G}.$$

Proof. Let us test $\delta_\eta^\alpha a$ against the element in $\mathcal{S}(G)$ equal to I at $[\eta] \in \widehat{G}$ and zero else, and truncate $a(\xi)$ up to $\langle\xi\rangle \leq k$ for a $k \in \mathbb{N}$. Then if $N \in \mathbb{N}$ is large enough, we get

$$\begin{aligned}\delta_\eta^\alpha a(\eta) &= \lim_{k\to\infty} \int_G d_\alpha(y) \Big[\sum_{\langle\xi\rangle\leq k} d_\xi \mathrm{Tr}(\xi(y)a(\xi)) \Big] \eta(y)^*\, dy \\ &= \sum_{[\xi]\in\widehat{G}} d_\xi \langle\xi\rangle^{-2N} \int_G \mathrm{Tr}\Big(\xi(y)a(\xi)\Big)(1-\Delta)^N [d_\alpha(y)\eta(y)^*]\, dy,\end{aligned}$$

where a trace norm estimate and $d_\xi = O(\langle\xi\rangle^{\frac{n}{2}})$ shows absolute convergence. □

The difference operators δ_ξ^α satisfy a Leibniz-type product rule on sequences in $\mathcal{S}'(\widehat{G})$. It was first observed by Ruzhansky, Turunen and Wirth [49]. Again, this will be needed. Let us put

$$d_{ij}(x) = d_{e_{ij}}(x) = \rho_{ij}(x^{-1}) - \delta_{ij} \quad \text{for all} \quad x \in G,$$

where e_{ij} is the $m \times m$ matrix with 1 at the i,j entry and zero else.

Lemma 3.2.2 (Ruzhansky, Turunen and Wirth [49])**.**

$$\delta_\xi^{e_{ij}}(ab) = \delta_\xi^{e_{ij}}(a)b + a\delta_\xi^{e_{ij}}(b) + \sum_{k=1}^m \delta_\xi^{e_{ik}}(a)\delta_\xi^{e_{kj}}(b) \quad \text{for any} \quad a,b \in \mathcal{S}'(\widehat{G}).$$

Proof. Since the product $\mathcal{S}'(\widehat{G}) \times \mathcal{S}'(\widehat{G}) \to \mathcal{S}'(\widehat{G}) : (a,b) \mapsto ab$ is separately continuous, and $\mathcal{S}(\widehat{G})$ is sequentially dense in $\mathcal{S}'(\widehat{G})$, it suffices to show that it holds for $a,b \in \mathcal{S}(\widehat{G})$. This is Corollary 3.2.7 and the remarks before it. Now, if $x,y \in G$, we have

$$d_{ij}(x) = d_{ij}(xy^{-1}) + d_{ij}(y) + \sum_{k=1}^m d_{kj}(xy^{-1})d_{ik}(y),$$

and so if we multiply by $\mathcal{F}^{-1}b(xy^{-1})\mathcal{F}^{-1}a(y)$, and integrate in y, we get

$$d_{ij}\mathcal{F}^{-1}(ab) = (d_{ij}\mathcal{F}^{-1}b) * \mathcal{F}^{-1}a + \mathcal{F}^{-1}b * (d_{ij}\mathcal{F}^{-1}a) + \sum_{k=1}^m (d_{kj}\mathcal{F}^{-1}b) * (d_{ik}\mathcal{F}^{-1}a),$$

and finally applying $\mathcal{F}$ gives the result. □

Let $P : C^\infty(G) \to C^\infty(G)$ be linear and continuous in the Frechet topology of $C^\infty(G)$. The Peter-Weyl expansion of $u \in C^\infty(G)$ converges in this topology, so we have

$$Pu(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\xi(x)p(x,\xi)\mathcal{F}u(\xi)) \quad \text{for all} \quad x \in G.$$

where $p(x,\xi) = \xi(x)^*(P\xi)(x)$ is a "matrix-symbol" of P. It clearly defines P uniquely, and we write $P = \mathrm{Op}(p)$. The map $p \mapsto \mathrm{Op}(p)$ is called the operator quantization map. Because only $[\xi] \in \widehat{G}$ matters in the expansion, we can view p as a mapping

$$p : G \times \widehat{G} \to \bigcup_{[\xi]\in\widehat{G}} \mathrm{Mat}(d_\xi, \mathbb{C}).$$

Proposition 3.2.7.

$$\sup_{(x,[\xi])\in G\times\widehat{G}} \langle\xi\rangle^{-k}||p(x,\xi)|| < \infty \quad \text{for some} \quad k \in \mathbb{N}_0.$$

Proof. Using the Sobolev embedding, we get some $N \in \mathbb{N}_0$ and $C_N > 0$ such that

$$\begin{aligned}\sup_{x\in G} ||\xi(x)^*(P\xi)(x)|| &\le d_\xi \max_{i,j} \sup_{x\in G} |P\xi_{ij}(x)| \\ &\le C_N d_\xi \max_{i,j} ||(I-\Delta)^{\frac{N}{2}}\xi_{ij}||_{L^2(G)} = C_N\sqrt{d_\xi}\langle\xi\rangle^N,\end{aligned}$$

and the Weyl asymptotic $d_\xi = O(\langle\xi\rangle^{\frac{n}{2}})$ gives the estimate. □

Definition 3.2.4. *Let $d \in \mathbb{R}$. Put $X^\beta = X_1^{\beta_1} \circ \cdots \circ X_n^{\beta_n}$ for $\beta \in \mathbb{N}_0^n$ if $\{X_j\}_{j=1}^n \subset \mathfrak{g}$. Write $p \in S^d(G \times \widehat{G}) = S^d$ of order $d \in \mathbb{R}$, if the following holds:*

1. *$p \in C^\infty(G; \mathcal{S}'(\widehat{G}))$.*
2. *Given any ordered basis $X = (X_1, \cdots, X_n)$ of $\mathfrak{g}$, we have*

$$\sup_{(x,[\xi])\in G\times\widehat{G}} \langle\xi\rangle^{|\alpha|-d}||\delta_\xi^\alpha X_x^\beta p(x,\xi)|| < \infty \quad \text{for any} \quad \alpha \in \mathbb{N}_0^{m\times m} \quad \text{and} \quad \beta \in \mathbb{N}_0^n.$$

The space S^d is given the Frechet topology induced by the above collection of semi-norms. Finally, if $\{p_j\}_{j=0}^\infty$ is a sequence with $p_j \in S^{d_j}$ and $d_j \to -\infty$ as $j \to \infty$, we write

$$p \sim \sum_{j=0}^\infty p_j \quad \text{if} \quad p - \sum_{j=0}^{k-1} p_j \in S^{\max_{j\ge k} d_j} \quad \text{for each} \quad k \in \mathbb{N}.$$

These matrix-symbols are called the Hörmander symbols.

The symbols characterizing operators in $\Psi(G)$ are precisely the Hörmander symbols. This of course seems natural, but was in fact only recently established:

Theorem 3.2.6 (Ruzhansky, Turunen and Wirth [49])**.** *If $d \in \mathbb{R}$, then*

$$P \in \Psi^d(G) \quad \text{if and only if} \quad P = \mathrm{Op}(p) \quad \text{for some} \quad p \in S^d.$$

As with the usual symbols on T^*G, we write $S^{-\infty} = \cap_{d\in\mathbb{R}} S^d$ for the residual symbols. The characterization guarantees asymptotic summability of symbols:

Corollary 3.2.8. *If $\{p_j\}_{j=0}^{\infty}$ is as above, there exist $p \in S^{\max_{j\geq 0} d_j}$ with $p \sim \sum_{j=0}^{\infty} p_j$.*

Proof. Using Theorem 3.1.2 and the characterization of $\Psi(G)$, we get a P such that

$$\Psi^{\max_{j\geq 0} d_j}(G) \ni \mathrm{Op}(p) = P \sim \sum_{j=0}^{\infty} \mathrm{Op}(p_j) \quad \text{for some} \quad p \in S^{\max_{j\geq 0} d_j},$$

and the asymptotic for p follows by applying the characterization again. □

The important fact about $\Psi(G)$ is that it admits a symbol calculus in terms of $\cup_{d\in\mathbb{R}} S^d$. It is isomorphic, via quantization, to $\cup_{d\in\mathbb{R}} S^d$ with a certain algebraic structure.

Definition 3.2.5. *Let $d_1, d_2 \in \mathbb{R}$. Given any two symbols $p \in S^{d_1}$ and $q \in S^{d_2}$, define:*

1. *Pointwise in $(x, [\eta]) \in G \times \widehat{G}$ a symbol $p \odot q$ by*

$$(p \odot q)(x, \eta) = \sum_{[\xi]\in\widehat{G}} d_\xi \int_G \mathrm{Tr}\Big(\xi(y^{-1}x)p(x,\xi)\Big)\eta(x^{-1}y)q(y,\eta)\, dy.$$

2. *Pointwise in $(x, [\eta]) \in G \times \widehat{G}$ a symbol $p^\dagger$ by*

$$p^\dagger(x, \eta) = \sum_{[\xi]\in\widehat{G}} d_\xi \int_G \mathrm{Tr}\Big(\xi(y^{-1}x)p(y,\xi)^*\Big)\eta(x^{-1}y)\, dy.$$

Note that both the sums in the above definition are uniformly absolutely convergent. To see this, do entry-wise integration by parts with $I - \Delta_y$ to summon powers of $\langle\xi\rangle^{-1}$. Then the sums are dominated by convergent sums scaling with $\langle\eta\rangle$, independently of x, which is because p and all of its derivatives in x grow polynomially in $\langle\xi\rangle$ at a fixed rate. It remains to show that $p \odot q$ and $p^\dagger$ are Hörmander symbols.

Theorem 3.2.7. *Let $d_1, d_2 \in \mathbb{R}$. Then the following holds:*

1. *The map $S^{d_1} \times S^{d_2} \to S^{d_1+d_2} : (p, q) \mapsto p \odot q$ is well-defined, and*

$$p \odot q \sim \sum_{N=0}^{\infty} \sum_{|\alpha|=N} \frac{1}{\alpha!} \delta_\xi^\alpha(p) \delta_x^\alpha(q).$$

2. *The map $S^{d_1} \to S^{d_1} : p \mapsto p^\dagger$ is well-defined, and*

$$p^\dagger \sim \sum_{N=0}^{\infty} \sum_{|\alpha|=N} \frac{1}{\alpha!} \delta_\xi^\alpha \delta_x^\alpha(p^*).$$

Both mappings are continuous with respect to the Frechet topologies on the symbol spaces, and, in particular, take bounded sets to bounded sets.

Proof. Let us expand $p(xy^{-1},\xi)$ and $q(xy^{-1},\xi)$ in $y^{-1}\in G$ with $(x,[\xi])\in G\times\widehat{G}$ fixed. Taking $N\in\mathbb{N}$ and using Proposition 3.2.6, labelling remainders R^p_α and R^q_α, we get

$$\begin{aligned} p(xy^{-1},\xi) &= \sum_{|\alpha|<N}\frac{1}{\alpha!}d_\alpha(y)\delta_x^\alpha p(x,\xi) + \sum_{|\alpha|=N}\frac{N}{\alpha!}d_\alpha(y)R^p_\alpha(x,\xi),\\ q(xy^{-1},\xi) &= \sum_{|\alpha|<N}\frac{1}{\alpha!}d_\alpha(y)\delta_x^\alpha q(x,\xi) + \sum_{|\alpha|=N}\frac{N}{\alpha!}d_\alpha(y)R^q_\alpha(x,\xi),\end{aligned}$$

where the remainders are

$$\begin{aligned} R^p_\alpha(x,\xi) &= \int_0^1 (1-t)^{N-1}D_x^\alpha p(\varrho(I+t(\rho(x)-I)),\xi)\,dt,\\ R^q_\alpha(x,\xi) &= \int_0^1 (1-t)^{N-1}D_x^\alpha q(\varrho(I+t(\rho(x)-I)),\xi)\,dt,\end{aligned}$$

and D^α are differential operators on G, depending on t and x, acting on the first entry. They arise from the partial derivatives, and contain the cutoff χ, in Proposition 3.2.6. Using the bi-invariance of the Haar measure, we compute

$$\begin{aligned} (p\odot q)(x,\eta) &= \sum_{[\xi]\in\widehat{G}} d_\xi\int_G \mathrm{Tr}\Big(\xi(y)p(x,\xi)\Big)\eta(y)^*q(xy^{-1},\eta)\,dy\\ &= \sum_{|\alpha|<N}\frac{1}{\alpha!}\delta_\eta^\alpha p(x,\eta)\delta_x^\alpha q(x,\eta) + \sum_{|\alpha|=N}\frac{N}{\alpha!}\delta_\eta^\alpha p(x,\eta)R^q_\alpha(x,\eta),\\ p^\dagger(x,\eta) &= \sum_{[\xi]\in\widehat{G}} d_\xi\int_G \mathrm{Tr}\Big(\xi(y)p(xy^{-1},\xi)^*\Big)\eta(y)^*\,dy\\ &= \sum_{|\alpha|<N}\frac{1}{\alpha!}\delta_\eta^\alpha\delta_x^\alpha[p(x,\eta)^*] + \sum_{|\alpha|=N}\frac{N}{\alpha!}\delta_\eta^\alpha R^p_\alpha(x,\eta)^*,\end{aligned}$$

and estimates follow by Lemma 3.2.2, taking δ_ξ^α through the integrals using Lemma 3.2.1. Explicitly, if $\{X_j\}_{j=1}^n$ is an ordered basis of $\mathfrak{g}$, and $\alpha,\gamma\in\mathbb{N}_0^{m\times m}$ and $\beta\in\mathbb{N}_0^n$, then

$$\begin{aligned} ||X_x^\beta\delta_\eta^\gamma R^q_\alpha(x,\eta)|| &\le \int_0^1(1-t)^{N-1}||X_x^\beta[D_x^\alpha(\delta_\eta^\gamma q)(\varrho(I+t(\rho(x)-I)),\eta)]||\,dt\\ &\le C^q_{\alpha,\beta,\gamma}\langle\eta\rangle^{d_2-|\gamma|}\int_0^1(1-t)^{N-1}\,dt,\\ ||X_x^\beta\delta_\eta^\gamma\delta_\eta^\alpha R^p_\alpha(x,\eta)^*|| &\le \int_0^1(1-t)^{N-1}||X_x^\beta[D_x^\alpha(\delta_\eta^{\gamma+\alpha}p)(\varrho(I+t(\rho(x)-I)),\eta)]^*||\,dt\\ &\le C^p_{\alpha,\beta,\gamma}\langle\eta\rangle^{d_1-|\gamma|-|\alpha|}\int_0^1(1-t)^{N-1}\,dt,\end{aligned}$$

where $C^p_{\alpha,\beta,\gamma}$ and $C^q_{\alpha,\beta,\gamma}$ are sums of finitely many semi-norms of p and q, respectively, and D^α is locally a sum of products of $\{X_j\}_{j=1}^n$, with coefficients depending on t and x. The integral is well-defined because χ is zero whenever $\varrho(I+t(\rho(x)-I))$ is not defined. This shows well-definedness, continuity, and the asymptotics. ☐

Next, we do the computations to show that $p \odot q$ and $p^\dagger$ are indeed the right symbols. Let p and q be symbols of Hörmander type.

Proposition 3.2.8.

$$\mathrm{Op}(p)\mathrm{Op}(q)u = \mathrm{Op}(p \odot q)u \quad \text{for any} \quad u \in C^\infty(G).$$

Proof. By the DCT and the FTT, computing pointwise in $x \in G$, we get

$$\begin{aligned}
\mathrm{Op}(p)\mathrm{Op}(q)u(x) &= \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\xi(x)p(x,\xi)\mathcal{F}\mathrm{Op}(q)u(\xi)) \\
&= \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}\Big(\xi(x)p(x,\xi)\int_G \Big[\sum_{[\eta]\in\widehat{G}} d_\eta \mathrm{Tr}(\eta(y)q(y,\eta)\mathcal{F}u(\eta))\Big]\xi(y)^*\,dy\Big) \\
&= \sum_{[\xi]\in\widehat{G}} d_\xi \int_G \Big[\sum_{[\eta]\in\widehat{G}} d_\eta \mathrm{Tr}(\eta(y)q(y,\eta)\mathcal{F}u(\eta))\Big]\mathrm{Tr}(\xi(x)p(x,\xi)\xi(y)^*)\,dy \\
&= \sum_{[\xi]\in\widehat{G}}\sum_{[\eta]\in\widehat{G}} d_\eta d_\xi \int_G \mathrm{Tr}(\xi(y^{-1}x)p(x,\xi))\mathrm{Tr}(\eta(y)q(y,\eta)\mathcal{F}u(\eta))\,dy \\
&= \sum_{[\eta]\in\widehat{G}} d_\eta \mathrm{Tr}(\eta(x)(p\odot q)(x,\eta)\mathcal{F}u(\eta)).
\end{aligned}$$

□

Proposition 3.2.9.

$$(\mathrm{Op}(p)u, v)_{L^2(G)} = (u, \mathrm{Op}(p^\dagger)v)_{L^2(G)} \quad \text{for any} \quad u, v \in C^\infty(G).$$

Proof. Again, by the DCT and the FTT, we get

$$\begin{aligned}
(\mathrm{Op}(p)u, v)_{L^2(G)} &= \int_G \Big[\sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\xi(y)p(y,\xi)\mathcal{F}u(\xi))\Big]\overline{v(y)}\,dy \\
&= \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}\Big(\Big[\int_G \xi(y)p(y,\xi)\overline{v(y)}\,dy\Big]\mathcal{F}u(\xi)\Big) \\
&= \int_G u(x)\Big[\sum_{[\xi]\in\widehat{G}} d_\xi \int_G \mathrm{Tr}(\xi(x^{-1}y)p(y,\xi))\overline{v(y)}\,dy\Big]\,dx \\
&= \int_G u(x)\Big[\sum_{[\xi]\in\widehat{G}}\sum_{[\eta]\in\widehat{G}} d_\xi d_\eta \int_G \mathrm{Tr}(\xi(x^{-1}y)p(y,\xi))\overline{\mathrm{Tr}(\eta(y)\mathcal{F}v(\eta))}\,dy\Big]\,dx \\
&= \int_G u(x)\overline{\Big[\sum_{[\eta]\in\widehat{G}} d_\eta \mathrm{Tr}(\eta(x)p^\dagger(x,\eta)\mathcal{F}v(\eta))\Big]}\,dx.
\end{aligned}$$

□

Note the repeated use of uniform dominated convergence to justify the interchanges, which are carefully performed in each step.

The calculus allows us to reformulate ellipticity entirely in terms of the global symbol. An operator $\mathrm{Op}(p)$ with $p \in S^d$ and $d \in \mathbb{R}$ will be elliptic if and only if p is:

Proposition 3.2.10. *Let $q_0 \in S^{-d}$. The following holds:*

1. *If $q_0 p - 1 \in S^{-1}$, then there is a $q_L \in S^{-d}$ such that $q_L \odot p - 1 \in S^{-\infty}$.*
2. *If $pq_0 - 1 \in S^{-1}$, then there is a $q_R \in S^{-d}$ such that $p \odot q_R - 1 \in S^{-\infty}$.*

Additionally, if both left and right parametrices exist, then $q_L - q_R \in S^{-\infty}$.

Proof. Note $q_0 \odot p - q_0 p \in S^{-1}$ in the left case and $p \odot q_0 - pq_0 \in S^{-1}$ in the right case. Either way, we write r for the residual $1 - q_0 \odot p$ or $1 - p \odot q_0$ in S^{-1}.

Left: Put $r = 1 - q_0 \odot p$. Define the sequence of symbols $q_j = r^{\odot j} \odot q_0$ for $j \in \mathbb{N}_0$. Then put $q \sim \sum_{j=0}^{\infty} q_j$ with $q \in S^{-d}$, and

$$\begin{aligned} S^{-N} &\ni \Big(q - \sum_{j=0}^{N-1} q_j\Big) \odot p - r^{\odot N} \\ &= q \odot p - \sum_{j=0}^{N-1} r^{\odot j} \odot (1 - r) - r^{\odot N} = q \odot p - 1. \end{aligned}$$

Right: Put $r = 1 - p \odot q_0$. Define the sequence of symbols $q_j = q_0 \odot r^{\odot j}$ for $j \in \mathbb{N}_0$. Then put $q \sim \sum_{j=0}^{\infty} q_j$ with $q \in S^{-d}$, and

$$\begin{aligned} S^{-N} &\ni p \odot \Big(q - \sum_{j=0}^{N-1} q_j\Big) - r^{\odot N} \\ &= p \odot q - \sum_{j=0}^{N-1} (1 - r) \odot r^{\odot j} - r^{\odot N} = p \odot q - 1. \end{aligned}$$

As this holds for every $N \in \mathbb{N}$, we have obtained inverses in the left and right cases. If both left q_L and right q_R parametrices exist, then

$$q_L - q_R = q_L \odot (1 - p \odot q_R) - (1 - q_L \odot p) \odot q_R \in S^{-\infty},$$

and similarly, any other inverse differs from either one by $S^{-\infty}$. □

Therefore $p \in S^d$ is called elliptic if it has a two-sided inverse $q_0 \in S^{-d}$ modulo S^{-1}. A necessary and sufficient condition for existence is the following:

Theorem 3.2.8 (Ruzhansky, Turunen and Wirth [49], and Ruzhansky and Wirth [47])**.** *A symbol $p \in S^d$ is elliptic in S^d if and only if there is a finite $F \subset \widehat{G}$ such that:*

1. *$p(x, \xi)$ is invertible for all $(x, [\xi]) \in G \times (\widehat{G} \setminus F)$.*
2. *The family of inverses satisfy*

$$\sup_{(x,[\xi]) \in G \times (\widehat{G} \setminus F)} \langle \xi \rangle^d ||p(x, \xi)^{-1}|| < \infty.$$

Lemma 3.2.3. *Let $\{p_Y\}_{j\in J}$ be bounded in S^d, each point-wise invertible as a matrix. Suppose that J is a compact space, x-derivatives of p_Y are continuous in (x, Y), and*

$$\sup_{Y\in J}\sup_{(x,[\xi])\in G\times\widehat{G}}\langle\xi\rangle^d||p_Y(x,\xi)^{-1}||<\infty.$$

Then $\{p_Y^{-1}\}_{Y\in J}$ is a bounded set in S^{-d}, and has the same continuity property.

Proof. Bounds on the $\mathfrak{g}$ derivatives in x can be established by the standard Leibniz rule. It suffices to get estimates with δ_ξ^α. By Lemma 3.2.2, we have

$$\delta_\xi^{e_{ij}}(p_Y^{-1})+\sum_{k=1}^m p_Y^{-1}\delta_\xi^{e_{ik}}(p_Y)\delta_\xi^{e_{kj}}(p_Y^{-1})=-p_Y^{-1}\delta_\xi^{e_{ij}}(p_Y)p_Y^{-1},$$

which is a linear system for $(\delta_\xi^{e_{ij}}(p_Y^{-1}))_{i,j}$ with the system matrix $(\delta_{ij}I+p_Y^{-1}\delta_\xi^{e_{ij}}(p_Y))_{i,j}$. By the bound on p_Y^{-1}, the system matrix is close to the identity for large enough $\langle\xi\rangle$, and can be inverted with norm uniformly bounded by 2, always independent of x and Y. Using the continuity hypothesis for the remaining $[\xi]$, we get $C_{e_{ij}}>0$ such that

$$||\delta_\xi^{e_{ij}}(p_Y^{-1})(x,\xi)||\leq C_{e_{ij}}\langle\xi\rangle^{-d-1}.$$

Suppose this holds with e_{ij} replaced by $\alpha\in\mathbb{N}_0^{m\times m}$ with $|\alpha|=k$ and $\langle\xi\rangle^{-d-1}$ by $\langle\xi\rangle^{-d-k}$. The Leibniz-like rule in Lemma 3.2.2 can be applied repeatedly to

$$\delta_\xi^{e_{ij}}\delta_\xi^\alpha(p_Yp_Y^{-1})=0,$$

which gives a sum of terms of the form $\delta_\xi^\beta(p_Y)\delta_\xi^\gamma(p_Y^{-1})$ with $|\beta|,|\gamma|\leq k+1\leq|\beta|+|\gamma|$. The leading terms here are, just as before, of the form $p_Y\delta_\xi^{e_{ij}}\delta_\xi^\alpha(p_Y^{-1})$ with $|\alpha|=k\in\mathbb{N}$, while the remaining terms have coefficients that can be estimated uniformly by $\langle\xi\rangle^{d-k}$. Similar to $k=1$, it can be solved for $\delta_\xi^{e_{ij}}\delta_\xi^\alpha(p_Y^{-1})$, and we get $C_{\alpha+e_{ij}}>0$ so that

$$||\delta_\xi^{e_{ij}}\delta_\xi^\alpha(p_Y^{-1})(x,\xi)||\leq C_{\alpha+e_{ij}}\langle\xi\rangle^{-d-k-1},$$

and the lemma follows by induction on k. □

Using this, we can prove the characterization of elliptic symbols in the above theorem. It is a simple application of the lemma with $J=\{0\}$:

Proof. Suppose that $p\in S^d$ is elliptic. So there exists a $q_0\in S^{-d}$ with $pq_0-1\in S^{-1}$. Then $(pq_0)(x,\xi)$ is invertible for $\langle\xi\rangle\geq R>0$, independent of $x\in G$. So is p, and

$$||p(x,\xi)^{-1}||\leq||q_0(x,\xi)||\,\big|\big|(p(x,\xi)q_0(x,\xi))^{-1}\big|\big|\leq C\langle\xi\rangle^{-d},$$

where $C>0$, and the inequality holds for $[\xi]\in\widehat{G}$ except the finite set with $\langle\xi\rangle<R$. Conversely, put $\chi_F(\xi)=1_F([\xi])I_{d_\xi}$ for all $[\xi]\in\widehat{G}$. So $\chi_F\in S^{-\infty}$, the lemma gives

$$(\chi_F+(1-\chi_F)p)^{-1}\in S^{-d},$$

and therefore $p\in S^d$ is elliptic. □

Let us investigate how $\Psi(G/H)$ relates to $\Psi(G)$ and to the symbolic calculus on G. To do so, we need the following theorem. It can be found in Connolly [8].

Theorem 3.2.9. *There is a metric on G/H such that π^* intertwines $\Delta_{G/H}$ and Δ_G. That is, $\Delta_G \pi^* = \pi^* \Delta_{G/H}$ on $C^\infty(G/H)$, where $\Delta_{G/H}$ is the Laplacian of this metric.*

As a consequence, the eigenfunctions of $\Delta_{G/H}$ must be closely related to those of Δ_G. Observe that if ξ_{ij} are the components of $[\xi] \in \widehat{G}$, with eigenvalue λ_ξ, then

$$\pi^* \Delta_{G/H} \Pi_{G/H} \xi_{ij} = \pi^* \Pi_{G/H} \Delta_G \xi_{ij} = \lambda_\xi \pi^* \Pi_{G/H} \xi_{ij},$$

and conversely, any eigenfunction of $\Delta_{G/H}$ pulls back via π to an eigenfunction of Δ_G. Hence the eigenspaces of $\Delta_{G/H}$ are $\mathrm{span}\{\Pi_{G/H}\xi_{ij}\}_{i,j=1}^{d_\xi}$ with associated eigenvalue λ_ξ. Therefore, if $s \in \mathbb{R}$ is arbitrary, we have

$$(I - \Delta_G)^{\frac{s}{2}} \pi^* = \pi^* (I - \Delta_{G/H})^{\frac{s}{2}} \quad \text{on} \quad C^\infty(G/H),$$

which also shows that $\Pi_{G/H} : C^\infty(G) \to C^\infty(G/H)$ is (well-defined and) continuous. This we see by taking $u \in C^\infty(G)$ and calculating pointwise in $x \in G$ to get

$$\begin{aligned}
(I - \Delta_G)^{\frac{s}{2}} \pi^* \Pi_{G/H} u(x) &= \sum_{[\xi] \in \widehat{G}} d_\xi \langle \xi \rangle^s \mathrm{Tr}\Big(\xi(x) \int_G \int_H u(yh) \xi(y)^* \, dh \, dy \Big) \\
&= \sum_{[\xi] \in \widehat{G}} d_\xi \langle \xi \rangle^s \mathrm{Tr}\Big(\int_H \xi(xh) \Big[\int_G u(y) \xi(y)^* \, dy \Big] \, dh \Big) \\
&= \int_H (I - \Delta_G)^{\frac{s}{2}} u(xh) \, dh.
\end{aligned}$$

Lemma 3.2.4. *The pullback map extends as an isometric map $\pi^* : H^s(G/H) \to H^s(G)$.*

Proof. Observe that if $f \in C^\infty(G/H)$, we have

$$\begin{aligned}
||\pi^* f||_{H^s(G)} &= ||(I - \Delta_G)^{\frac{s}{2}} \pi^* f||_{L^2(G)} \\
&= ||\pi^* (I - \Delta_{G/H})^{\frac{s}{2}} f||_{L^2(G)} \\
&= ||(I - \Delta_{G/H})^{\frac{s}{2}} f||_{L^2(G/H)},
\end{aligned}$$

which implies that π^* extends to an isometry of $H^s(G/H)$ into $H^s(G)$ □

Lemma 3.2.5. *The projection extends as a continuous map $\Pi_{G/H} : H^s(G) \to H^s(G/H)$.*

Proof. Observe that if $u \in C^\infty(G)$, we have

$$\begin{aligned}
||\pi^* \Pi_{G/H} u||_{H^s(G)} &= ||(I - \Delta_G)^{\frac{s}{2}} \pi^* \Pi_{G/H} u||_{L^2(G)} \\
&= ||\pi^* \Pi_{G/H} (I - \Delta_G)^{\frac{s}{2}} u||_{L^2(G)} \\
&\leq ||(I - \Delta_G)^{\frac{s}{2}} u||_{L^2(G)},
\end{aligned}$$

and the result follows from the previous lemma. □

Lemma 3.2.6. *If $d \in \mathbb{R}$, then*

$$\pi^* \Pi_{G/H} P \in \Psi^d(G) \quad \text{for any} \quad P \in \Psi^d(G).$$

Proof. The characterization of $\Psi(G)$ gives a Hörmander symbol $p \in S^d$ with $P = \mathrm{Op}(p)$. In view of this, if $u \in C^\infty(G)$, we can write

$$\pi^* \Pi_{G/H} \mathrm{Op}(p) u(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \mathrm{Tr}\Big(\xi(x) \Big[\int_H \xi(h) p(xh, \xi)\, dh \Big] \mathcal{F}u(\xi) \Big) \quad \text{for all} \quad x \in G,$$

and hence $\pi^* \Pi_{G/H} \mathrm{Op}(p) = \mathrm{Op}(q)$, where

$$q(x, \xi) = \int_H \xi(h) p(xh, \xi)\, dh \quad \text{for all} \quad (x, [\xi]) \in G \times \widehat{G},$$

It remains to show that $q \in S^d$. Let $\alpha, \beta \in \mathbb{N}_0^n$ and observe that

$$\begin{aligned}
\delta_\xi^\alpha q(x, \xi) &= \sum_{[\eta] \in \widehat{G}} d_\eta \int_G d_\alpha(y) \mathrm{Tr}\Big(\eta(y) \Big[\int_H \eta(h) p(xh, \eta)\, dh \Big] \Big) \xi(y)^*\, dy \\
&= \sum_{[\eta] \in \widehat{G}} d_\eta \int_H \int_G d_\alpha(y) \mathrm{Tr}(\eta(yh) p(xh, \eta)) \xi(y)^*\, dy\, dh \\
&= \sum_{[\eta] \in \widehat{G}} d_\eta \int_H \xi(h) \Big[\int_G d_\alpha(y) \mathrm{Tr}(\eta(y) p(xh, \eta)) \xi(y)^*\, dy \Big]\, dh \\
&= \int_H \xi(h) \delta_\xi^\alpha p(xh, \xi)\, dh.
\end{aligned}$$

Furthermore, if $(X_1, \cdots, X_n)$ is an ordered basis of right-invariant vector fields on G, then X_x^β can be taken through the integral, and we get estimates

$$\langle \xi \rangle^{|\alpha| - d} ||\delta_\xi^\alpha X_x^\beta q(x, \xi)|| \le \int_H ||\xi(h)|| \Big[\langle \xi \rangle^{|\alpha| - d} ||\delta_\xi^\alpha X_x^\beta p(xh, \xi)|| \Big]\, dh.$$

It follows that $q \in S^d$, because any vector field can be expressed in this basis. □

Theorem 3.2.10 (Ruzhansky and Turunen [48]). *If $d \in \mathbb{R}$, then*

$$\Pi_{G/H} P \pi^* \in \Psi^d(G/H) \quad \text{for any} \quad P \in \Psi^d(G).$$

Proof. We show that the conditions of the characterization in Theorem 3.1.15 are satisfied. Since $\pi : G \to G/H$ is a surjective submersion, smooth vector fields X on G have lifts. That is, there is a (generally not unique) vector field $\widetilde{X}$ on G/H such that $\widetilde{X}\pi^* = \pi^* X$. Thus, if $g \in C^\infty(G)$ and X is a smooth vector field with lift $\widetilde{X}$, we compute

$$\begin{aligned}
[X, \Pi_{G/H} P \pi^*] &= \Pi_{G/H} [\widetilde{X}, (\pi^* \Pi_{G/H} P)] \pi^*, \\
[g, \Pi_{G/H} P \pi^*] &= \Pi_{G/H} [\pi^* g, (\pi^* \Pi_{G/H} P)] \pi^*,
\end{aligned}$$

and these have the right boundedness properties by Lemma 3.2.4, 3.2.5 and 3.2.6. □

Combining all the above results, we obtain a "lifting calculus" from $\Psi(G/H)$ to $\Psi(G)$. It is based on the notion of a lift of an operator. Let $d \in \mathbb{R}$.

Definition 3.2.6. *A lift of $A \in \Psi^d(G/H)$ is an operator $\widetilde{A} \in \Psi^d(G)$ that projects to A:*

1. $\pi^* \Pi_{G/H} \widetilde{A} = \widetilde{A}$.
2. $\Pi_{G/H} \widetilde{A} \pi^* = A$.

Theorem 3.2.11 (Ruzhansky and Turunen [48], Ruzhansky, Turunen and Wirth [49])**.** *Suppose that $A \in \Psi(G/H)$ lifts to $\widetilde{A}$ and $B \in \Psi(G/H)$ lifts to $\widetilde{B}$. Then:*

1. *$\widetilde{A}\widetilde{B}$ is a lift of AB.*
2. *$\widetilde{A}^*$ is a lift of A^*.*

Furthermore, if $\widetilde{A}$ is also elliptic, the following holds:

1. *$\widetilde{A}$ has a parametrix $\widetilde{Q}$ such that $\pi^* \Pi_{G/H} \widetilde{Q} = \widetilde{Q}$.*
2. *A is elliptic with parametrix $Q = \Pi_{G/H} \widetilde{Q} \pi^*$.*

Proof. Of course, $\Pi_{G/H} \widetilde{A}\widetilde{B} \pi^* = \Pi_{G/H} \widetilde{A} \pi^* \Pi_{G/H} \widetilde{B} \pi^* = AB$, and $\pi^* \Pi_{G/H} \widetilde{A}\widetilde{B} = \widetilde{A}\widetilde{B}$. Take $f, g \in C^\infty(G/H)$ and $u, v \in C^\infty(G)$, and compute

$$
\begin{aligned}
(Af, g)_{L^2(G/H)} &= \int_{G/H} \Big[\int_H \widetilde{A}(f \circ \pi)(xh)\, dh \Big] \overline{g(xH)}\, d\mu(xH) \\
&= \int_{G/H} f(xH) \int_H \overline{\widetilde{A}^*(g \circ \pi)(xh)}\, dh\, d\mu(xH) = (f, \Pi_{G/H} \widetilde{A}^* \pi^* g)_{L^2(G/H)},
\end{aligned}
$$

and also

$$
\begin{aligned}
(\widetilde{A}u, v)_{L^2(G)} &= \int_G \Big[\int_H \widetilde{A}u(xh)\, dh \Big] \overline{v(x)}\, dx \\
&= \int_G u(x) \overline{\Big[\int_H \widetilde{A}^* v(xh)\, dh \Big]}\, dx = (u, \pi^* \Pi_{G/H} \widetilde{A}^* v)_{L^2(G)}.
\end{aligned}
$$

Hence $\Pi_{G/H} \widetilde{A}^* \pi^* = A^*$ and $\pi^* \Pi_{G/H} \widetilde{A}^* = \widetilde{A}^*$ hold, which shows that $\widetilde{A}^*$ is a lift of A^*. Finally, suppose that $\widetilde{A}$ is an elliptic operator in $\Psi(G)$. Let $\widetilde{Q}_0$ be any parametrix to $\widetilde{A}$. Then $\widetilde{Q}_0 \widetilde{A} - \widetilde{A} \widetilde{Q}_0 \in \Psi^{-\infty}(G)$, and $\pi^* \Pi_{G/H} \widetilde{Q}_0 \widetilde{A} - \widetilde{A} \widetilde{Q}_0 \in \Psi^{-\infty}(G)$, and so

$$
\pi^* \Pi_{G/H} \widetilde{Q}_0 \widetilde{A} - I \in \Psi^{-\infty}(G).
$$

Thus we can take $\widetilde{Q} = \pi^* \Pi_{G/H} \widetilde{Q}_0$, which is valid as both a left and a right parametrix:

$$
\begin{aligned}
AQ - I &= \Pi_{G/H} \widetilde{A} \pi^* \Pi_{G/H} \widetilde{Q} \pi^* - I = \Pi_{G/H} (\widetilde{A}\widetilde{Q} - I) \pi^* \in \Psi^{-\infty}(G/H), \\
QA - I &= \Pi_{G/H} \widetilde{Q} \pi^* \Pi_{G/H} \widetilde{A} \pi^* - I = \Pi_{G/H} (\widetilde{Q}\widetilde{A} - I) \pi^* \in \Psi^{-\infty}(G/H),
\end{aligned}
$$

and this completes the proof. □

Theorem 3.2.12 (Ruzhansky and Turunen [48])**.** *If $d \in \mathbb{R}$, then*

$$
P \in \Psi^d(G/H) \quad \text{has a lift} \quad \widetilde{P} \in \Psi^d(G).
$$

3.2.2 Functional Calculus of Matrix-Symbols

In the Lie group setting, a functional calculus is available for the matrix-symbol algebra, and relies also on a sort of parameter-ellipticity, invented by Wirth and Ruzhansky [47]. Let Λ be a closed sector in $\mathbb{C}$, and let $p \in S^d$ be a matrix-symbol with $d \geq 0$.

Definition 3.2.7. *The symbol p is said to be parameter-elliptic w.r.t. Λ if*

1. *$p(x,\xi) - \lambda I$ is invertible for all $(x,[\xi]) \in G \times \widehat{G}$ when $\lambda \in \Lambda$.*
2. *The family of inverses satisfy the estimates*

$$\sup_{\lambda\in\Lambda} \sup_{(x,[\xi])\in G\times\widehat{G}} (\langle\xi\rangle^d + |\lambda|)||(p(x,\xi) - \lambda I)^{-1}|| < \infty.$$

The inequality ensures that such p has its spectra bounded uniformly away from 0. Thus there is a disc of radius $R > 0$ containing no eigenvalues of $p(x,\xi)$.

Theorem 3.2.13 (Wirth and Ruzhansky [47])**.** *Let p be parameter-elliptic w.r.t. Λ. Suppose that Λ' is a closed sector in $\mathbb{C}$ with $R > 0$ such that the following holds:*

1. *Λ'_R contains the union of all spectra of p in its interior.*
2. *Λ contains $\Gamma = \partial\Lambda'_R$ except for some disc about the origin.*

Let $f : \Lambda'_R \to \mathbb{C}$ be holomorphic, continuous up to $\overline{\Lambda'_R}$, and

$$\sup_{\lambda\in\Lambda'_R} |\lambda|^{-r}|f(\lambda)| < \infty \quad \text{for some} \quad r < 0.$$

Then "the function f of p" denoted $f(p)$, defined below, is a well-defined symbol in S^{rd}.

$$f(p) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda I - p)^{-1}\, d\lambda,$$

where Γ is viewed as a counter-clockwise contour about the union of all the spectra of p. Also, if J is compact, $\{p_Y\}_{Y\in J} \subset S^d$ bounded and uniformly parameter-elliptic w.r.t Λ, and the x-derivatives of p_Y are continuous in (x,Y), then $\{f(p_Y)\}_{Y\in J}$ is bounded in S^{rd}.

Of course, the above theorem can be extended to work for $r \geq 0$ by the usual trick. In particular, f can be the power of any $z \in \mathbb{C}$, with branch cut along $(-\infty, 0]$.

Lemma 3.2.7 (Wirth and Ruzhansky [47])**.** *Let p consist of positive-definite matrices. Suppose that they together satisfy*

$$\sup_{(x,[\xi])\in G\times\widehat{G}} \langle\xi\rangle^d||p(x,\xi)^{-1}|| < \infty.$$

Then p is parameter-elliptic w.r.t. $(-\infty, 0]$, and $p^z \in S^{\mathrm{Re}\,(z)d}$.

Proof. The hypothesis implies that the spectrum of $\langle\xi\rangle^{-d}p(x,\xi)$ is in $[\frac{1}{c}, c]$ for a $c > 0$. Since $p(x,\xi) - \lambda I$ is normal, the spectral radius theorem gives

$$||(p(x,\xi) - \lambda I)^{-1}|| \leq \Big(\frac{1}{c}\langle\xi\rangle^d + |\lambda|\Big)^{-1} \quad \text{for any} \quad \lambda \in (-\infty, 0],$$

and we then have $p^z \in S^{\mathrm{Re}\,(z)d}$ by the above theorem. ☐

Using the above theorem, we prove a variant of the Calderon-Vaillancourt theorem. It is a modified version of the one found in Ruzhansky and Wirth [47].

Lemma 3.2.8. *Let $d \in \mathbb{R}$ and $\{p_Y\}_{Y\in J} \subset S^d$ be a set bounded in the topology of S^d. Suppose that J is a compact space, and that x-derivatives of p_Y are continuous in (x, Y). Then $\mathrm{Op}(p_Y) : H^s(G) \to H^{s-d}(G)$ are bounded uniformly in $Y \in J$ for any $s \in \mathbb{R}$.*

Proof. By the continuity of the symbolic product, it suffices to prove the case $s = d = 0$. But in this case, there is some $C > 1$ such that

$$\sup_{(x,[\xi])\in G\times\widehat{G}} ||p_Y(x,\xi)|| \leq C - 1,$$

which ensures that $C^2 I - p_Y^* p_Y \geq \frac{1}{c} > 0$. By Lemma 3.2.7, we have

$$q_Y = \sqrt{C^2 I - p_Y^* p_Y} \in S^0,$$

and because of Theorem 3.2.13, the set $\{q_Y\}_{Y\in J} \subset S^0$ must be a bounded subset of S^0. Forming the product $q_Y^\dagger \odot q_Y$, we then get

$$q_Y^\dagger \odot q_Y - (C^2 I - p_Y^\dagger \odot p_Y) = r_Y \in S^{-1},$$

and $\{r_Y\}_{Y\in J} \in S^{-1}$ is of course also bounded in S^{-1}, by continuity of the product. Combining this, we can for any $u \in C^\infty(G)$ estimate

$$\begin{aligned}
||\mathrm{Op}(p_Y)u||^2_{L^2(G)} &= (u, \mathrm{Op}(p_Y)^*\mathrm{Op}(p_Y)u)_{L^2(G)} \\
&= C^2||u||^2_{L^2(G)} - ||\mathrm{Op}(q_Y)u||^2_{L^2(G)} - (u, \mathrm{Op}(r_Y)u)_{L^2(G)} \\
&\leq C^2||u||^2_{L^2(G)} + ||u||_{L^2(G)}||\mathrm{Op}(r_Y)u||_{L^2(G)},
\end{aligned}$$

where $\mathrm{Op}(r_Y) : L^2(G) \to L^2(G)$ is bounded uniformly in $Y \in J$, as we shall see next. Apply induction in $k \in \mathbb{N}$ to get

$$||\mathrm{Op}(r_Y)u||^{2^k} \leq ||u||^{2^{k-1}} ||\mathrm{Op}(r_Y^\dagger \odot r_Y)^{2^{k-1}} u||.$$

Then, eventually, $\{(r_Y^\dagger \odot r_Y)^{\odot 2^{k-1}}\}_{Y\in J} \subset S^{-\frac{n}{2}-1}$, and this subset is bounded in $S^{-\frac{n}{2}-1}$. Therefore, for such a fixed k, we define

$$w_Y = (r_Y^\dagger \odot r_Y)^{\odot 2^{k-1}} \quad \text{if} \quad 2^k > \frac{n}{2} + 1$$

and estimate the L^2-norm

$$\begin{aligned}
\int_G |\mathrm{Op}(w_Y)u(x)|^2\, dx &\leq \Big(\int_G \sum_{[\xi]\in\widehat{G}} d_\xi |\mathrm{Tr}(w_Y(x,\xi)^* w_Y(x,\xi))|\, dx\Big) ||u||^2_{L^2(G)} \\
&\leq \Big(\int_G \sum_{[\xi]\in\widehat{G}} d_\xi^2 ||w_Y(x,\xi)||^2\, dx\Big) ||u||^2_{L^2(G)} \\
&\leq (C')^2 \Big(\sum_{[\xi]\in\widehat{G}} d_\xi^2 \langle\xi\rangle^{2(-\frac{n}{2}-1)}\Big) ||u||^2_{L^2(G)},
\end{aligned}$$

where $||w_Y(x,\xi)|| \leq C'\langle\xi\rangle^{-\frac{n}{2}-1}$ holds with uniform constant $C' > 0$. ∎

Lemma 3.2.9 (Wirth and Ruzhansky [47]). *Suppose that* $a \in \mathrm{Mat}(\widehat{G})$, *and*

$$\sup_{[\xi]\in\widehat{G}} \langle\xi\rangle^{-d}||a(\xi)|| < \infty.$$

Then, if $q \in C^\infty(G)$, *there is a* $C > 0$, *depending only on* n, *such that*

$$\sup_{[\xi]\in\widehat{G}} \langle\xi\rangle^{-d}||(\mathcal{F}q\mathcal{F}^{-1}a)(\xi)|| \leq C||q||_{C^{\lceil |d|\rceil+\lceil\frac{n}{2}\rceil}(G)}\Big[\sup_{[\xi]\in\widehat{G}} \langle\xi\rangle^{-d}||a(\xi)||\Big].$$

Proof. Let us define $A : H^d(G) \to L^2(G) : u \mapsto u * \mathcal{F}^{-1}a$, convolution by $\mathcal{F}^{-1}a \in \mathcal{D}'(G)$ On the Fourier side it is $u \mapsto a\mathcal{F}u$. By the Parseval identiy, A is well-defined, and

$$||A|| = \sup_{[\xi]\in\widehat{G}} \langle\xi\rangle^{-d}||a(\xi)||.$$

Denote by M_f the multiplication by $f \in C^\infty(G)$. Put $q_y(x) = q(x^{-1}y)$ for all $x, y \in G$. Again, $\mathcal{F}(AM_{q_x}u) = (\mathcal{F}q\mathcal{F}^{-1}a)\mathcal{F}u$, and by Sobolev embedding, we estimate

$$\begin{aligned}
\int_G |(AM_{q_x}u)(x)|^2\,dx &\leq \int_G \sup_{y\in G} |(AM_{q_y}u)(x)|^2\,dx \\
&\leq C\int_G\int_G |(I-\Delta_y)^{\frac{1}{2}\lceil\frac{n}{2}\rceil}AM_{q_y}u(x)|^2\,dy\,dx \\
&= C\int_G\int_G |AM_{(I-\Delta_y)^{\frac{1}{2}\lceil\frac{n}{2}\rceil}q_y}u(x)|^2\,dx\,dy \\
&\leq C\Big[\int_G ||M_{(I-\Delta_y)^{\frac{1}{2}\lceil\frac{n}{2}\rceil}q_y}||^2_{B(H^d(G))}\,dy\Big]||A||^2_{B(H^d(G),L^2(G))}||u||^2_{H^d(G)},
\end{aligned}$$

and so we get

$$\sup_{[\xi]\in\widehat{G}} \langle\xi\rangle^{-d}||\mathcal{F}q\mathcal{F}^{-1}a(\xi)|| \leq C\sup_{y\in G}||M_{(I-\Delta_y)^{\frac{1}{2}\lceil\frac{n}{2}\rceil}q_y}||\Big[\sup_{[\xi]\in\widehat{G}} \langle\xi\rangle^{-d}||a(\xi)||\Big].$$

The proof is completed by using that $||M_f||_{B(H^d(G))} \leq C'||f||_{C^{\lceil |d|\rceil}(G)}$. □

In a similar way, if p is an operator symbol satisfying mild bounds on the derivatives, then one can show L^2-boundedness of $\mathrm{Op}(p)$ using the same type of argument.

Proposition 3.2.11 (Ruzhansky and Turunen [48]. Ruzhansky, Turunen and Wirth [49]). *Let* $P : C^\infty(G) \to C^\infty(G)$ *be continuous and linear with matrix-symbol* p *such that*

$$\sup_{(x,[\xi])\in G\times\widehat{G}} ||(I-\Delta_x)^{\frac{1}{2}\lceil\frac{n}{2}\rceil}p(x,\xi)|| < \infty.$$

Then P *extends to a bounded operator* $P : L^2(G) \to L^2(G)$, *and*

$$||P||_{B(L^2(G))} \leq C\sup_{(x,[\xi])\in G\times\widehat{G}} ||(I-\Delta_x)^{\frac{1}{2}\lceil\frac{n}{2}\rceil}p(x,\xi)||,$$

where C *is a dimensional constant independent of* A.

Finally, we will show how to form explicit asymptotic sums of Hörmander symbols. Let $X = (X_1, \cdots, X_n)$ be a basis of $\mathfrak{g}$, and let $\beta \in \mathbb{N}_0^n$ and $\alpha \in \mathbb{N}_0^{m\times m}$ be multi-indices. Take an approximate identity $\{\psi_\epsilon\}_{\epsilon>0} \subset C^\infty(G)$ for the convolution product $*$ on G. That is, if $f \in L^2(G)$, we have

$$||\psi_\epsilon * f - f||_{L^2(G)} \to 0 \quad \text{as} \quad \epsilon \to 0,$$

and same is true of $f \in H^s(G)$ in the $H^s(G)$-norm for any $s \in \mathbb{R}$.

It will be necessary to use the following lemma, based on Ruzhansky and Wirth [47]. Some differences appear in the argument presented here.

Lemma 3.2.10. *Let $p \in S^d$ with $d \in \mathbb{R}$. Put $R(x) = \mathcal{F}_\xi^{-1}[p(x,\xi)] \in \mathcal{D}'(G)$ for any $x \in G$. Then we have that*

$$R \in C^\infty(G, H^{-d-\lceil \frac{n}{2} \rceil}(G)).$$

Proof. Moving derivatives under $\mathcal{F}_\xi^{-1}$, we see that $R \in C^\infty(G, \mathcal{D}'(G))$ in the weak*-sense. Using the Parseval identity, if $x \in G$, we have

$$\begin{aligned} ||R(x)||_{H^{-d-k}(G)} &= \sum_{[\xi]\in\widehat{G}} d_\xi \langle\xi\rangle^{-2d-2\lceil \frac{n}{2}\rceil} \mathrm{Tr}\Big(p(x,\xi)^* p(x,\xi)\Big) \\ &\leq \sum_{[\xi]\in\widehat{G}} d_\xi^2 \langle\xi\rangle^{-2\lceil \frac{n}{2}\rceil} \Big[\langle\xi\rangle^{-d} ||p(x,\xi)||\Big]^2, \end{aligned}$$

and $G \to H^{-d-\lceil \frac{n}{2}\rceil}(G) : x \mapsto R(x)$ is differentiable, the X^β derivatives are just

$$X_x^\beta R(x) = \mathcal{F}_\xi^{-1}[X_x^\beta p(x,\xi)],$$

which we see by a similar estimate with a difference quotient. □

Theorem 3.2.14. *Let $\{p_j\}_{j=0}^\infty$ be a sequence with $p \in S^{d_j}$ and $d_j \searrow -\infty$ as $j \to \infty$. Then we can construct $p \sim \sum_{j=0}^\infty p_j$ by setting*

$$p(x,\xi) = \sum_{j=0}^\infty p_j(x,\xi)(I - \mathcal{F}\psi_{\epsilon_j}(\xi)) \quad \text{for all} \quad (x,[\xi]) \in G \times \widehat{G},$$

where the sum converges absolutely in $C^\infty(G, \mathcal{F}H^{-d_0-\lceil \frac{n}{2}\rceil}(G))$.

Proof. Define the distributions $R_j(x) = \mathcal{F}_\xi^{-1}[p_j(x,\xi)]$ for any $x \in G$ as in Lemma 3.2.10. Then we have

$$R_j \in C^\infty(G, H^{-d_j-\lceil \frac{n}{2}\rceil}(G)).$$

Pick $\{\epsilon_j\}_{j=0}^\infty$ such that

$$\sup_{x\in G} ||X_x^\beta R_j(x) - \psi_{\epsilon_j} * (X_x^\beta R_j(x))||_{H^{-d_j-\lceil \frac{n}{2}\rceil}(G)} < \frac{1}{2^{j+1}} \quad \text{when} \quad |\beta| \leq j.$$

But these estimates imply that the sum defining p is uniformly absolutely convergent. To see this, take $N \in \mathbb{N}_0$ so large that $d_j \leq d_0$ for $j \geq N$, and estimate

$$\sup_{x\in G} \sum_{j\geq N}^{\infty} ||X_x^\beta R_j(x) - \psi_{\epsilon_j} * X_x^\beta R_j(x)||_{H^{-d_0-\lceil \frac{n}{2}\rceil}(G)} \leq \sum_{j\geq N}^{\infty} \frac{1}{2^{j+1}} \leq 1,$$

where we use that the inclusion $H^{-d_j-\lceil \frac{n}{2}\rceil}(G) \hookrightarrow H^{-d_0-\lceil \frac{n}{2}\rceil}(G)$ has exactly unit norm. In particular, p is well-defined, and the sum in p converges in $C^\infty(G, \mathcal{F}H^{-d_0-\lceil \frac{n}{2}\rceil}(G))$. To see the asymptotic property, define r_N by

$$r_N(x,\xi) = \sum_{j=N}^{\infty} p_j(x,\xi)(I - \mathcal{F}\psi_{\epsilon_j}(\xi)),$$

and so, if $k \in \mathbb{N}$, we have

$$p(x,\xi) - \sum_{j=0}^{k-1} p_j(x,\xi) = \sum_{j=k}^{N-1} p_j(x,\xi) - \sum_{j=k}^{N-1} p_j(x,\xi)\mathcal{F}\psi_{\epsilon_j}(\xi) + r_N(x,\xi),$$

where the first term is in S^{d_k}, and the second is smoothing by the product rule for δ_ξ^α. It remains to estimate the r_N term. So doing, we are free to make N arbitrarily large. Using Lemma 3.2.9 and the Parseval identity, we get

$$\begin{aligned}\langle\xi\rangle^{|\alpha|-d_k} ||\delta_\xi^\alpha X_x^\beta r_N(x,\xi)|| &\leq C_{\alpha,\beta}\Big[\sup_{[\xi]\in\widehat{G}} \langle\xi\rangle^{|\alpha|-d_k} ||X_x^\beta r_N(x,\xi)||\Big] \\ &\leq C_{\alpha,\beta}\Big[\sum_{j=N}^{\infty} ||X_x^\beta R_j(x) - \psi_{\epsilon_j} * X_x^\beta R_j(x)||_{H^{|\alpha|-d_k}(G)}\Big] \leq C_{\alpha,\beta},\end{aligned}$$

where N is chosen so that $d_N + \lceil \frac{n}{2}\rceil < d_k - |\alpha|$, we use that

$$\begin{aligned}\langle\xi\rangle^{|\alpha|-d_k} ||X_x^\beta r_N(x,\xi)|| &\leq \sum_{j=N}^{\infty} \langle\xi\rangle^{|\alpha|-d_k} ||X_x^\beta p_j(x,\xi)(I - \mathcal{F}\psi_{\epsilon_j}(\xi))|| \\ &\leq \sum_{j=N}^{\infty} \Big|\Big|\mathcal{F}_\xi^{-1}\Big[X_x^\beta p_j(x,\xi)(I - \mathcal{F}\psi_{\epsilon_j}(\xi))\Big]\Big|\Big|_{H^{|\alpha|-d_k}(G)},\end{aligned}$$

and $C_{\alpha,\beta} > 0$ are some constants. □

4

Complexifications of Real-Analytic Manifolds

Questions about "holomorphic extensions" of functions on a real-analytic manifold M are meaningless without a notion of enveloping manifold, where such extensions can live. The prototypical example is that of the (complex-valued) real-analytic functions on $\mathbb{R}^n$, which naturally extend into connected neighbourhoods of $\mathbb{R}^n$ in $\mathbb{C}^n$ via series expansion. However, if we are given any real-analytic manifold M, there is no clear analogue of $\mathbb{C}^n$. In this case, we may try to complexify the transition functions and patch them together. Although rough, this idea turns out to work. It leads to the Bruhat-Whitney theorem, and this is the starting point for the study of complexifications.

Definition 4.0.1. *Let Y be a complex manifold with the (almost) complex structure J. A real-analytic submanifold X of Y is totally real in Y if $2\dim_{\mathbb{R}} X = \dim_{\mathbb{R}} Y$, and*

$$J_x(T_xX) \cap T_xX = \{0\} \quad \text{for any} \quad x \in X.$$

Theorem 4.0.1 (Bruhat and Whitney [62]. See also Leichtnam, Golse and Stenzel [54])**.** *Let M be a real-analytic manifold. The following holds:*

1. *There is a complex manifold $M_{\mathbb{C}}$ and a real-analytic embedding $\iota : M \to M_{\mathbb{C}}$, where $\iota(M)$ is totally real in $M_{\mathbb{C}}$, and $\dim_{\mathbb{R}} M_{\mathbb{C}} = 2\dim M$.*
2. *There is a unique anti-holomorphic involution on an open subset $W \subset M_{\mathbb{C}}$, where $\iota(M) \subset W$, and all points of $\iota(M)$ are fixed by the involution.*
3. *Let $M'_{\mathbb{C}}$ be another such manifold, ι' a totally real embedding of M in $M'_{\mathbb{C}}$. Then there are open $U \subset M_{\mathbb{C}}$ and $U' \subset M'_{\mathbb{C}}$ with $\iota(M) \subset U$ and $\iota'(M) \subset U'$, and a biholomorphism $\kappa : U \to U'$ such that $\iota' = \kappa \circ \iota$.*

$$\begin{array}{ccc} & & U' \\ & \overset{\iota'}{\nearrow} & \uparrow{\scriptstyle \kappa} \\ M & \xrightarrow{\ \iota\ } & U \end{array}$$

Here the third point states that the germ of the complexification about M is unique, while the other two state that M in $M_{\mathbb{C}}$ looks like $\mathbb{R}^n$ in $\mathbb{C}^n$ locally:

Proposition 4.0.1. *Let Y be a complex manifold, X a real-analytic submanifold of Y. Then the following are equivalent:*

1. *X is totally real in Y.*
2. *There is a holomorphic chart $\varphi : U \to \mathbb{C}^n$ of Y about any $x \in X$ such that*

$$X \cap U = \{y \in U \mid \operatorname{Im}\varphi(y) = 0\} \quad \text{and} \quad \operatorname{Re}\varphi : X \cap U \to \mathbb{R}^n \quad \text{is a chart of} \quad X.$$

It follows that we have an analogue of the principle of unique analytic continuation. This we see by locally reducing to the model case of $\mathbb{R}^n$ in $\mathbb{C}^n$.

Corollary 4.0.1. *Let Z be a complex manifold, X also connected and totally real in Y. If $f : X \to Z$ is real-analytic, there is a unique holomorphic $f_{\mathbb{C}} : W \to Z$ with $f_{\mathbb{C}}|_X = f$, where $W \supset X$ is a connected open subset of Y containing X.*

On a real-analytic manifold M there can be real-analytic objects beyond functions. For example, vector fields, differential forms, metrics, or even some higher order tensors. It is natural to ask if these somehow extend holomorphically into a complexification $M_{\mathbb{C}}$, where $M_{\mathbb{C}}$ carries the (almost) complex structure J, and $\iota(M)$ is totally real inside $M_{\mathbb{C}}$. Let us illustrate how this happens. We always identify M with $\iota(M)$.

Proposition 4.0.2. *Let T be any real-analytic section of $(T^*M)^{\otimes k}$ for some $k \in \mathbb{N}$. There is a neighbourhood $W \subset M_{\mathbb{C}}$ of M, and a unique holomorphic section $T_{\mathbb{C}}$ of $T^*_{k,0}W$, which extends T in the sense that at any point $x \in M$ we have*

$$(T_{\mathbb{C}})_x\Big(\frac{I_x - iJ_x}{2}\xi_1, \cdots, \frac{I_x - iJ_x}{2}\xi_k\Big) = T_x(\xi_1, \cdots, \xi_k) \quad \text{for any} \quad \xi_1, \cdots, \xi_k \in T_xM.$$

It has the following additional properties:

1. *If T is symmetric/anti-symmetric, then so is $T_{\mathbb{C}}$.*
2. *If T is non-vanishing, then W can be chosen so that $T_{\mathbb{C}}$ is non-vanishing.*

Proof. Using total reality, pick a holomorphic chart $\varphi : U \to \mathbb{C}^n$ of $M_{\mathbb{C}}$ about $x \in M$, which makes $M \hookrightarrow M_{\mathbb{C}}$ look like the inclusion of $\mathbb{R}^n$ in $\mathbb{C}^n$, and write

$$T|_{U\cap M} = \sum_{j_1,\cdots,j_k=1}^{n} T_{j_1,\cdots,j_k}\, d\mathrm{Re}\,\varphi_{j_1} \otimes \cdots \otimes d\mathrm{Re}\,\varphi_{j_k}.$$

The components $T_{j_1,\cdots,j_k} : U \cap M \to \mathbb{C}$ are real-analytic, and extend by Corollary 4.0.1. Let $(T_{\mathbb{C}})_{j_1,\cdots,j_k} : U_{\mathbb{C}} \to \mathbb{C}$ be their extension to $U_{\mathbb{C}} \subset U$ with $U_{\mathbb{C}} \cap M = U \cap M$, and

$$T_{\mathbb{C}}|_{U_{\mathbb{C}}} = \sum_{j_1,\cdots,j_k=1}^{n} (T_{\mathbb{C}})_{j_1,\cdots,j_k}\, d\varphi_{j_1} \otimes \cdots \otimes d\varphi_{j_k}.$$

This can be done for a collection of such charts, forming a locally finite covering of M. Since the compatibility relations between coefficients in different charts are real-analytic, and depend only on the transition functions, they extend at least as far as the coefficients. Also, any identity imposed on the coefficients continues by uniqueness to the extension, and therefore symmetry/anti-symmetry of the coefficients is preserved upon continuation. Hence the $(T_{\mathbb{C}})|_{U_{\mathbb{C}}}$ join together uniquely to a holomorphic section. □

Corollary 4.0.2. *Suppose that M has a real-analytic metric g, with Laplacian $\Delta = \Delta_g$. Then there is an open neighbourhood W of M in $M_{\mathbb{C}}$ such that $g_{\mathbb{C}}|_W$ is non-degenerate. If $f : W \to \mathbb{C}$ is holomorphic, $\Delta(f|_M)$ has a unique holomorphic extension $\Delta_{\mathbb{C}} f$ to W. In a holomorphic chart $\varphi = (z^1, \cdots, z^n) : U \to \mathbb{C}^n$ as above, it is given by*

$$(\Delta_{\mathbb{C}} f)|_{U\cap W} = \frac{1}{\sqrt{\det g_{\mathbb{C}}}} \sum_{i=1}^{n} \frac{\partial}{\partial z^i}\Big(\sum_{j=1}^{n} (g_{\mathbb{C}})^{ij} \sqrt{\det g_{\mathbb{C}}}\, \frac{\partial f}{\partial z^j}\Big).$$

A rudimentary distance measure exists for $M_{\mathbb{C}}$ when M is compact and Riemannian. It is obtained by holomorphically extending the map $\exp : T_xM \to M$ at every $x \in M$, which is real-analytic if both the manifold and the Riemannian metric are real-analytic. Using local trivializations, we get an $\epsilon > 0$ so that at each $x \in M$, it extends to

$$\exp_x : \{\xi \in T_xM \otimes_{\mathbb{R}} \mathbb{C} \mid \sqrt{g_x(\xi, \xi)} < \epsilon\} \to M_{\mathbb{C}},$$

where g_x has been extended (in the natural way) to a sesquilinear form on $T_xM \otimes_{\mathbb{R}} \mathbb{C}$. But then we may construct a "tubular neighbourhood" $T^{\epsilon}M$ of the zero section in TM, and consider the "imaginary time" exponential map on this special tube neighbourhood. That is, we put

$$T^{\epsilon}M = \{(x, \xi) \in TM \mid \sqrt{g_x(\xi, \xi)} < \epsilon\},$$

and on this set consider the map

$$(x, \xi) \mapsto \exp_x(i\xi).$$

Proposition 4.0.3 (Leichtnam, Golse and Stenzel [54]. See also Lempert and Szöke [42])**.** *There is an $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ the following holds:*

1. *The map Φ below is real-analytic diffeomorphism onto its image M_ϵ in $M_{\mathbb{C}}$:*

$$\Phi : T^{\epsilon}M \to M_{\mathbb{C}} : (x, \xi) \mapsto \exp_x(i\xi).$$

2. *The map π below is a real-analytic fibration with totally real fibers in M_ϵ:*

$$\pi : M_\epsilon \to M : \exp_x(i\xi) \mapsto x.$$

Proof. Using the standard identification $T_{(x,\xi)}TM \cong T_xM \oplus T_\xi(T_xM)$ at $(x, \xi) \in TM$. Evaluated at $(x, 0) \in T^{\epsilon}M$, the differential of Φ is

$$(d\Phi)_{(x,0)}(\delta x, \delta\xi) = \delta x + J\delta\xi \quad \text{for any} \quad (\delta x, \delta\xi) \in T_xM \oplus T_xM.$$

It follows that $d\Phi$ has full rank on the zero section in $T^{\epsilon}M$, because M is totally real. Therefore, by the compactness of M, Φ is a diffeomorphism for $\epsilon_0 > 0$ small enough. Consequently, π is equal to Φ^{-1} followed by the tangent bundle projection $TM \to M$. Then, we see that at any $x \in M \subset M_\epsilon$, we have

$$T_x\pi^{-1}(x) = \ker d\pi_x = J_xT_xM,$$

and by continuity, for z in a neighbourhood in M_ϵ of each such x, we also get

$$T_z\pi^{-1}(x) \cap J_z(T_z\pi^{-1}(x)) = \ker d\pi_z \cap J_z(\ker d\pi_z) = \{0\},$$

whence by compactness, there is an $\epsilon_0 > 0$ such that all fibers of π are totally real. □

Corollary 4.0.3. *Shrinking ϵ_0, then for all $\epsilon \in (0, \epsilon_0)$ the following also holds:*

1. *g extends to a non-degenerate, symmetric holomorphic section of $T^*_{2,0}M_\epsilon$.*
2. *If it exists, vol_g extends to a non-vanishing holomorphic $(n, 0)$-form on M_ϵ.*

The extension is unique in either case.

4.1 Complexifications of Riemannian Manifolds

If we wish to measure how far an extension of a function $f \in C^\omega(M)$ reaches into $M_\mathbb{C}$, then we will need a coordinate-invariant notion of distance inside this enveloping space. It is natural to ask for a metric on $M_\mathbb{C}$ such that inclusion map $M \hookrightarrow M_\mathbb{C}$ is an isometry, but there are in general many choices [42]. Guillemin and Stenzel [16] build a special one. They turn M_ϵ into a Kähler manifold for $\epsilon > 0$ small enough.

Before we proceed, we make a few basic observations. Let N be a complex manifold. The Dolbeault lemma, the $\overline{\partial}$-analogue of the Poincaré lemma for N, implies the following: A real $\omega \in \Omega^{1,1}(N)$ is closed if and only if any $x \in N$ has a neighbourhood U such that

$$\omega|_U = i\partial\overline{\partial}\rho \quad \text{for some} \quad \rho \in C^\infty(U).$$

On the other hand, if $\rho \in C^2(N)$, we associate $i\partial\partial\rho \in \Omega^{1,1}(N)$ with the Levi form $\mathrm{Lev}(\rho)$. In any local holomorphic coordinates $(z^j) : U \to \mathbb{C}^n$, it is defined by

$$\mathrm{Lev}(\rho)|_U = \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z^i \partial \overline{z}^j}\, dz^i \otimes d\overline{z}^j,$$

and it is checked that these patch together to a smooth section $\mathrm{Lev}(\rho)$ of $T^*_{1,0}N \otimes T^*_{0,1}N$. The function ρ is said to be strictly plurisubharmonic if the Levi form is positive-definite.

Now we define the notion of a Kähler manifold. We tacitly extend all tensors $\mathbb{C}$-linearly. An easily readable introduction to Kähler manifolds can be found in Moroianu [44].

Definition 4.1.1. *A Kähler manifold is a manifold N with complex structure tensor J. It carries a Riemannian metric h, and a symplectic form ω, compatible with J via*

$$h(JX, Y) = \omega(X, Y) \quad \text{for all} \quad X, Y \in C^\infty(N, TN).$$

The symplectic form ω above is a real form $\omega \in \Omega^{1,1}(N)$. It is called the Kähler form. A function ρ for ω in some chart U as above is said to be a local Kähler potential for ω. In local holomorphic coordinates $(z^j) : U \to \mathbb{C}^n$, we have

$$\begin{aligned}\omega|_U &= \sum_{i,j=1}^{n} h\Big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j}\Big)\Big[(J^* dz^i) \otimes d\overline{z}^j + (J^* d\overline{z}^i) \otimes dz^j\Big] \\ &= i \sum_{i,j=1}^{n} h\Big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j}\Big)\, dz^i \wedge d\overline{z}^j,\end{aligned}$$

and shrinking U, if necessary, we also have

$$h\Big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j}\Big) = \frac{\partial^2\rho}{\partial z^i \partial \overline{z}^j} \quad \text{for some} \quad \rho \in C^\infty(U),$$

which means that $\mathrm{Lev}(\rho)$ is somehow "half" of h.

4.1.1 Grauert Tube Construction

The construction by Guillemin and Stenzel creates "tubes" about the core M in $M_{\mathbb{C}}$. Each tube carries a Kähler potential ρ, measuring distance in the imaginary directions. We go through the existence part of their construction.

Theorem 4.1.1 (Guillemin and Stenzel [16]. See alternatively Lempert and Szöke [42])**.** *On an open $W \supset M$ in $M_{\mathbb{C}}$ there is a real-analytic solution $\rho : W \to [0, \infty)$ of*

$$\begin{cases} (i\partial\overline{\partial}\sqrt{\rho})^{\wedge n} = 0 \quad \text{in} \quad W \setminus M, \\ \mathrm{Lev}(\rho)|_M = \frac{1}{2} g, \end{cases}$$

where W is invariant under the unique conjugation on $M_{\mathbb{C}}$, and so carries its restriction. The solution has the following properties:

1. *ρ is invariant under the conjugation on W.*
2. *ρ is strictly plurisubharmonic on W.*
3. *$\rho|_M = 0$ and $d\rho|_M = 0$.*

Actually, ρ is unique. This fact is not proved in [16, 17], but is in Stenzel's thesis [57]. When W is given the Kähler form $i\partial\overline{\partial}\rho$, $\mathrm{Lev}(\rho)|_M = \frac{1}{2} g$ forces $M \hookrightarrow W$ to be isometric. This is because if h is the (real) Hermitian metric associated with $i\partial\overline{\partial}\rho$, then

$$\mathrm{Lev}(\rho)|_M = \frac{1}{2} h|_M.$$

Proof. Let V be a neighbourhood of the diagonal in $M \times M$ on which distance is defined. That is, on this neighbourhood V, the geodesic distance function $r : V \to [0, \infty)$ exists, and we may take V to be invariant under reflections $M \times M \to M \times M : (p, q) \mapsto (q, p)$. Take $(p, q) \in V$. Put $\exp_q^{-1} p = v \in T_q M$. By the Gauss lemma, we have

$$|v|_q = |d_v \exp_q(v) v|_p,$$

and, if $p \neq q$, we can apply it one more time to calculate the x-derivative of r in (p, q). That is, taking $u \in T_p M$, we compute

$$\begin{aligned} (d_x r)(p, q) u &= g_q\Big(\frac{v}{|v|_q}, d_x[\exp_q^{-1}(x)] u\Big) \\ &= |v|_q^{-1} g_p(d_v \exp_q(v) v, u) = g_p\Big(\frac{d_v \exp_q(v) v}{|d_v \exp_q(v) v|_p}, u\Big), \end{aligned}$$

where we have used $r(p, q) = |\exp_q^{-1} p|_q$. It follows that $(d_x r)(p, q)$ is a unit covector. Varying $q \in V_p \setminus \{p\}$ for a neighbourhood V_p of p with $\{p\} \times V_p \subset V$, we get a map

$$V_p \setminus \{p\} \to S_p^* M : q \mapsto (d_x r)(p, q),$$

and $q \mapsto (d_x r)(p, q)$ is not of full rank; it maps an open set into the unit cosphere in $T_p^* M$. So, in any coordinates (x^i) and (y^j) about p and q, respectively, we must have

$$\det\Big(\frac{\partial^2 r}{\partial y^j \partial x^i}\Big)_{i,j=1}^n (p, q) = 0.$$

The idea is to extend r^2 to an open $V_{\mathbb{C}} \subset M_{\mathbb{C}} \times M_{\mathbb{C}}$ containing the diagonal of $M \times M$. It extends to such a unique holomorphic $r^2_{\mathbb{C}} : V_{\mathbb{C}} \to \mathbb{C}$ because r^2 is real-analytic on V. Let $M_{\mathbb{C}} \to M_{\mathbb{C}} : z \mapsto \overline{z}$ be the unique anti-holomorphic involution we know exists on $M_{\mathbb{C}}$. By considering M component-wise, we may take $V_{\mathbb{C}}$ to be connected, with $V_{\mathbb{C}} \cap M = V$, and also invariant under both reflection $(z, w) \mapsto (w, z)$ and conjugation $(z, w) \mapsto (\overline{z}, \overline{w})$. Since r is real and symmetric, this gives

$$r^2_{\mathbb{C}}(z, w) = r^2_{\mathbb{C}}(w, z) \quad \text{and} \quad \overline{r^2_{\mathbb{C}}(z, w)} = r^2_{\mathbb{C}}(\overline{z}, \overline{w}) \quad \text{if} \quad (z, w) \in V_{\mathbb{C}}.$$

Note that $(r^2_{\mathbb{C}})^{-1}(0)$ is a complex hypersurface in $V_{\mathbb{C}}$. Thus $V_{\mathbb{C}} \setminus (r^2_{\mathbb{C}})^{-1}(0)$ is connected, and the two square roots of $r^2_{\mathbb{C}}$ are holomorphic on this set. Take $(z, w) \in V_{\mathbb{C}} \setminus (r^2_{\mathbb{C}})^{-1}(0)$. If (z^i) and (w^j) are holomorphic coordinates about z and w, respectively, then

$$\det \Big(\frac{\partial^2 \sqrt{\pm r^2_{\mathbb{C}}}}{\partial z^i \partial w^j} \Big)^n_{i,j=1} (z, w) = 0.$$

Choose a small neighbourhood W of M in $M_{\mathbb{C}}$ such that $(z, \overline{z}) \in V_{\mathbb{C}}$ whenever $z \in W$. Let $\varphi = (\varphi_i) : U \to \mathbb{C}^n$ be holomorphically extended geodesic coordinates about $z_0 \in M$, and with $U \subset W$ invariant under the involution on W. If $p, q \in M \cap U$, we have

$$r^2(p, q) = \sum_{i,j=1}^n g_{ij}(z_0)(\varphi_i(p) - \varphi_i(q))(\varphi_j(p) - \varphi_j(q)) + R(p, q),$$

where R is real-analytic with extension $R_{\mathbb{C}}$, and $R(z_0, z_0) = 0$ and $(d_x d_y R)(z_0, z_0) = 0$. This formula extends to U. So if $z, w \in U$, we get

$$r^2_{\mathbb{C}}(z, w) = \sum_{i,j=1}^n g_{ij}(z_0)(\varphi_i(z) - \varphi_i(w))(\varphi_j(z) - \varphi_j(w)) + R_{\mathbb{C}}(z, w),$$
$$r^2_{\mathbb{C}}(z, \overline{z}) = \sum_{i,j=1}^n g_{ij}(z_0)(\varphi_i(z) - \overline{\varphi_i(z)})(\varphi_j(z) - \overline{\varphi_j(z)}) + R_{\mathbb{C}}(z, \overline{z}),$$

where $r^2_{\mathbb{C}}(z, \overline{z})$ is strictly negative for $z \in W \setminus M$, if the neighbourhood W is small enough. By possibly shrinking W, we may put

$$\sqrt{\rho} : W \to i[0, \infty) : z \mapsto i \sqrt{-\frac{1}{4} r^2_{\mathbb{C}}(z, \overline{z})},$$

and (by abuse $\varphi = (z^i)$) see that

$$\mathrm{Lev}(\rho)|_{W \cap U} = \frac{1}{2} \sum_{i,j=1}^n \Big[g_{ij}(z_0) + \frac{1}{2} \frac{\partial^2 R_{\mathbb{C}}}{\partial z^i \partial \overline{z}^j} \Big] dz^i \otimes d\overline{z}^j,$$

which implies that $d\rho|_M = 0$, $\mathrm{Lev}(\rho)|_M = \frac{1}{2} g$, and ρ is strictly plurisubharmonic on W. In the geodesic coordinates, it is valid to put $w = \overline{z}$ for $z \in (W \cap U) \setminus M$ to get

$$\det \Big(\frac{\partial^2 \sqrt{\rho}}{\partial z^i \partial \overline{z}^j} \Big)^n_{i,j=1} (z, \overline{z}) = 0,$$

and therefore $(i \partial \overline{\partial} \sqrt{\rho})^{\wedge n} = 0$ holds on $W \setminus M$. □

Now let W be as above, and pick $\epsilon_0 > 0$ so small that $M_\epsilon \subset W$ whenever $\epsilon \in (0, \epsilon_0)$. The imaginary time exponential Φ maps to W, and M_ϵ carries the Kähler potential $\rho|_{M_\epsilon}$, where the strict plurisubharmonicity of ρ ensures that $\omega = i\partial\overline{\partial}\rho$ is always a Kähler form. This gives each of the "tubes" M_ϵ the structure of a Kähler manifold.

Corollary 4.1.1.

$$(\rho \circ \Phi)(x, \xi) = -|\xi|_x^2 \quad \text{for any} \quad (x, \xi) \in T^\epsilon M.$$

Proof. Observe that

$$r^2(\exp_x(\xi), \exp_x(-\xi)) = 4|\xi|_x^2,$$

which extends holomorphically from $\xi \in T_x^\epsilon M$ to those $\xi \in T_x M \otimes_{\mathbb{R}} \mathbb{C}$ with $g_x(\xi, \xi) < \epsilon^2$. The unique holomorphic extension evaluated in $i\xi$ for $\xi \in T_x^\epsilon M$ then gives

$$4(\rho \circ \Phi)(x, \xi) = r_{\mathbb{C}}^2(\exp_x(i\xi), \exp_x(-i\xi)) = -4|\xi|_x^2,$$

and this is exactly the desired equality. □

There is an alternative form of the (Monge-Ampere) equation for $\sqrt{\rho}$ in Theorem 4.1.1. Using the 2-form ω, we construct the unique vector field Ξ on W induced by $-\mathrm{Im}\,\overline{\partial}\rho$. That is, we put $i(\Xi)\omega = -\mathrm{Im}\,\overline{\partial}\rho$. It leads to an equivalent equation for ρ.

Corollary 4.1.2 (Stenzel [16]). *On $W \setminus M$, we have*

$$(i\partial\overline{\partial}\sqrt{\rho})^{\wedge n} = 0 \quad \text{if and only if} \quad \Xi\rho = 2\rho.$$

Proof. It will follow from expressing $(\partial\overline{\partial}\sqrt{\rho})^{\wedge n}$ in terms of derivatives of ρ instead of $\sqrt{\rho}$. Observe that on $W \setminus M$, we have

$$\rho\sqrt{\rho}\partial\overline{\partial}\sqrt{\rho} = \frac{1}{2}\rho\partial\overline{\partial}\rho - \frac{1}{4}(\partial\rho \wedge \overline{\partial}\rho),$$

and also

$$0 = i(\Xi)(d\rho \wedge \omega^{\wedge n}) = (\Xi\rho)\omega^{\wedge n} - d\rho \wedge i(\Xi)(\omega^{\wedge n}),$$

where we used the product rule, and the anti-derivation property of the interior product. Combining the above observations, we get

$$\begin{aligned}
(\partial\overline{\partial}\sqrt{\rho})^{\wedge n} &= \Big(\frac{1}{2\sqrt{\rho}}\Big)^n (\partial\overline{\partial}\rho)^{\wedge n} - \frac{n}{4\rho\sqrt{\rho}}\Big(\frac{1}{2\sqrt{\rho}}\Big)^{n-1}(\partial\rho \wedge \overline{\partial}\rho) \wedge (\partial\overline{\partial}\rho)^{\wedge(n-1)} \\
&= \Big(\frac{-i}{2\sqrt{\rho}}\Big)^n \Big(\omega^{\wedge n} + \frac{n}{2\rho} d\rho \wedge \mathrm{Im}\,\overline{\partial}\rho \wedge \omega^{\wedge(n-1)}\Big) \\
&= \Big(\frac{-i}{2\sqrt{\rho}}\Big)^n \Big(\omega^{\wedge n} - \frac{n}{2\rho} d\rho \wedge (i(\Xi)\omega \wedge \omega^{\wedge(n-1)})\Big) \\
&= \Big(\frac{-i}{2\sqrt{\rho}}\Big)^n \Big(\omega^{\wedge n} - \frac{1}{2\rho} d\rho \wedge i(\Xi)(\omega^{\wedge n})\Big) \\
&= \Big(\frac{-i}{2\sqrt{\rho}}\Big)^n \Big(1 - \frac{1}{2\rho}\Xi(\rho)\Big)\omega^{\wedge n},
\end{aligned}$$

which is zero if and only if $\Xi\rho = 2\rho$, because $\omega^{\wedge n}$ is non-vanishing. □

The Grauert tubes relative to $M_{\mathbb{C}}$ are those M_ϵ contained in the neighbourhood W. They are the images of $T^\epsilon M$ under the imaginary time exponential map.

Definition 4.1.2. *The Grauert tube of radius $\epsilon \in (0, \epsilon_0)$ is the image $M_\epsilon \subset W$ of*

$$\Phi : T^\epsilon M \to M_{\mathbb{C}} : (x, \xi) \mapsto \exp_x(i\xi).$$

Besides each M_ϵ carrying a Kähler potential, Φ is a symplectomorphism onto M_ϵ. This is meant relative to the canonical symplectic structure on TM coming from T^*M. In fact, even more is true. The canonical 1-form corresponds exactly to $-\mathrm{Im}\,\overline{\partial}\rho$ under Φ, and pullback via Φ to $T^\epsilon M$ of the structure on M_ϵ yields a unique structure on $T^\epsilon M$, where the Kähler form $\Phi^*(i\partial\overline{\partial}\rho)$ exactly equals the canonical symplectic 2-form on $T^\epsilon M$. This was a main result of both Guillemin and Stenzel [16] and Lempert and Szöke [42]. Of course, T^*M and TM are here naturally identified via g.

Theorem 4.1.2 (Lempert and Szöke [42]. See also Golse, Leichtnam and Stenzel [54])**.** *Let $\epsilon \in (0, \epsilon_0)$. The following holds:*

1. *The 1-form $\Phi^*(-\mathrm{Im}\,\overline{\partial}\rho)$ is the canonical 1-form on $T^\epsilon M$.*
2. *The 2-form $\Phi^*(i\partial\overline{\partial}\rho)$ is the canonical symplectic 2-form on $T^\epsilon M$.*

There is a complex structure on $T^\epsilon M$ such that $\Phi : T^\epsilon M \to M_\epsilon$ is a biholomorphism. This structure is unique, independent of $M_{\mathbb{C}}$, and has the following properties:

1. *The map $T^\epsilon M \to T^\epsilon M : (x, \xi) \mapsto (x, -\xi)$ is anti-holomorphic.*
2. *If $\gamma : \mathbb{R} \to M$ is any geodesic on M, it produces a holomorphic map*

$$\{(t, s) \in \mathbb{C} \,|\, |s| < \epsilon\} \to M_\epsilon : (t, s) \mapsto \exp_{\gamma(t)}(is\, d\gamma(t))$$

3. *It admits a real-analytic Kähler potential*

$$\varrho : T^\epsilon M \to \mathbb{R} : (x, \xi) \mapsto -|\xi|_x^2.$$

4. *Relative to this structure, $\Phi : T^\epsilon M \to M_\epsilon$ is a Kähler isomorphism.*

The structure is said to be adapted, and $T^\epsilon M$ is a Bruhat-Whitney complexification of M, which is a Kähler manifold such that the symplectic structure is the canonical structure.

Give M_ϵ the complex structure from $M_{\mathbb{C}}$, and the symplectic structure from $T^\epsilon M$. According to the above, these are compatible; $T^\epsilon M$ is Kähler, with potential

$$\varrho = \Phi^*\rho,$$

and we automatically have

$$M_\epsilon = \{z \in M_{\mathbb{C}} \,|\, -i\sqrt{\rho}(z) < \epsilon\}.$$

In this way, we may view $T^\epsilon M$ as a particular Bruhat-Whitney complexification of M, where the complex structure comes from the exponential map extended relative to $M_{\mathbb{C}}$, and $T^\epsilon M$ is an open subset with C^ω boundary inside a compact real-analytic manifold. The latter fact can be obtained by just one-point compactifying each of the fibers of TM, which results in a sphere bundle over M containing $T^\epsilon M$ as such a subset.

4.1.2 Examples of Grauert Tubes

In general, it is impossible to obtain $\sqrt{\rho}$ explicitly. However, simple examples do exist. The most important being the sphere. Those below are discussed by Zelditch [65].

Example 4.1.1 (Torus $\mathbb{T}^n$)**.** Equip the n-torus $\mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n$ with the usual flat metric. The Bruhat-Whitney complexification is

$$\mathbb{T}^n_{\mathbb{C}} \cong \mathbb{C}^n/\mathbb{Z}^n.$$

The complexified geodesic at $([z], \xi) \in \mathbb{T}^n_{\mathbb{C}} \times \mathbb{C}^n \cong T\mathbb{T}^n_{\mathbb{C}}$ is

$$t \mapsto [z + t\xi],$$

and the Grauert tube function is

$$\sqrt{\rho} : \mathbb{T}^n_{\mathbb{C}} \to i[0, \infty) : [z] \mapsto i|\mathrm{Im}\,(z)|.$$

Example 4.1.2 (Sphere $\mathbb{S}^n$)**.** Equip the n-sphere $\mathbb{S}^n$ with the induced metric from $\mathbb{R}^n$. The Bruhat-Whitney complexification is

$$\mathbb{S}^n_{\mathbb{C}} = \Big\{(z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} \,\Big|\, \sum_{j=1}^{n+1} z_j^2 = 1\Big\}.$$

The complexified geodesic at $(z, \xi) \in T\mathbb{S}^n_{\mathbb{C}} \cong \{(z, \xi) \in T^*\mathbb{C}^{n+1} \,|\, \xi \cdot z = 0\}$ is

$$t \mapsto \cos(t\sqrt{\xi \cdot \xi})z + \frac{\sin(t\sqrt{\xi \cdot \xi})}{\sqrt{\xi \cdot \xi}}\xi,$$

and the Grauert tube function is

$$\sqrt{\rho} : \mathbb{S}^n_{\mathbb{C}} \to i[0, \infty) : z \mapsto \frac{i}{2}\cosh^{-1}(|z|^2).$$

Example 4.1.3 (Hyperbolic space $\mathbb{H}^n$)**.** Equip $\mathbb{H}^n$ with the $(n, 1)$ Lorentz metric $(\cdot, \cdot)_L$. This is an example of a non-compact manifold that admits an analogous construction. The Bruhat-Whitney complexification is

$$\mathbb{H}^n_{\mathbb{C}} = \Big\{(z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} \,\Big|\, -z_{n+1}^2 + \sum_{j=1}^{n} z_j^2 = -1\Big\}.$$

According to Kan and Daowei [35], the Grauert tube of $\mathbb{H}^n$ in $\mathbb{H}^n_{\mathbb{C}}$ is not an "entire" tube. Rather, it is the $\mathrm{Re}\,(z_{n+1}) > 0$ component of

$$\mathbb{H}^n_{\frac{\pi}{\sqrt{2}}} = \{z \in \mathbb{H}^n_{\mathbb{C}} \,|\, |\mathrm{Re}\,(z)|^2 + |\mathrm{Im}\,(z)|^2 < 1\},$$

and if $\cos^{-1}$ takes values in $[0, \pi)$, the analogous Grauert tube function is

$$\sqrt{\rho} : \mathbb{H}^n_{\frac{\pi}{\sqrt{2}}} \to i\Big[0, \frac{\pi}{\sqrt{2}}\Big) : z \mapsto \frac{i}{\sqrt{2}}\cos^{-1}(|\mathrm{Re}\,(z)|^2 + |\mathrm{Im}\,(z)|^2).$$

4.2 Complexifications of Lie Groups

Any Lie group G naturally carries a real-analytic structure. See for example Warner [40]. It admits a complexification $G_{\mathbb{C}}$, which, in fact, can be taken to be a complex Lie group. Here the prototypical (compact) example is the unitary group $\mathrm{U}(m)$ of dimension $m \in \mathbb{N}$, which is embedded as a totally real Lie subgroup inside the complex Lie group $\mathrm{GL}(m, \mathbb{C})$. Also, $G_{\mathbb{C}}$ can be made to satisfy a universal property that fixes it up to biholomorphism. We shall go through the construction. Let $\mathfrak{g}$ be the (real) Lie algebra of G.

Theorem 4.2.1. *Let G be a connected real Lie group with almost complex structure J. The following are equivalent:*

1. *(G, J) is a complex Lie group.*
2. *The Lie bracket is J-linear, and J commutes with left translations.*

Proof. Let us show (1) $\Rightarrow$ (2). By hypothesis, J commutes with left and right translation. Thus, J defines an automorphism of $\mathfrak{g}$, and it commutes with the adjoint representation. In particular, if $X, Y \in \mathfrak{g}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}\exp(t\,\mathrm{ad}(Y))JX &= \mathrm{Ad}(\exp(tY))JX \\ &= J\mathrm{Ad}(\exp(tY))X = J\exp(t\,\mathrm{ad}(Y))X,\end{aligned}$$

and then, taking the derivative in t at $t = 0$, we get $\mathrm{ad}(Y)JX = J\mathrm{ad}(Y)X$, as desired. Now, we show (2) $\Rightarrow$ (1). If the Lie bracket is J-linear, then if $X, Y \in \mathfrak{g}$, we have

$$\begin{aligned}N_J(X, Y) &= [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] \\ &= [X, Y] + J^2[X, Y] + J^2[X, Y] - J^2[X, Y] = 0,\end{aligned}$$

and so, by the Newlander-Nirenberg theorem, G admits a complex structure inducing J. Again, we have

$$\begin{aligned}\mathrm{Ad}(\exp(Y))JX &= \exp(\mathrm{ad}(Y))JX \\ &= J\exp(\mathrm{ad}(Y))X = J\mathrm{Ad}(\exp(Y))X,\end{aligned}$$

and since G is connected, this actually implies that J commutes with $\mathrm{Ad}(g)$ for all $g \in G$. It follows that it commutes with right translations

$$dR_g(JX) = dL_g\mathrm{Ad}(g^{-1})(JX) = JdR_g(X),$$

and thus, left and right multiplication by a fixed $g \in G$ is holomorphic as a map $G \to G$. This means $G \times G \to G : (g, h) \mapsto gh$ is separately holomorphic, and hence also jointly. Since this map is a submersion, its pre-image of $\{e\}$ is a complex submanifold of $G \times G$. But it is just the graph of $G \to G : g \mapsto g^{-1}$. So inversion is also holomorphic. □

Corollary 4.2.1. *Let H be a real Lie subgroup $H \hookrightarrow G$ of a complex Lie group (G, J). If the Lie algebra of H is closed under J, then H is a complex Lie subgroup of G.*

Proof. The almost complex structure J on TG restricts to TH and the above applies. Since $J|_{TH}$ already commutes with right translations, connectedness is not required. □

4.2.1 Universal Complexification

Let us assume that G is a real Lie group, which is given its natural real-analytic structure, and G is not necessarily compact nor connected.

Definition 4.2.1. *A universal complexification of G is a set $G_{\mathbb{C}}$ and a map $\iota : G \to G_{\mathbb{C}}$. The set $G_{\mathbb{C}}$ is a complex Lie group, and $\iota : G \to G_{\mathbb{C}}$ is a real Lie group homomorphism.*

$$\begin{array}{ccc} & & G_{\mathbb{C}} \\ & \overset{\iota}{\nearrow} & \big\downarrow \phi_{\mathbb{C}} \\ G & \xrightarrow{\ \phi\ } & H \end{array}$$

If $\phi : G \to H$ is a real Lie group homomorphism from G to some complex Lie group H, then there is a unique holomorphic homomorphism $\phi_{\mathbb{C}} : G_{\mathbb{C}} \to H$ such that $\phi = \phi_{\mathbb{C}} \circ \iota$.

Theorem 4.2.2. *If G is connected, then it admits a universal complexification.*

Proof. Let $\pi : \widetilde{G} \to G$ be a universal Lie group covering of G, where $\widetilde{G}$ has Lie algebra $\widetilde{\mathfrak{g}}$, and $\widetilde{G}_{\mathbb{C}}$ a simply-connected complex Lie group with the complex Lie algebra $\widetilde{\mathfrak{g}}_{\mathbb{C}} = \widetilde{\mathfrak{g}} \otimes_{\mathbb{R}} \mathbb{C}$. Take $\widetilde{\iota} : \widetilde{G} \to \widetilde{G}_{\mathbb{C}}$ to be the real homomorphism induced by $\widetilde{\mathfrak{g}} \hookrightarrow \widetilde{\mathfrak{g}} \otimes_{\mathbb{R}} \mathbb{C} : \widetilde{X} \mapsto \widetilde{X} \otimes_{\mathbb{R}} 1$. Let H be a complex Lie group with Lie algebra $\mathfrak{h}$, and $\phi : G \to H$ a real homomorphism. Then $d(\phi \circ \pi) : \widetilde{\mathfrak{g}} \to \mathfrak{h}$ extends canonically to a Lie algebra homomorphism of $\widetilde{\mathfrak{g}}_{\mathbb{C}}$ into $\mathfrak{h}$, and is thus the differential of a unique holomorphic homomorphism

$$\widetilde{\phi}_{\mathbb{C}} : \widetilde{G}_{\mathbb{C}} \to H \quad \text{with} \quad \phi \circ \pi = \widetilde{\phi}_{\mathbb{C}} \circ \widetilde{\iota}.$$

Let K be the intersection of every $\ker(\widetilde{\phi}_{\mathbb{C}})$ of such homomorphisms ϕ (up to equivalence). It follows that K is closed and normal, $\widetilde{G}_{\mathbb{C}}/K$ is a complex Lie group, and $\widetilde{\iota}(\ker(\pi)) \subset K$. So if $\pi_K : \widetilde{G}_{\mathbb{C}} \to \widetilde{G}_{\mathbb{C}}/K$ is the natural quotient, we have

$$\phi \circ \pi = \phi_{\mathbb{C}} \circ (\pi_K \circ \widetilde{\iota}),$$

where $\phi_{\mathbb{C}} : \widetilde{G}_{\mathbb{C}}/K \to H$ is the holomorphic homomorphism obtained by factoring out K. Let us therefore put $G_{\mathbb{C}} = \widetilde{G}_{\mathbb{C}}/K$. The inverse of the natural isomorphism $G \to \widetilde{G}/\ker(\pi)$, which comes from π, can be composed with $\pi_K \circ \widetilde{\iota}$ on the right after factoring out $\ker(\pi)$. This gives a continuous homomorphism $\iota : G \to G_{\mathbb{C}}$ with the right property; $\phi = \phi_{\mathbb{C}} \circ \iota$, and $\phi_{\mathbb{C}}$ is always unique because $\widetilde{\phi}_{\mathbb{C}}$ is unique. □

In fact, the connectedness assumption can be removed by a more involved argument. See Neeb [45] or Hochschild [26].

Theorem 4.2.3 (Neeb [45])**.** *Any Lie group G admits a universal complexification.*

Corollary 4.2.2. *There exists a unique anti-holomorphic involution, $G_{\mathbb{C}} \to G_{\mathbb{C}} : z \mapsto \overline{z}$, which has the property of being an endomorphism that restricts to the identity on $\iota(G)$.*

Proof. Give $H = G_{\mathbb{C}}$ the conjugate complex structure (the abstract group is unchanged). Applying the universal property to $\iota : G \to H = G_{\mathbb{C}}$ gives the map. □

Corollary 4.2.3. *If $\xi : G \to \mathrm{GL}(m, \mathbb{C})$ is a representation of G of dimension $m \in \mathbb{N}$, then there is a unique holomorphic representation $\xi_{\mathbb{C}} : G_{\mathbb{C}} \to \mathrm{GL}(m, \mathbb{C})$ so that $\xi = \xi_{\mathbb{C}} \circ \iota$.*

The universal property ensures that the pair $(G_{\mathbb{C}}, \iota)$ is unique up to biholomorphisms, and we say that it is a complexification of G if the homomorphism $\iota : G \to G_{\mathbb{C}}$ is injective. It is standard that compact G admit a complexification. See e.g. Hall [18].

Theorem 4.2.4. *If G is compact, it has a complexification $(G_{\mathbb{C}}, \iota)$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. This complexification has the following properties:*

1. *The homomorphism $d\iota : \mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$ maps $\mathfrak{g}$ onto a totally real subspace of $\mathfrak{g}_{\mathbb{C}}$.*
2. *The homomorphism $\iota : G \to G_{\mathbb{C}}$ is a real-analytic embedding of G into $G_{\mathbb{C}}$.*
3. *The image $\iota(G)$ is totally real, and is a maximal compact subgroup of $G_{\mathbb{C}}$.*
4. *If G is connected, then so is $G_{\mathbb{C}}$.*

Proof. By Corollary 3.2.1 G is isomorphic to a closed subgroup of $\mathrm{U}(m)$ for some $m \in \mathbb{N}$. So we may take $G \subset \mathrm{U}(m)$ and $\mathfrak{g} \subset \mathfrak{u}(m)$. Polar decomposition gives a map

$$\Phi : \mathrm{U}(m) \times \mathfrak{u}(m) \to \mathrm{GL}(m, \mathbb{C}) : (g, X) \mapsto \exp(iX)g,$$

and we may put

$$G_{\mathbb{C}} = \{\exp(iX)g \,|\, (g, X) \in G \times \mathfrak{g}\}.$$

The map Φ is a known diffeomorphism. Thus $G_{\mathbb{C}}$ is a closed submanifold of $\mathrm{GL}(m, \mathbb{C})$, and we get the complexification by taking ι to be the inclusion, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$.

Lemma 4.2.1. *The (matrix) tangent spaces are of the form $T_z G_{\mathbb{C}} = z\mathfrak{g}_{\mathbb{C}}$ for any $z \in G_{\mathbb{C}}$.*

Proof. Write $z = \exp(iX)g$, and note that $d\Phi(g, X)(gY, 0) = \exp(iX)gY \in z\mathfrak{g}_{\mathbb{C}}$ if $Y \in \mathfrak{g}$. Next, fix $\delta X \in \mathfrak{g}$, and put $\gamma_s(t) = \eta^{-1}(s, t)\partial_s \eta(s, t)$, where η is given by

$$\eta : \mathbb{R} \times \mathbb{R} \to G_{\mathbb{C}} : (s, t) \mapsto \Phi(g, t(X + s\delta X)).$$

Then $\eta^{-1}\partial_t \eta$ runs in $\mathfrak{g}_{\mathbb{C}}$, and γ_0 solves

$$\begin{cases} \partial_t \gamma_0 = \partial_s(\eta^{-1}\partial_t \eta) + [\gamma_0, \eta^{-1}\partial_t \eta] & \text{in} \quad \mathfrak{g}_{\mathbb{C}}, \\ \gamma_0(0) = 0. \end{cases}$$

It follows that $\gamma_0(t) \in \mathfrak{g}_{\mathbb{C}}$ for all $t \in \mathbb{R}$, and

$$d\Phi(g, X)(0, \delta X) = \partial_s \eta(0, 1) = z\gamma_0(1) \in z\mathfrak{g}_{\mathbb{C}},$$

which together with the first observation implies the lemma. ☐

Lemma 4.2.2. *The map $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}} : Z \mapsto zZz^{-1}$ is well-defined and injective if $z \in G_{\mathbb{C}}$.*

Proof. Write $z = \exp(iX)g$, put $\gamma(t) = \exp(itX)g$ for all $t \in \mathbb{R}$, and

$$\begin{cases} \partial_t(\gamma^{-1} Z \gamma) + [\gamma^{-1}\partial_t \gamma, \gamma^{-1} Z \gamma] = 0 & \text{in} \quad \mathfrak{g}_{\mathbb{C}}, \\ \gamma^{-1}(0) Z \gamma(0) \in \mathfrak{g}_{\mathbb{C}}. \end{cases}$$

As a consequence, we deduce that $\gamma^{-1}(t) Z \gamma(t) \in \mathfrak{g}_{\mathbb{C}}$ for all $t \in \mathbb{R}$. Therefore $z^{-1} Z z \in \mathfrak{g}_{\mathbb{C}}$, and since conjugation is clearly injective, we see that $zZz^{-1} \in \mathfrak{g}_{\mathbb{C}}$ holds as well. ☐

Once we show that $G_{\mathbb{C}}$ is a real Lie subgroup of $\mathrm{GL}(m,\mathbb{C})$, it is a complex subgroup. This is because its Lie algebra is closed under the almost complex structure on $\mathrm{GL}(m,\mathbb{C})$, which is just given by $T_z\mathrm{GL}(m,\mathbb{C}) \to T_z\mathrm{GL}(m,\mathbb{C}) : Z \mapsto z(iz^{-1}Z)$ for any $z \in \mathrm{GL}(m,\mathbb{C})$. But $T_eG_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, so this is clear.

Lemma 4.2.3. *The manifold $G_{\mathbb{C}}$ is a complex Lie subgroup of* $\mathrm{GL}(m,\mathbb{C})$*.*

Proof. Let us take two generic elements $z = \exp(iX)g \in G_{\mathbb{C}}$ and $w = \exp(iY)h \in G_{\mathbb{C}}$, and form paths $\alpha(t) = \exp(itX)g$, $\beta(t) = \exp(itY)h$ and $\eta(t) = \alpha(t)\beta(t)^{-1}$ for all $t \in \mathbb{R}$. An easy computation shows that

$$\eta^{-1}\partial_t\eta = \beta(\alpha^{-1}\partial_t\alpha - \beta^{-1}\partial_t\beta)\beta^{-1},$$

which implies $\eta(t)^{-1}\partial_t\eta(t) \in \mathfrak{g}_{\mathbb{C}}$ for all $t \in \mathbb{R}$, because conjugation takes $\mathfrak{g}_{\mathbb{C}}$ back to itself. Therefore $\mathrm{GL}(m,\mathbb{C}) \to T\mathrm{GL}(m,\mathbb{C}) : z \mapsto z(\eta(t)^{-1}\partial_t\eta(t))$ is tangent to $G_{\mathbb{C}}$ for any $t \in \mathbb{R}$. It follows that η runs in $G_{\mathbb{C}}$, because $G_{\mathbb{C}}$ is closed in $\mathrm{GL}(m,\mathbb{C})$, and $\gamma = \eta$ solves

$$\begin{cases} \partial_t\gamma = \gamma\eta^{-1}\partial_t\eta \quad \text{in} \quad \mathrm{GL}(m,\mathbb{C}), \\ \gamma(0) \in G. \end{cases}$$

In particular, $zw^{-1} \in G_{\mathbb{C}}$. □

Lemma 4.2.4. *The subgroup G is maximally compact in $G_{\mathbb{C}}$.*

Proof. Let $H \subset G_{\mathbb{C}}$ be a subgroup containing G properly. Take $z = \exp(iX)g \in H \setminus G$. Then $X \neq 0$ and $\exp(iX)$ is self-adjoint and positive-definite with a non-unit eigenvalue. Therefore $\{\exp(ikX)\}_{k=1}^{\infty} \subset H \setminus G$ can not have a convergent subsequence in $\mathrm{GL}(m,\mathbb{C})$, and so H can not be compact. □

Lemma 4.2.5. *The pair $(G_{\mathbb{C}}, \iota)$ is a universal complexification of G.*

Proof. Let $\phi : G \to H$ be a real Lie group homomorphism into a complex Lie group H. Recall that such ϕ are automatically real-analytic. By the above, G is totally real in $G_{\mathbb{C}}$. Then ϕ extends uniquely to a holomorphic $\phi_{\mathbb{C}} : U \to H$ on an open $U \subset G_{\mathbb{C}}$ with $G \subset U$. Since $\phi_{\mathbb{C}}$ is holomorphic on U, but a homomorphism on G, the extension satisfies

$$\phi_{\mathbb{C}}(zw) = \phi_{\mathbb{C}}(z)\phi_{\mathbb{C}}(w) \quad \text{for all} \quad z, w \in U_0,$$

where U_0 is a connected neighbourhood of the identity component G_0 with $U_0U_0 \subset U$. Pick a symmetric open V of $0 \in \mathfrak{g}$ with the property that

$$\exp(iX)U_0 \cap U_0 \neq \emptyset \quad \text{and} \quad \exp(iX) \in U_0 \quad \text{if} \quad X \in V,$$

and for $k \in \mathbb{N}_0$ consider

$$\exp(ikX)U_0 \to H : z \mapsto \phi_{\mathbb{C}}(\exp(iX))^k\phi_{\mathbb{C}}(\exp(-ikX)z).$$

These join together to a unique continuation on the component of $G_{\mathbb{C}}$ that contains G_0. It follows by translating that $\phi_{\mathbb{C}}$ extends to all of $G_{\mathbb{C}}$, and is a homomorphism. □

Finally, by construction, $G_{\mathbb{C}}$ is connected if G is, having just as many components. Combining the results above completes the theorem. □

4.2.2 Cartan Polar Decomposition

The diffeomorphism that was constructed above is known as the Cartan decomposition. Henceforth G is assumed to be compact.

Corollary 4.2.4. *Let $(G_{\mathbb{C}}, \iota)$ be the complexification of G in the proof of Theorem 4.2.4. Then $G_{\mathbb{C}}$ is diffeomorphic to $G \times \mathfrak{g}$ via*

$$G \times \mathfrak{g} \to G_{\mathbb{C}} : (g, X) \mapsto \exp(iX)\iota(g).$$

Theorem 4.2.5 (Hall [18])**.** *Let $H \subset G$ be a closed subgroup. The following holds:*

1. *The complexification $H_{\mathbb{C}}$ identifies naturally with a closed subgroup of $G_{\mathbb{C}}$.*
2. *The manifold G/H has a natural real-analytic embedding in $G_{\mathbb{C}}/H_{\mathbb{C}}$.*

Note that $G_{\mathbb{C}}/H_{\mathbb{C}}$ always admits a complex structure. See the references in Hall [18]. It is the unique one making the natural action holomorphic. That is, the action

$$G_{\mathbb{C}} \times G_{\mathbb{C}}/H_{\mathbb{C}} \to G_{\mathbb{C}}/H_{\mathbb{C}} : (w, zH_{\mathbb{C}}) \mapsto wzH_{\mathbb{C}}.$$

This is the assumed structure in the second point, given the first point.

Proof. Let $(H_{\mathbb{C}}, \iota_H)$ and $(G_{\mathbb{C}}, \iota_G)$ be complexifications (ι_H and ι_G are both embeddings). If $\iota : H \to G$ is the natural inclusion, we obtain $\iota_{\mathbb{C}} : H_{\mathbb{C}} \to G_{\mathbb{C}}$ such that

$$\iota_{\mathbb{C}} \circ \iota_H = \iota_G \circ \iota.$$

The injectiveness of the derivatives of ι_H and ι_G together imply that $d\iota_{\mathbb{C}}$ is also injective.

$$\begin{array}{ccccc} & & & & H_{\mathbb{C}} \\ & & \overset{\iota_H}{\nearrow} & & \downarrow{\scriptstyle \iota_{\mathbb{C}}} \\ H & \overset{\iota}{\hookrightarrow} & G & \xrightarrow{\iota_G} & G_{\mathbb{C}} \end{array}$$

Observe that $\iota_{\mathbb{C}}(\exp(iY)\iota_H(h)) = \exp(i\, d\iota_{\mathbb{C}}(Y))\iota_G(h)$ holds for any $h \in H$ and $Y \in \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H. Therefore, by the Cartan decomposition, $\iota_{\mathbb{C}}$ is injective. It also follows from this that $\iota_{\mathbb{C}}(H_{\mathbb{C}})$ is closed in $G_{\mathbb{C}}$, and that $\iota_{\mathbb{C}}(H_{\mathbb{C}}) \cap \iota_G(G) = \iota_G(H)$.

$$\begin{array}{ccc} G & \xrightarrow{\iota_G} & G_{\mathbb{C}} \\ \downarrow{\scriptstyle \pi_H} & \searrow & \downarrow{\scriptstyle \pi_{\iota_{\mathbb{C}}(H_{\mathbb{C}})}} \\ G/H & \xrightarrow{\kappa} & G_{\mathbb{C}}/\iota_{\mathbb{C}}(H_{\mathbb{C}}) \end{array}$$

Therefore $\kappa : G/H \to G_{\mathbb{C}}/\iota_{\mathbb{C}}(H_{\mathbb{C}}) : gH \mapsto \iota_G(g)\iota_{\mathbb{C}}(H_{\mathbb{C}})$ is a well-defined injective map, and is real-analytic, because of the natural quotients π_H and $\pi_{\iota_{\mathbb{C}}(H_{\mathbb{C}})}$ in the above diagram. The quotients intertwine left multiplication and left actions, so if $g \in G$, we have

$$\ker(d\pi_H)_g = \operatorname{span}\{(d\iota(X))_g \mid X \in \mathfrak{h}\} = \ker d(\pi_{\iota_{\mathbb{C}}(H_{\mathbb{C}})} \circ \iota_G)_g,$$

and hence $d\pi_{\iota_{\mathbb{C}}(H_{\mathbb{C}})} \circ d\iota_G = d\kappa \circ d\pi_H$ forces $d\kappa$ to be injective. Thus κ is an immersion. Because G/H is compact, it is actually an embedding. □

In addition, we may observe that the embedding constructed above is totally real. This follows from the neat relation between the kernels in the proof.

Corollary 4.2.5. *The natural embedding $G/H \hookrightarrow G_{\mathbb{C}}/H_{\mathbb{C}}$ is totally real.*

Proof. Overloading notation, J denotes the almost complex structures of $G_{\mathbb{C}}$ and $G_{\mathbb{C}}/H_{\mathbb{C}}$. Since $\iota_G : G \to G_{\mathbb{C}}$ is totally real, then by the proof above, if $g \in G$, we have

$$\ker(d\pi_{\iota_{\mathbb{C}}(H_{\mathbb{C}})})_{\iota_G(g)} = J_{\iota_G(g)}(d\iota_G)_g(\ker(d\pi_H)_g) \oplus (d\iota_G)_g(\ker(d\pi_H)_g).$$

Suppose that $X_g, Y_g \in T_gG$ are some tangent vectors such that

$$J_{\kappa(gH)} d\kappa_{gH}((d\pi_H)_g(X_g)) = d\kappa_{gH}((d\pi_H)_g(Y_g)).$$

Then, because $\pi_{\iota_{\mathbb{C}}(H_{\mathbb{C}})}$ is holomorphic, we have

$$J_{\iota_G(g)}(d\iota_G)_g(X_g) - (d\iota_G)_g(Y_g) \in \ker(d\pi_{\iota_{\mathbb{C}}(H_{\mathbb{C}})})_{\iota_G(g)}.$$

It follows that $X_g, Y_g \in \ker(d\pi_H)_g$. □

Proposition 4.2.1. *The quotient $G_{\mathbb{C}}/\iota(G)$ is simply-connected.*

Proof. Let $\gamma : [0,1] \to G_{\mathbb{C}}/\iota(G)$ be a continuous loop. It lifts via the natural quotient, which is a proper surjective submersion, and the lift is continuous. So, we get

$$\gamma(t) = \exp(iX(t))\iota(G) \quad \text{for all} \quad t \in [0,1],$$

where $[0,1] \to \mathfrak{g} : t \mapsto X(t)$ is continuous, and a loop by uniqueness of the decomposition. But $\mathfrak{g}$ is simply connected, so it is homotopic to $X(0)$. Thus γ is homotopic to $\gamma(0)$. □

Furthermore, $G_{\mathbb{C}}$ retains the property of unimodularity that G automatically enjoys. It follows from the fact that the modular function extends holomorphically.

Proposition 4.2.2. *The complexification $G_{\mathbb{C}}$ is unimodular.*

Proof. Using the universal property, $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ extends to $\mathrm{Ad} : G_{\mathbb{C}} \to \mathrm{Aut}(g_{\mathbb{C}})$. Therefore, if the compact group G is connected, then

$$\det \mathrm{Ad}(g) = 1 \quad \text{for all} \quad g \in G_{\mathbb{C}},$$

and the non-connected case follows by translation from the identity component. □

The complexification $G_{\mathbb{C}}$ is, properly understood, the *maximal* Grauert tube of G. Given an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, the tubes in $G_{\mathbb{C}}$ are easily described:

Proposition 4.2.3 (Hall [19]). *The Grauert tube function $\sqrt{\rho}$ of G is defined on all $G_{\mathbb{C}}$. It is simply given by*

$$\sqrt{\rho} : G_{\mathbb{C}} \to i[0,\infty) : \exp(iX)g \mapsto i|X|_{\mathfrak{g}},$$

and the tube G_ϵ, of any radius $\epsilon \in (0,\infty)$, is just

$$G_\epsilon = \left\{ \exp(iX) \,\middle|\, |X|_{\mathfrak{g}} < \epsilon \right\} G.$$

Analogous to Corollary 4.2.4, we have the map $(x,\xi) \mapsto \exp_x(i\xi)$ in the general setting. It coincides with the Lie exponential, and motivates the notion of an entire tube.

Definition 4.2.2. *Let $M_{\mathbb{C}}$ be a (choice of) Bruhat-Whitney complexification of (M,g). Then $M_{\mathbb{C}}$ is called entire if $\exp_x$ extends holomorphically to $T_xM \otimes_{\mathbb{R}} \mathbb{C}$ for each $x \in M$.*

Finally, there is a result relating integration on $G_{\mathbb{C}}$ to $G \times \mathfrak{g}$ via the Corollary 4.2.4. Identify G with $\iota(G) \subset G_{\mathbb{C}}$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus J\mathfrak{g}$, where J is the complex structure on $G_{\mathbb{C}}$. Any $\mathrm{Ad}(G)$-invariant inner product $(\cdot,\cdot)_{\mathfrak{g}}$ on $\mathfrak{g}$ extends to $\mathfrak{g}_{\mathbb{C}}$ by setting

$$(X_1 + JY_1, X_2 + JY_2)_{\mathfrak{g}_{\mathbb{C}}} = (X_1, X_2)_{\mathfrak{g}} + (Y_1, Y_2)_{\mathfrak{g}} \quad \text{for all} \quad X_1, X_2, Y_1, Y_2 \in \mathfrak{g},$$

which is real-valued, and $\mathrm{Ad}(G)$-invariant (not $\mathrm{Ad}(G_{\mathbb{C}})$), by holomorphy of multiplication. It determines a left-invariant metric on $G_{\mathbb{C}}$, which in turn determines a measure dz. Similarly, let dx be the measure on G induced by the chosen inner product on $\mathfrak{g}$.

Theorem 4.2.6 (Hall [19])**.** *If $f \in C_c(G_{\mathbb{C}})$, then*

$$\int_{G_{\mathbb{C}}} f(z)\, dz = \int_G \Big[\int_{\mathfrak{g}} f(\exp(iX)g) \frac{dX}{\Theta(X)^2} \Big]\, dg,$$

where $\Theta : \mathfrak{g} \to (0,\infty)$ is the $\mathrm{Ad}(G)$-invariant function defined by

$$\frac{1}{\Theta(X)^2} = \det\Big(\frac{\sin\, \mathrm{ad}(X)}{\mathrm{ad}(X)} \Big) \quad \text{for all} \quad X \in \mathfrak{g}.$$

Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$. Pick positive roots R^+ for $\mathfrak{a}$ relative to $\mathfrak{g}_{\mathbb{C}}$. Restricted to $\mathfrak{a}$, the function Θ is also expressed by

$$\Theta(X) = \prod_{\alpha \in R^+} \frac{\alpha(X)}{\sinh(\alpha(X))} \quad \text{for all} \quad X \in \mathfrak{a}.$$

5

Fourier Integral Operators

Almost all the operators that are encountered in the studty of PDE are instances of FIO. The class of FIO contains solution operators to many hyperbolic systems (propagators), but also trace operators, pseudo-differential operators and more. See e.g. Duistermaat [10]. It turns out that they provide a unified geometric framework for describing operators, where the singular structure of the kernel is encoded by a conic Lagrangian submanifold. This can be exploited using techniques of microlocal analysis to obtain precise descriptions of the trajectories of singularities propagated by solution operators to hyperbolic systems. In fact, we will later see that $e^{it\sqrt{-\Delta}}$ is related to our questions of holomorphic extension. This operator is the half-wave propagator, the solution operator to

$$\begin{cases} \left(\frac{\partial}{\partial t} - i\sqrt{-\Delta}\right)u = 0 & \text{on} \quad \mathbb{R} \times M, \\ u|_{t=0} = u_0 \in C^\infty(M), \end{cases}$$

where $u \in C^\infty(\mathbb{R} \times M)$, and M is a compact Riemannian manifold with its Laplacian Δ. This is a prototypical example of an FIO, and is perhaps also one of the most important. If X and Y are (smooth) manifolds, a general FIO is a continuous linear operator

$$A : C^\infty(Y, \Omega^{\frac{1}{2}}) \to \mathcal{D}'(X, \Omega^{\frac{1}{2}}),$$

and hence is defined uniquely by its kernel $K(A)$, which must locally be of a certain form. It is described by a collection of *phase functions* that are defined locally on $T^*(X \times Y) \setminus 0$, and which patch together to parametrize a certain closed conic Lagrangian submanifold. Usually, the phases are real-valued and positively homogeneous in the cotangent variable, but they can also be complex-valued (just as long as the imaginary part is non-negative). It turns out that $e^{it\sqrt{-\Delta}}$ becomes such an FIO upon Wick rotation $t \mapsto it$ for small t, and all functions in the image then extend holomorphically into a complexification of M. The resulting operator, which is called $\mathcal{P}_\epsilon$, will become a central object of study later. Because of this, we have included this chapter, with special focus on FIO.

The theory of FIO with real phase is due to Hörmander and Duistermaat [27, 30]. Later, Melin and Sjöstrand expanded this work to the theory of complex phase FIO [43], and then Sjöstrand finally developed a theory of "Cauchy Integral Operators" in [52], which was used to study propagation of analytic singularities in the complex domain. However, Sjöstrand's operators in [52] are not complex phase FIO in the original sense. The theory is given a modern exposition in the final volume of Hörmander's series [34]. It is without doubt a very powerful tool, but also very difficult to learn.

5.1 Real and Complex Phase FIO

The construction of FIO begins with the definition of regular real and complex phases, which enter into the local oscillatory integrals that are the whole foundation of the theory. They form parametrizations of conic Lagrangian submanifolds of the cotangent bundle, and the microlocal properties of FIO are determined by these. Let $n, N \in \mathbb{N}$.

A phase function ϕ is a special smooth function on a conic subset of $\mathbb{R}^n \times \mathbb{R}^N \setminus \{0\}$. The word "conic" means invariance under $(x,\theta) \mapsto (x, t\theta)$ for any $t \in (0,\infty)$.

Definition 5.1.1. *Let $N \in \mathbb{N}$ be fixed, and let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^N \setminus \{0\}$ be an open conic set. A real-valued $\phi \in C^\infty(\Gamma)$ is said to be a regular real phase if the following holds:*

1. *ϕ has no critical points.*
2. *ϕ is positively homogeneous of degree 1 in the second variable.*
3. *$\{d_{(x,\theta)}\partial_{\theta_j}\phi(x,\theta)\}_{j=1}^N$ is $\mathbb{R}$-linearly independent for each $(x,\theta) \in \Gamma_\phi$, where*

$$\Gamma_\phi = \{(x,\theta) \in \Gamma \,|\, d_\theta\phi(x,\theta) = 0\}.$$

There is an analogue in the complex case where we always take ϕ to be real-analytic. A more general situation is described by Melin and Sjöstrand in [43].

Definition 5.1.2. *Let $N \in \mathbb{N}$ be fixed, and let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^N \setminus \{0\}$ be an open conic set. A (complex) $\phi \in C^\omega(\Gamma)$ is said to be a regular complex phase if the following holds:*

1. *ϕ has no critical points.*
2. *ϕ is positively homogeneous of degree 1 in the second variable.*
3. *ϕ extends holomorphically to $\phi_\mathbb{C}$ on an open cone $\Gamma_\mathbb{C}$ containing Γ, where*

$$\Gamma_\mathbb{C} \subset \mathbb{C}^n \times \mathbb{C}^N \setminus \{0\} \quad \text{and} \quad \overline{\Gamma_\mathbb{C}} = \Gamma_\mathbb{C}.$$

4. *$\{d_{(x,\theta)}\partial_{\theta_j}\phi_\mathbb{C}(x,\theta)\}_{j=1}^N$ is $\mathbb{C}$-linearly independent for each $(x,\theta) \in \Gamma_\phi$, where*

$$\Gamma_\phi = \{(x,\theta) \in \Gamma_\mathbb{C} \,|\, d_\theta\phi_\mathbb{C}(x,\theta) = 0\}.$$

Furthermore, ϕ is said to be of positive type if $\operatorname{Im}\phi \geq 0$.

In either case, by the implicit function theorem, Γ_ϕ is a submanifold of dimension n. It is a complex submanifold for complex ϕ, by the holomorphic version of the theorem. In the real case, we put

$$\Gamma_\phi \to \Lambda_\phi \subset \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} : (x,\theta) \mapsto (x, d_x\phi(x,\theta)),$$

and in the complex case, we similarly put

$$\Gamma_\phi \to \Lambda_\phi \subset \mathbb{C}^n \times \mathbb{C}^n \setminus \{0\} : (x,\theta) \mapsto (x, d_x\phi_\mathbb{C}(x,\theta)).$$

The subsets Λ_ϕ, being the images of the mappings above, are of course always conic. Let us show that they are immersions, and that Λ_ϕ is real (or complex) Lagrangian.

Lemma 5.1.1. *Let ϕ be a regular phase function. The following holds:*

1. *If ϕ is real, Λ_ϕ is an immersed Lagrangian submanifold.*
2. *If ϕ is complex, Λ_ϕ is an immersed complex Lagrangian submanifold.*

Proof. Assume that ϕ is a real phase function. The argument for complex phase is similar. Take any point $(x,\theta) \in \Gamma_\phi$, and suppose that

$$(\delta x, d_x d_x \phi(x,\theta)\delta x + d_\theta d_x \phi(x,\theta)\delta\theta) = (0,0) \quad \text{for some} \quad (\delta x, \delta\theta) \in T_{(x,\theta)}\Gamma_\phi.$$

Then $\delta x = 0$ and $(d_\theta d_x \phi)(x,\theta)\delta\theta = 0$. The tangent vector must satisfy

$$d_x d_\theta \phi(x,\theta)\delta x + d_\theta d_\theta \phi(x,\theta)\delta\theta = 0,$$

which implies

$$d_\theta(d_{(x,\theta)}\phi)(x,\theta)\delta\theta = 0.$$

Thus $\delta\theta = 0$ by regularity. It follows that the differential is injective, so Λ_ϕ is immersed. In order for Λ_ϕ to be Lagrangian, the canonical 1-form must vanish

$$d_x\phi(x,\theta)\delta x = 0 \quad \text{for all} \quad (\delta x, \delta\theta) \in T_{(x,\theta))}\Gamma_\phi.$$

However, we have $d_\theta\phi(x,\theta) = 0$ by construction, and $\phi(x,\theta) = 0$ holds by homogeneity. The tangent vectors are therefore forced to satisfy

$$d_x\phi(x,\theta)\delta x + d_\theta\phi(x,\theta)\delta\theta = 0,$$

which shows what we want. In the complex case, just replace ϕ by $\phi_{\mathbb{C}}$ in the calculations, and take the tangent vectors out of the $(1,0)$ holomorphic tangent space. □

In the complex case, we will often need to isolate the "real part" of the sets Λ_ϕ or Γ_ϕ. These are defined by

$$(\Lambda_\phi)_{\mathbb{R}} = \Lambda_\phi \cap (\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}) \quad \text{and} \quad (\Gamma_\phi)_{\mathbb{R}} = \Gamma_\phi \cap (\mathbb{R}^n \times \mathbb{R}^N \setminus \{0\}).$$

Theorem 5.1.1 (Melin and Sjöstrand [43]. The notation used here is slightly different)**.** *If ϕ is a regular complex phase of positive type, then $(\Lambda_\phi)_{\mathbb{R}}$ is precisely the image of $(\Gamma_\phi)_{\mathbb{R}}$.*

The "regular" part of the above definitions of phase functions is not always required. It is possible to have degenerate phases, satisfying only the first two points.

Definition 5.1.3. *Let $N \in \mathbb{N}$ be fixed, and let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^N \setminus \{0\}$ be an open conic set. A $\phi \in C^\infty(\Gamma)$ is said to be a (general) phase function if the following holds:*

1. *ϕ has no critical points.*
2. *ϕ is positively homogeneous in the second variable.*

As before, it is of positive type if $\operatorname{Im}\phi \geq 0$.

A phase function allows us to define special distributions called oscillatory integrals. In the literature, these are often abusively treated as actual integrals, but they are not. The following theorem shows exactly how they are formed.

Theorem 5.1.2. *Let $N \in \mathbb{N}$ be fixed, and let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^N \setminus \{0\}$ be an open conic set. Assume that*

1. *ϕ is a (general) phase of positive type on Γ.*
2. *$a \in S^d(\mathbb{R}^n \times \mathbb{R}^N)$ is an amplitude of order $d \in \mathbb{R}$.*
3. *$\chi \in C_0^\infty(\mathbb{R}^N)$ equals 1 in a neighbourhood of the ball $\{\theta \in \mathbb{R}^N \mid |\theta| \leq 1\}$.*

Then the limit below exists, and the map defines a continuous linear functional:

$$I_\phi(a) : C_0^\infty(\mathbb{R}^n) \to \mathbb{C} : \varphi \mapsto \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} \chi(\epsilon\theta) a(x,\theta) \varphi(x)\, dx\, d\theta.$$

In addition, the following holds:

1. *The limit is independent of the choice of χ.*
2. *The distribution $I_\phi(a) \in \mathcal{D}'(\mathbb{R}^n)$ is supported in the base of Γ.*

Proof. Using a cutoff in θ near 0, we may assume that $a(x,\theta)$ vanishes whenever $|\theta| \leq 1$. Since ϕ has no critical points in Γ, we can define

$$L = \frac{1}{i} \frac{d_{(x,\theta)}\phi}{|d_{(x,\theta)}\phi|^2} \cdot d_{(x,\theta)} \quad \text{on} \quad \Gamma,$$

and away from $\theta = 0$, coefficients in the transposed operator L^t define symbols in (x,θ). Consequently, if $k \in \mathbb{N}$, it lowers the order

$$|(L^t)^k(\chi(\epsilon\theta)a(x,\theta)\varphi(x))| \leq C_k \langle\theta\rangle^{d-k} \max_{|\alpha| \leq k} \sup_{y \in \mathbb{R}^n} |\partial_y^\alpha \varphi(y)|,$$

where $C_k > 0$ is a constant depending on k but not $|\epsilon| \leq 1$. It holds especially for $\epsilon = 0$. This is because ϕ is positively homogeneous in θ, and $a(x,\theta)$ is assumed zero for $|\theta| \leq 1$. Then, if $d - k < -N$, we can integrate by parts and use the DCT to get

$$\begin{aligned} I_\phi(a)\varphi &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^n} e^{i\phi(x,\theta)} a_\epsilon(x,\theta)\varphi(x)\, dx\, d\theta \\ &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^n} e^{i\phi(x,\theta)} (L^t)^k(\chi(\epsilon\theta)a_\epsilon(x,\theta)\varphi(x))\, dx\, d\theta \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^n} e^{i\phi(x,\theta)} (L^t)^k(a(x,\theta)\varphi(x))\, dx\, d\theta, \end{aligned}$$

which shows existence, and independence of both χ and $k \in \mathbb{N}$, as long as $d - k < -N$. Finally, the map is a continuous functional, because

$$|I_\phi(a)\varphi| \leq C_k \Big[\int_{\mathrm{supp}(\varphi)} \int_{\mathbb{R}^N} \langle\theta\rangle^{d-k}\, d\theta\, dx \Big] \max_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |\partial_x^\alpha \varphi(x)|,$$

and thus $I_\phi(a) \in \mathcal{D}'(\mathbb{R}^n)$, with $\operatorname{supp} I_\phi(a)$ contained in the base of Γ. □

The above distribution $I_\phi(a)$ is the so-called oscillatory integral attached to ϕ and a. Its microlocal properties are controlled by Λ_ϕ and $\operatorname{ess\,supp}(a)$.

Theorem 5.1.3. *In extension of Theorem 5.1.2, the following holds:*

1. *If ϕ is a real phase, then*

$$\operatorname{WF} I_\phi(a) \subset \Lambda_\phi.$$

2. *If ϕ is a complex phase of positive type, then*

$$\operatorname{WF} I_\phi(a) \subset (\Lambda_\phi)_{\mathbb{R}}.$$

Proof. As before, by a cutoff in θ near 0, we can make $a(x,\theta) = 0$ provided that $|\theta| \le 1$. Pick any $(x_0, \xi_0) \notin \Lambda_\phi$. We can choose $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi(x_0) = 1$ such that:

1. There is a closed cone $\Gamma_0 \subset \mathbb{R}^N \setminus \{0\}$ (possibly empty), where

$$\{(x,\theta) \in \Gamma \cap (\operatorname{supp}(\varphi) \times \mathbb{R}^N \setminus \{0\}) \,|\, d_\theta \phi(x,\theta) = 0\} \subset \operatorname{supp}(\varphi) \times \Gamma_0.$$

2. There is a closed cone $\Gamma_\xi \subset \mathbb{R}^N \setminus \{0\}$ containing ξ_0, where

$$d_x\phi(x,\theta) \notin \Gamma_\xi \quad \text{if} \quad (x,\theta) \in \operatorname{supp}(\varphi) \times \Gamma_0.$$

Using a conic cutoff in θ, we may then assume that $\operatorname{cone\,supp}(a) \subset \operatorname{supp}(\varphi) \times \Gamma_0$ holds. This is because $d_\theta\phi(x,\theta) \ne 0$ holds for $(x,\theta) \in \operatorname{cone\,supp}(a) \subset \operatorname{supp}(\varphi) \times (\mathbb{R}^N \setminus \{0\}) \setminus \Gamma_0$, and integration by parts with $|d_\theta\phi|^{-2} d_\theta\phi \cdot d_\theta$ on this set gives a convergent integral in θ. On the other hand, $|d_x\phi(x,\theta) - \xi| \ge C(|\xi| + |\theta|)$ if $(x,\theta) \in \operatorname{supp}(\varphi) \times \Gamma_0$ and $\xi \in \Gamma_{\xi_0}$, where $C > 0$ depends only on the cones and φ. So we can for $\xi \in \Gamma_{\xi_0}$ define

$$L_\xi = \frac{1}{i} \frac{d_x\phi - \xi}{|d_x\phi - \xi|^2} \cdot d_x \quad \text{on} \quad \operatorname{supp}(\varphi) \times \Gamma_0.$$

The Leibniz rule gives a constant $C_k > 0$ independent of $\epsilon \le 1$ such that

$$|(L_\xi^t)^k(\chi(\epsilon\theta)a(x,\theta)\varphi(x))| \le C_k \frac{\langle\theta\rangle^d}{(|\xi| + |\theta|)^k} \max_{|\alpha| \le k} \sup_{y \in \mathbb{R}^n} |\partial_y^\alpha \varphi(y)|.$$

Then, if $N' \in \mathbb{N}$ and $k > d + N + N'$, we can integrate by parts and use the DCT to get

$$\begin{aligned}
\langle I_\phi(a)(x), e^{-ix\cdot\xi}\varphi(x)\rangle &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^n} e^{i(\phi(x,\theta) - x\cdot\xi)} \chi(\epsilon\theta) a(x,\theta)\varphi(x)\, dx\, d\theta \\
&= \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^n} e^{i(\phi(x,\theta) - x\cdot\xi)} (L_\xi^t)^k (\chi(\epsilon\theta) a(x,\theta)\varphi(x))\, dx\, d\theta \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^n} e^{i(\phi(x,\theta) - x\cdot\xi)} (L_\xi^t)^k (a(x,\theta)\varphi(x))\, dx\, d\theta,
\end{aligned}$$

where the last integral is absolutely convergent, and

$$\sup_{\xi \in \Gamma_{\xi_0}} \langle\xi\rangle^{N'} |\langle I_\phi(a)(x), e^{-ix\cdot\xi}\varphi(x)\rangle| \le C_k \sup_{\xi \in \Gamma_{\xi_0}} \int_{|\theta| \ge 1} \frac{\langle\xi\rangle^{N'}\langle\theta\rangle^d}{(|\xi| + |\theta|)^k}\, d\theta < \infty.$$

Therefore $(x_0, \xi_0) \notin \operatorname{WF} I_\phi(a)$. □

The connection between Λ_ϕ and $I_\phi(a)$ is revealed by the following cornerstone theorem. This is the theorem on the invariance of phase - the theory of global FIO rests on it.

Definition 5.1.4. *The notation $I_\phi(a)$ is used for distributions as in Theorem 5.1.2.*

Theorem 5.1.4 (Hörmander [27], Melin and Sjöstrand [43]. See also Duistermaat [10]). *Let $N, \widetilde{N} \in \mathbb{N}$ be fixed, and let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^N$ and $\widetilde{\Gamma} \subset \mathbb{R}^n \times \mathbb{R}^{\widetilde{N}}$ be two open conic sets. Let $\phi \in C^\infty(\Gamma)$ and $\widetilde{\phi} \in C^\infty(\widetilde{\Gamma})$ be regular real phases with the properties:*

1. *There is a point $(x_0, \xi_0) \in \Lambda_\phi \cap \Lambda_{\widetilde{\phi}} \neq \emptyset$ such that*

$$\xi_0 = d_x\phi(x_0, \theta_0) = d_x\widetilde{\phi}(x_0, \widetilde{\theta}_0) \quad \text{with} \quad (x_0, \theta_0) \in \Gamma_\phi \quad \text{and} \quad (x_0, \widetilde{\theta}_0) \in \widetilde{\Gamma}_{\widetilde{\phi}}.$$

2. *There is an open set U containing (x_0, ξ_0) such that*

$$\Lambda_\phi \cap U = \Lambda_{\widetilde{\phi}} \cap U.$$

Then there exists an open cone $\Gamma' \subset \Gamma$, containing (x_0, θ_0), with the following property: Given any $d \in \mathbb{R}$ and

$$a \in S^d(\mathbb{R}^n \times \mathbb{R}^N) \quad \text{with} \quad (x_0, \theta_0) \in \operatorname{cone\,supp}(a) \subset \Gamma',$$

then there is an open cone $\widetilde{\Gamma}' \subset \widetilde{\Gamma}$ such that $I_\phi(a) = I_{\widetilde{\phi}}(\widetilde{a})$, where

$$\widetilde{a} \in S^{d + \frac{N - \widetilde{N}}{2}}(\mathbb{R}^n \times \mathbb{R}^{\widetilde{N}}) \quad \text{with} \quad (x_0, \widetilde{\theta}_0) \in \operatorname{cone\,supp}(\widetilde{a}) \subset \widetilde{\Gamma}'.$$

The same is true if $\phi \in C^\omega(\Gamma)$ and $\widetilde{\phi} \in C^\omega(\widetilde{\Gamma})$ are regular complex phases of positive type, but where we ask for $(x_0, \xi_0) \in (\Lambda_\phi)_\mathbb{R} \cap (\Lambda_{\widetilde{\phi}})_\mathbb{R} \neq \emptyset$ to hold in the first condition instead. Also, the same holds if each space of symbols above is replaced by its PHG analogue.

In order to globalize in the complex case, we must complexify the cotangent bundle. Let X be a real-analytic manifold, and $X_\mathbb{C}$ a Bruhat-Whitney complexification of X. Then $T^*_{1,0}X_\mathbb{C} \setminus 0$ is holomorphically symplectic, and complexifies $T^*X \setminus 0$ via

$$T^*X \setminus 0 \hookrightarrow T^*_{1,0}X_\mathbb{C} \setminus 0 : (x, \xi) \mapsto \Big(x, \frac{1}{2}(I_x - iJ_x)^*\xi\Big).$$

Definition 5.1.5. *Let $\Lambda_\mathbb{C} \subset T^*_{1,0}X_\mathbb{C} \setminus 0$ be any conic complex Lagrangian submanifold. The real part $\Lambda_\mathbb{R}$ of $\Lambda_\mathbb{C}$ is defined by*

$$\Lambda_\mathbb{R} = \Lambda_\mathbb{C} \cap \frac{1}{2}(I - iJ)T^*X \setminus 0.$$

It is customary to say that $\Lambda_\mathbb{C}$ is closed if $\Lambda_\mathbb{R}$ is closed (see the conventions used in [43]). Also, $\Lambda_\mathbb{C}$ is said to be positive if every point of $\Lambda_\mathbb{R}$ is contained in a Λ_ϕ of the form

$$\Gamma_\phi \to \Lambda_\phi \subset \Lambda_\mathbb{C} : (x, \theta) \mapsto d\kappa^t_\mathbb{C}(x, d_x\phi_\mathbb{C}(x, \theta)),$$

where ϕ is some regular complex phase of positive type on an open cone $\Gamma \subset \kappa(U) \times \mathbb{R}^N$, and $\kappa_\mathbb{C} : U_\mathbb{C} \to \mathbb{C}^n$ is the holomorphic extension to $X_\mathbb{C}$ of a chart $\kappa : U \to \mathbb{R}^n$ of X.

5.1.1 The Global Calculus

The local result on the invariance of phase in oscillatory integrals leads to global FIO, where the local pieces Λ_ϕ parametrize a global conic Lagrangian submanifold.

Let X, Y and Z be three smooth manifolds of dimension n_X, n_Y and n_Z, respectively. The class of Fourier Integral Distributions (FID) is defined in the following way:

Definition 5.1.6. *Let Λ be any fixed closed conic Lagrangian submanifold of $T^*X \setminus 0$. An order $d \in \mathbb{R}$ FID $u \in I^d(X, \Lambda)$ is a $u \in \mathcal{D}'(X, \Omega^{\frac{1}{2}})$ of the form*

$$\langle u, \varphi \rangle = \sum_{j \in \mathbb{N}} I_{\phi_j}(a_j)(\Phi_j(\varphi|_{U_j})) + \langle v, \varphi \rangle \quad \text{for all} \quad \varphi \in C_0^\infty(X, \Omega^{\frac{1}{2}}),$$

where $\{U_j\}_{j\in\mathbb{N}}$ are precompact charts of X forming a locally finite cover of the base of Λ. The chart maps $\kappa_j : U_j \to \mathbb{R}^{n_X}$ enter into the remaining components:

1. *$v \in C^\infty(X, \Omega^{\frac{1}{2}})$.*
2. *$\Phi_j : C^\infty(U, \Omega^{\frac{1}{2}}) \to C^\infty(\kappa(U))$ is the trivialization that is induced by κ_j.*
3. *Each $\phi_j \in C^\infty(\Gamma_j)$ is a real phase on an open conic set $\Gamma_j \subset \kappa_j(U_j) \times \mathbb{R}^{N_j}$. It must be regular, inducing a diffeomorphism*

$$\Gamma_{\phi_j} \to \Lambda_{\phi_j} \subset \Lambda : (x, \theta) \mapsto d\kappa_j^t(x, d_x\phi_j(x, \theta)),$$

where Λ_{ϕ_j} is open in Λ_j.

4. *Each a_j is an amplitude defined on $\kappa_j(U_j) \times \mathbb{R}^{N_j}$ with $\operatorname{cone\,supp} a_j \subset \Gamma_j$. It may be rapidly decaying, but must satisfy*

$$a_j \in S^{d - \frac{N_j}{2} + \frac{n_X}{4}}(\kappa_j(U_j) \times \mathbb{R}^{N_j}),$$

where $d \in \mathbb{R}$ is the order of u.

Also, put $u \in I^d_{\mathrm{phg}}(X, \Lambda)$ if the above holds with $a_j \in S^{d-\frac{N_j}{2}+\frac{n_X}{4}}_{\mathrm{phg}}(\kappa_j(U_j) \times \mathbb{R}^{N_j})$ instead.

Thus u looks like $I_\phi(a)$, with ϕ parametrizing Λ locally, near points in X under Λ. The definition is independent of the charts by Theorem 5.1.4, and the form of $\mathrm{WF}\, I_\phi(a)$. In general, Λ may have multiple rays in $T^*X \setminus 0$ emanating from the same base point, and several ϕ_j may be needed to parametrize Λ near such a point.

Let Λ be any closed conic Lagrangian (embedded) submanifold of $T^*(X \times Y) \setminus 0$. Associated to Λ is a set C, defined by

$$C = \Lambda' = \Big\{((x,\xi),(y,\eta)) \in T^*X \times T^*Y \,\Big|\, (x, y, \xi, -\eta) \in \Lambda\Big\}.$$

Definition 5.1.7. *If $A : C_0^\infty(Y, \Omega^{\frac{1}{2}}) \to \mathcal{D}'(X, \Omega^{\frac{1}{2}})$ is continuous and linear, we write*

$$A \in I^d(X, Y; C) \quad \text{if} \quad K(A) \in I^d(X \times Y, \Lambda) \quad \text{for} \quad d \in \mathbb{R}.$$

The analogous space of real phase PHG FIO is denoted $I^d_{\mathrm{phg}}(X, Y; C)$.

On the other hand, there is a notion of complex phase FID, using the same notation. The difference is that X, Y and Z are real-analytic when dealing with complex phases, and we always consider closed conic *positive* complex Lagrangian submanifolds instead. So fix Bruhat-Whitney complexifications $X_{\mathbb{C}}$, $Y_{\mathbb{C}}$ and $Z_{\mathbb{C}}$, of X, Y and Z, respectively. This avoids the technicalities of almost analytic extensions in [43].

Definition 5.1.8. *Let $\Lambda_{\mathbb{C}}$ be a closed conic positive Lagrangian submanifold of $T^*_{1,0}X_{\mathbb{C}}\setminus 0$. An order $d \in \mathbb{R}$ FID $u \in I^d(X, \Lambda_{\mathbb{C}})$ is a $u \in \mathcal{D}'(X, \Omega^{\frac{1}{2}})$ of the form*

$$\langle u, \varphi\rangle = \sum_{j\in\mathbb{N}} I_{\phi_j}(a_j)(\Phi_j(\varphi|_{U_j})) + \langle v, \varphi\rangle \quad \text{for all} \quad \varphi \in C_0^\infty(X, \Omega^{\frac{1}{2}}),$$

where $\{U_j\}_{j\in\mathbb{N}}$ are locally finite and precompact charts of X that cover the base of $\Lambda_{\mathbb{R}}$. The chart maps $\kappa_j : U_j \to \mathbb{R}^{n_X}$ enter into the remaining components:

1. *$v \in C^\infty(X, \Omega^{\frac{1}{2}})$.*
2. *$\Phi_j : C^\infty(U, \Omega^{\frac{1}{2}}) \to C^\infty(\kappa(U))$ is the trivialization that is induced by κ_j.*
3. *Each $\phi_j \in C^\omega(\Gamma_j)$ is a complex phase on an open cone $\Gamma_j \subset \kappa_j(U_j) \times \mathbb{R}^{N_j}$. It must be positive and regular, inducing a diffeomorphism*

$$\Gamma_{\phi_j} \to \Lambda_{\phi_j} \subset \Lambda_{\mathbb{C}} : (x, \theta) \mapsto d(\kappa_j)^t_{\mathbb{C}}(x, d_x(\phi_j)_{\mathbb{C}}(x, \theta)),$$

where Λ_ϕ is open in $\Lambda_{\mathbb{C}}$, and $(\kappa_j)_{\mathbb{C}}$ is the holomorphic extension of κ_j.

4. *Each a_j is an amplitude defined on $\kappa_j(U_j) \times \mathbb{R}^{N_j}$ with* $\operatorname{cone\,supp} a_j \subset \Gamma_j$. *It may be rapidly decaying, but must satisfy*

$$a_j \in S^{d - \frac{N_j}{2} + \frac{n_X}{4}}(\kappa_j(U_j) \times \mathbb{R}^{N_j}),$$

where $d \in \mathbb{R}$ is the order of u.

Also, put $u \in I^d_{\text{phg}}(X, \Lambda_{\mathbb{C}})$ if the above holds with $a_j \in S_{\text{phg}}^{d - \frac{N_j}{2} + \frac{n_X}{4}}(\kappa_j(U_j) \times \mathbb{R}^{N_j})$ instead.

The notation $I^d(X, \Lambda_{\mathbb{C}})$ makes no distinction between real and complex phase FID. Given Λ or $\Lambda_{\mathbb{C}}$ (with the subscript $\mathbb{C}$) it should be clear from the context what is meant, and as before, Theorems 5.1.4 and 5.1.3 apply, so $I^d(X, \Lambda_{\mathbb{C}})$ is independent of the charts. The holomorphic diffeomorphisms need not cover all of $\Lambda_{\mathbb{C}}$, only parametrize it near $\Lambda_{\mathbb{R}}$. In the complex case, only the germ of $\Lambda_{\mathbb{C}}$ about $\Lambda_{\mathbb{R}}$ really matters.

Let now $\Lambda_{\mathbb{C}}$ be a closed conic positive Lagrangian submanifold of $T^*_{1,0}(X_{\mathbb{C}} \times Y_{\mathbb{C}}) \setminus 0$. Analogous to the real case, we associate to $\Lambda_{\mathbb{C}}$ a set $C_{\mathbb{C}}$, defined by

$$C_{\mathbb{C}} = \Lambda'_{\mathbb{C}} = \Big\{((x, \xi), (y, \eta)) \in T^*_{1,0}X_{\mathbb{C}} \times T^*_{1,0}Y_{\mathbb{C}} \,\Big|\, (x, y, \xi, -\eta) \in \Lambda\Big\}.$$

Definition 5.1.9. *If $A : C_0^\infty(Y, \Omega^{\frac{1}{2}}) \to \mathcal{D}'(X, \Omega^{\frac{1}{2}})$ is continuous and linear, we write*

$$A \in I^d(X, Y; C_{\mathbb{C}}) \quad \text{if} \quad K(A) \in I^d(X \times Y, \Lambda_{\mathbb{C}}) \quad \text{for} \quad d \in \mathbb{R}.$$

The analogous space of complex phase PHG FIO is denoted $I^d_{\text{phg}}(X, Y; C_{\mathbb{C}})$.

It is implicit in the above and [10, 27] that it is necessary to "cut" cones of frequencies. By this we mean using an amplitude to isolate a cone (e.g. where Theorem 5.1.4 applies). The construction is simple. Pick $\chi_0 \in C_0^\infty(\mathbb{R})$ such that

$$\chi_0(t) = \begin{cases} 1 & \text{if} \quad |t| \geq 1, \\ 0 & \text{if} \quad |t| \leq \frac{1}{2}. \end{cases}$$

Suppose that Γ' is an open cone with closure lying in an open cone $\Gamma \subset V \times (\mathbb{R}^N \setminus 0)$, where $V \subset \mathbb{R}^n$ is open, and the closure of Γ' may or may not have compact base in V. Let $\chi \in C^\infty(V \times \mathbb{S}^{N-1})$ equal 1 on $\Gamma' \cap (V \times \mathbb{S}^{N-1})$ but 0 off $\Gamma \cap (V \times \mathbb{S}^{N-1})$, and

$$a(x,\theta) = \chi_0(|\theta|)\chi(x, \frac{\theta}{|\theta|}) \quad \text{for any} \quad (x,\theta) \in \mathbb{R}^n \times \mathbb{R}^N,$$

which is then a symbol $a \in S^0_{\mathrm{phg}}(V \times \mathbb{R}^N)$ with $\operatorname{cone\,supp}(a) \subset \Gamma$.

The wavefront set of an FID is by their very definition contained inside Λ (or $\Lambda_\mathbb{C}$). Let Λ and $\Lambda_\mathbb{C}$ be submanifolds as in the above definitions, and let $d \in \mathbb{R}$.

Proposition 5.1.1. *The following holds:*

1. *If $u \in I^d(X, \Lambda)$, then*

$$\mathrm{WF}(u) \subset \Lambda.$$

2. *If $u \in I^d(X, \Lambda_\mathbb{C})$, then*

$$\mathrm{WF}(u) \subset \Lambda_\mathbb{R}.$$

Proof. Apply Theorem 5.1.3 to the individual oscillatory integrals making up u. □

It is customary to talk about C instead of Λ, because of the wavefront relation for A. By this we mean the relation between wavefront sets in the following proposition.

Proposition 5.1.2. *Let $A \in I^d(X, Y; C)$ be a real phase FIO attached to the set $C = \Lambda'$. Then if $\Gamma \subset T^*Y \setminus 0$ is a closed cone with $\Gamma \cap \mathrm{WF}'_X(A) = \emptyset$, we have*

$$\mathrm{WF}(Au) \subset \Big[C \circ \mathrm{WF}(u)\Big] \cup \mathrm{WF}'_X(A) \quad \text{if} \quad u \in \mathcal{E}'_\Gamma(Y, \Omega^{\frac{1}{2}}).$$

Proof. Combine Proposition 5.1.1 and Theorem 2.1.2. □

In many applications, A is a propagator, so A and C depend on a parameter $t \in [0, \infty)$, and the relation shows how singularities in the initial data are carried by C.

Definition 5.1.10. *Let C and $C_\mathbb{C}$ be obtained from Λ and $\Lambda_\mathbb{C}$ as $C = \Lambda'$ and $C_\mathbb{C} = \Lambda'_\mathbb{C}$. Then C is said to be a homogeneous canonical relation from T^*Y to T^*X if*

$$C \subset T^*X \setminus 0 \times T^*Y \setminus 0,$$

and likewise for $C_\mathbb{C}$ if this is true of $\Lambda_\mathbb{R}$, and $C_\mathbb{C}$ is said to be a positive if $\Lambda_\mathbb{C}$ is positive.

The homogeneous canonical relations are important because they are well-behaved. They map smooth functions to smooth functions continuously, and extend by duality.

Theorem 5.1.5 (Hörmander [27]). *Let $A \in I^d(X, Y; C)$. The following holds:*

1. *If $\Lambda \cap (T^*X \times 0_Y) = \emptyset$, then A restricts to a continuous linear operator*

$$A : C_0^\infty(Y, \Omega^{\frac{1}{2}}) \to C^\infty(X, \Omega^{\frac{1}{2}})$$

2. *If $\Lambda \cap (0_X \times T^*Y) = \emptyset$, then A extends to a continuous linear operator*

$$A : \mathcal{E}'(Y, \Omega^{\frac{1}{2}}) \to \mathcal{D}'(X, \Omega^{\frac{1}{2}})$$

The same holds if X and Y are real-analytic, $A \in I^d(X, Y; C_{\mathbb{C}})$, and Λ is replaced by $\Lambda_{\mathbb{R}}$.

Note that the hypotheses of (1) and (2) hold if C is a homogeneous canonical relation.

Proof. By the decomposition of $K(A)$, it suffices to consider $A : C_0^\infty(\mathbb{R}^{n_Y}) \to \mathcal{D}'(\mathbb{R}^{n_X})$, where $K(A) = I_\phi(a)$, ϕ is a regular real (or complex positive) phase on an open cone Γ, and we may assume that $\operatorname{cone supp}(a) \subset \Gamma$. The amplitude is of order $d - \frac{N}{2} + \frac{n_X+n_Y}{4}$. In the local picture, the hypothesis of (1) ensures that

$$d_{(y,\theta)}\phi(x, y, \theta) \neq 0 \quad \text{when} \quad (x, y, \theta) \in \Gamma,$$

and, in that case, we can form the differential operator

$$L = \frac{1}{i} \frac{d_{(y,\theta)}\phi}{|d_{(y,\theta)}\phi|^2} \cdot d_{(y,\theta)} \quad \text{on} \quad \Gamma.$$

Take $\psi \in C_0^\infty(\mathbb{R}^{n_X})$ and $\varphi \in C_0^\infty(\mathbb{R}^{n_Y})$, and integrate by parts to get

$$\begin{aligned} \langle I_\phi(a), \psi \otimes \varphi \rangle &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^{n_X}} \int_{\mathbb{R}^{n_Y}} e^{i\phi(x,y,\theta)} a_\epsilon(x, y, \theta)\psi(x)\varphi(y)\, dy\, dx\, d\theta \\ &= \int_{\mathbb{R}^{n_X}} \psi(x) \Big[\int_{\mathbb{R}^N} \int_{\mathbb{R}^{n_Y}} e^{i\phi(x,y,\theta)} (L^t)^k (a(x, y, \theta)\varphi(y))\, dy\, d\theta \Big]\, dx, \end{aligned}$$

which means $A\varphi$ is the bracketed term, independent of $k \in \mathbb{N}$, if $k > d + \frac{N}{2} + \frac{n_X+n_Y}{4}$. Then, by the DCT, derivatives of any order go through the integrals, so $A\varphi$ is smooth. In particular, if $\alpha \in \mathbb{N}_0^{n_X}$ and $K \subset\subset \mathbb{R}^{n_X}$ is compact, we get

$$\begin{aligned} \sup_{x \in K} |\partial_x^\alpha A\varphi(x)| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^N} \int_{\mathbb{R}^{n_Y}} |\partial_x^\beta [e^{i\phi(x,y,\theta)}] \partial_x^{\alpha-\beta} (L^t)^k (a(x, y, \theta)\varphi(y))|\, dy\, d\theta \\ &\leq C \max_{|\beta| \leq k} \sup_{y \in \mathbb{R}^{n_Y}} |\partial_y^\beta \varphi(y)|, \end{aligned}$$

when $k > |\alpha| + d + \frac{N}{2} + \frac{n_X+n_Y}{4}$. So A is continuous between the respective topologies. The second point (2) follows from (1). This is because we can put

$$\langle Au, \psi \rangle = \langle u, A^t \psi \rangle \quad \text{if} \quad u \in \mathcal{E}'(\mathbb{R}^{n_Y}),$$

where the transpose A^t satisfies the first condition (swap x and y). □

The FIO, like pseudo-differential operators, admit formal L^2 (or even L^2_{loc})-adjoints. However, the associated homogeneous (positive) canonical relation C (or $C_{\mathbb{C}}$) is changed. It turns out to be natural to define the adjoint canonical relation

$$C^\dagger = \Big\{((y,\eta),(x,\xi)) \in T^*Y \setminus 0 \times T^*X \setminus 0 \,\Big|\, ((x,\xi),(y,\eta)) \in C\Big\},$$

and, if X and Y are real-analytic, its Hermitian analogue

$$C^\dagger_{\mathbb{C}} = \Big\{(\overline{(y,\eta)},\overline{(x,\xi)}) \in T^*_{1,0}Y_{\mathbb{C}} \setminus 0 \times T^*_{1,0}X_{\mathbb{C}} \setminus 0 \,\Big|\, ((x,\xi),(y,\eta)) \in C_{\mathbb{C}}\Big\},$$

where $(x,\xi) \mapsto \overline{(x,\xi)}$ is the conjugation on $T^*_{1,0}X_{\mathbb{C}}$ relative to T^*X.

The result on real and complex phase adjoints can be found in Hörmander [27, 28]. But the complex case is given extremely brief treatment in [28].

Theorem 5.1.6 (Hörmander [27, 28])**.** *If $A \in I^d(X,Y,C)$, then A^* exists, and*

$$A^* \in I^d(Y,X,C^\dagger).$$

The same holds if X and Y are real-analytic, $A \in I^d(X,Y;C_{\mathbb{C}})$, with $C^\dagger$ replaced by $C^\dagger_{\mathbb{C}}$. Finally, it also holds if each space of FIO above is replaced by its PHG analogue.

Proof. By the decomposition of $K(A)$, we just need to find the formal adjoint of $I_\phi(a)$, where ϕ is a regular real (or complex positive) phase parametrizing a Λ_ϕ in Λ (or $\Lambda_{\mathbb{C}}$). Take $u \in C_0^\infty(\mathbb{R}^{n_Y})$ and $v \in C_0^\infty(\mathbb{R}^{n_X})$. By the FTT and DCT, we have

$$\begin{aligned}
(I_\phi(a)u,v)_{L^2(\mathbb{R}^{n_X})} &= \int_{\mathbb{R}^{n_X}} \Big[\lim_{\epsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{n_Y}} e^{i\phi(x,y,\theta)}\,\chi(\epsilon\theta)a(x,y,\theta)u(y)\,dy\,d\theta\Big]\overline{v(x)}\,dx\\
&= \lim_{\epsilon\to 0}\int_{\mathbb{R}^{n_Y}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{n_X}} e^{i\phi(x,y,\theta)}\,\chi(\epsilon\theta)a(x,y,\theta)u(y)\overline{v(x)}\,dx\,d\theta\,dy\\
&= \int_{\mathbb{R}^{n_Y}} u(y)\overline{\Big[\lim_{\epsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{n_X}} e^{-i\overline{\phi(x,y,\theta)}}\,\chi(\epsilon\theta)\overline{a(x,y,\theta)}v(x)\,dx\,d\theta\Big]}\,dy,
\end{aligned}$$

where $\chi \in C_0^\infty(\mathbb{R}^N)$ is a real-valued cutoff equal to 1 in some neighbourhood of $0 \in \mathbb{R}^n$. Then the formal adjoint exists, because

$$(I_\phi(a)u,v)_{L^2(\mathbb{R}^{n_X})} = (u, I_{\phi^\dagger}(\overline{a})v)_{L^2(\mathbb{R}^{n_Y})},$$

and $\phi^\dagger(y,x,\theta) = -\overline{\phi(x,y,\theta)}$ is a regular real (or complex positive) phase associated to it. So, if ϕ is real, we have that $\Gamma_{\phi^\dagger} = \Gamma_\phi$, and $\phi^\dagger$ parametrizes

$$\Gamma_{\phi^\dagger} \to \Lambda_{\phi^\dagger} : (y,x,\theta) \mapsto \Big((y,x), -d_{y,x}\phi(x,y,\theta)\Big),$$

which, after accounting for the minus, corresponds exactly to $\Lambda'_{\phi^\dagger} = (\Lambda'_\phi)^\dagger$, as required. On the other hand, if ϕ is complex, its holomorphic extension parametrizes

$$\Gamma_{\phi^\dagger} \to \Lambda_{\phi^\dagger} : (y,x,\theta) \mapsto \Big((y,x), -d_{y,x}\overline{\phi_{\mathbb{C}}(\overline{x},\overline{y},\overline{\theta})}\Big),$$

which corresponds to $\Lambda'_{\phi^\dagger} = (\Lambda'_\phi)^\dagger$, because $\Gamma_{\phi^\dagger}$ is obtained by conjugating Γ_ϕ. $\square$

Perhaps the most important property of FIO is their (global) composition calculus. A composition of FIO is an FIO if their homogeneous canonical relations are compatible. This is a classical result due to Hörmander [27] (real) and Melin-Sjöstrand (complex) [43]. In order to formulate it, we need to define the real diagonal subset

$$\Delta_{1,2} = T^*X \times \mathrm{diag}(T^*Y) \times T^*Z,$$

and, if X, Y and Z are real-analytic, its complexified analogue

$$(\Delta_{1,2})_{\mathbb{C}} = T^*_{1,0}X_{\mathbb{C}} \times \mathrm{diag}(T^*_{1,0}Y_{\mathbb{C}}) \times T^*_{1,0}Z_{\mathbb{C}},$$

which clearly contains $\Delta_{1,2}$ in this case.

The FIO composition calculus is contained in the following theorem. Let $d_1, d_2 \in \mathbb{R}$. Let C_1 and C_2 be two such relations from T^*Y to T^*X and T^*Z to T^*Y, respectively, and, in the real-analytic case, let $(C_1)_{\mathbb{C}}$ and $(C_2)_{\mathbb{C}}$ be their positive analogues.

Theorem 5.1.7 (Hörmander [27], Melin and Sjöstrand [43]. See also Duistermaat [10]). *Suppose that $A \in I^{d_1}(X,Y;C_1)$ and $B \in I^{d_2}(Y,Z;C_2)$ are two properly supported FIO. Assume that the following holds:*

1. *$C_1 \times C_2$ intersects $\Delta_{1,2}$ transversally.*
2. *The natural projection below is injective and proper:*

$$(C_1 \times C_2) \cap \Delta_{1,2} \to T^*X \setminus 0 \times T^*Z \setminus 0.$$

Then $C_1 \circ C_2$ has the structure of a homogeneous canonical relation from T^*Z to T^*X, and the composition of A and B satisfies

$$A \circ B \in I^{d_1+d_2}(X,Z;C_1 \circ C_2).$$

However, if X, Y and Z are real-analytic, a similar result holds in the complex case. Suppose that $A \in I^{d_1}(X,Y;(C_1)_{\mathbb{C}})$ and $B \in I^{d_2}(Y,Z;(C_2)_{\mathbb{C}})$ are properly supported FIO. Assume that the following holds:

1. *$(C_1)_{\mathbb{C}} \times (C_2)_{\mathbb{C}}$ intersects $(\Delta_{1,2})_{\mathbb{C}}$ transversally at $((C_1)_{\mathbb{R}} \times (C_2)_{\mathbb{R}}) \cap \Delta_{1,2}$.*
2. *The natural projection below is injective and proper:*

$$((C_1)_{\mathbb{R}} \times (C_2)_{\mathbb{R}}) \cap \Delta_{1,2} \to T^*X \setminus 0 \times T^*Z \setminus 0.$$

Then $(C_1)_{\mathbb{C}} \circ (C_2)_{\mathbb{C}}$ is a homogeneous positive canonical relation going from T^*Z to T^*X. It has the property that

$$((C_1)_{\mathbb{C}} \circ (C_2)_{\mathbb{C}})_{\mathbb{R}} = (C_1)_{\mathbb{R}} \circ (C_2)_{\mathbb{R}},$$

and the composition of A and B satisfies

$$A \circ B \in I^{d_1+d_2}(X,Z;(C_1)_{\mathbb{C}} \circ (C_2)_{\mathbb{C}}).$$

Finally, everything still holds if each space of FIO above is replaced by its PHG analogue.

The structure of the FIO simplifies greatly when the canonical relation C is a graph. That is, C is of the form

$$C = \{(\gamma(y,\eta),(y,\eta)) \mid (y,\eta) \in T^*Y \setminus 0\},$$

where the map $\gamma : T^*Y \setminus 0 \to T^*X \setminus 0$ is a (fiber-wise) homogeneous symplectomorphism. This type of C is called a canonical graph.

Henceforth, X and Y are compact, and we fix smooth positive 1-densities on them. As the simplest example, $C = \operatorname{diag}(T^*X \setminus 0)$, and Theorem 5.1.4 implies

$$I^d(X, X, \operatorname{diag}(T^*X \setminus 0)) = \Psi^d(X).$$

Then any FIO from X to itself can be left and right composed with elements of $\Psi(X)$, and the result is still an FIO with the same canonical relation. Only the order is changed.

Theorem 5.1.8 (A variant of the Egorov theorem. See Hörmander [34] or Treves [60])**.** *Let C arise from the graph of γ. Given $A \in I^d(X,Y,C)$ and $d_0 \in \mathbb{R}$, then*

$$A^*PA \in \Psi^{d_0+2d}(Y) \quad \text{for any} \quad P \in \Psi^{d_0}(X).$$

The following holds:

1. *If $\sigma_{2d}([A^*A]) = [a]$ and $\sigma_{d_0}(P) = [p]$, then*

$$\sigma_{d_0+2d}(A^*PA) = [(p \circ \gamma)a].$$

2. *If P and A^*A are elliptic, then so is A^*PA.*

The same holds if X, Y and the map γ are real-analytic, and $A \in I^d(X,Y;C_{\mathbb{C}})$ instead, where $C_{\mathbb{C}}$ arises from the graph of the holomorphic extension of γ.

An example of natural FIO are the solution operators to certain hyperbolic systems. These have canonical relations arising from graphs of Hamiltonian flows:

Theorem 5.1.9 (Treves [60], Hörmander [34], alternatively Shubin [51] for simplicity)**.** *Let $P \in \Psi^1_{\mathrm{phg}}(X)$ be formally self-adjoint and elliptic with classical principal symbol p. Assume that p is positive on $T^*M \setminus 0$. (This automatically guarantees that P is elliptic.) Then given any $u_0 \in C^\infty(X)$, there is a unique solution $u \in C^\infty(\mathbb{R} \times X)$ to*

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = iP_x u(t,x) \quad \text{for all} \quad (t,x) \in \mathbb{R} \times X, \\ u|_{\{0\}\times X} = u_0. \end{cases}$$

The map $C^\infty(X) \to C^\infty(X) : u_0 \mapsto u(t,\cdot)$ is a continuous linear operator

$$e^{itP} \in I^0_{\mathrm{phg}}(X, X, C_t) \quad \text{for any} \quad t \in \mathbb{R},$$

where C_t is the graph of the Hamiltonian flow γ_t generated by making p the Hamiltonian. In particular, by dualizing, we have

$$\mathrm{WF}(e^{itP}u_0) \subset \gamma_t(\mathrm{WF}(u_0)) \quad \text{if} \quad u_0 \in \mathcal{D}'(X).$$

5.1.2 Boundedness on L^2 and Sobolev Spaces

Finally, we turn to the question of bounded extensions of FIO on L^2 and Sobolev spaces. The situation is then much more complicated than that for pseudo-differential operators, but it is possible to get bounded realizations in some easy special cases.

Definition 5.1.11. *Let C be a (real) homogeneous canonical relation from T^*Y to T^*X. Consider the left/right natural projections:*

$$\begin{array}{ccc} & C & \\ \swarrow & & \searrow \\ T^*X \setminus 0 & & T^*Y \setminus 0 \end{array}$$

The following nomenclature is adopted:

1. *C is non-degenerate if both projections have maximal rank everywhere.*
2. *C is a local canonical graph if both projections are local diffeomorphisms.*

Let now C be a non-degenerate homogeneous canonical relation as in Definition 5.1.11. In general, if $n_X > n_Y$, the left projection is an immersion, and the right is a submersion. The property of being a local canonical graph is only possible if $n_X = n_Y$.

Proposition 5.1.3 (Hörmander [27])**.** *Let C, defined above, be a local canonical graph. Then any $A \in I^0(X, Y; C)$ realizes a bounded linear operator*

$$A : L^2(Y) \to L^2(X).$$

Proof. The idea is to break C into small pieces that are graphs of symplectomorphisms. So if $c_0 = ((x_0, \xi_0), (y_0, \eta_0)) \in C$, we obtain an open conic subset Γ_{c_0} of C containing c_0, and a symplectomorphism $\gamma : \Gamma_{(y_0,\eta_0)} \to \Gamma_{(x_0,\xi_0)}$ such that $\gamma(y_0, \eta_0) = (x_0, \xi_0)$, and

$$C \cap \Gamma_{c_0} = \{(\gamma(y, \eta), (y, \eta)) \,|\, (y, \eta) \in \Gamma_{(y_0,\eta_0)}\},$$

where $\Gamma_{(y_0,\eta_0)}$ is an open cone about (y_0, η_0) and $\Gamma_{(x_0,\xi_0)}$ is an open cone about (x_0, ξ_0). Using sufficiently small charts and amplitude cutoffs, A decomposes into a finite sum

$$A = \sum_{j=1}^{N} A_j + R \quad \text{with} \quad A_j \in I^0(X, Y; C_j) \quad \text{and} \quad K(R) \in C^\infty(X \times Y),$$

where C_j arises from a symplectomorphism γ_j from a closed cone in $T^*Y \setminus 0$ to $T^*X \setminus 0$. Then, we have

$$\begin{aligned} ||Au||^2_{L^2(X)} &\leq \sum_{j=1}^{N} (A_j^* A_j u, u)_{L^2(Y)} + ||Ru||^2_{L^2(X)} \\ &\leq \Big[\sum_{j=1}^{N} ||A_j^* A_j||^2_{B(L^2(Y))} + ||R||^2_{B(L^2(X), L^2(Y))} \Big] ||u||^2_{L^2(X)}, \end{aligned}$$

where $A_j^* A_j \in \Psi^0(Y)$, because the adjoint of C_j is the graph of the inverse map. □

Just like for pseudo-differential operators, it leads to realizations on Sobolev spaces. Put smooth metrics on X and Y and form their Laplacians Δ_X and Δ_Y, respectively.

Corollary 5.1.1 (Duistermaat [10])**.** *Let C be exactly as before, a local canonical graph. Then, if $s, d \in \mathbb{R}$, any $A \in I^d(X, Y; C)$ realizes a bounded linear operator*

$$A : H^s(Y) \to H^{s-d}(X).$$

Proof. By the real phase composition calculus, we have that

$$(I - \Delta_X)^{\frac{s-d}{2}} A (I - \Delta_Y)^{-\frac{s}{2}} \in I^0(X, Y; C),$$

where C is unchanged, and Proposition 5.1.3 applies. □

6

Transforms and Holomorphic Function Spaces

Complexifying a compact real-analytic Riemannian manifold (M, g) gives a manifold $M_{\mathbb{C}}$, where holomorphic extensions of $f \in C^{\omega}(M)$ exist on neighbourhoods of M inside $M_{\mathbb{C}}$. But how do holomorphic functions on such neighbourhoods relate to functions on M? Using the heat propagator $e^{\frac{t}{2}\Delta}$ on M, we can generate many real-analytic functions, and it turns out (see the paper [54]) that all of these extend to the same Grauert tube. This idea leads to the Segal-Bargmann (holomorphically extended heat kernel) transform, which answers the question partially for compact Lie groups and homogeneous spaces. It maps $L^2(M)$ onto a holomorphic "heat-kernel weighted L^2-space" on the entire tube, and restrictions of functions from this space are of the form $e^{\frac{t}{2}\Delta}u$ for some $u \in L^2(M)$. The resulting correspondence with $L^2(M)$ is unitary. See Hall [18, 21] and Stenzel [58]. On $\mathbb{R}^n$ the Segal-Bargmann transform is for $f \in C^{\infty}(\mathbb{R}^n)$ and $t > 0$ given by

$$\mathcal{C}_t f(z) = \frac{1}{(2\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{(z-x)^2}{2t}} f(x)\, dx \quad \text{for all} \quad z \in \mathbb{C}^n.$$

The Segal-Bargmann transform has been studied extensively by Hall and collaborators, where the underlying manifold is either a Lie group, or symmetric space of complex type. In these cases, analysis of the transform is facilitated by Lie group representation theory, which has no analogue for Riemannian manifolds. See [18, 19, 21, 23, 24, 25, 20, 59]. Although the general Riemannian case is less understood, the transform is still injective, and Golse, Leichtnam and Stenzel [54] are able to obtain some explicit inversion formulae. However, there is no general Hilbert space structure on the image that makes it unitary, because, according to Stenzel [56], this depends on existence of an entire Grauert tube. In order to overcome this limitation, Stenzel introduces the Poisson transform [55, 56], which is a complex-phase FIO, and so has well-understood Sobolev mapping properties. On $\mathbb{R}^n$ this transform is for $f \in C^{\infty}(\mathbb{R}^n)$ and $\epsilon > 0$ given by

$$\mathcal{P}_{\epsilon} f(z) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i((z-y)\cdot\xi - i\epsilon|\xi|)} f(x)\, dx\, đ\xi \quad \text{for all} \quad z \in \mathbb{R}^n_{\epsilon}.$$

The Poisson transform, as defined by Stenzel [56], is at this time a very recent invention. But the theory surrounding it has been applied in great profusion by Zelditch and others. Applications include nodal hypersurfaces of eigenfunctions, quantum ergodicity/limits, and complex geodesics and pluripotential theory on Grauert tubes. See [65, 64, 67, 66]. See [22] for an introduction to the Segal-Bargmann transform in mathematical physics.

Understanding these transformations is a step towards answering the above question. They reveal the link between holomorphic functions on M_{ϵ} and their restrictions to M, and we hope that this may eventually shed light on the bigger, more involved question: When does a $P \in \Psi(G)$ preserve spaces of $f \in C^{\omega}(M)$ that extend to M_{ϵ}?

6.1 The Poisson Transform

In order to begin, we need some very basic results from the theory of complex manifolds. If N is a Kähler manifold, it is automatically orientable by the virtue of being symplectic. In fact, if N has Kähler metric h with associated Kähler form ω, we have

$$\mathrm{vol}_h = \frac{1}{n!}\omega^{\wedge n},$$

and with this, we get the formal L^2-adjoints of the Dolbeault operators

$$\partial^* = - * \bar{\partial} * \quad \text{and} \quad \bar{\partial}^* = - * \partial *,$$

where $*$ is the Hodge star operator relative to vol_h, and $d = \partial + \bar{\partial}$ is the splitting of d. On a Kähler manifold, they are tied via the so-called Kähler identities to

$$\Delta_h = dd^* + d^*d,$$

where d^* is the formal L^2-adjoint of d relative to vol_h. It sends k-forms back to k-forms. Of course, on 0-forms it it is just the usual scalar Laplacian Δ_h, so there is no confusion. We record two theorems that we will need. They appear in [44] and [29], respectively.

Theorem 6.1.1 (See Moroianu [44])**.**

$$\frac{1}{2}\Delta_h = \partial\partial^* + \partial^*\partial = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

The above implies that holomorphic functions on N are harmonic with respect to Δ_h. This is because $\bar{\partial}^*$ vanishes on scalars, and holomorphic functions are annihilated by $\bar{\partial}$. Another fact that will become important is the following:

Theorem 6.1.2. *If $f \in \mathcal{O}(N)$ and $K \subset\subset N$, there is a $C_K > 0$ such that*

$$\sup_{z \in K} |f(z)|^2 \leq C_K \int_M |f|^2 \, \omega^{\wedge n}.$$

Consequently, L^2-convergence implies locally uniform convergence for $\mathcal{O}(N)$ functions. In particular, the set of L^2-holomorphic functions on N is a closed subspace of $L^2(N)$. Here $L^2(N)$ is always relative to the positive measure induced by vol_h.

The most important special case is of course when N is the interior of the tube $\overline{M_\epsilon}$. Then h is the Kähler metric coming from the Kähler potential on a slightly larger tube, and we obtain a volume on ∂M_ϵ by contracting vol_h with the outward unit normal field. It also arises from the pullback of h onto ∂M_ϵ from this larger tube that contains M_ϵ, and then ∂M_ϵ carries a metric which induces a Laplacian on it.

In this way, we get Sobolev spaces on ∂M_ϵ of any order $s \in \mathbb{R}$ relative to this structure. The same ideas work if (N, h) is a compact smooth Riemannian manifold-with-boundary. On the manifold (M, g), we use the notation $\Delta = \Delta_g$ for the Laplacian associated to g. This Δ is different from the Kähler Laplacian Δ_h on M_ϵ (or N).

The formulation in Stenzel's papers [55, 56] features a special space denoted $\mathcal{O}^s(\partial M_\epsilon)$. It appears in Boutet de Monvel's theorem, and requires a bit of theory to understand. Specifically, we need the solution operator to the Dirichlet Laplace problem on $\overline{M_\epsilon}$.

Assume therefore now that (N, h) is a smooth Riemannian manifold-with-boundary. In all our cases, N is the closure of an open subset N° in a compact smooth manifold, where $N = N^\circ \sqcup \partial N$, the topological boundary ∂N is a co-dimension one hypersurface. This "enveloping manifold" $\widetilde{N}$ carries a metric coinciding with h on N.

Definition 6.1.1. *Given any $s \in \mathbb{R}$, we define the H^s-extendible distributions*

$$\bar{H}^s(N^\circ) = \{u|_{N^\circ} \in \mathcal{D}'(N^\circ) \,|\, u \in H^s(\widetilde{N})\},$$

and equip this space with the norm

$$||v||_{\bar{H}^s(N^\circ)} = \inf_{u|_{N^\circ}=v} ||u||_{H^s(\widetilde{N})} \quad \text{if} \quad v \in \bar{H}^s(N^\circ).$$

These spaces of $s \in \mathbb{R}$ extendible distributions are ingredients in the following theorem. It states that the Dirichlet problem for Δ_h on N is uniquely solvable in them:

Theorem 6.1.3 (Boutet de Monvel [4]. See also Grubb [15] for a better explanation)**.** *Let $s \in \mathbb{R}$. There are bounded, linear, and mutually inverse bijections*

$$K_\gamma : H^{s-\frac{1}{2}}(\partial N) \to \bar{H}^s(N^\circ) \cap \ker(\Delta_h|_{\bar{H}^s(N^\circ)}),$$
$$\gamma_0 : \bar{H}^s(N^\circ) \cap \ker(\Delta_h|_{\bar{H}^s(N^\circ)}) \to H^{s-\frac{1}{2}}(\partial N).$$

If $f \in H^{s-\frac{1}{2}}(\partial N)$, they together uniquely solve

$$\begin{cases} \Delta_h(K_\epsilon f) = 0 & \text{in} \quad N, \\ \gamma_0(K_\epsilon f) = f & \text{on} \quad \partial N. \end{cases}$$

The operator γ_0 acts as the restriction operator onto ∂N for any function smooth on $\overline{N}$, and K_γ is a Poisson operator (see [15]), restricting to a continuous linear map

$$K_\gamma : C^\infty(\partial N) \to C^\infty(N).$$

Relative to fixed positive 1-densities on N and ∂N, it has the following properties:

1. *K_γ admits a unique formal L^2-adjoint K_γ^*.*
2. *$K_\gamma^* K_\gamma \in \Psi^{-1}_{\mathrm{phg}}(\partial N)$ has classical principal symbol positive on $T^*N \setminus 0$.*

In the case that (N°, h) is Kähler, we can use the above theorem to define $\mathcal{O}^s(\partial N)$. Naturally, the boundary ∂N carries the Riemannian structure induced from N.

Definition 6.1.2. *In the above setting, if (N°, h) is Kähler, we put*

$$\mathcal{O}^s(\partial N) = \gamma_0\{\, u \in \bar{H}^{s+\frac{1}{2}}(N^\circ) \cap C^\infty(N^\circ) \,|\, \overline{\partial}(u|_{N^\circ}) = 0 \,\} \subset H^s(\partial N).$$

By the continuity of K_γ, and $\gamma_0 K_\gamma = I$, we have that $\mathcal{O}^s(\partial N)$ is closed in $H^s(\partial N)$. It consists of the "boundary values" on ∂N of holomorphic functions on N°.

6.1.1 Boutet de Monvel's Continuation Theorem

Choose $\epsilon_0 > 0$ such that M_ϵ exists as a Grauert tube about (M, g) whenever $\epsilon \in (0, \epsilon_0)$. Every eigenfunction of $-\Delta$ belongs to $C^\omega(M)$, because $-\Delta$ is real-analytic and elliptic. In fact, all of them extend holomorphically to M_ϵ:

Theorem 6.1.4 (Zelditch [65]). *Let ϕ be any eigenfunction of the Laplacian Δ on M. Then ϕ extends holomorphically on to any M_ϵ with $\epsilon < \epsilon_0$.*

Proof. Let r be the geodesic distance on a neighbourhood V of the diagonal in $M \times M$. Recall that, in the construction of the Grauert tubes, we had

$$g_p((d_x r)(p,q), (d_x r)(p,q)) = 1 \quad \text{for all} \quad (p,q) \in V,$$

which we can extend holomorphically, and evaluate meaningfully in $(z, \overline{z})$ for all $z \in M_\epsilon$. But in any holomorphic chart (z^i) of M_ϵ, this just says that

$$\sum_{i,j=1}^{n} (g_{\mathbb{C}})^{ij} \frac{\partial\sqrt{\rho}}{\partial z^i} \frac{\partial\sqrt{\rho}}{\partial z^j} = 1.$$

Therefore the C^ω-boundary ∂M_ϵ is non-characteristic for (the principal symbol of) $\Delta_{\mathbb{C}}$, and Zerner's theorem applies to locally extend ϕ beyond ∂M_ϵ as long as $\epsilon < \epsilon_0$. □

Of course, extensions for different eigenvalues are not a priori orthogonal in $L^2(M_\epsilon)$. But they will form a Schauder basis for $HL^2(M_\epsilon)$ when $\epsilon > 0$ is small enough.

Definition 6.1.3.

$$HL^2(M_\epsilon) = L^2(M_\epsilon) \cap \mathcal{O}(M_\epsilon).$$

The extensions turn out to be related (via their growth) to the Poisson propagator. Let $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots$ be the eigenvalues of $-\Delta$, counted with multiplicity, and let $\{\phi_k\}_{k=0}^\infty$ be a corresponding real-valued ONB for $L^2(M)$ of eigenfunctions for $-\Delta$. Then the Poisson propagator is defined for $t > 0$ by

$$e^{-t\sqrt{-\Delta}} : C^\infty(M) \to C^\infty(M) : u \mapsto \sum_{k=0}^{\infty} e^{-t\sqrt{\lambda_k}} (u, \phi_k)\phi_k,$$

where the sum converges in $C^\infty(M)$, and the kernel $P_t(\cdot,\cdot) = P(t,\cdot,\cdot)$ is

$$P_t = \sum_{k=0}^{\infty} e^{-t\sqrt{\lambda_k}} \phi_k \otimes \phi_k \quad \text{in} \quad \mathcal{D}'(M \times M),$$

which converges in the space of distributions, and is easily read off from the above series. In fact, we see that $P \in C^\omega((0,\infty) \times M \times M)$, since it solves

$$\Big(\frac{\partial^2}{\partial t^2} + \frac{1}{2}(\Delta_x + \Delta_y)\Big) P = 0,$$

which is elliptic with real-analytic coefficients. So P must be real-analytic in all variables. Therefore P_t also extends holomorphically to $M_\epsilon \times M_\epsilon$ for some $\epsilon > 0$.

All of the above is tied together by the following theorem, which is difficult to prove. It expresses the extension of the Poisson propagator as a complex-phase FIO.

Theorem 6.1.5 (Boutet de Monvel [3, 5]. See Stenzel [55, 56] and Zelditch [64, 65]). *There is an $\epsilon_0' \in (0, \epsilon_0)$ such that for any $\epsilon \in (0, \epsilon_0')$, the following holds:*

1. *The map $x \mapsto P_\epsilon(x, y)$ extends holomorphically to M_ϵ for each fixed $y \in M$.*
2. *The kernel $P_\epsilon|_{\partial M_\epsilon \times M}$ induces a complex-phase PHG FIO S_ϵ of order $-\frac{n-1}{4}$.*
3. *S_ϵ defines a homeomorphism $S_\epsilon : H^s(M) \to \mathcal{O}^{s+\frac{n-1}{4}}(\partial M_\epsilon)$ for any $s \in \mathbb{R}$.*

Here $S_\epsilon \in I_{\text{phg}}^{-\frac{n-1}{4}}(\partial M_\epsilon, M; C_\epsilon)$ is well-defined as an operator $S_\epsilon : C^\infty(M) \to C^\infty(\partial M_\epsilon)$. The positive homogeneous canonical relation C_ϵ arises from the graph of

$$T^*M \setminus 0 \to T^*(\partial M_\epsilon) \setminus 0 : (x, \xi) \mapsto \frac{|\xi|_x}{\epsilon} \alpha_{(x,\xi)},$$

where the cotangent vector $\alpha_{(x,\xi)}$ is

$$\alpha_{(x,\xi)} = (\iota^*_{\partial M_\epsilon} \operatorname{Im} \overline{\partial} \rho)_{\exp_x(i\epsilon|\xi|_x^{-1}\xi^\sharp)}.$$

The classical principal symbol of $S_\epsilon^* S_\epsilon \in \Psi_{\text{phg}}^{-\frac{n-1}{2}}(M)$ is $T^*M \to [0, \infty) : (x, \xi) \mapsto |\xi|_x^{-\frac{n-1}{2}}$.

Choose now $\epsilon_0' < \epsilon_0$ so that the above Boutet de Monvel theorem continuation holds. Pick $\epsilon \in (0, \epsilon_0')$, let h be the Kähler metric on the closure of M_ϵ, and take $s \in \mathbb{R}$.

Definition 6.1.4. *If $f \in H^{s-\frac{n+1}{4}}(M)$, define $\mathcal{P}_\epsilon f$ to be the extension of $e^{-\epsilon\sqrt{-\Delta}} f$ to M_ϵ.*

Suppose that K_ϵ is the Poisson operator solving the Dirichlet problem for Δ_h on $\overline{M_\epsilon}$. By Theorem 6.1.3 it is a linear homeomorphism

$$K_\epsilon : H^{s-\frac{1}{2}}(\partial M_\epsilon) \to \bar{H}^s(M_\epsilon) \cap \ker(\Delta_h|_{\bar{H}^s(M_\epsilon)}),$$

and so, by Theorem 6.1.5, $\mathcal{P}_\epsilon = K_\epsilon S_\epsilon$ realizes a linear homeomorphism

$$\mathcal{P}_\epsilon : H^{s-\frac{n+1}{4}}(M) \to K_\epsilon \mathcal{O}^{s-\frac{1}{2}}(\partial M_\epsilon).$$

In the special case that $s = 0$, we have

$$K_\epsilon \mathcal{O}^{-\frac{1}{2}}(\partial M_\epsilon) = HL^2(M_\epsilon).$$

This is because holomorphic functions on M_ϵ are automatically harmonic for Δ_h on M_ϵ, and therefore $HL^2(M_\epsilon) \subset K_\epsilon H^{-\frac{1}{2}}(\partial M_\epsilon)$. But $\gamma_0 HL^2(M_\epsilon) = \mathcal{O}^{-\frac{1}{2}}(\partial M_\epsilon)$ by definition. So it follows that $K_\epsilon : \mathcal{O}^{-\frac{1}{2}}(\partial M_\epsilon) \to HL^2(M_\epsilon)$ is a well-defined linear homeomorphism. The composite $\mathcal{P}_\epsilon : L^2(M) \to HL^2(M_\epsilon)$ is the Poisson transform defined by Stenzel [56]. Summarizing, we have the following result:

Theorem 6.1.6 (Stenzel [56]). *The map $\mathcal{P}_\epsilon : H^{-\frac{n+1}{4}}(M) \to HL^2(M_\epsilon)$ is well-defined. Furthermore, it is a linear homeomorphism.*

Definition 6.1.5. *The map $\mathcal{P}_\epsilon$ is called the Poisson transform.*

An operator between Hilbert spaces has both a left and a right polar decomposition. That is, if H_1 and H_2 are Hilbert spaces and $A \in B(H_1, H_2)$, then

$$A = P_L U_L = U_R P_R,$$

where the operators $U_L \in B(H_1, H_2)$ and $U_R \in B(H_1, H_2)$ are both partial isometries, and $P_L \in B(H_2)$ and $P_R \in B(H_1)$ are positive operators on H_2 and H_1, respectively. These can be chosen uniquely such that

$$\ker(U_R) = \ker(P_R) \quad \text{and} \quad \operatorname{img}(U_L) = \overline{\operatorname{img}(P_L)}.$$

Also, U_R is an isometry if A is injective with dense range.

This decomposition can be applied to the Poisson transform in the case that $s = 0$. Using functional calculus, we can form the "right positive part" of $\mathcal{P}_\epsilon$ as follows:

Corollary 6.1.1 (Stenzel [56])**.** *There is a well-defined, positive operator*

$$\mathcal{Q}_\epsilon = (S_\epsilon^*(K_\epsilon^* K_\epsilon)S_\epsilon)^{-\frac{1}{2}} \in \Psi_{\text{phg}}^{\frac{n+1}{4}}(M),$$

where K_ϵ^* is understood to be the formal L^2-adjoint of K_ϵ as is stated in Theorem 6.1.3. It has the property that $(K_\epsilon S_\epsilon)\mathcal{Q}_\epsilon$ extends to a unitary map

$$\mathcal{P}_\epsilon \mathcal{Q}_\epsilon : L^2(M) \to HL^2(M_\epsilon).$$

Proof. Using Theorems 6.1.3 and 6.1.5 and the Egorov theorem, we get

$$S_\epsilon^*(K_\epsilon^* K_\epsilon)S_\epsilon \in \Psi_{\text{phg}}^{-\frac{n+1}{2}}(M),$$

which is then elliptic, and its classical principal symbol is strictly positive on $T^*M \setminus 0$. So by Theorem 3.1.14 it is meaningful to take powers of it. We show that it is invertible. Now, if we view $\mathcal{P}_\epsilon$ as a bounded map into $L^2(M_\epsilon)$, we can form its Hilbert adjoint $\mathcal{P}_\epsilon^*$. Then $\mathcal{P}_\epsilon^* \mathcal{P}_\epsilon : H^{-\frac{n+1}{4}}(M) \to H^{-\frac{n+1}{4}}(M)$ is injective with dense range, and

$$\mathcal{P}_\epsilon^* \mathcal{P}_\epsilon|_{C^\infty(M)} = (I - \Delta)^{\frac{n+1}{4}} S_\epsilon^*(K_\epsilon^* K_\epsilon)S_\epsilon \in \Psi^0(M).$$

But this shows that $\mathcal{P}_\epsilon^* \mathcal{P}_\epsilon$ is a Fredholm operator on $H^{-\frac{n+1}{4}}(M)$, and must be bijective. Therefore we have a bijective and continuous, hence invertible, operator

$$S_\epsilon^*(K_\epsilon^* K_\epsilon)S_\epsilon : C^\infty(M) \to C^\infty(M).$$

It follows that $\mathcal{Q}_\epsilon$ is well-defined, elliptic and invertible. It is clearly positive on $C^\infty(M)$. Taking any $u \in C^\infty(M)$, we have

$$\begin{aligned} ||(K_\epsilon S_\epsilon)\mathcal{Q}_\epsilon u||_{L^2(M_\epsilon)}^2 &= ((K_\epsilon S_\epsilon)^*(K_\epsilon S_\epsilon)\mathcal{Q}_\epsilon u, \mathcal{Q}_\epsilon u)_{L^2(M)} \\ &= (\mathcal{Q}_\epsilon^{-1} u, \mathcal{Q}_\epsilon u)_{L^2(M)} \\ &= ||u||_{L^2(M)}^2, \end{aligned}$$

and so $(K_\epsilon S_\epsilon)\mathcal{Q}_\epsilon$ extends to a unitary map $\mathcal{P}_\epsilon \mathcal{Q}_\epsilon : L^2(M) \to HL^2(M_\epsilon)$. ◻

6.1.2 Holomorphic Fourier Expansions

The Fourier series of the restriction of a function from $HL^2(M_\epsilon)$ extends naturally to M_ϵ. This is a consequence of the fact that $\mathcal{P}_\epsilon$ is bounded from $H^{-\frac{n+1}{4}}(M)$ onto $HL^2(M_\epsilon)$, and that it takes each ϕ_k to its extension $\widetilde{\phi}_k$ on M_ϵ multiplied by $e^{-\epsilon\sqrt{\lambda_k}}$.

Theorem 6.1.7 (Stenzel [56]). *If $f \in HL^2(M_\epsilon)$, then*

$$f = \sum_{k=0}^{\infty} (f|_M, \phi_k)_{L^2(M)} \widetilde{\phi}_k \quad \text{in} \quad HL^2(M_\epsilon).$$

Proof. Using that $\mathcal{P}_\epsilon$ is surjective, we obtain $u \in H^{-\frac{n+1}{4}}(M)$ such that $f|_M = e^{-\epsilon\sqrt{\Delta}}u$. Observe that if $k \in \mathbb{N}_0$, we have

$$\mathcal{P}_\epsilon(\phi_k) = e^{-\epsilon\sqrt{\lambda_k}} \widetilde{\phi}_k,$$

and also

$$(f|_M, \phi_k)_{L^2(M)} = \langle e^{-\epsilon\sqrt{\Delta}}u, \overline{\phi_k}\rangle = e^{-\epsilon\sqrt{\lambda_k}} \langle u, \overline{\phi_k}\rangle.$$

Now, $\sum_{k=0}^{N} \langle u, \overline{\phi_k}\rangle \phi_k$ converges to $u \in H^{-\frac{n+1}{4}}(M)$ in the norm of $H^{-\frac{n+1}{4}}(M)$ as $N \to \infty$. Therefore, we may combine the above to get

$$\begin{aligned} f = \mathcal{P}_\epsilon u &= \lim_{N\to\infty} \mathcal{P}_\epsilon \Big(\sum_{k=0}^{N} \langle u, \overline{\phi_k}\rangle \phi_k \Big) \\ &= \lim_{N\to\infty} \sum_{k=0}^{N} e^{\epsilon\sqrt{\lambda_k}} (f|_M, \phi_k)_{L^2(M)} \mathcal{P}_\epsilon(\phi_k) \\ &= \lim_{N\to\infty} \sum_{k=0}^{N} (f|_M, \phi_k)_{L^2(M)} \widetilde{\phi}_k, \end{aligned}$$

where the limit is taken in $L^2(M_\epsilon)$. □

Although the theorem looks simple and somehow standard, it is a quite recent result. In fact, the boundedness property of $\mathcal{P}_\epsilon$ is key here, and this is highly non-trivial to prove. See also Lebeau [39] for another approach.

Proposition 6.1.1 (Stenzel [56]). *The map $HL^2(M_\epsilon) \to L^2(M) : f \mapsto f|_M$ is bounded. Furthermore, it is injective with dense range.*

Proof. The full eigenspace $\operatorname{span}\{\phi_k\}_{k=0}^{\infty}$ is dense in $L^2(M)$, and M is totally real in M_ϵ. Thus R_ϵ must be injective ($\mathcal{O}(M_\epsilon)$ is determined by restrictions to M) with dense range. Finally, by Theorem 6.1.2, we get

$$\begin{aligned} ||f|_M||^2_{L^2(M)} &\le \Big(\int_M \omega_0 \Big) \sup_{x\in M} |f(x)|^2 \\ &\le \Big(\int_M \omega_0 \Big) C_M \, ||f||^2_{L^2(M_\epsilon)}, \end{aligned}$$

where $C_M > 0$ is a constant, and ω_0 is a fixed positive 1-density on M. □

Henceforth the "restriction map onto M" is the map $\mathcal{R}_\epsilon$ above in Proposition 6.1.1. It is understood both on $HL^2(M_\epsilon)$, but also as a map

$$\mathcal{R}_\epsilon : \mathcal{O}(M_\epsilon) \to C^\omega(M) : f \mapsto f|_M.$$

There is a close relationship between the transform $(\mathcal{P}_\epsilon \mathcal{Q}_\epsilon)^{-1}$ and the restriction map $\mathcal{R}_\epsilon$. Under certain circumstances, the two differ by a factor of a positive operator

$$\mathcal{R}_\epsilon \mathcal{P}_\epsilon \mathcal{Q}_\epsilon = P : L^2(M) \to L^2(M),$$

which we can express explicitly in terms of the (Schauder) basis $\{\widetilde{\phi}_k\}_{k=0}^\infty$ for $HL^2(M_\epsilon)$. It turns out to be equivalent to the existence of

$$\{\phi_k\}_{k=0}^\infty \quad \text{such that} \quad \{\widetilde{\phi}_k\}_{k=0}^\infty \quad \text{is orthogonal in} \quad HL^2(M_\epsilon).$$

According to Stenzel [56], it is unlikely that this very nice property could hold in general, but Stenzel [56] gives an example where it does hold. No counterexamples are known yet. To prove these statements, we need the following lemma.

Lemma 6.1.1. *Let $A, B \in \Psi(M)$ be of positive order, formally self-adjoint, and elliptic. If $[A, B] = 0$, then there exists an ONB for $L^2(M)$ of joint eigenfunctions for A and B. Additionally, if A and B commute with complex conjugation, they can be made real-valued.*

Proof. Realized as unbounded operators on $L^2(M)$, both become closed and self-adjoint, where the domain is exactly the Sobolev space corresponding to the order of the operator. The eigenspaces $\{E_\lambda\}_{\lambda \in \sigma(A)}$ for A are mutually orthogonal, and we have

$$E_\lambda \subset C^\infty(M) \quad \text{and} \quad \dim E_\lambda < \infty \quad \text{for each} \quad \lambda \in \sigma(A),$$

where their orthogonal sum is

$$\bigoplus_{\lambda \in \sigma(A)} E_\lambda = L^2(M).$$

If B commutes with A on $C^\infty(M)$, then E_λ is invariant under B, that is $B(E_\lambda) \subset E_\lambda$. Consequently, $B|_{E_\lambda} : E_\lambda \to E_\lambda$ is finite-dimensional self-adjoint, and $B|_{E_\lambda}$ diagonalizes. Thus E_λ decomposes into an orthogonal sum of finite-dimensional eigenspaces for $B|_{E_\lambda}$, and these consist of eigenfunctions for A with eigenvalue λ. This shows the first statement. Given the extra condition, each E_λ can be spanned by real-valued eigenfunctions of A, and $B|_{E_\lambda}$ becomes a real symmetric matrix that diagonalizes via a real orthogonal matrix. In this case, E_λ decomposes via real-valued eigenfunctions for both A and B. □

The claims above are encapsulated in the next proposition, which we shall now prove. It makes use of holomorphic Fourier expansions.

Theorem 6.1.8 (Stenzel [56])**.** *The following are equivalent:*

1. *The unitary part of $\mathcal{P}_\epsilon^{-1}$ is the unitary part of $\mathcal{R}_\epsilon$.*
2. *$e^{-\epsilon\sqrt{-\Delta}} \mathcal{Q}_\epsilon : L^2(M) \to L^2(M)$ is a positive operator.*
3. *$[e^{-\epsilon\sqrt{-\Delta}}, \mathcal{Q}_\epsilon] = 0$.*
4. *There is a choice of $\{\phi_k\}_{k=0}^\infty$ such that $\{\widetilde{\phi}_k\}_{k=0}^\infty$ is orthogonal in $HL^2(M_\epsilon)$.*

Proof. The easiest way is to show the circle of implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1).

(1) ⇒ (2). $e^{-\epsilon\sqrt{-\Delta}}\mathcal{Q}_\epsilon : L^2(M) \to L^2(M)$ is bounded and injective with dense range. This follows from Proposition 6.1.1, because it equals $\mathcal{R}_\epsilon(\mathcal{P}_\epsilon\mathcal{Q}_\epsilon)$ by the definition of $\mathcal{P}_\epsilon$. It then has a unique polar decomposition

$$e^{-\epsilon\sqrt{-\Delta}}\mathcal{Q}_\epsilon = PU \quad \text{with} \quad \mathcal{R}_\epsilon = PU(\mathcal{P}_\epsilon\mathcal{Q}_\epsilon)^{-1},$$

and so it is positive if and only if $(\mathcal{P}_\epsilon\mathcal{Q}_\epsilon)^{-1}$ is the unitary part of $\mathcal{R}_\epsilon$.

(2) ⇒ (3). On $L^2(M)$, the operator $e^{-\epsilon\sqrt{-\Delta}}\mathcal{Q}_\epsilon$ is positive, in particular self-adjoint. Then, since $\mathcal{Q}_\epsilon$ and $e^{-\epsilon\sqrt{-\Delta}}$ are symmetric on $L^2(M)$, their Hilbert adjoints are

$$\mathcal{Q}_\epsilon^*|_{C^\infty(M)} = \mathcal{Q}_\epsilon(I-\Delta)^{-\frac{n+1}{4}}|_{C^\infty(M)},$$
$$(e^{-\epsilon\sqrt{-\Delta}})^*|_{C^\infty(M)} = (I-\Delta)^{+\frac{n+1}{4}}e^{-\epsilon\sqrt{-\Delta}}|_{C^\infty(M)},$$

and therefore, by the above, we get

$$\begin{aligned} e^{-\epsilon\sqrt{-\Delta}}\mathcal{Q}_\epsilon|_{C^\infty(M)} &= (e^{-\epsilon\sqrt{-\Delta}}\mathcal{Q}_\epsilon)^*|_{C^\infty(M)} \\ &= (\mathcal{Q}_\epsilon)^*(e^{-\epsilon\sqrt{-\Delta}})^*|_{C^\infty(M)} \\ &= \mathcal{Q}_\epsilon e^{-\epsilon\sqrt{-\Delta}}|_{C^\infty(M)}. \end{aligned}$$

(3) ⇒ (4). Applying Lemma 6.1.1 to $\mathcal{Q}_\epsilon$ and $e^{-\epsilon\sqrt{-\Delta}}$, we obtain a joint eigenbasis. That is, a joint ON eigenbasis $\{\phi_k\}_{k=0}^\infty$ with eigenvalues $\{\eta_k\}_{k=0}^\infty \subset (0,\infty)$ given for $\mathcal{Q}_\epsilon$. If $l \neq k$ (eigenvalues may be equal), then by Corollary 6.1.1, we have

$$\begin{aligned} 0 = (\phi_k, \phi_l)_{L^2(M)} &= ((\mathcal{P}_\epsilon\mathcal{Q}_\epsilon)\phi_k, (\mathcal{P}_\epsilon\mathcal{Q}_\epsilon)\phi_l)_{HL^2(M_\epsilon)} \\ &= (e^{-\epsilon\sqrt{\lambda_k}}\eta_k\widetilde{\phi}_k, e^{-\epsilon\sqrt{\lambda_l}}\eta_l\widetilde{\phi}_l)_{HL^2(M_\epsilon)}. \end{aligned}$$

(4) ⇒ (1). Take such an ON eigenbasis $\{\phi_k\}_{k=0}^\infty$ with orthogonal extensions $\{\widetilde{\phi}_k\}_{k=0}^\infty$. Taking any $f \in HL^2(M_\epsilon)$, then, with series convergence in $L^2(M)$, we have

$$\mathcal{R}_\epsilon f = \sum_{k=0}^\infty \frac{1}{||\widetilde{\phi}_k||}\Big(f, \frac{\widetilde{\phi}_k}{||\widetilde{\phi}_k||}\Big)\phi_k \quad \text{and} \quad \mathcal{P}_\epsilon^{-1}f = \sum_{k=0}^\infty \frac{e^{\epsilon\sqrt{\lambda_k}}}{||\widetilde{\phi}_k||}\Big(f, \frac{\widetilde{\phi}_k}{||\widetilde{\phi}_k||}\Big)\phi_k,$$

which is meaningful, because there is a $C > 0$ such that

$$C \leq ||\mathcal{P}_\epsilon\phi_k|| = e^{-\epsilon\sqrt{\lambda_k}}||\widetilde{\phi}_k||.$$

Immediately, this implies that the unitary part of $\mathcal{P}_\epsilon^{-1}$ must be the unitary part of $\mathcal{R}_\epsilon$. The unitary part is just the abstract change of basis

$$HL^2(M_\epsilon) \to L^2(M) : f \mapsto \sum_{k=0}^\infty \Big(f, \frac{\widetilde{\phi}_k}{||\widetilde{\phi}_k||}\Big)\phi_k.$$

This concludes the proof. □

The example due to Stenzel considers a compact Lie group G with Lie subgroup H. It says that tubes of G/H are well-behaved if constructed from certain types of metric. Let $\pi : G \to G/H$ be the natural projection in the following proposition.

Proposition 6.1.2 (Stenzel [56])**.** *Suppose that G acts only by isometries on $M = G/H$, and that G is given a left-invariant metric such that $\Delta_G \pi^* = \pi^* \Delta$ holds on $C^\infty(M)$. Then the statements of Theorem 6.1.8 hold for $\epsilon > 0$ small enough.*

Proof. The Riemannian measure on G is left-invariant. Therefore it is a multiple of dx. The action by isometries is a real-analytic map

$$G \times G/H \to G/H : (x, yH) \mapsto x \cdot yH = xyH,$$

and it therefore extends to a holomorphic map $G_\epsilon \times M_\epsilon \to M_\epsilon$ for some small $\epsilon > 0$. By the tube construction, it preserves the Kähler potential, and thus the Kähler metric. Observe that if ϕ_λ is an eigenfunction of $\Delta_{G/H}$ with eigenvalue λ, we have

$$\Delta_G(\pi^* \phi_\lambda) = \pi^*(\Delta \phi_\lambda) = \lambda \pi^* \phi_\lambda.$$

So $\pi^* \phi_\lambda$ is an eigenfunction of Δ_G with eigenvalue λ. Let $\lambda \neq \mu$ be different eigenvalues. Using the orthogonality, and bi-invariance of dx, then if $y \in G$ is fixed, we get

$$\begin{aligned} 0 &= \int_G (\pi^* \phi_\lambda)(x) \overline{(\pi^* \phi_\mu)(x)}\, dx \\ &= \int_G (\phi_\lambda \circ \pi)(xy) \overline{(\phi_\mu \circ \pi)(xy)}\, dx \\ &= \int_G \phi_\lambda(x \cdot yH) \overline{\phi_\mu(x \cdot yH)}\, dx, \end{aligned}$$

and since G acts by isometries, the above identity extends from $yH \in G/H$ to any $z \in M_\epsilon$. It follows from the FTT that

$$\begin{aligned} (\widetilde{\phi}_\lambda, \widetilde{\phi}_\mu)_{HL^2(M_\epsilon)} &= \int_G \Big[\int_{M_\epsilon} \widetilde{\phi}_\lambda(x \cdot z) \overline{\widetilde{\phi}_\mu(x \cdot z)}\, \omega^{\wedge n}(z) \Big]\, dx \\ &= \int_{M_\epsilon} \Big[\int_G \widetilde{\phi}_\lambda(x \cdot z) \overline{\widetilde{\phi}_\mu(x \cdot z)}\, dx \Big]\, \omega^{\wedge n}(z) = 0, \end{aligned}$$

where $\widetilde{\phi}_\lambda$ and $\widetilde{\phi}_\mu$ are bounded on M_ϵ, so the interchange is justified. □

The above is not artificial; it occurs in "nature" by way of the following proposition. Although the types of allowed geometries are still rather restricted.

Proposition 6.1.3 (Bergery and Bourguignon [1])**.** *Suppose that G/H carries a metric. If G acts by isometries, then it admits a left-invariant metric so that the following holds:*

1. *$\pi : G \to G/H$ is a Riemannian submersion with totally geodesic fibers.*
2. *π^* intertwines $\Delta_{G/H}$ and Δ_G. That is, $\Delta_G \pi^* = \pi^* \Delta_{G/H}$ on $C^\infty(G/H)$.*

The above theorem is somewhat related and similar to Theorem 3.2.9 in certain ways. But that theorem gives G/H a metric rather than G.

Corollary 6.1.2. *In Proposition 6.1.3, $(\mathcal{P}_\epsilon \mathcal{Q}_\epsilon)^{-1}$ is the unitary part of $\mathcal{R}_\epsilon$.*

6.2 The Segal-Bargmann Transform

On a compact Lie group G, the Segal-Bargmann transform has a very tractable structure. We shall go through the steps as they are presented in Hall's paper [18] on the subject. Fix an orthonormal basis $\{X_j\}_{j=1}^n$ for $\mathfrak{g}$ carrying an $\mathrm{Ad}(G)$-invariant inner product $(\cdot,\cdot)_{\mathfrak{g}}$. It is well-known [8] that

$$\Delta_G = \sum_{j=1}^n X_j^2,$$

where Δ_G is the Laplacian with respect to the $(\cdot,\cdot)_{\mathfrak{g}}$-induced bi-invariant metric on G. It is independent of the choice of $\{X_j\}_{j=1}^n$, depending only on the inner product on $\mathfrak{g}$. This Laplacian extends to an operator $(\Delta_G)_{\mathbb{C}}$ on holomorphic functions on $G_{\mathbb{C}}$ (or G_ϵ). If $u \in \mathcal{O}(G_{\mathbb{C}})$ (or $u \in \mathcal{O}(G_\epsilon)$), and $z = \exp(iY)x$ with $x \in G$, we can write

$$\begin{aligned}(\Delta_G)_{\mathbb{C}}u(z) &= \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(-\lambda_\xi \xi(z)\mathcal{F}u(\xi))\\ &= \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}\Big(-\lambda_\xi\xi(x)\int_G u(\exp(iY)y)\xi(y)^*\,dy\Big),\end{aligned}$$

which converges absolutely uniformly for Y in compact subsets of $\mathfrak{g}$ (or $\{Y\in\mathfrak{g}\,|\,|Y|_{\mathfrak{g}}<\epsilon\}$). Thus $(\Delta_G)_{\mathbb{C}}u$ is well-defined and holomorphic on $G_{\mathbb{C}}$ (or G_ϵ). It is clearly bi-G-invariant. It is bi-$G_{\mathbb{C}}$-invariant for $u \in \mathcal{O}(G_{\mathbb{C}})$. To see left-invariance, take $w \in G$, note that

$$\mathrm{Tr}\Big(\xi(w^{-1}x)\int_G u(y)\xi(y)^*\,dy\Big) = \mathrm{Tr}\Big(\xi(x)\int_G u(w^{-1}y)\xi(y)^*\,dy\Big),$$

and both sides are holomorphic in $w \in G_{\mathbb{C}}$, hence equal. The right-invariance is similar. The expansion leads to a rudimentary calculus of $(\Delta_G)_{\mathbb{C}}$ acting on holomorphic functions. If $f \in C^0(\mathbb{C})$ has no more than polynomial growth, then we can replace $-\lambda_\xi$ with $f(-\lambda_\xi)$. For example, if $s\in\mathbb{R}$, we form $(I-(\Delta_G)_{\mathbb{C}})^{\frac{s}{2}}$ by replacing $-\lambda_\xi$ with $\langle\xi\rangle^s$ in the expansion. We will also need the following lemma.

Lemma 6.2.1. *If* $[\xi]\in\widehat{G}$, *then*

$$||d\xi(X)|| \le \sqrt{\lambda_\xi}|X|_{\mathfrak{g}} \quad \text{for all} \quad X\in\mathfrak{g}.$$

Proof. Write $X\in\mathfrak{g}$ as $X=\sum_{k=1}^n (X,X_k)_{\mathfrak{g}}X_k$. Note that $d\xi(X)\in\mathfrak{u}(d_\xi)$ is skew-adjoint. If u is any vector in the representation space of ξ, then we have

$$\begin{aligned}||d\xi(X)u|| &\le \sum_{k=1}^n |(X,X_k)_{\mathfrak{g}}|\,||d\xi(X_k)u||\\ &\le |X|_{\mathfrak{g}}\Big[\sum_{k=1}^n (d\xi(X_k)u, d\xi(X_k)u)\Big]^{\frac{1}{2}} = \sqrt{(-d\xi(\Delta_G)u,u)}|X|_{\mathfrak{g}} = \sqrt{\lambda_\xi}|X|_{\mathfrak{g}}||u||,\end{aligned}$$

where we have used that $d\xi(\Delta_G) = -\lambda_\xi I$ (by definition of λ_ξ). ☐

6.2.1 Holomorphic Peter-Weyl Expansions

Using the Cartan decomposition, fix the left G-invariant Grauert tubes $\{G_k\}_{k=1}^{\infty}$ in $G_{\mathbb{C}}$. Of course, each G_k is pre-compact, and

$$G_1 \subset \cdots \subset G_k \subset G_{k+1} \subset \cdots \subset \cup_{k=1}^{\infty} G_k = G_{\mathbb{C}}.$$

Definition 6.2.1. *Let $HL^2(G_{\mathbb{C}}, \mu)$ be all the $L^2(G_{\mathbb{C}}, \mu)$ holomorphic functions on $G_{\mathbb{C}}$. Here μ is a left Haar measure on $G_{\mathbb{C}}$ scaled by a strictly positive left G-invariant weight.*

The sup-norm on any compact subset is bounded by the L^2-norm by Theorem 6.1.2. Therefore $HL^2(G_{\mathbb{C}}, \mu)$ is always a closed subspace of $L^2(G_{\mathbb{C}}, \mu)$.

Theorem 6.2.1 (Hall [18])**.** *If $f \in HL^2(G_{\mathbb{C}}, \mu)$, then*

$$f = \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}[\xi \mathcal{F}_G f(\xi)] \quad \text{in} \quad HL^2(G_{\mathbb{C}}, \mu).$$

Proof. Observe that the map $G \to \mathbb{C} : x \mapsto f(xz)$ is smooth on G for any fixed $z \in G_{\mathbb{C}}$. Then for each $x \in G$ and $k \in \mathbb{N}$ we get uniform convergence over $z \in G_k$ of

$$\begin{aligned} f(xz) &= \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}\Big(\xi(x) \int_G f(yz)\xi(y)^* \, dy\Big) \\ &= \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}\Big(\xi(xz) \int_G f(y)\xi(y)^* \, dy\Big), \end{aligned}$$

where we make use of the holomorphy of f on $G_{\mathbb{C}}$, and that G is totally real inside $G_{\mathbb{C}}$. If $[\xi], [\eta] \in \widehat{G}$ are inequivalent, we have that

$$\begin{aligned} 0 &= \int_{G_k} \Big[\int_G \operatorname{Tr}\Big(\xi(x)[\xi(z)A]\Big) \overline{\operatorname{Tr}\Big(\eta(x)[\eta(z)B]\Big)} \, dx \Big] \, d\mu(z) \\ &= \int_G \int_{G_k} \operatorname{Tr}(\xi(xz)A) \overline{\operatorname{Tr}(\eta(xz)B)} \, d\mu(z) \, dx \\ &= \int_{G_k} \operatorname{Tr}(\xi(z)A) \overline{\operatorname{Tr}(\eta(z)B)} \, d\mu(z), \end{aligned}$$

where A and B are endomorphisms of the representation spaces of ξ and η, respectively. It follows from this orthogonality that

$$\sum_{[\xi] \in \widehat{G}} d_\xi^2 \int_{G_k} \Big| \operatorname{Tr}\Big(\xi(z) \int_G f(y)\xi(y)^* \, dy\Big) \Big|^2 \, d\mu(z) = ||f|_{G_k}||^2_{L^2(G_k, \mu)} < \infty,$$

and the terms in the above series increase with k, so the MCT allows us to take $k \to \infty$. Thus each term is in $HL^2(G_{\mathbb{C}})$, and the series converges in $L^2(G_{\mathbb{C}})$. □

Contained in the above is of course the fact that each term in the sum is in $L^2(G_{\mathbb{C}}, \mu)$. This is interesting in its own right.

Corollary 6.2.1. *If $f \in HL^2(G_{\mathbb{C}}, \mu)$, then*

$$\operatorname{Tr}[\xi \mathcal{F}_G f(\xi)] \in HL^2(G_{\mathbb{C}}, \mu) \quad \text{for each} \quad [\xi] \in \widehat{G}.$$

6.2.2 Hall's Isometry and Inversion Theorems

Associated to $(\cdot,\cdot)_{\mathfrak{g}}$ there is the $\mathrm{Ad}(G)$-invariant extended inner product $(\cdot,\cdot)_{\mathfrak{g}_{\mathbb{C}}}$ on $\mathfrak{g}_{\mathbb{C}}$, and with respect to this, a bi-G-invariant (by definition, and using Ad) operator

$$\Delta_{G_{\mathbb{C}}} = \sum_{j=1}^{n} X_j^2 + \sum_{j=1}^{n} (JX_j)^2,$$

which is the left $G_{\mathbb{C}}$-invariant Laplacian for the left $G_{\mathbb{C}}$-invariant metric induced by $(\cdot,\cdot)_{\mathfrak{g}_{\mathbb{C}}}$. It is also very important to note that the operators $(\Delta_G)_{\mathbb{C}}$ and $\Delta_{G_{\mathbb{C}}}$ are not the same. The former is only defined on holomorphic functions (”holomorphic extension” of Δ_G), while the latter acts on $\mathcal{D}'(G_{\mathbb{C}})$. The difference is essentially that of

$$(\Delta_{\mathbb{R}})_{\mathbb{C}} = \frac{d^2}{dz^2} \quad \text{versus} \quad \Delta_{\mathbb{R}_{\mathbb{C}}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Both Laplacians Δ_G and $\Delta_{G_{\mathbb{C}}}$ are independent of $\{X_j\}_{j=1}^n$, and depend only on $(\cdot,\cdot)\mathfrak{g}$. Fix the left Haar measure dz on $G_{\mathbb{C}}$ so it coincides with the Riemannian volume measure. In this setup, we have the following central theorem on heat kernels:

Theorem 6.2.2 (Hall [18], Stein [53], Nelson [46]. See also Gangolli [14] and Hall [19]). *Let δ_e be the unit point measure at the identity $e \in G \subset G_{\mathbb{C}}$.*

1. *There exists a solution $\rho \in C^\infty((0,\infty)\times G)$ (where $\rho_t = \rho(t,\cdot)$) to*

$$\begin{cases} \frac{\partial}{\partial t}\rho(t,x) = \frac{1}{2}(\Delta_G)_x \rho(t,x) & \text{for all} \quad (t,x) \in (0,\infty)\times G, \\ \lim_{t\to 0}\rho_t = \delta_e & \text{in} \quad \mathcal{D}'(G). \end{cases}$$

2. *There exists a solution $\mu \in C^\infty((0,\infty)\times G_{\mathbb{C}})$ (where $\mu_t = \mu(t,\cdot)$) to*

$$\begin{cases} \frac{\partial}{\partial t}\mu(t,z) = \frac{1}{4}(\Delta_{G_{\mathbb{C}}})_z \mu(t,z) & \text{for all} \quad (t,z) \in (0,\infty)\times G_{\mathbb{C}}, \\ \lim_{t\to 0}\mu_t = \delta_e & \text{in} \quad \mathcal{D}'(G_{\mathbb{C}}). \end{cases}$$

Here the solution ρ to the first system above is unique, while μ is in general not unique. But if we put $\nu_t(z) = \int_G \mu_t(xz)\,dx$ for all $z \in G_{\mathbb{C}}$, then for $t > 0$ we obtain the following:

1. *ρ_t is strictly positive, real-valued, $\int_G \rho_t(x)\,dx = 1$, and*

$$\rho_t(x) = \sum_{[\xi]\in\widehat{G}} d_\xi e^{-\frac{1}{2}t\lambda_\xi}\operatorname{Tr}\xi(x) \quad \text{for all} \quad x \in G.$$

2. *μ_t can be taken non-negative, real-valued, $\int_{G_{\mathbb{C}}} \mu_t(z)\,dz = 1$, and*

$$\nu_t(\exp(iY)x) = \frac{e^{-|\tau|^2 t}}{(\pi t)^{\frac{n}{2}}}\Theta(Y)e^{-\frac{|Y|^2}{t}} \quad \text{for all} \quad (x,Y) \in G\times\mathfrak{g}.$$

Here τ is half of the sum of the positive roots for some maximal abelian subalgebra of $\mathfrak{g}$. The number n is the dimension of G.

In the following we will make extensive use of ρ_t, μ_t and ν_t as in Theorem 6.2.2. Henceforth we fix these functions at a particular time $t > 0$.

Lemma 6.2.2 (Hall [18]). *If $\epsilon > 0$, then*

$$\sup_{z \in G_\epsilon} |\mathrm{Tr}\, \xi(z)| \le d_\xi e^{\epsilon \sqrt{\lambda_\xi}} \quad \text{for any} \quad [\xi] \in \widehat{G}.$$

Proof. Let $z \in G_\epsilon$ decompose as $z = \exp(iY)x$ where $x \in G$ and $Y \in \mathfrak{g}$ with $|Y|_{\mathfrak{g}} < \epsilon$. Since $\xi(x)$ is unitary, and $i\, d\xi(Y)$ is self-adjoint, we get from Lemma 6.2.1 that

$$\begin{aligned} ||\xi(z)|| &= ||\xi(\exp(iY))\xi(x)|| \\ &= ||\exp(i\, d\xi(Y))|| \le e^{||i\, d\xi(Y)||} \le e^{\sqrt{\lambda_\xi}|Y|_{\mathfrak{g}}}, \end{aligned}$$

and so, we get

$$\sup_{z \in G_\epsilon} |\mathrm{Tr}\, \xi(z)| \le d_\xi \sup_{z \in G_\epsilon} ||\xi(z)|| \le d_\xi e^{\epsilon\sqrt{\lambda_\xi}}.$$

□

Corollary 6.2.2. *The heat kernel ρ_t has a unique holomorphic extension to all of $G_{\mathbb{C}}$. It is given by the locally uniformly convergent sum*

$$\rho_t(z) = \sum_{[\xi] \in \widehat{G}} d_\xi e^{-\frac{t}{2}\lambda_\xi} \mathrm{Tr}\, \xi(z) \quad \text{for all} \quad z \in G_{\mathbb{C}}.$$

Proof. Taking $\epsilon = k \in \mathbb{N}$ and using $d_\xi = O(\langle \xi \rangle^{\frac{n}{2}})$, we get $C > 0$ such that

$$\begin{aligned} |d_\xi e^{-\frac{t}{2}\lambda_\xi} \mathrm{Tr}\, \xi(z)| &\le d_\xi^2 e^{k\sqrt{\lambda_\xi} - \frac{t}{2}\lambda_\xi} \\ &\le C(1 + \lambda_\xi^2)^{\frac{n}{2}} e^{k\sqrt{\lambda_\xi} - \frac{t}{2}\lambda_\xi}, \end{aligned}$$

and it follows that we have uniform convergence in all $z \in G_k$ of the above defined series. Hence it converges to a holomorphic function on all of $G_{\mathbb{C}}$. □

This corollary allows us to define the (generalized) Segal-Bargmann transform on G. It is essentially just $e^{\frac{t}{2}\Delta_G}$ followed by analytic continuation.

Definition 6.2.2. *If $f \in L^2(G)$, let $\mathcal{C}_t f$ be the holomorphic extension of $f * \rho_t$ to $G_{\mathbb{C}}$. The map $L^2(G) \ni f \mapsto \mathcal{C}_t f$ is the (generalized) Segal-Bargmann transform on G.*

Lemma 6.2.3 (Hall [18]. See also Nelson [46]). *The following holds.*

1. *If ξ is any finite-dimensional representation of G, then*

$$\int_G \xi(x)\rho_t(x)\, dx = \exp\Big(\frac{t}{2} d\xi(\Delta_G)\Big).$$

2. *If $\xi_{\mathbb{C}}$ is any finite-dimensional representation of $G_{\mathbb{C}}$, then*

$$\int_{G_{\mathbb{C}}} \xi_{\mathbb{C}}(z)\mu_t(z)\, dz = \exp\Big(\frac{t}{4} d\xi_{\mathbb{C}}(\Delta_{G_{\mathbb{C}}})\Big).$$

Proof. To see that the first point is true, we simply differentiate $\xi(\rho_t)$ with respect to t, and then obtain a differential equation that can only be solved by the right hand side. Doing this, we get

$$\begin{aligned}\frac{d}{dt}\xi(\rho_t) = \frac{1}{2}\xi(\Delta_G\rho_t) &= \frac{1}{2}\sum_{j=1}^{n}\int_G \xi(x)X_j^2\rho_t(x)\,dx \\ &= \frac{1}{2}\sum_{j=1}^{n}\Big(\int_G \xi(x)\rho_t(x)\,dx\Big)d\xi(X_j)^2 \\ &= \frac{1}{2}\xi(\rho_t)d\xi(\Delta_G),\end{aligned}$$

and this leads to

$$\begin{cases}\frac{d}{dt}\xi(\rho_t) = \frac{1}{2}\xi(\rho_t)d\xi(\Delta_G) \quad \text{for all} \quad t\in(0,\infty),\\ \lim_{t\to 0}\xi(\rho_t) = I,\end{cases}$$

which of course has the unique solution $\exp(\frac{t}{2}d\xi(\Delta_G))$. This establishes the first point. Now, to see that the second point holds, we do the same, but avoid convergence issues. Using the explicit expression for ν_t in Theorem 6.2.2, we see that

$$\sup_{t\in(0,T]}\int_{G_{\mathbb{C}}}||\xi_{\mathbb{C}}(z)||\mu_t(z)\,dz \le C\sup_{t\in(0,T]}\int_{\mathfrak{g}} e^{||d\xi_{\mathbb{C}}||\,|Y|}\Big[\frac{\nu_t(\exp(iY))}{\Theta(Y)^2}\Big]\,dY < \infty,$$

where $|\cdot|_{\mathfrak{g}}$ refers to the norm on $\mathfrak{g}$, extended to $\mathfrak{g}_{\mathbb{C}}$, and $C>0$ is independent of $T>0$. Multiplying a left G-invariant cutoff onto $\xi_{\mathbb{C}}$, a similar estimate shows $\lim_{t\to 0}\xi_{\mathbb{C}}(\mu_t) = I$. Also, if $f\in C_0^\infty(G_{\mathbb{C}})$, then by the FTT, we furthermore have

$$\sup_{t\in(0,T]}\int_{G_{\mathbb{C}}}||\xi_{\mathbb{C}}(z)||\,|(\mu_t * f)(z)|\,dz < \infty.$$

Using that $G_{\mathbb{C}}$ is unimodular, there is a right-invariant differential operator A on $G_{\mathbb{C}}$, which corresponds to $\Delta_{G_{\mathbb{C}}}$, such that $\Delta_{G_{\mathbb{C}}}\mu_t * f = \mu_t * Af$ and $\xi_{\mathbb{C}}(Af) = d\xi_{\mathbb{C}}(\Delta_{G_{\mathbb{C}}})\xi_{\mathbb{C}}(f)$. Combining all the above with this fact (checked by a computation), we get

$$\begin{aligned}\frac{d}{dt}\Big[\xi_{\mathbb{C}}(\mu_t)\Big]\xi_{\mathbb{C}}(f) &= \frac{d}{dt}\Big[\xi_{\mathbb{C}}(\mu_t * f)\Big] \\ &= \frac{1}{4}\xi_{\mathbb{C}}(\Delta_{G_{\mathbb{C}}}\mu_t * f) \\ &= \frac{1}{4}\xi_{\mathbb{C}}(\mu_t)d\xi_{\mathbb{C}}(\Delta_{G_{\mathbb{C}}})\xi_{\mathbb{C}}(f).\end{aligned}$$

Thus if f is positive with small support near $e\in G_{\mathbb{C}}$, then $\xi_{\mathbb{C}}(f)$ is invertible, and

$$\begin{cases}\frac{d}{dt}\xi_{\mathbb{C}}(\mu_t) = \frac{1}{4}\xi_{\mathbb{C}}(\mu_t)d\xi_{\mathbb{C}}(\Delta_{G_{\mathbb{C}}}) \quad \text{for all} \quad t\in(0,\infty),\\ \lim_{t\to 0}\xi_{\mathbb{C}}(\mu_t) = I,\end{cases}$$

which has the unique solution $\exp(\frac{t}{4}d\xi_{\mathbb{C}}(\Delta_{G_{\mathbb{C}}}))$. □

The Segal-Bargmann transform is unitary from ordinary to holomorphic L^2-spaces. This is the most important property of $\mathcal{C}_t$.

Theorem 6.2.3 (Hall [18]). *The transform $\mathcal{C}_t : L^2(G) \to HL^2(G_{\mathbb{C}}, \nu_t)$ is well-defined. Furthermore, it is a surjective isometry.*

Proof. Take $[\xi], [\eta] \in \widehat{G}$ and endomorphisms A and B of the spaces of ξ and η, respectively. It follows from the definition of ρ_t, and analytic continuation in $z \in G_{\mathbb{C}}$, that

$$\int_G \mathrm{Tr}(\xi(x)A)\rho_t(x^{-1}z)\,dx = e^{-\frac{t}{2}\lambda_\xi}\mathrm{Tr}(\xi(z)A).$$

It suffices to show isometricity on functions of the form $G \to \mathbb{C} : x \mapsto \mathrm{Tr}(\xi(x)A)$ only. This is because their span is dense in $L^2(G)$ by Theorem 3.2.2. In that case, we have

$$\begin{aligned}\int_G \mathrm{Tr}(\xi(x)A)\overline{\mathrm{Tr}(\eta(x)B)}\rho_t(x)\,dx &= \mathrm{Tr}\Big[\Big(\int_G (\xi\otimes\overline{\eta})(x)\rho_t(x)\,dx\Big)(A\otimes\overline{B})\Big]\\ &= \mathrm{Tr}\Big[\exp\Big(\frac{t}{2}d(\xi\otimes\overline{\eta})(\Delta_G)\Big)(A\otimes\overline{B})\Big],\end{aligned}$$

where $\overline{\eta}$ is the contragredient of η, and

$$d(\xi\otimes\overline{\eta})(\Delta_G) = -(\lambda_\xi+\lambda_\eta) + \frac{1}{2}d(\xi\otimes\eta^\dagger)(\Delta_{G_{\mathbb{C}}}),$$

and $\eta^\dagger$ is the anti-holomorphic representation given by $\eta^\dagger : G_{\mathbb{C}} \to \mathrm{GL}(d_\eta, \mathbb{C}) : z \mapsto \overline{\eta}(\overline{z})$. To see the above identity, we note that $d\overline{\eta}(\Delta_G) = -\lambda_\eta I$, and write

$$\begin{aligned}d(\xi\otimes\eta^\dagger)(\Delta_{G_{\mathbb{C}}}) &= \sum_{j=1}^n \Big[\Big(d\xi(X_j)\otimes I + I\otimes d\eta^\dagger(X_j)\Big)^2 + \Big(d\xi(JX_j)\otimes I + I\otimes d\eta^\dagger(JX_j)\Big)^2\Big]\\ &= \sum_{j=1}^n \Big[\Big(d\xi(X_j)\otimes I + I\otimes d\overline{\eta}(X_j)\Big)^2 + \Big(i\,d\xi(X_j)\otimes I - I\otimes i\,d\overline{\eta}(X_j)\Big)^2\Big]\\ &= 2(\lambda_\xi+\lambda_\eta) + 2\sum_{j=1}^n \Big(d\xi(X_j)\otimes I + I\otimes d\overline{\eta}(X_j)\Big)^2,\end{aligned}$$

Using this, we continue the computation to get

$$\begin{aligned}\mathrm{Tr}\Big[\exp\Big(\frac{t}{4}d(\xi\otimes\eta^\dagger)(\Delta_{G_{\mathbb{C}}})\Big)(A\otimes\overline{B})\Big] &= \mathrm{Tr}\Big[\Big(\int_{G_{\mathbb{C}}}[\xi(z)\otimes\overline{\eta}(\overline{z})]\mu_t(z)\,dz\Big)(A\otimes\overline{B})\Big]\\ &= \int_{G_{\mathbb{C}}} \mathrm{Tr}(\xi(z)A)\overline{\mathrm{Tr}(\eta(z)B)}\mu_t(z)\,dz,\end{aligned}$$

and finally multiplying by $e^{-\frac{t}{2}(\lambda_\xi+\lambda_\eta)}$ shows isometry as a map $L^2(G,\rho_t) \to L^2(G_{\mathbb{C}}, \mu_t)$. Since $\mathcal{C}_t$ commutes with the left regular representation, we also have

$$||\mathcal{C}_t f||^2_{L^2(G_{\mathbb{C}},\nu_t)} = \int_G\int_{G_{\mathbb{C}}} |\mathcal{C}_t f(y^{-1}z)|^2\mu_t(z)\,dz\,dy = \int_G\int_G |f(y^{-1}x)|^2\rho_t(x)\,dx\,dy,$$

where an interchange of integrals and $\int_G \rho_t(y)\,dy = 1$ gives the desired isometry property. It is surjective because of the holomorphic Peter-Weyl expansion. □

Also, there is a characterization the image of any order $s \geq 0$ Sobolev space under $\mathcal{C}_t$. It is given in terms of the aptly named holomorphic Sobolev spaces.

Definition 6.2.3. *Let $s \in \mathbb{R}$. Define*

$$HH^s(G_{\mathbb{C}}, \nu_t) = \{u \in \mathcal{O}(G_{\mathbb{C}}) \,|\, (I - (\Delta_G)_{\mathbb{C}})^{\frac{s}{2}} u \in HL^2(G_{\mathbb{C}}, \nu_t)\}.$$

These inherit the Hilbert space structure by just forcing $(I - (\Delta_G)_{\mathbb{C}})^{\frac{s}{2}}$ to be unitary. Note then that if $s' > s \geq 0$, we have

$$HH^s(G_{\mathbb{C}}, \nu_t) \subset HH^{s'}(G_{\mathbb{C}}, \nu_t),$$

which can be seen by writing $(I - (\Delta_G)_{\mathbb{C}})^{\frac{s}{2}}$ out in its holomorphic Peter-Weyl expansion.

Theorem 6.2.4 (Hall and Lewkeeratiyutkul [20])**.** *(Slightly more general.) Let $s \geq 0$. Then $\mathcal{C}_t : H^s(G) \to HH^s(G_{\mathbb{C}}, \nu_t)$ is a well-defined surjective isometry.*

Proof. Observe first that $[(I - \Delta_G)^{\frac{s}{2}}, e^{\frac{t}{2}\Delta_G}] = 0$, both understood as operators on $C^\infty(G)$. But $\mathcal{C}_t$ is just $e^{\frac{t}{2}\Delta_G}|_{L^2(G)}$ followed by holomorphic extension. This implies that

$$(I - (\Delta_G)_{\mathbb{C}})^{\frac{s}{2}} \mathcal{C}_t|_{H^s(G)} = \mathcal{C}_t (I - \Delta_G)^{\frac{s}{2}}|_{H^s(G)},$$

and it follows that

$$\mathcal{C}_t(H^s(G)) \subset HH^s(G_{\mathbb{C}}, \nu_t).$$

As an unbounded operator into $L^2(G)$ with the domain $H^s(G)$, $(I - \Delta_G)^{\frac{s}{2}}$ is self-adjoint, and hence so is $\mathcal{C}_t (I - \Delta_G)^{\frac{s}{2}} \mathcal{C}_t^{-1} : \mathcal{C}_t(H^s(G)) \to HL^2(G_{\mathbb{C}}, \nu_t)$, by the L^2-unitarity of $\mathcal{C}_t$. The latter operator coincides with

$$\mathcal{C}_t (I - \Delta_G)^{\frac{s}{2}} \mathcal{C}_t^{-1}|_{\mathcal{C}_t(H^s(G))} = (I - (\Delta_G)_{\mathbb{C}})^{\frac{s}{2}}|_{\mathcal{C}_t(H^s(G))},$$

and if this extends symmetrically to $HH^s(G_{\mathbb{C}}, \nu_t)$, maximality forces domains to coincide. To see that this is the case, take $f, g \in HH^s(G_{\mathbb{C}}, \nu_t)$, and use Theorem 4.2.6 to get

$$\begin{aligned} ((I - (\Delta_G)_{\mathbb{C}})^{\frac{s}{2}} f, g)_{L^2(G_{\mathbb{C}}, \nu_t)} &= \int_{G_{\mathbb{C}}} [(I - (\Delta_G)_{\mathbb{C}})^{\frac{s}{2}} f](z) \overline{g(z)} \nu_t(z)\, dz \\ &= \int_{\mathfrak{g}} \Big[\int_G (I - \Delta_G)^{\frac{s}{2}} (f_Y)(x) \overline{(g_Y)(x)}\, dx \Big] \frac{\nu_t(\exp(iY))}{\Theta(Y)^2}\, dY \\ &= \int_{\mathfrak{g}} \Big[\int_G (f_Y)(x) \overline{(I - \Delta_G)^{\frac{s}{2}} (g_Y)(x)}\, dx \Big] \frac{\nu_t(\exp(iY))}{\Theta(Y)^2}\, dY, \end{aligned}$$

where we write $f_Y(x) = f(\exp(iY)x)$ and $g_Y(x) = g(\exp(iY)x)$ for any $(x, Y) \in G \times \mathfrak{g}$, and use that for such holomorphic functions f on $G_{\mathbb{C}}$, we have

$$[(I - (\Delta_G)_{\mathbb{C}})^{\frac{s}{2}} f](\exp(iY)x) = (I - \Delta_G)^{\frac{s}{2}} (f_Y)(x).$$

This we may apply again to g_Y to recombine the integrals above and get the symmetry. It is an isometry because $\mathcal{C}_t$ intertwines $(1 - \Delta_G)^{\frac{s}{2}}$ and $(1 - (\Delta_G)_{\mathbb{C}})^{\frac{s}{2}}$ on $H^s(G)$. ☐

The following result shows that $HH^s(G_\mathbb{C}, \nu_t)$ captures a growth condition at infinity. It is rather lengthy to prove, so we refer to Hall and Lewkeeratiyutkul [20].

Theorem 6.2.5 (Hall and Lewkeeratiyutkul [20])**.** *Let $s = 2k$ for some number $k \in \mathbb{N}$. The space $HH^s(G_\mathbb{C}, \nu_t)$ consists exactly of those $f \in HL^2(G_\mathbb{C}, \nu_t)$ satisfying*

$$\int_{\mathfrak{g}} \int_G |f(\exp(iY)x)|^2 \Big[(1+|Y|^2)^s \frac{\nu_t(\exp(iY))}{\Theta(Y)^2}\Big]\, dx\, dY < \infty.$$

Finally, we prove the simplest inversion formula for $\mathcal{C}_t$ in terms of ρ_t and the sets G_k. There are others, but these would require yet more theory.

Theorem 6.2.6 (Hall [18])**.** *The map $\mathcal{C}_t^{-1} : HL^2(G_\mathbb{C}, \nu_t) \to L^2(G)$ has an explicit form. It is given by the L^2-limit*

$$\mathcal{C}_t^{-1} g = \lim_{k\to\infty} \mathcal{C}_t^{-1}(1_{G_k} g) \quad \text{for any} \quad g \in HL^2(G_\mathbb{C}, \nu_t),$$

where the cut-off is

$$\mathcal{C}_t^{-1}(1_{G_k} g)(x) = \int_{G_k} g(z)\overline{\rho_t(x^{-1}z)}\nu_t(z)\, dz \quad \text{for a.e} \quad x \in G.$$

Proof. Note that $\mathcal{C}_t : L^2(G) \to L^2(G_\mathbb{C}, \nu_t)$ is an isometry, so its adjoint is a left inverse. Taking $f \in L^2(G)$, we see that

$$\begin{aligned}(\mathcal{C}_t f, 1_{G_k} g)_{L^2(G_\mathbb{C},\nu_t)} &= \int_{G_k} \Big[\int_G f(x)\rho_t(x^{-1}z)\, dx\Big]\overline{g(z)}\nu_t(z)\, dz \\ &= \int_G f(x)\overline{\Big[\int_{G_k} g(z)\overline{\rho_t(x^{-1}z)}\nu_t(z)\, dz\Big]}\, dx,\end{aligned}$$

and letting $k \to \infty$ gives the formula. □

The limit above is unavoidable, since it is not possible to directly replace G_k by $G_\mathbb{C}$. In fact, $G_\mathbb{C} \to \mathbb{C} : z \mapsto \rho_t(x^{-1}z)$ can never belong to $L^2(G_\mathbb{C}, \nu_t)$ regardless of $x \in G$. Otherwise, there must exist some $f \in L^2(G)$ such that its restriction to G equals $f * \rho_t$, but this gives an identity for the convolution, which is impossible.

7

Operators Preserving ϵ-Extendible Functions

In this chapter, we use the theory from the previous chapters to investigate our question. Let (M, g) be a compact real-analytic Riemannian manifold with Grauert tubes $\{M_\epsilon\}_{\epsilon<\epsilon_0}$. If $P \in \Psi(M)$, we seek a $\widetilde{P}$ such that the following diagram commutes:

$$\begin{array}{ccc} H_1 & \xrightarrow{\widetilde{P}} & H_2 \\ \Big\downarrow{\scriptstyle \mathcal{R}_\epsilon} & & \Big\downarrow{\scriptstyle \mathcal{R}_\epsilon} \\ C^\infty(M) & \xrightarrow{P} & C^\infty(M) \end{array}$$

where $H_1, H_2 \subset \mathcal{O}(M_\epsilon)$ are some holomorphic function spaces, $\mathcal{R}_\epsilon$ is the restriction map. That is, we seek necessary or sufficient conditions for the existence of such an operator, and we would also like the two function spaces H_1 and H_2 above to be Hilbert spaces. This would make the machinery of functional analysis available in any further analysis. An operator P with this property will be called (by slight abuse of language) ϵ-extendible. The next question is then, if P is also elliptic, does it admit an ϵ-extendible parametrix, and if so, is it possible to somehow construct this parametrix from P?

7.1 Algebras of ϵ-Extendible Operators

Let G_ϵ be the tube of radius $\epsilon > 0$ about a compact (possibly disconnected) Lie group G, which has the Lie algebra $\mathfrak{g}$, always carrying a fixed $\mathrm{Ad}(G)$-invariant inner product $(\cdot,\cdot)_\mathfrak{g}$. It is possible to define order $s \in \mathbb{R}$ "holomorphic Sobolev spaces" on G_ϵ, as follows:

Definition 7.1.1. *Let $s \in \mathbb{R}$. Define*

$$HH^s(G_\epsilon) = \{u \in \mathcal{O}(G_\epsilon) \,|\, (I - (\Delta_G)_\mathbb{C})^{\frac{s}{2}} u \in HL^2(G_\epsilon)\}.$$

Of course, these are Hilbert spaces given the inner product inherited from $L^2(M_\epsilon)$. That is, we put

$$(u, v)_{HH^s(G_\epsilon)} = ((I - (\Delta_G)_\mathbb{C})^{\frac{s}{2}} u, (I - (\Delta_G)_\mathbb{C})^{\frac{s}{2}} v)_{L^2(M_\epsilon)} \quad \text{if} \quad u, v \in HH^s(G_\epsilon).$$

The space $HH^s(G_\epsilon)$ is different from $HH^s(G_\mathbb{C}, \nu_t)$. It may contain non-entire functions, and it is defined with respect to the finite volume measure dz on G_ϵ with no weight. Unfortunately, the above definition does not carry over easily to Grauert tubes M_ϵ of M. How should we even define $(I - \Delta_\mathbb{C})^{\frac{s}{2}}$, analogous to $(I - (\Delta_G)_\mathbb{C})^{\frac{s}{2}}$, on M_ϵ when $s \notin 2\mathbb{N}_0$? One way to overcome this problem is to use the Poisson transform.

7.1.1 On a Compact Riemannian Manifold M

Choose $\epsilon_0' < \epsilon_0$ so that the Boutet de Monvel theorem holds for $\epsilon \in (0, \epsilon_0')$. Put $n = \dim M$. It will be necessary to use the following lemma repeatedly.

Lemma 7.1.1. *Let $d, d' \in \mathbb{N}$ and let $P \in \mathrm{Diff}^d(M)$ be elliptic and formally self-adjoint. Assume that P has classical principal symbol p positive on $T^*M \setminus 0$, and*

$$\sigma(P) \subset [0, \infty).$$

Then, if $A \in \Psi^{d'}(M)$ with $d' \in \mathbb{R}$ and $[A, P] = 0$, the following holds:

1. $[A, P^{\frac{1}{d}}] = 0$.
2. $[A, e^{-\epsilon P^{\frac{1}{d}}}] = 0$.

Proof. By hypothesis, P is parameter-elliptic w.r.t. any closed sector inside $\mathbb{C} \setminus (0, \infty)$. Therefore the two operators are defined, given appropriate contours in the complex plane. That is, we can choose $R > 0$ so that if $u \in C^\infty(M)$, we have

$$P^{\frac{1}{d}} u = \frac{1}{2\pi i} \int_{\Gamma_R} \lambda^{\frac{1-d}{d}} (\lambda I - P)^{-1} P u \, d\lambda,$$

and

$$e^{-tP^{\frac{1}{d}}} u = \frac{1}{2\pi i} \int_{\Gamma_R'} e^{-t\lambda^{\frac{1}{d}}} (\lambda I - P)^{-1} u \, d\lambda + \frac{1}{2\pi i} \int_{R\mathbb{S}^1} (\lambda I - P)^{-1} u \, d\lambda,$$

where $\lambda \mapsto \lambda^{\frac{1-d}{d}}$ is defined using the principal logarithm with branch cut along $(-\infty, 0]$, and Γ_R and Γ_R' are keyhole contours, $R\mathbb{S}^1$ encircles no eigenvalues other than possibly 0. The commutators are easily calculated to be

$$\begin{aligned}
[A, P^{\frac{1}{d}}]u &= A\Big(\frac{1}{2\pi i} \int_{\Gamma_R} \lambda^{\frac{1-d}{d}} (\lambda I - P)^{-1} P u \, d\lambda\Big) - P^{\frac{1}{d}} A u \\
&= \frac{1}{2\pi i} \int_{\Gamma_R} \lambda^{\frac{1-d}{d}} \Big[A(\lambda I - P)^{-1} P - (\lambda I - P)^{-1} P A\Big] u \, d\lambda = 0,
\end{aligned}$$

and also

$$\begin{aligned}
[A, e^{-\epsilon P^{\frac{1}{d}}}]u &= A\Big(\frac{1}{2\pi i} \int_{\Gamma_R'} e^{-\epsilon\lambda^{\frac{1}{d}}} (\lambda I - P)^{-1} u \, d\lambda\Big) - e^{-\epsilon P^{\frac{1}{d}}} A u + 0 \\
&= \frac{1}{2\pi i} \int_{\Gamma_R'} e^{-\epsilon\lambda^{\frac{1}{d}}} \Big[A(\lambda I - P)^{-1} - (\lambda I - P)^{-1} A\Big] u \, d\lambda = 0.
\end{aligned}$$

□

In the above lemma, $I + P$ is always invertible, and $(I + P)^s$ is defined for any $s \in \mathbb{R}$. Of course, in that case, A also commutes with any such power:

$$[A, (I + P)^s] = 0.$$

Our approach is to indirectly define spaces as images under $\mathcal{P}_\epsilon$ of any order $s \in \mathbb{R}$. These will coincide with the holomorphic Sobolev spaces on G_ϵ when $M = G$.

Definition 7.1.2.

$$HH^s(M_\epsilon) = \mathcal{P}_\epsilon H^{s-\frac{n+1}{4}}(M).$$

Using the inverse $\mathcal{P}_\epsilon^{-1}$, the image space is made to inherit the Hilbert space structure. It is equipped with the induced inner product

$$(u,v)_{HH^s(M_\epsilon)} = (\mathcal{P}_\epsilon(I-\Delta)^{\frac{s}{2}}\mathcal{P}_\epsilon^{-1}u, \mathcal{P}_\epsilon(I-\Delta)^{\frac{s}{2}}\mathcal{P}_\epsilon^{-1}v)_{L^2(M_\epsilon)} \quad \text{if} \quad u,v \in HH^s(G_\epsilon),$$

and $\mathcal{P}_\epsilon : H^{s-\frac{n+1}{4}}(M) \to HH^s(M_\epsilon)$ is automatically a bounded isomorphism in this way. This is clear from

$$\begin{aligned} ||\mathcal{P}_\epsilon u||_{HH^s(M_\epsilon)} &= ||\mathcal{P}_\epsilon(I-\Delta)^{\frac{s}{2}}u||_{L^2(M_\epsilon)} \\ &\leq ||\mathcal{P}_\epsilon||_{B(H^{-\frac{n+1}{4}}(M), HL^2(M_\epsilon))} ||u||_{H^{s-\frac{n+1}{4}}(M)}. \end{aligned}$$

Let us show that if $M = G$ and $\Delta = \Delta_G$ the definition agrees with that for Lie groups.

Proof. Take any $u \in C^\infty(G)$, and for any $z \in G_\epsilon$ write

$$\begin{aligned} \mathcal{P}_\epsilon(I-\Delta_G)^{\frac{s}{2}}u(z) &= \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\mathcal{P}_\epsilon\xi(z)\mathcal{F}(I-\Delta_G)^{\frac{s}{2}}u) \\ &= \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}\Big(\langle\xi\rangle^s\xi(z)\int_G u(x)e^{-\epsilon\sqrt{\lambda_\xi}}\xi(x)^*\,dx\Big) \\ &= \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}\Big(\langle\xi\rangle^s\xi(z)\int_G (e^{-\epsilon\sqrt{-\Delta}}u)(x)\xi(x)^*\,dx\Big), \end{aligned}$$

where we use that the Peter-Weyl expansion of u converges in the topology of $C^\infty(G)$, and thus the first sum converges in $HL^2(G_\epsilon)$, hence uniformly on compact subsets of G_ϵ. This shows that

$$\mathcal{P}_\epsilon(I-\Delta_G)^{\frac{s}{2}}u = (I-(\Delta_G)_\mathbb{C})^{\frac{s}{2}}\mathcal{P}_\epsilon u,$$

which, on the left hand side, must extend to $u \in H^{s-\frac{n+1}{4}}(M)$ just by using the continuity. To see equality for $u \in H^{s-\frac{n+1}{4}}(M)$, pick $\{u_k\}_{k=1}^\infty \subset C^\infty(M)$ such that

$$u_k \to u \quad \text{in} \quad H^{-\frac{n+1}{4}}(M) \quad \text{as} \quad k \to \infty,$$

and note that

$$(I-(\Delta_G)_\mathbb{C})^{\frac{s}{2}}\mathcal{P}_\epsilon u_k \to (I-(\Delta_G)_\mathbb{C})^{\frac{s}{2}}\mathcal{P}_\epsilon u \quad \text{uniformly on} \quad K \subset\subset G_\epsilon \quad \text{as} \quad k \to \infty,$$

while the same type of convergence is implied by HL^2-convergence on the left hand side. Therefore, the limits agree as merely holomorphic functions on G_ϵ when $u \in H^{s-\frac{n+1}{4}}(M)$. It follows that the definitions agree, justifying the notation. □

The first interesting result in this section follows from the construction of $\mathcal{P}_\epsilon$ itself. Those operators commuting with Δ preserve ϵ-extendible functions:

Proposition 7.1.1. *Suppose that $d' \in \mathbb{R}$ and $A \in \Psi^{d'}(M)$ commutes with Δ on $C^\infty(M)$. Then if $s \in \mathbb{R}$ the following diagram commutes and consists of bounded operators:*

$$\begin{array}{ccc} HH^s(M_\epsilon) & \xrightarrow{\mathcal{P}_\epsilon A \mathcal{P}_\epsilon^{-1}} & HH^{s-d'}(M_\epsilon) \\ \downarrow \mathcal{R}_\epsilon & & \downarrow \mathcal{R}_\epsilon \\ H^s(M) & \xrightarrow{A} & H^{s-d'}(M) \end{array}$$

Proof. By the construction, $\mathcal{R}_\epsilon \mathcal{P}_\epsilon|_{H^{s-\frac{n+1}{4}}(M)} = e^{-\epsilon\sqrt{-\Delta}}|_{H^{s-\frac{n+1}{4}}(M)}$ holds for any $s \in \mathbb{R}$, and so by Lemma 7.1.1, if $u \in C^\infty(M)$ this implies

$$\mathcal{R}_\epsilon \mathcal{P}_\epsilon A u = e^{-\epsilon\sqrt{-\Delta}} A u = A e^{-\epsilon\sqrt{-\Delta}} u = A \mathcal{R}_\epsilon \mathcal{P}_\epsilon u,$$

which then extends by continuity to all $u \in H^s(M)$. Therefore the diagram commutes. As a special case, if $u \in HH^s(M_\epsilon)$, we have

$$(I-\Delta)^{\frac{s}{2}} \mathcal{R}_\epsilon u = \mathcal{R}_\epsilon \mathcal{P}_\epsilon (I-\Delta)^{\frac{s}{2}} \mathcal{P}_\epsilon^{-1} u,$$

and we obtain the estimates

$$\begin{aligned} ||\mathcal{R}_\epsilon u||_{H^s(M)} &\leq ||(I-\Delta)^{-\frac{s}{2}}||_{B(H^s(M),L^2(M))} ||(I-\Delta)^{\frac{s}{2}} \mathcal{R}_\epsilon u||_{L^2(M)} \\ &\leq ||(I-\Delta)^{-\frac{s}{2}}||_{B(H^s(M),L^2(M))} ||\mathcal{R}_\epsilon||_{B(L^2(M_\epsilon),L^2(M))} ||u||_{HH^s(M_\epsilon)}. \end{aligned}$$

Therefore the $\mathcal{R}_\epsilon$ are also bounded in the diagram ☐

Proposition 7.1.2. *Suppose that $d' \in \mathbb{R}$. Let $\{\eta_k\}_{k=0}^\infty$ be a sequence of complex numbers. If there is a $C > 0$ with $|\eta_k| \leq C\langle\lambda_k\rangle^{d'}$ for all $k \in \mathbb{N}_0$, we can meaningfully define*

$$A : C^\infty(M) \to C^\infty(M) : u \mapsto \sum_{k=0}^\infty \eta_k (u, \phi_k)_{L^2(M)} \phi_k.$$

Then A is continuous, $[A, \Delta] = 0$, and the conclusion of Proposition 7.1.1 holds.

Proof. Taking any $s \in \mathbb{R}$ and $u \in H^s(M)$, we have

$$\begin{aligned} ||Au||^2_{H^{s-d'}(M)} &= \sum_{k=0}^\infty \langle\lambda_k\rangle^{2(s-d')} |\eta_k|^2 |(u, \phi_k)|^2 \\ &\leq C \sum_{k=0}^\infty |((I-\Delta)^{-\frac{s}{2}} u, \phi_k)|^2 = C||u||^2_{H^s(M)}, \end{aligned}$$

and so by the Sobolev embedding theorem, A is continuous from $C^\infty(M)$ to itself. ☐

Corollary 7.1.1. *Δ admits a parametrix satisfying the above diagram.*

Proof. Put $\eta_k = \frac{1}{\lambda_k}$ when $\lambda_k \neq 0$ and zero otherwise. ☐

The continuation theorem was announced by Boutet de Monvel in the old paper [3]. But no proof was provided. Only the case of the Laplacian was fully proved in [55, 65]. To our knowledge, the general statement in [3] remains unproven at the time of writing. It will imply that the above results have analogues for other elliptic differential operators, where the Grauert tubes are replaced by other neighbourhoods M_ϵ^Φ of M in $M_\mathbb{C}$.

Let $d \in \mathbb{N}$ and let $P \in \mathrm{Diff}^d(M)$ be real-analytic with classical principal symbol p. Suppose that P is formally self-adjoint, elliptic and that p is always positive on $T^*M \setminus 0$. Then P is semi-lower bounded, and we assume that this P has no negative eigenvalues. In that case, p generates a real-analytic complete Hamiltonian flow $\varphi_t : T^*M \to T^*M$. This flow satisfies

$$(\pi_x \varphi_{t^{d-1}})(x,\xi) = (\pi_x \varphi_1)(x, t\xi) \quad \text{for all} \quad (x,\xi) \in T^*M \quad \text{and} \quad t \in \mathbb{R},$$

where π_x is the cotangent bundle projection

$$\pi_x : T^*M \to M : (x,\xi) \mapsto x.$$

Conjecture 7.1.1 (Boutet de Monvel [3]. The statement here is slightly more special)**.** *Let P_ϵ denote the kernel of $e^{-\epsilon P^{\frac{1}{d}}}$, and put*

$$\Phi_x(\xi) = (\pi_x \varphi_1)(x,\xi) \quad \text{for any} \quad (x,\xi) \in T^*M.$$

Then there is a maximal $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ the following holds:

1. *The map Φ_x extends holomorphically to*

$$\Phi_x : \{\xi \in T_x^*M \otimes_\mathbb{R} \mathbb{C} \,|\, |p(x,\xi)|^{\frac{1}{d}} < \epsilon\} \to M_\mathbb{C}.$$

2. *The extended Φ_x combine into a real-analytic diffeomorphism*

$$\Phi : \{(x,\xi) \in T^*M \,|\, p(x,\xi)^{\frac{1}{d}} < \epsilon\} \to M_\epsilon^\Phi \subset M_\mathbb{C} : (x,\xi) \mapsto \Phi_x(i\xi).$$

3. *The image M_ϵ^Φ is open in $M_\mathbb{C}$ with (orientable) C^ω-boundary ∂M_ϵ^Φ.*

This image M_ϵ^Φ generalizes the Grauert tube construction in the following sense:

1. *M_ϵ^Φ admits a unique Kähler structure such that Φ is a symplectomorphism.*
2. *M_ϵ^Φ admits a global Kähler potential for this structure.*
3. *All eigenfunctions of P extend holomorphically to M_ϵ^Φ.*

Then there is a maximal $\epsilon_0' \in (0, \epsilon_0]$ such that for any $\epsilon \in (0, \epsilon_0')$ the following holds:

1. *The map $x \mapsto P_\epsilon(x,y)$ extends holomorphically to M_ϵ^Φ for each fixed $y \in M$.*
2. *The kernel $P_\epsilon|_{\partial M_\epsilon^\Phi \times M}$ induces a complex-phase PHG FIO S_ϵ of order $-\frac{n-1}{4}$.*
3. *S_ϵ defines a homeomorphism $S_\epsilon : H^s(M) \to \mathcal{O}^{s+\frac{n-1}{4}}(\partial M_\epsilon^\Phi)$ for any $s \in \mathbb{R}$.*

Assuming this conjecture to be true, we make a second conjecture in extension of it. The existence radius of S_ϵ should be proportional to that of the eigenfunctions:

Conjecture 7.1.2. *The above conjecture holds with $\frac{\epsilon_0'}{\epsilon_0} \in (0,1]$ independent of P.*

Let us momentarily assume that just the first (and most plausible) conjecture holds. Pick $\epsilon \in (0, \epsilon_0')$, let h be the Kähler metric on the closure of M_ϵ^Φ, and take $s \in \mathbb{R}$.

Definition 7.1.3. *If $f \in H^{s-\frac{n+1}{4}}(M)$, define $\mathcal{W}_\epsilon f$ to be the extension of $e^{-\epsilon P^{\frac{1}{d}}} f$ to M_ϵ^Φ.*

Suppose that K_ϵ is the Poisson operator solving the Dirichlet problem for Δ_h on $\overline{M_\epsilon^\Phi}$. By Theorem 6.1.3 it is a linear homeomorphism

$$K_\epsilon : H^{s-\frac{1}{2}}(\partial M_\epsilon^\Phi) \to \bar{H}^s(M_\epsilon^\Phi) \cap \ker(\Delta_h|_{\bar{H}^s(M_\epsilon^\Phi)}),$$

and $\mathcal{W}_\epsilon = K_\epsilon S_\epsilon$ realizes a well-defined linear homeomorphism

$$\mathcal{W}_\epsilon : H^{s-\frac{n+1}{4}}(M) \to K_\epsilon \mathcal{O}^{s-\frac{1}{2}}(\partial M_\epsilon^\Phi).$$

Definition 7.1.4. *Let $s \in \mathbb{R}$. Define*

$$HH^s(M_\epsilon^\Phi) = \mathcal{W}_\epsilon H^{s-\frac{n+1}{4}}(M).$$

Note that $I + P \in \Psi^d(M)$ is invertible, and the inverse always belongs to $\Psi^{-d}(M)$. So each space is a Hilbert space with the inner product induced by the norm

$$||u||_{HH^s(M_\epsilon^\Phi)} = ||\mathcal{W}_\epsilon (I+P)^{\frac{s}{d}} \mathcal{W}_\epsilon^{-1} u||_{L^2(M_\epsilon^\Phi)} \quad \text{if} \quad u \in HH^s(M_\epsilon^\Phi),$$

and, automatically, $\mathcal{W}_\epsilon$ becomes a bounded isomorphism onto this space.

Proposition 7.1.3. *Suppose that $d' \in \mathbb{R}$ and $A \in \Psi^{d'}(M)$ commutes with P on $C^\infty(M)$. Then if $s \in \mathbb{R}$ the following diagram commutes and consists of bounded operators:*

$$\begin{array}{ccc} HH^s(M_\epsilon^\Phi) & \xrightarrow{\mathcal{W}_\epsilon A \mathcal{W}_\epsilon^{-1}} & HH^{s-d'}(M_\epsilon^\Phi) \\ \big\downarrow \mathcal{R}_\epsilon & & \big\downarrow \mathcal{R}_\epsilon \\ H^s(M) & \xrightarrow{\quad A \quad} & H^{s-d'}(M) \end{array}$$

Proof. As in Proposition 7.1.1, $\mathcal{R}_\epsilon \mathcal{W}_\epsilon|_{H^{s-\frac{n+1}{4}}(M)} = e^{-\epsilon P^{\frac{1}{d}}}|_{H^{s-\frac{n+1}{4}}(M)}$ holds for any $s \in \mathbb{R}$, and so by Lemma 7.1.1, if $u \in C^\infty(M)$ this implies

$$\mathcal{R}_\epsilon \mathcal{W}_\epsilon A u = e^{-\epsilon P^{\frac{1}{d}}} A u = A e^{-\epsilon P^{\frac{1}{d}}} u = A \mathcal{R}_\epsilon \mathcal{W}_\epsilon u,$$

which then extends by continuity to all $u \in H^s(M)$. Therefore the diagram commutes. In particular, if $u \in HH^s(M_\epsilon)$, we have

$$(I+P)^{\frac{s}{d}} \mathcal{R}_\epsilon u = \mathcal{R}_\epsilon \mathcal{W}_\epsilon (I+P)^{\frac{s}{d}} \mathcal{W}_\epsilon^{-1} u,$$

and so we get

$$\begin{aligned} ||\mathcal{R}_\epsilon u||_{H^{s'}(M)} &\leq ||(I+P)^{-\frac{s'}{d}}||_{B(H^{s'}(M), L^2(M))} ||(I+P)^{\frac{s'}{d}} \mathcal{R}_\epsilon u||_{L^2(M)} \\ &\leq ||(I+P)^{-\frac{s}{d}}||_{B(H^{s'}(M), L^2(M))} ||\mathcal{R}_\epsilon||_{B(L^2(M_\epsilon^\Phi), L^2(M))} ||u||_{HH^{s'}(M_\epsilon^\Phi)}, \end{aligned}$$

which shows that the $\mathcal{R}_\epsilon$ are bounded in the diagram □

The most interesting implication of the above is the existence of special parametrices. Let $d' \in \mathbb{N}$ and suppose that $A \in \mathrm{Diff}^{d'}(M)$ is real-analytic, elliptic and formally normal. In that case, let a be the classical principal symbol of A, and put

$$P = A^* A \in \Psi^d_{\mathrm{phg}}(M) \quad \text{with} \quad d = 2d',$$

which then has the unique classical principal symbol $p = |a|^2$, always positive on $T^*M \setminus 0$. This P is elliptic, formally self-adjoint, and can not have negative eigenvalues.

Theorem 7.1.1. *Let M^Φ_ϵ and $\mathcal{W}_\epsilon$ be obtained from Conjecture 7.1.1 with $\epsilon \in (0, \epsilon_0)$. Then A has a parametrix $B \in \Psi^{-d'}(M)$ so that if $s \in \mathbb{R}$ the diagram below commutes:*

$$\begin{array}{ccc} HH^s(M^\Phi_\epsilon) & \xrightarrow{\mathcal{W}_\epsilon B \mathcal{W}_\epsilon^{-1}} & HH^{s-d'}(M^\Phi_\epsilon) \\ \big\downarrow \mathcal{R}_\epsilon & & \big\downarrow \mathcal{R}_\epsilon \\ H^s(M) & \xrightarrow{\quad B \quad} & H^{s-d'}(M) \end{array}$$

Proof. If A is invertible, the inverse commutes with P on $C^\infty(M)$, and so we are done. Assume therefore that A is not invertible. But then 0 is necessarily an eigenvalue of P. The spectrum of P consists of eigenvalues. Let it be $\{\eta_k\}_{k=0}^\infty$, counted with multiplicity. Using now Theorem 3.1.12, $\int_\Gamma \frac{1}{\lambda}(\lambda I - P)^{-1}\, d\lambda$ converges in $B(H^k(M))$ for every $k \in \mathbb{N}$, where Γ is a circle lying outside the spectrum of P, enclosing 0 but no other eigenvalues. By the Sobolev embedding theorem, the integral converges in the topology of $C^\infty(M)$, and we obtain a continuous operator

$$B : C^\infty(M) \to C^\infty(M) : u \mapsto -\frac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda}(\lambda I - P)^{-1} A^* u \, d\lambda.$$

The theorem follows if we show that B is a parametrix commuting with P on $C^\infty(M)$. Take $u \in C^\infty(M)$, and use the formal normality of A to write

$$\begin{aligned} [P, B]u &= -\frac{1}{2\pi i} P\Big(\int_\Gamma \frac{1}{\lambda}(\lambda I - P)^{-1} A^* u \, d\lambda \Big) - BPu \\ &= -\frac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda} \Big[P(\lambda I - P)^{-1} A^* - (\lambda I - P)^{-1} A^* P \Big] u \, d\lambda = 0. \end{aligned}$$

So, if $\{\psi_k\}_{k=0}^\infty$ is an ON eigenbasis associated to $\{\eta_k\}_{k=0}^\infty$, we have

$$\begin{aligned} ABu = BAu &= -\frac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda}(\lambda I - P)^{-1} P u \, d\lambda \\ &= \Big[\frac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda} \, d\lambda \Big] u - \frac{1}{2\pi i} \int_\Gamma (\lambda I - P)^{-1} \Big[\sum_{k=0}^\infty (u, \psi_k)_{L^2(M)} \psi_k \Big] \, d\lambda \\ &= Iu - \sum_{k=0}^\infty (u, \psi_k)_{L^2(M)} \Big[\frac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda - \eta_k} \, d\lambda \Big] \psi_k, \end{aligned}$$

where the remainder operator is simply the projection onto the finite-dimensional kernel. It follows that $B \in \Psi^{-d'}(M)$, and that it is a parametrix for A. □

7.1.2 On a Compact Lie Group G

It is possible to form a subalgebra of $\Psi(G)$ with the desired properties for any $\epsilon > 0$. That is, operators with bounded extensions between holomorphic Sobolev spaces on G_ϵ, and with an analogous notion of, and confditions for, ellipticity within the subalgebra. We will show that the subalgebra is at least non-trivial. Let $d \in \mathbb{R}$.

Definition 7.1.5. *Define S^d_ϵ to be those $p \in S^d$ satisfying the following two conditions:*

1. *The map $G \to \mathrm{Mat}(d_\xi, \mathbb{C}) : x \mapsto p(x,\xi)$ is real-analytic for every $[\xi] \in \widehat{G}$. Each of these extend holomorphically to G_ϵ.*
2. *Write $p_Y(x,\xi) = p(\exp(iY)x, \xi)$ for every $(x,[\xi]) \in G \times \widehat{G}$ when $|Y|_\mathfrak{g} < \epsilon$. Then $\{p_Y\}_{|Y|_\mathfrak{g}<\epsilon}$ is a bounded subset of S^d.*

Take $p \in S^d_\epsilon$. Let us adopt the notation p_Y as it is appears above in the second point. If $u \in C^\omega(G)$ extends to G_ϵ, then since G is totally real in $G_\mathbb{C}$, we have

$$\begin{aligned}\mathrm{Op}(p)u(\exp(iY)x) &= \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\xi(\exp(iY)x)p_Y(x,\xi)\mathcal{F}u(\xi)) \\ &= \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}\Big(\xi(x)p_Y(x,\xi)\int_G u(\exp(iY)y)\xi(y)^*\,dy\Big),\end{aligned}$$

where the sum is uniformly absolutely convergent on $x \in G$ and $Y \in \mathfrak{g}$ with $|Y| \le r < \epsilon$. This gives us the following simple result.

Proposition 7.1.4. *If $u \in C^\omega(G)$ extends holomorphically to G_ϵ, then so does $\mathrm{Op}(p)u$.*

Theorem 7.1.2. *Let $d_1 \in \mathbb{R}$ and $d_2 \in \mathbb{R}$. If $p \in S^{d_1}_\epsilon$ and $q \in S^{d_2}_\epsilon$, then $p \odot q \in S^{d_1+d_2}_\epsilon$.*

Proof. If $[\eta] \in \widehat{G}$ is fixed, then $p(x,\eta)$ extends holomorphically in $x \in G$ to $z \in G_\epsilon$ via

$$(p\odot q)(z,\eta) = \sum_{[\xi]\in\widehat{G}} d_\xi \int_G \mathrm{Tr}\Big(\xi(y)p(z,\xi)\Big)\eta(y)^* q(zy^{-1},\eta)\,dy,$$

and, if $|Y|_\mathfrak{g} < \epsilon$, this also shows that

$$\begin{aligned}(p\odot q)_Y(x,\eta) &= (p\odot q)(\exp(iY)x,\eta) \\ &= (p_Y \odot q_Y)(x,\eta).\end{aligned}$$

Then, because $\{p_Y\}_{|Y|_\mathfrak{g}<\epsilon} \subset S^{d_1}$ and $\{p_Y\}_{|Y|_\mathfrak{g}<\epsilon} \subset S^{d_2}$ are bounded sets by definition, and the symbolic product is continuous, $\{(p\odot q)_Y\}_{|Y|_\mathfrak{g}<\epsilon}$ must be a bounded set in $S^{d_1+d_2}$. This shows that $p \odot q \in S^{d_1+d_2}_\epsilon$. □

It follows from the above that $\Psi_\epsilon(G) = \cup_{d\in\mathbb{R}} \mathrm{Op}\, S^d_\epsilon$ is actually a subalgebra of $\Psi(G)$. The holomorphic extension of $\mathrm{Op}(p)u$ can also be expressed in terms of $\{\mathrm{Op}(p_Y)\}_{|Y|_\mathfrak{g}<\epsilon}$. In fact, with x and Y as above, we have

$$\mathrm{Op}(p)u(z) = \mathrm{Op}(p_Y)(L_{\exp(iY)}u)(x) \quad \text{if} \quad z = \exp(iY)x \in G_\epsilon.$$

Next, we prove the holomorphic mapping property using the Cartan decomposition. It mirrors precisely the standard Sobolev mapping property of $\Psi(G)$.

Proposition 7.1.5. *If $p \in S^d_\epsilon$ and $s \in \mathbb{R}$, then $\mathrm{Op}(p) : HH^s(G_\epsilon) \to HH^{s-d}(G_\epsilon)$ exists. That is, there is a unique bounded operator making the diagram below commute:*

$$\begin{array}{ccc} HH^s(G_\epsilon) & \xrightarrow{\widetilde{\mathrm{Op}(p)}} & HH^{s-d}(G_\epsilon) \\ \downarrow{\scriptstyle \mathcal{R}_\epsilon} & & \downarrow{\scriptstyle \mathcal{R}_\epsilon} \\ H^s(G) & \xrightarrow{\mathrm{Op}(p)} & H^{s-d}(G) \end{array}$$

Proof. Of course, $(I - \Delta_G)^{\frac{s}{2}} \in \Psi^s_\epsilon(G)$ for all $s \in \mathbb{R}$, so it is enough to prove $s = d = 0$. But then we just use the boundedness of $\{p_Y\}_{|Y|_\mathfrak{g} < \epsilon}$ and Lemma 3.2.8 to write

$$\begin{aligned} \int_{G_\epsilon} |\mathrm{Op}(p)u(z)|^2 \, dz &= \int_{|Y|_\mathfrak{g} < \epsilon} \left[\int_G |[\mathrm{Op}(p)u](\exp(iY)x)|^2 \, dx \right] \frac{dY}{\Theta(Y)^2} \\ &\leq C \int_{|Y|_\mathfrak{g} < \epsilon} \left[\int_G |u(\exp(iY)x)|^2 \, dx \right] \frac{dY}{\Theta(Y)^2} \\ &= C \int_{G_\epsilon} |u(z)|^2 \, dz, \end{aligned}$$

where $C > 0$ is a constant. □

Let us make a very simple observation. Take $p \in S^{d_1}_\epsilon$ and $q \in S^{d_2}_\epsilon$ for some $d_1, d_2 \in \mathbb{R}$. But $(p \odot q)_Y = p_Y \odot q_Y$ so $\{p_Y \odot q_Y - p_Y q_Y\}_{|Y|_\mathfrak{g} < \epsilon} \subset S^{d_1 + d_2 - 1}$ is always a bounded set, and we must have

$$p \odot q - pq \in S^{d_1 + d_2 - 1}_\epsilon.$$

It does not appear possible to include more terms, because these involve the δ_x operator, which contains a cutoff, so higher order terms may not extend holomorphically.

Before we show that each S^d_ϵ is non-trivial, we show our claim about elliptic elements. To this end, we must expand some theory from the functional calculus of matrix symbols. Let us reuse X, α, β and $\{\psi_\epsilon\}_{\epsilon > 0} \subset C^\infty(G)$ from that section. Also, we put

$$\mathfrak{g}_\epsilon = \{Y \in \mathfrak{g} \,|\, |Y| < \epsilon\}$$

Lemma 7.1.2. *Let $p \in S^d_\epsilon$ for $d \in \mathbb{R}$, and $R(z) = \mathcal{F}^{-1}_\xi[p(z,\xi)] \in \mathcal{D}'(G)$ for any $z \in G_\epsilon$, where $(x, Y) \mapsto p(\exp(iY)x, \xi)$ has been extended by continuity to $x \in G$ and all $|Y| \leq \epsilon$. It is alternatively viewed as a map*

$$G \times \overline{\mathfrak{g}_\epsilon} \to \mathcal{D}'(G) : (x, Y) \mapsto R(\exp(iY)x) = R(x, Y).$$

Then, we have

$$R \in C^\infty(G \times \overline{\mathfrak{g}_\epsilon}, H^{-d-\lceil \frac{n}{2} \rceil}(G))$$

Proof. The proof is the same as for Lemma 3.2.10, but just with a parameter $|Y|_\mathfrak{g} \leq \epsilon$. □

Lemma 7.1.3. *Suppose $\{p_j\}_{j=0}^\infty$ is a sequence with $p \in S_\epsilon^{d_j}$ and $d_j \searrow -\infty$ as $j \to \infty$. Then we can construct $p \in S_\epsilon^{d_0}$ with $p \sim \sum_{j=0}^\infty p_j$ such that*

$$p - \sum_{j=0}^{k-1} p_j \in S_\epsilon^{d_k} \quad \text{for each} \quad k \in \mathbb{N}.$$

Proof. The proof mimics that of Theorem 3.2.14, making use of uniform convergence. Construct R_j from p_j like just as in Lemma 7.1.2. Then $R_j \in C^\infty(G \times \overline{\mathfrak{g}_\epsilon}, H^{-d_j - \lceil \frac{n}{2} \rceil}(G))$. Pick $\{\epsilon_j\}_{j=0}^\infty$ such that

$$\sup_{(x,Y)\in G\times\overline{\mathfrak{g}_\epsilon}} ||X_x^\beta R_j(x,Y) - \psi_{\epsilon_j} * (X_x^\beta R_j(x,Y))||_{H^{-d_j-\lceil \frac{n}{2}\rceil}(G)} < \frac{1}{2^{j+1}} \quad \text{when} \quad |\beta| \leq j.$$

Take $N \in \mathbb{N}_0$ with $-d_j - \lceil \frac{n}{2} \rceil \geq 0$ for $j \geq N$, and observe that

$$\begin{aligned} ||p_j(z,\xi)(1 - \mathcal{F}\psi_{\epsilon_j}(\xi))||^2 &\leq ||R_j(x,Y) - \psi_{\epsilon_j} * (R_j(x,Y))||_{L^2(G)}^2 \\ &\leq ||R_j(x,Y) - \psi_{\epsilon_j} * (R_j(x,Y))||_{H^{-d_j-\lceil \frac{n}{2}\rceil}(G)}^2 < \frac{1}{4^{j+1}}, \end{aligned}$$

where the estimates are uniform in $z = \exp(iY)x$ for $(x,Y) \in G_\epsilon \times \overline{\mathfrak{g}_\epsilon}$ and $[\xi] \in \widehat{G}$. Consequently, we have

$$\sum_{j=N}^\infty ||p_j(z,\xi)(1 - \mathcal{F}\psi_{\epsilon_j}(\xi))|| < 1.$$

It follows that $p \in S_\epsilon^{d_0}$ if we put

$$p(z,\xi) = \sum_{j=0}^\infty p_j(z,\xi)(1 - \mathcal{F}\psi_{\epsilon_j}(\xi)) \quad \text{for any} \quad (z,[\xi]) \in G_\epsilon \times \widehat{G},$$

which converges absolutely uniformly (in matrix norm) on $z \in G_\epsilon$ for each fixed $[\xi] \in \widehat{G}$, and $x \mapsto p_Y(x,\xi)$ is differentiable, where X_x^β falls onto $p_j(\exp(iY)x,\xi)$ under the sum. The semi-norm estimates on p_Y are obtained exactly as in the proof of Theorem 3.2.14: Let r_N denote the sum starting at $j = N$. Take $k \in \mathbb{N}$, and write

$$p(z,\xi) - \sum_{j=0}^{k-1} p_j(z,\xi) = \sum_{j=k}^{N-1} p_j(z,\xi) - \sum_{j=k}^{N-1} p_j(z,\xi)\mathcal{F}\psi_{\epsilon_j}(\xi) + r_N(z,\xi),$$

where the first term is in $S_\epsilon^{d_k}$, and the second belongs to $S_\epsilon^{-\infty}$ by the product rule for δ_ξ^α. It suffices to show that $r_N \in S^{d_k}$ for large $N \in \mathbb{N}_0$. But this is clear, because

$$\begin{aligned} \langle\xi\rangle^{|\alpha|-d_k} ||\delta_\xi^\alpha X_x^\beta (r_N)_Y(x,\xi)|| &\leq C_{\alpha,\beta}\Big[\sup_{[\xi]\in\widehat{G}} \langle\xi\rangle^{|\alpha|-d_k} ||X_x^\beta (r_N)_Y(x,\xi)||\Big] \\ &\leq C_{\alpha,\beta}\Big[\sum_{j=M}^\infty ||X_x^\beta R_j(x,Y) - \psi_{\epsilon_j} * X_x^\beta R_j(x,Y)||_{H^{|\alpha|-d_k}(G)}\Big], \end{aligned}$$

where N is chosen so that $d_N + \lceil \frac{n}{2} \rceil < d_k - |\alpha|$, and $C_{\alpha,\beta} > 0$ is a constant. □

The above means that we can sum symbols in the $\{S^d_\epsilon\}_{d\in\mathbb{R}}$ classes asymptotically. This allows us to construct parametrices in S^{-d}_ϵ for elliptic symbols in S^d_ϵ.

Proposition 7.1.6. *Let $q_0 \in S^{-d}_\epsilon$. The following holds:*

1. *If $q_0 p - 1 \in S^{-1}_\epsilon$, then there is a $q_L \in S^{-d}_\epsilon$ such that $q_L \odot p - 1 \in S^{-\infty}_\epsilon$.*
2. *If $pq_0 - 1 \in S^{-1}_\epsilon$, then there is a $q_R \in S^{-d}_\epsilon$ such that $p \odot q_R - 1 \in S^{-\infty}_\epsilon$.*

Finally, if both left and right parametrices exist, then $q_L - q_R \in S^{-\infty}_\epsilon$.

Proof. This is proved exactly as in Proposition 3.2.10, but using Lemma 7.1.3 instead.

Left: Put $r = 1 - q_0 \odot p$. Define the sequence of symbols $q_j = r^{\odot j} \odot q_0$ for $j \in \mathbb{N}_0$. Then put $q \sim \sum_{j=0}^\infty q_j$ with $q \in S^{-d}_\epsilon$, and write

$$S^{-N}_\epsilon \ni \Big(q - \sum_{j=0}^{N-1} q_j\Big) \odot p - r^{\odot N} = q \odot p - 1.$$

Right: Put $r = 1 - p \odot q_0$. Define the sequence of symbols $q_j = q_0 \odot r^{\odot j}$ for $j \in \mathbb{N}_0$. Then put $q \sim \sum_{j=0}^\infty q_j$ with $q \in S^{-d}_\epsilon$, and write

$$S^{-N}_\epsilon \ni p \odot \Big(q - \sum_{j=0}^{N-1} q_j\Big) - r^{\odot N} = p \odot q - 1.$$

In either case, it holds for any $N \in \mathbb{N}$, and therefore the residual must be in $S^{-\infty}_\epsilon$. The last statement is clear. □

Proposition 7.1.7. *The space S^d_ϵ contains non-trivial symbols for each $d \in \mathbb{R}$.*

Proof. By the Segal-Bargmann transformation, $HL^2(G_\mathbb{C}, \nu_t)$ must be infinite-dimensional. Therefore so is the space of the restrictions of these functions to G_ϵ, and hence also $\mathcal{O}(G_\epsilon)$. Let S^d_{inv} be the symbols of degree $d \in \mathbb{R}$ bi-invariant operators on G (depend only on $[\xi]$). Then, we have that

$$\mathcal{O}(G_{\epsilon'})|_{G_\epsilon} \otimes S^d_{\mathrm{inv}} \subset S^d_\epsilon \quad \text{when} \quad \epsilon' > \epsilon,$$

and this space is infinite-dimensional, the symbols depend on both z and $[\xi]$. □

The proof of the ellipticity condition for matrix-symbols carries over directly to S^d_ϵ. It is natural that the following holds:

Theorem 7.1.3 (This is a specialization of the Ruzhansky-Turunen-Wirth condition)**.** *A symbol $p \in S^d_\epsilon$ has a parametrix in S^d_ϵ if and only if there is a finite $F \subset \widehat{G}$ such that:*

1. *$p(z,\xi)$ is invertible for all $(z,[\xi]) \in G_\epsilon \times (\widehat{G} \setminus F)$.*
2. *The family of inverses satisfy*

$$\sup_{(z,[\xi])\in G_\epsilon \times (\widehat{G}\setminus F)} \langle \xi \rangle^d ||p(z,\xi)^{-1}|| < \infty.$$

Proof. Suppose that p has a parametrix in S^d_ϵ. That is, a $q_0 \in S^{-d}_\epsilon$ with $pq_0 - 1 \in S^{-1}_\epsilon$. Then we get a $C > 0$ so that, uniformly in $(z, [\xi]) \in G_\epsilon \times \widehat{G}$, we have

$$||p(z,\xi)q_0(z,\xi) - I|| \leq C\langle\xi\rangle^{-1},$$

and $(pq_0)(z,\xi)$ is invertible for $\langle\xi\rangle \geq R > 0$, independent of $z \in G_\epsilon$. So is p, and

$$||p(z,\xi)^{-1}|| \leq ||q_0(z,\xi)|| \, \left|\left|(p(z,\xi)q_0(z,\xi))^{-1}\right|\right| \leq C'\langle\xi\rangle^{-d},$$

where $C' > 0$, and the inequality holds for $[\xi] \in \widehat{G}$ except the finite set with $\langle\xi\rangle < R$. Conversely, if the two points above hold, then consider the bounded set $\{p_Y\}_{Y \in \overline{\mathfrak{g}_\epsilon}} \subset S^d$. It fulfils the hypotheses of Lemma 3.2.3 with $J = \overline{\mathfrak{g}_\epsilon}$ and the above estimate over $z \in G_\epsilon$. Thus, if we put $\chi_F(\xi) = 1_F([\xi])I_{d_\xi}$ for all $[\xi] \in \widehat{G}$, then $\chi_F \in S^{-\infty}_\epsilon$, it gives

$$(\chi_F + (1 - \chi_F)p)^{-1} \in S^{-d}_\epsilon,$$

and so $p \in S^d_\epsilon$ has a parametrix in S^d_ϵ. ☐

The spaces $\mathrm{Op}\, S^k_\epsilon$ contain all the real-analytic differential operators of degree $k \in \mathbb{N}$. This is actually very simple to prove.

Proposition 7.1.8. *If $P \in \mathrm{Diff}^k(G)$ is real-analytic, then $P \in \mathrm{Op}\, S^k_\epsilon$ for some $\epsilon > 0$.*

Proof. Let $(X_1, \cdots, X_n)$ be an ordered basis (of left-invariant vector fields on G) for $\mathfrak{g}$. In any sufficiently small chart U of G, we have

$$P|_U = \sum_{|\alpha| \leq k} g_\alpha X^\alpha,$$

where $g_\alpha \in C^\omega(U)$ extend holomorphically into $\exp(i\mathfrak{g}_\epsilon)U$ for an $\epsilon > 0$ depending on U. It follows that if $z = \exp(iY)x \in \exp(i\mathfrak{g}_\epsilon)U$, we have

$$\begin{aligned}\xi(z)^{-1}P\xi(z) &= \xi(x)^{-1}\xi(\exp(iY))^{-1} \sum_{|\alpha| \leq k} g_\alpha(z) X^\alpha \xi(\exp(iY)x) \\ &= \sum_{|\alpha| \leq k} g_\alpha(\exp(iY)x)\Big[\xi(x)^{-1}X^\alpha\xi(x)\Big],\end{aligned}$$

and by taking a finite cover of G by such charts, we obtain $\epsilon > 0$ such that $P \in \mathrm{Op}\, S^d_\epsilon$. This is because the symbol p of P in the neighbourhood above is

$$p(\exp(iY)x, \xi) = \sum_{|\alpha| \leq k} g_\alpha(\exp(iY)x)p_\alpha(\xi),$$

where $p_\alpha \in S^{|\alpha|}$ is the matrix-symbol of $X^\alpha \in \Psi^{|\alpha|}(G)$, independent of $(x, Y) \in G \times \mathfrak{g}_\epsilon$, and $\epsilon > 0$ is chosen so $(x, Y) \mapsto g_\alpha(\exp(iY)x)$ has x-derivatives uniformly bounded in Y. Thus $\{p_Y\}_{Y \in \mathfrak{g}_\epsilon}$ is bounded in S^k. ☐

Corollary 7.1.2. *If P above is left-invariant, then $P \in \mathrm{Op}\, S^k_\epsilon$ for all $\epsilon > 0$.*

Proof. In this case, the coefficients g_α are globally defined and constant. ☐

As a consequence, elliptic left-invariant operators admit a parametrix in such a space. More generally, only the leading term in p has to belong to such an operator:

Proposition 7.1.9. *Suppose that p of $\operatorname{Op}(p) \in \Psi^k(G)$ is of the form*

$$p = p_k + \sum_{|\alpha| \leq k-1} g_\alpha p_\alpha,$$

where $p_\alpha \in S^{|\alpha|}$ and $p_k \in S^k$ are independent of $x \in G$, the leading $p_k \in S^k$ is elliptic, and $g_\alpha \in C^\omega(G)$ extends holomorphically to a bounded function on G_ϵ for a fixed $\epsilon > 0$. Then P has a parametrix in $\operatorname{Op} S_\epsilon^{-k}$ for this particular $\epsilon > 0$.

Proof. Observe that

$$\sup_{(x,Y) \in G \times \mathfrak{g}_\epsilon} \langle \xi \rangle^{-k} ||p_Y(x,\xi) - p_k(\xi)|| = O_{\lambda_\xi \to \infty}(\langle \xi \rangle^{-1}),$$

and so, the symbols $p_Y(x,\xi)$ are invertible for all $(x,Y) \in G \times \mathfrak{g}_\epsilon$ if $\langle \xi \rangle$ is large enough. This is because $p_k(\xi)$ is invertible for large $\langle \xi \rangle$ by the characterization in Theorem 3.2.8. Furthermore, it gives a finite $F \subset \widehat{G}$ and a $C > 0$ such that

$$||\langle \xi \rangle^k p_k(\xi)^{-1}|| \leq C \quad \text{for all} \quad [\xi] \in \widehat{G} \setminus F,$$

and for $\langle \xi \rangle$ suitably large, we can then write

$$\begin{aligned}
\langle \xi \rangle^k ||p_Y(x,\xi)^{-1}|| &= \left|\left| \Big(I + \sum_{|\alpha| \leq k-1} g_\alpha(\exp(iY)x) p_k(\xi)^{-1} p_\alpha(\xi) \Big)^{-1} \langle \xi \rangle^k p_k(\xi)^{-1} \right|\right| \\
&\leq \Big(1 - \sum_{|\alpha| \leq k-1} |g_\alpha(\exp(iY)x)| \, ||p_k(\xi)^{-1} p_\alpha(\xi)|| \Big)^{-1} ||\langle \xi \rangle^k p_k(\xi)^{-1}|| \\
&\leq \Big(1 - \sum_{|\alpha| \leq k-1} C |g_\alpha(\exp(iY)x)| \langle \xi \rangle^{-k} ||p_\alpha(\xi)|| \Big)^{-1} C,
\end{aligned}$$

where the last sum is made smaller than $\frac{1}{2}$ for all $(x,Y) \in G \times \mathfrak{g}_\epsilon$ if $\langle \xi \rangle$ is large enough. Therefore the conditions of Theorem 7.1.3 are fulfilled. □

In other words, the "leading term" in p determines if P has the property that we seek. These observations indicate that $\operatorname{Op} S_\epsilon^d$ contain many non-trivial operators.

Example 7.1.1. Operators in $\Psi(\mathbb{T}^n)$, acting on $u \in C^\infty(\mathbb{T}^n)$, are of the form

$$\operatorname{Op}(p)u(x) = \sum_{k \in \mathbb{Z}^n} e^{ik \cdot x} p(x,k) \mathcal{F}_{\mathbb{T}^n} u(k) \quad \text{for all} \quad x \in \mathbb{T}^n,$$

and we may take p to be the symbol

$$p(x,k) = |k|^2 + \sum_{j=1}^n g_j(x) k_j + f(x),$$

where $\{g_j\}_{j=1}^n$ and f are bounded holomorphic on $\mathbb{T}_\epsilon^n$ (2π-periodic on the polystrip $\mathbb{R}_\epsilon^n$). Quantization of this particular symbol gives the Laplacian $\Delta_{\mathbb{T}^n}$ plus lower order terms. The result above says that p has a parametrix in $S_\epsilon^2(\mathbb{T}^n \times \mathbb{Z}^n)$, e.g. $p(x,k)^{-1}$ for large k, which is not a sum of products of functions in x and k separately.

Finally, we note how $\mathcal{C}_t$ and the restriction $\mathcal{R} : \mathcal{O}(G_{\mathbb{C}}) \to C^\omega(G) : f \mapsto f|_G$ interact. It is of interest, because of the earlier observation we made about conjugating Δ_G by $\mathcal{P}_\epsilon$. If $p \in S^d$ and $s \geq d$, consider the diagram:

$$\begin{array}{ccc} HH^s(G_{\mathbb{C}}, \nu_t) & \xrightarrow{\mathcal{C}_t \mathrm{Op}(p) \mathcal{C}_t^{-1}} & HH^{s-d}(G_{\mathbb{C}}, \nu_t) \\ \big\downarrow \mathcal{R}_\infty & & \big\downarrow \mathcal{R}_\infty \\ H^s(G) & \xrightarrow{\quad A \quad} & H^{s-d}(G) \end{array}$$

The question is, can we find A so the diagram commutes, and does A belong to $\Psi^d(G)$? Unlike when conjugating by $\mathcal{P}_\epsilon$, we can form explicit expressions when conjugating by $\mathcal{C}_t$. Taking $u \in HH^s(G_{\mathbb{C}}, \nu_t)$ and any $[\xi] \in \widehat{G}$, we can write

$$\begin{aligned} \mathcal{F}\mathcal{C}_t^{-1} u(\xi) &= \int_G \mathcal{C}_t^{-1} \Big[\sum_{[\eta] \in \widehat{G}} d_\eta \mathrm{Tr}\left(\eta \mathcal{F}\mathcal{R}_\infty u(\eta)\right) \Big](x) \xi(x)^* \, dx \\ &= \sum_{[\eta] \in \widehat{G}} d_\eta \int_G \mathcal{C}_t^{-1} \Big[\mathrm{Tr}\left(\eta \mathcal{F}\mathcal{R}_\infty u(\eta)\right) \Big](x) \xi(x)^* \, dx \\ &= \sum_{[\eta] \in \widehat{G}} d_\eta e^{\frac{\lambda_\eta}{2} t} \int_G \mathrm{Tr}(\eta(x) \mathcal{F}\mathcal{R}_\infty u(\eta)) \xi(x)^* \, dx = e^{\frac{\lambda_\xi}{2} t} \mathcal{F}\mathcal{R}_\infty u(\xi), \end{aligned}$$

where the interchanges are justified by isometry of $\mathcal{C}_t$ and L^2-convergence of the sums. Therefore, if $z \in G_{\mathbb{C}}$, we have

$$\begin{aligned} \mathcal{C}_t \mathrm{Op}(p) \mathcal{C}_t^{-1} u(z) &= \sum_{[\xi] \in \widehat{G}} d_\xi \mathrm{Tr} \Big[\Big(\int_G \Big(\sum_{[\eta] \in \widehat{G}} d_\eta e^{-\frac{\lambda_\eta}{2} t} \mathrm{Tr}\, \eta(x^{-1} z) \Big) \xi(x) p(x, \xi) \, dx \Big) \mathcal{F}\mathcal{C}_t^{-1} u(\xi) \Big] \\ &= \sum_{[\xi] \in \widehat{G}} d_\xi \mathrm{Tr} \Big(\xi(z) \Big[e^{\frac{\lambda_\xi t}{2}} (c_t \odot p)(z, \xi) \Big] \mathcal{F}\mathcal{R}_\infty u(\xi) \Big), \end{aligned}$$

where the outer sums converge in $L^2(G_{\mathbb{C}}, \nu_t)$, hence uniformly on compact subsets of $G_{\mathbb{C}}$. Here, we have put

$$c_t(x, \xi) = e^{-\frac{\lambda_{x i}}{2} t} I_{d_\xi} \quad \text{for all} \quad (x, [\xi]) \in G \times \widehat{G},$$

which, by the functional calculus, is (at the least) a Hörmander class symbol $c_t \in S^{-\infty}$, and the symbolic product $c_t \odot p$ extends holomorphically in the first variable to all $G_{\mathbb{C}}$. It is given by

$$(c_t \odot p)(z, \xi) = \sum_{[\eta] \in \widehat{G}} d_\eta \int_G \mathrm{Tr} \left(\eta(y^{-1} z) e^{-\frac{\lambda_\eta}{2} t} \right) \xi(z^{-1} y) p(y, \xi) \, dy$$

This suggests A should be the operator with symbol $c_t^{-1}(c_t \odot p)$. But there is a problem. The growth of c_t^{-1} is exponential, so $c_t^{-1}(c_t \odot p)$ may not even be an operator symbol, and the last sum above is well-defined only because $\mathcal{F}u$ decays fast enough to balance it. But even if $c_t^{-1}(c_t \odot p)$ defines a symbol in S^d, we do not in general have $c_t^{-1}(c_t \odot p) = p$, and so A does not equal $\mathrm{Op}(p)$ in general (except when $\mathrm{Op}(p)$ is a bi-invariant operator). It seems then that $\mathcal{C}_t$ is not the right transformation for our purposes in general.

7.2 Towards a Proof of Boutet de Monvel's Conjecture

In [55] Stenzel provides a very detailed proof of Conjecture 7.1.1 in the case that $P = \Delta$, and so far, it seems to be clearest proof available in the literature, besides Zelditch [65]. The case $P = \Delta$ is prototypical; it is reasonable to think that it holds more generally. However, neither the the proof in [55] nor in [65] can be generalized straightforwardly, because some of the tools that are used along the way so far only work for the Laplacian, or are only available for second-order operators.

The idea in [55] is to express P_ϵ as the Laplace transform of an analytic amplitude. This leads to a nice workable expression for the kernel in terms of a complex phase FIO, and the Laplace transform gives a clear description of its branched holomorphic extension. To obtain these formulations, Stenzel subordinates the Poisson kernel to the heat kernel, and uses the Minakshisundaram-Pleijel asymptotics to construct the analytic amplitude, where several of the results obtained in Golse, Leichtnam and Stenzel [54] are invoked. The argument is completely global, making use of the subordination formula

$$e^{-\epsilon\sqrt{-\Delta}} = \frac{\epsilon}{2\sqrt{\pi}} \int_0^\infty e^{t\Delta} e^{-\frac{\epsilon^2}{4t}} t^{-\frac{3}{2}}\, dt,$$

or more generally (for $d > 1$)

$$e^{-\epsilon P^{\frac{1}{d}}} = \frac{1}{2\pi} \int_0^\infty e^{-tP} \mathcal{F}_{\lambda\to t}(e^{-\epsilon(i\lambda)^{\frac{1}{d}}})(t)\, dt,$$

which converges as an improper Riemann integral in the norm of $B(H^k(M))$ for $k \in \mathbb{N}$. It relates the Poisson kernel P_ϵ of $-\sqrt{-\Delta}$ to the heat kernel E_t of Δ via

$$\begin{aligned} P_\epsilon(z,y) &= \frac{\epsilon}{\sqrt{4\pi}} \int_0^\infty E_t(z,y) e^{-\frac{\epsilon^2}{4t}} t^{-\frac{3}{2}}\, dt \\ &= \epsilon(4\pi)^{-\frac{n+1}{2}} \mathcal{L}[a(\cdot,z,y)]\Big(\frac{1}{4}(r_{\mathbb{C}}^2(z,y)+\epsilon^2)\Big) + R_\epsilon(z,y), \end{aligned}$$

where a is a suitable analytic amplitude obtained from E_t, $\mathcal{L}$ is the Laplace transform, and $R_\epsilon(z,y)$ is a smooth kernel that is also holomorphic in z and ϵ in a certain domain. The details here are not so important, these can of course all be found in the paper [55], but the point is that it allows Stenzel to use a global contour deformation to prove:

Theorem 7.2.1 (Stenzel [55]). *Let $u \in \mathcal{O}(U)$ for some neighbourhood U containing M_ϵ. Consider the map*

$$(0,\infty) \times M \mapsto \mathbb{C} : (t,x) \mapsto e^{-t\sqrt{-\Delta}}(u|_M)(x).$$

Then the following holds:

1. *It extends holomorphically in $x \in M$ to $x \in M_\epsilon$ for each $t > 0$.*
2. *It extends holomorphically in $t > 0$ to $t \in D(0,\epsilon) \setminus i(-\epsilon, 0]$ for each $x \in M_\epsilon$.*
3. *The joint extension is smooth in $(t,x) \in D(0,\epsilon) \setminus i(-\epsilon,0] \times M_\epsilon$.*

The question is, does this approach generalize? If so, how exactly does it generalize? Unfortunately, the kernel asymptotics for e^{-tP} are in general not known.

An alternative is to go through the very construction of the propagator as an FIO. This is what Zelditch describes as the approach via the "Hörmander parametrix" in [65]. In this way, the Poisson kernel is represented near the diagonal by

$$\int_{T_y^* M} e^{i(g_y(\xi,\exp_y^{-1} x)+it|\xi|_y)} a_t(x,y,\xi)\,d\xi,$$

where a_t is a suitable analytic symbol depending real-analytically on the time variable t. It seems that this approach is by far the best adapted for generalization to other symbols, because the "off-diagonal" results on the (half) wave kernel in [55] generalize quite well, and therefore the proof would be reduced to a study of such integrals near the diagonal. The "off-diagonal" part of the results in [55] makes use of the following:

Theorem 7.2.2 (Duistermaat and Guillemin [11]. This is stronger than Theorem 5.1.9)**.** *Let $P \in \Psi^d_{\mathrm{phg}}(M)$ be an elliptic, self-adjoint, and positive operator of positive order $d > 0$. So P has classical principal symbol p positive on $T^*M \setminus 0$, and no negative eigenvalues. Consider the parametrization of kernels (the wave group of P):*

$$U : \mathbb{R} \to \mathcal{D}'(M \times M) : t \mapsto K(e^{itP^{\frac{1}{d}}}).$$

Then, viewed as an element of $\mathcal{D}'(\mathbb{R} \times M \times M)$, it satisfies

$$U \in I^{-\frac{1}{4}}_{\mathrm{phg}}(\mathbb{R} \times M, M; C) \quad \text{with} \quad \mathrm{WF}'(K(U)) = C,$$

and C is the (real) homogeneous canonical relation

$$C = \Big\{ \Big((t, -p(y,\eta)^{\frac{1}{d}}), \Phi_t(y,\eta), (y,\eta) \Big) \,\Big|\, (t,(y,\eta)) \in \mathbb{R} \times T^*M \setminus 0 \Big\},$$

where Φ_t is the complete Hamiltonian flow generated by the Hamiltonian $p^{\frac{1}{d}}$ on $T^*M \setminus 0$. Thus, $\operatorname{sing\,supp}(K(U))$ *consists exactly of (t,x,y) such that x and y can be joined by Φ_t. The same is true of the analytic singular support of $K(U)$ if P is analytic.*

As a consequence, the kernel must be expressed in terms of certain phase functions. That is, near the diagonal, $U(t)$ must be of the form

$$\int_{T_y^* M} e^{i\psi(t,x,y,\xi)} a(t,x,y,\xi) u(y)\,d\xi,$$

where ψ is a real phase function locally parametrizing C', and a is an analytic amplitude. The important detail here is that C is the real part of a complex Lagrangian submanifold. Namely, the one that is obtained by extending Φ holomorphically about $\mathbb{R} \times T^*M \setminus 0$. Then ψ should be real-analytic, with a holomorphic extension parametrizing it locally, and at least part of this submanifold (lying over $\mathrm{Im}\,(t) \geq 0$) should be of positive type, which would imply that $K(U)$ extends holomorphically to a domain controlled by the ψ. The positivity could be checked intrinsically without ψ (see Melin and Sjöstrand [43]), but it is yet unclear (to me) how to use any of this to get results like in [55].

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