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Neutron charge radius from intrinsic quark flavour generation

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Abstract – The finite, non-zero mean square neutron charge radius is understood in the present work to have a topological origin from an intrinsic neutron configuration in the Lie group $U(3)$. We introduce up and down quark orbits in the configuration for the neutron mass eigenstate. From reciprocal Gaussian curvatures we infer a mean square charge radius of -0.1075 square fermis a few standard deviations away from the world average of -0.1161 square fermis.

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Introduction. – Determinations of the mean square neutron charge radius has been under long discussion [1] until a certain level of authoritative values were reached [2,3]. However, the question has recently regained interest [4–6]. The value $\langle r_n^2 \rangle = (-0.1161 \pm 0.0022) \text{ fm}^2$ cited by the Particle Data Group, PDG [7], just manages to connect new investigations which state $\langle r_n^2 \rangle = (-0.106^{+0.007}_{-0.005}) \text{ fm}^2$, $(-0.105^{+0.005}_{-0.006}) \text{ fm}^2$ [4,5] and $\langle r_n^2 \rangle = (-0.122 \pm 0.004_{(\text{stat})} \pm 0.010_{(\text{sys})}) \text{ fm}^2$ [6]. The new results lie on opposite sides of the uncertainty interval of [7] with the PDG value based primarily on [2,3,8]. The uncertainties may be underestimated or the theoretical backgrounds behind the different methods may not be fully compatible. After submission Heacock *et al.* found $\langle r_n^2 \rangle = (-0.1100 \pm 0.0092) \text{ fm}^2$ by interferometry [9].

The basic idea in the present theoretical work is to interpret the mean square radius as a manifestation of an intrinsic curvature in the neutron configuration. It yields $\langle r_n^2 \rangle = (-0.1075 \pm 0.0020) \text{ fm}^2$, a value that results from reciprocal Gaussian curvatures of u and d quark tracks on the intrinsic configuration space, the Lie group $U(3)$. The u quark tracks have positive curvature on the average and the d quark tracks have negative curvature. It should be stressed that our conception of quarks is as generated by scattering on baryons rather than as fundamental fields constituting baryons [10].

In earlier works the mean square charge radius is derived from the concept of a neutron-electron scattering length b_{ne} . This scattering length contribution is then

transformed to $\langle r_n^2 \rangle$ by the relation [3]

$$\langle r_n^2 \rangle = \frac{3m_e a_0}{m_n} b_{ne}, \quad (1)$$

where m_e is the electron mass, m_n is the neutron mass and a_0 is the Bohr radius [7].

One way to determine b_{ne} (and actually the first one used [11]) is to scatter thermal neutrons on a contained gaseous target. The scattering will then result from the incoming neutrons interacting with atoms in the target. The atoms consist of both electrons and nuclei. The hard problem is to extract the minute interaction of a possible intrinsic neutron charge distribution with the electric fields in the atoms from the much larger strong interaction between the incoming neutrons and the atomic nuclei in the target. It has even been discussed whether the effect should be ascribed to weak interactions [12]. Around 1986, however, the theoretical framework was more or less settled [1] and the contribution from the neutron-electron scattering length was grouped in a class of 0.1%-level “corrections” to the measured total scattering lengths, whereas weak interaction terms were estimated to the vanishing level of 10^{-35} relative to the two leading contributions from strong interactions and from interaction of the incoming neutron magnetic dipole moment with the atomic magnetic moment [1].

As it became clear that the electric contribution is *not* vanishingly small—and its sign consistently was negative—its origin was interpreted as an intrinsic neutron charge distribution being negative on the outskirts of the neutron interior. Already from the beginning (virtual) mesons were introduced for this purpose [11]

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Fig. 1: Quark flavour tracks on the toroidal part of the intrinsic neutron configuration space, the Lie group $U(3)$. The Gaussian curvature is negative on the inner part of the torus. The torus embeds in the laboratory space two out of three dimensions of the Abelian part of $U(3)$. The two dimensions are excited by flavour generators T_u, T_d (26) with integral curves [13] of the vector fields (blue and green, respectively) excited by the momentum form applied to the intrinsic neutron wave function as a ground state of (19).

and this interpretation is still alive in [14] which yields $\langle r_n^2 \rangle = -0.119 \text{ fm}^2$ as a center value within the interval $(-0.13 \dots -0.108) \text{ fm}^2$.

Quark models were introduced [15] using approximate harmonic oscillators for an interior neutron wave function of three flavour quarks, two negatively charged d quarks and one positively charged u quark. The d quarks are more likely to be in a relative spin-one state suggested to give a repelling force driving the d 's further from the center of mass. For a prior critique of quark model calculations not even giving the right sign, cf. [16]. The recent work [6] also relies on the quark concept, now by introducing quark flavour-dependent form factors for electron-deuteron scattering.

In the present work we introduce flavour quarks from an intrinsic neutron configuration and find $\langle r_n^2 \rangle = (-0.1075 \pm 0.0020) \text{ fm}^2$ with the uncertainty from a purely theoretical calculation relying only on a length scale $\pi a = r_e$ set by the very accurate classical electron radius $r_e = 2.8179403262(13) \text{ fm}$ [7,17,18]. We get our value for $\langle r_n^2 \rangle$ from average curvatures of quark flavour tracks with negatively charged d flavours giving negative curvature contributions which on the average more than outweigh the positive curvature contributions from u flavours, see fig. 1. The flavour tracks are unfolded from an intrinsic configuration for the neutron state, see table 1. The accuracy can be improved by expansion of the intrinsic neutron wave function on higher-order contributions than those included for the results in table 1.

Before we describe our model, we sketch the theoretical foundation for the most common experimental techniques.

Scattering length and form factor towards $\langle r_n^2 \rangle$. – In this section we give a short presentation of the concept of scattering length and the concept of form factor. For a

Table 1: Expansion of the neutronic ground state on Slater determinants (33) with expansion coefficients for the most important contributions representing 99.4% of the probability. The last column shows the reciprocal Gaussian curvatures of the individual contributions. The expansion coefficients are found from a diagonalisation of the Hamiltonian in (19) by a Rayleigh-Ritz method [10,19] with 75 terms corresponding to $0 \leq p < r \leq N$ and $0 < q \leq N$ with $N = 5$.

Component pqr	Coefficient c_{pqr}	Probability c_{pqr}^2	r^2 fm^2
011	0.69342275	0.48083511	-0.1072039923
012	0.49859432	0.248596296	-0.107371106
013	0.05871873	0.003447889	-0.1083696661
021	0.23871329	0.056984035	-0.1073274076
022	0.12232589	0.014963623	-0.1102520727
023	-0.01284764	0.000165062	-0.1087722956
112	0.41766359	0.174442874	-0.1083344645
113	0.04649025	0.002161343	-0.1099410075
122	0.11337581	0.012854074	-0.1067497261
Totals	–	0.994450307	-0.107501387
($N = 3$)	–	0.997013839	-0.107493495)

thorough description of electromagnetic neutron-atom interactions we recommend Sears [1] in which the different contributions to the scattering length are defined. For the extraction of the square of the charge radius from deuteron form factors we recommend Filin *et al.* [5]. For a textbook presentation of the relation between charge radius and form factor we recommend Sakurai [20]. We follow Sears and Sakurai for our resumes here.

Sears imagines an incoming plane wave beam of neutrons interacting with the scattering medium through a potential V and leaving the medium again as scattered plane waves in various directions. He uses a Born approximation [21] to write down a scattering amplitude

$$F = -\frac{m_n}{2\pi\hbar^2} \int e^{-i\mathbf{p}' \cdot \mathbf{r}/\hbar} V(\mathbf{r}) e^{i\mathbf{p} \cdot \mathbf{r}/\hbar} d\mathbf{r}, \quad (2)$$

where $\mathbf{p}$ and $\mathbf{p}'$ are the momenta of the incoming and outgoing neutrons, respectively. Sears writes the potential as a sum over four contributions,

$$V = V_N + V_M + V_E + V_P, \quad (3)$$

where the first two terms are the large contributions mentioned in the introduction from strong interactions and the electromagnetic interaction from the neutron magnetic moment. The last two small terms are electromagnetic in origin with the third ascribed to an electrostatic energy arising from an intrinsic structure of the neutron and the fourth term arises from a possible electric polarisability of the neutron. Provided these potentials are independent on the neutron momentum, the integral in (2) may be written

as a Fourier transform,

$$F = -\frac{m_n}{2\pi\hbar^2} V(\mathbf{q}), \quad V(\mathbf{q}) = \int e^{i\mathbf{q}\cdot\mathbf{r}/\hbar} V(\mathbf{r}) d\mathbf{r} \quad (4)$$

with $\mathbf{q} = \mathbf{p} - \mathbf{p}'$ being the momentum change of the scattering neutron and $\mathbf{r}$ is its varying position throughout the process. The linearity in (3) gives four scattering amplitudes to be considered individually from a theoretical point of view,

$$F = F_N + F_M + F_E + F_P. \quad (5)$$

It turns out that all four terms are of a form to sum up as

$$F = -be^{i\mathbf{q}\cdot\mathbf{R}/\hbar}, \quad (6)$$

with $\mathbf{R}$ being the position of the atomic nucleus in the target and with b being the total *scattering length*

$$b = b_N + b_M + b_E + b_P. \quad (7)$$

It is b_E which is of interest here¹. It reads [1]

$$b_E = \frac{2m_n Ze}{4\pi\epsilon_0\hbar^2} \left[\frac{ze}{q^2} - i\frac{le}{q} \boldsymbol{\sigma} \cdot \frac{\mathbf{q}}{q} - \varepsilon \right] [1 - f_A(\mathbf{q})], \quad q = |\mathbf{q}|. \quad (9)$$

Here f_A is a form factor for the atom. For vanishing neutron charge z and vanishing neutron electric dipole length l we have

$$b_E = -b_I Z [1 - f_A(\mathbf{q})], \quad (10)$$

where

$$b_I = \frac{2m_n e}{4\pi\epsilon_0\hbar^2} \varepsilon = \frac{1}{3} \frac{m_n}{m_e} \frac{\langle r_n^2 \rangle}{a_0} \quad (11)$$

in Sears' notation with $\varepsilon \equiv \frac{1}{6} e \langle r_n^2 \rangle$ (cf. (18)) parametrising a neutron charge distribution term derived by Foldy [1,22, 23]. Sears calls b_I the *intrinsic neutron-electron scattering length*. We recognise the relation in (1) between b_{ne} and the mean square charge radius $\langle r_n^2 \rangle$.

The form factor $f_A(\mathbf{q})$ is a correction factor on the differential scattering cross-section of point particle scattering. It collects the effect of the target having an interior/intrinsic electric structure. One may think of unpolarised electrons scattering on atomic nuclei of atomic number Z [20]

$$\left(\frac{d\sigma}{d\Omega} \right) = \left(\frac{d\sigma}{d\Omega} \right)_{\text{point}} |f_A(\mathbf{q})|^2. \quad (12)$$

For scattering on a point particle

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{point}} = \left(\frac{Z\alpha\hbar}{2c} \right)^2 \frac{E^2}{|\mathbf{p}|^4 \sin^4(\theta/2)} (1 - \beta^2 \sin^2(\theta/2)) \quad (13)$$

¹The earliest data are expressed in interaction strengths as analogous depths V_0 (Fermi convention) [11] of corresponding radial square well potentials of range equal to the classical electron radius r_e [17,18]. These depths correspond to scattering lengths

$$b_{ne} \approx \frac{1}{3} r_e^3 \frac{2m_n}{\hbar^2} V_0. \quad (8)$$

See appendix A for a derivation of the conversion.

with fine structure constant α , electron energy E , scattering angle θ and $\beta = |\mathbf{p}|c/E$.

As mentioned, the form factor is interpreted as originating in a charge distribution $\varrho(\mathbf{x})$ in the target wherefore the Coulomb potential at distance $|\mathbf{x}|$ from the nucleus is substituted by an integral over the interior charge density

$$-\frac{Ze}{4\pi\epsilon_0|\mathbf{x}|} \rightarrow \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\varrho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad (14)$$

which leads in the case of electron-neutron scattering to a *neutron form factor* [20]

$$f_n(\mathbf{q}) = \frac{1}{-Ze} \int \varrho(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}/\hbar} d^3x. \quad (15)$$

In this interpretation thus, the form factor represents Fourier components of the charge distribution of the neutron target when in the scattering of an electron a momentum $\mathbf{q} = \mathbf{p} - \mathbf{p}'$ is exchanged with the neutron.

For not too large momentum exchanges, the exponential in the form factor can be expanded,

$$e^{-i\mathbf{q}\cdot\mathbf{x}/\hbar} = 1 - \frac{i\mathbf{q}\cdot\mathbf{x}}{\hbar} - \frac{1}{2} \left(\frac{\mathbf{q}\cdot\mathbf{x}}{\hbar} \right)^2 + \dots \quad (16)$$

and for a spherical charge distribution, the expansion integrates to yield²

$$f_n(\mathbf{q}) = 1 - \frac{\langle r^2 \rangle}{6\hbar^2} |\mathbf{q}|^2 + \dots \quad (17)$$

with $\langle r^2 \rangle$ defining the *mean square radius* that is at our focus,

$$\langle r^2 \rangle = \int r^2 \varrho d^3x. \quad (18)$$

Now, what is determined *experimentally* is the form factor as of (12) leading (possibly through subtractions of other contributions as of (7)) to the electric scattering length as of (10). The results for $\langle r_n^2 \rangle$ are derived from (17) and (11), respectively, based on the underlying interpretation.

From the experiments one *cannot* distinguish between the effect of an interior neutron charge distribution or an intrinsic neutron structure. We use the terms *interior* and *intrinsic* to represent, respectively, a spatial distribution within a certain (small, but finite) domain of laboratory space (the interior of a neutron) *or* a neutronic structure represented by an intrinsic configuration, *i.e.*, a configuration excited at a *point* in laboratory space where the neutron is hit and the intrinsic configuration space “comes to life” in its entirety.

Baryon configurations. – We consider baryons as stationary states on an intrinsic configuration space, the

²It is customary to write the neutron form factor as a sum of electric and magnetic terms [6], but we shall not go into that here.

Lie group $U(3)$ [24]³

$$\frac{\hbar c}{a} \left[-\frac{1}{2}\Delta + \frac{1}{2}d^2(e, u) \right] \Psi(u) = \mathcal{E}\Psi(u), \quad u \in U(3). \quad (19)$$

The configuration space $U(3)$ contains all the three gauge groups of the standard model which motivates to use it for baryonic configurations; baryons namely feel all the three quantum interactions with gauge groups $SU(3)$ for the strong interactions and $U(1) \times SU(2)$ for the electroweak interactions. $U(3)$ has nine generators analogous to momentum, angular momentum and Laplace-Runge-Lenz operators in the laboratory space, which may explain its origin in scattering processes in the laboratory space which kinematically excite intrinsic degrees of freedom. The intrinsic degrees of freedom may be thought of as generalised spin. The electron spin was first realized as an intrinsic degree of freedom in 1926 by Uhlenbeck and Goudsmit [28]. We have our unitary configuration variable u generated by a Hermitian operator χ

$$u = e^{i\chi}, \quad \chi = \theta_j T_j + (\alpha_j S_j + \beta_j M_j)/\hbar, \quad (20)$$

$$j = 1, 2, 3, \quad \theta_j, \alpha_j, \beta_j \in \mathbb{R}.$$

Here the toroidal generators T_j , (e.g., $T_2 = \text{diag}(0, 1, 0)$) equate the momentum generators p_j in laboratory space

$$iT_j = \frac{\partial}{\partial \theta_j} = \frac{a}{-i\hbar} p_j, \quad e^{i\theta_j T_j} = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}), \quad (21)$$

the spin generators S_j equate the angular momentum generators—in coordinate representation [29], ($x_j = a\theta_j$)

$$S_1 = a\theta_2 p_3 - a\theta_3 p_2 = \hbar \lambda_7 = \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad (22)$$

and the generators M_j mix spin and flavour (25), e.g.,

$$M_1/\hbar = \theta_2 \theta_3 + \frac{a^2}{\hbar^2} p_2 p_3 = \lambda_6. \quad (23)$$

The M_j 's share commutators with quantum analogues of the Laplace-Runge-Lenz vector [30] $\mathbf{M} = c_1(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) + c_2/r \cdot \mathbf{r}$, which is a constant of motion in Kepler orbits and in the hydrogen atom [29]

$$[M_i, M_j] = [S_i, S_j] = -i\hbar \varepsilon_{ijk} S_k. \quad (24)$$

Note the minus sign as in body fixed intrinsic coordinates for spin in nuclear physics.

The lambdas are the Gell-Mann matrices [29]. The spectrum of $\mathbf{S}^2$ is well-known [31] to be $s(s+1)\hbar^2$ with half odd integer s in multiples of $\hbar$ for our intrinsic case. The spectrum of the positive definite $\mathbf{M}^2$ is [24]

$$\mathbf{M}^2/\hbar^2 = \frac{4}{3} \left(n + \frac{3}{2} \right)^2 - s(s+1) - 3 - \frac{1}{3}y^2 - 4i_3^2, \quad (25)$$

³This is a radical reinterpretation of the Kogut-Susskind Hamiltonian [25] of lattice gauge theory [26] using Manton's action [27] as a potential and taking u to concern the full baryonic dynamics.

Fig. 2: The intrinsic potential is periodic in the dynamical eigenangles $\theta_j = x_j/a$. Figure from [10].

where n is a non-negative integer, y is baryonic hypercharge and i_3 is the three-component of baryonic isospin. To generate u and d flavour orbits we use

$$T_u = \frac{2}{3}T_1 - T_3, \quad T_d = -\frac{1}{3}T_1 - T_3 \quad (26)$$

with quark charge coefficients on T_1 and common coefficients on T_3 . The same generators have been used to derive u and d valence quark parton distributions for the proton [24] as well as the proton spin structure function and the proton magnetic dipole moment [10] in good agreement with experiments.

To solve (19) we first factorise the wave function

$$\Psi(u) = \tau(\theta_1, \theta_2, \theta_3) \Upsilon(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3). \quad (27)$$

This is possible because the potential in (19) only depends on the eigenvalues $e^{i\theta_j}$ of u [32] through the three dynamical eigenangles θ_j ($[a\theta_i, p_j] = -i\hbar\delta_{ij}$),

$$\frac{1}{2}d^2(e, u) = \frac{1}{2}\text{Tr } \chi^2 = \sum_{j=1}^3 w(\theta_j) \quad (28)$$

with (fig. 2)

$$w(\theta) = \frac{1}{2}(\theta - n \cdot 2\pi)^2, \quad \theta \in [(2n-1)\pi, (2n+1)\pi], \quad n \in \mathbb{Z}. \quad (29)$$

To suit the potential we use a polar decomposition of the Laplacian on $U(3)$ [33]

$$\Delta = \sum_{j=1}^3 \frac{1}{J^2} \frac{\partial}{\partial \theta_j} J^2 \frac{\partial}{\partial \theta_j} - \sum_{\substack{1 \leq i < j \leq 3 \\ k \neq i, j}} \frac{(S_k^2 + M_k^2)/\hbar^2}{8 \sin^2 \frac{1}{2}(\theta_i - \theta_j)}, \quad (30)$$

where the van de Monde determinant [34]

$$J = \prod_{1 \leq i < j \leq 3} 2 \sin \frac{1}{2}(\theta_i - \theta_j). \quad (31)$$

We then introduce a measure scaled wave function

$$\Phi(u) = J\Psi(u) = R\Upsilon. \quad (32)$$

With (30) inserted, eq. (19) solves analogously to solving the hydrogen atom [21] by integrating over the off-toroidal

degrees of freedom —now with three “radial” dimensions and Υ taking the role of spherical harmonics.

Since the labelling of the eigenangles is arbitrary, the toroidal wave function τ in (27) should be symmetric in these. The van de Monde determinant J (31) is antisymmetric in the eigenangles and likewise will $R = J\tau$ have to be. We therefore expand R on Slater determinants antisymmetric in the three toroidal eigenangles which we interpret as colour degrees of freedom. For unflavoured baryons of neutral charge we expand R on the complete set

$$f_{pqr} = \begin{vmatrix} \cos p\theta_1 & \cos p\theta_2 & \cos p\theta_3 \\ \sin q\theta_1 & \sin q\theta_2 & \sin q\theta_3 \\ \cos r\theta_1 & \cos r\theta_2 & \cos r\theta_3 \end{vmatrix}, \quad R = \sum_{pqr} c_{pqr} f_{pqr}, \quad (33)$$

where p, q, r are integers $0 \leq p < r \in \mathbb{N}$ and $0 < q \in \mathbb{N}$.

Flavour orbit curvatures towards $\langle r_n^2 \rangle$. — In [10] we described how colour fields are generated from the intrinsic wave function by use of the exterior derivative, the momentum form [13]. We found quark colour fields in the laboratory space

$$\psi_j(u) = dR_u(\partial_j) \equiv \frac{d}{dt} R(ue^{itT_j})|_{t=0}, \quad \partial_j = uiT_j \quad (34)$$

and gluon fields

$$G^{(k)}(u) = d\Phi_u(\partial_k), \quad \partial_k = u \frac{i\lambda_k}{2}, \quad k = 1, 2, \dots, 8 \quad (35)$$

to transform under the fundamental and the adjoint representation of $SU(3)$, respectively, and found that local gauge transformations in the laboratory space equate coordinate translations in the configuration space.

Similarly we generate the u and d quark flavour fields by derivation along the directions given in (26). We used this to derive u and d valence quark parton distributions for the proton in [24]. Here we focus on the neutron and consider the geometry of the orbits generated.

With two linearly independent generators in each of the two T_q 's (26), their orbits will run along the same two-dimensional torus, winding their helical ways in opposite orientation (fig. 1).

We want to ascribe curvatures with physical dimensions to these orbits. For that we use the length scale a in (19). We determine the length scale from the classical electron radius r_e [17,18] by taking [24]

$$\pi a = r_e. \quad (36)$$

Here the factor π indicates a mapping to the flat laboratory space from the intrinsic torus⁴.

We now embed in the laboratory space the intrinsic torus on which u and d vector fields are excited according to the common coefficient $h = -1$ and the individual fractional quark charges $e_q = \frac{2}{3}, -\frac{1}{3}$, respectively.

⁴In [24] we describe how the classical electron radius enters baryonic physics via the neutron to proton decay. We write on the creation of the electron charge in this decay as a “peel off” from the neutron, leaving a “scarred” nucleon: the proton.

Fig. 3: Experimental determinations of the square of the neutron charge radius. Figure from [7].

The quark charges are in units of the elementary charge $e = 1.602176634 \cdot 10^{-19} \text{ C}$ [7]. Generalising [35] to independent angular velocities, we consider 2D surfaces $\mathbf{x}$ parametrised by the two excited degrees of freedom θ_1 and θ_3 for our flavour cases

$$\mathbf{x}(\theta_1, \theta_3) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (c + a \cos e_q \theta_1) \cos(-\theta_3) \\ (c + a \cos e_q \theta_1) \sin(-\theta_3) \\ a \sin e_q \theta_1 \end{pmatrix}. \quad (37)$$

The same kind of embedding was used in [24] where a first edition of fig. 1 was presented. First, one calculates the metric components, then one can find the local Gaussian curvature K at each point. The result for $c = a$ is

$$K_q = \frac{\cos e_q \theta_1}{a e_q (a + a \cos e_q \theta_1)}. \quad (38)$$

See appendix B for details. We have flavour curvature expectation values

$$\langle K_q \rangle = \int_0^\pi \int_0^\pi \int_0^\pi R^* K_q R \, d\theta_1 d\theta_2 d\theta_3. \quad (39)$$

For the leading component f_{011} , we find $\langle K_u \rangle = 0.15582 \dots \text{ fm}^{-2}$ and $\langle K_d \rangle = -1.32875 \dots \text{ fm}^{-2}$. We interpret the mean square of the neutron charge radius as an average over three colours of the reciprocal of the sum of the average Gaussian flavour curvatures for one u quark and two d quarks,

$$\langle r_n^2 \rangle = \frac{1}{3} \cdot \frac{1}{\langle K_u \rangle + 2\langle K_d \rangle}. \quad (40)$$

The result $(-0.1075 \pm 0.0020) \text{ fm}^2$ shown in table 1 seems promising in comparison with the value $(-0.1161 \pm 0.0022) \text{ fm}^2$ from the Particle Data Group [7] and even more promising compared to [4,5] and the recent [9]. Note that all components in table 1 yield surprisingly consistent

values for their $\langle r_n^2 \rangle$ contributions. This supports the idea of $\langle r_n^2 \rangle$ as a well-defined entity of intrinsic origin. Our value is some standard deviations away from the PDG value [7]. However, experimental determinations have shown quite a variation throughout the years, see fig. 3, and we consider the matter ripe for further investigation.

Conclusion. – We assumed the neutron structure to be of intrinsic origin determined by a mass Hamiltonian on the intrinsic configuration space $U(3)$. This space has local curvatures of varying signs. We assumed the mean square neutron charge radius to be a manifestation of the average Gaussian curvature of quark flavour tracks —generated in scattering by the momentum form— and weighted by the square of the wave function. The value derived is intriguingly close to values established experimentally in various experiments. We look forward to further experimental investigations for which we think the time is ripe.

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Appendix A: scattering length from potential depth. – The scattering length is defined in effective range theory [36]. Using the Wronskian theorem one can relate the phase shifts under scattering by two different potentials to the Wronskian of the solutions to the radial Schrödinger equation including the respective potentials

$$\frac{\hbar^2}{2m_n} \left[-\frac{d^2}{dr^2} + V(r) + \frac{l(l+1)}{r^2} \right] y_l(r) = E y_l(r). \quad (\text{A.1})$$

The shape of the potential V was not known except that it was supposed to be of finite range. To compare experimental results the convention was [11] to model by square wells of various depths $V_0 < 0$ but all of range equal to the classical electron radius r_e

$$V(r) = \begin{cases} V_0, & r \leq r_e, \\ 0, & r > r_e. \end{cases} \quad (\text{A.2})$$

We follow Messiah [36] and consider solutions $\hat{y}_l$ and y_l for two different potentials. Asymptotically with $E = \hbar^2 k^2 / (2m_n)$

$$\begin{aligned} y_l(r) &\sim \sin\left(kr - l\frac{\pi}{2} + \delta_l\right), \\ \hat{y}_l(r) &\sim \sin\left(kr - l\frac{\pi}{2} + \hat{\delta}_l\right). \end{aligned} \quad (\text{A.3})$$

The Wronskian theorem yields

$$\int_{r_1}^{r_2} W(y_l, \hat{y}_l) = - \int_{r_1}^{r_2} \hat{y}_l (U - \hat{U}) y_l \, dr \quad (\text{A.4})$$

with $U = \frac{2m_n}{\hbar^2} V$ and $\hat{U} = \frac{2m_n}{\hbar^2} \hat{V}$. Asymptotically

$$\lim_{r \rightarrow \infty} W(y_l, \hat{y}_l) = k \sin(\delta_l - \hat{\delta}_l) \quad (\text{A.5})$$

and thus⁵

$$\sin(\delta_l - \hat{\delta}_l) = -\frac{2m_n}{\hbar^2 k} \int_0^\infty \hat{y}_l (V - \hat{V}) y_l \, dr. \quad (\text{A.6})$$

The *scattering length* b is defined from the phase shift δ_0 in S -wave scattering at vanishing incident energy as

$$b \equiv - \lim_{k \rightarrow 0} \frac{\delta_0}{k}. \quad (\text{A.7})$$

Choosing $\hat{V} = 0$ there is no scattering and one has $\hat{\delta}_l = 0$ and $\hat{y}_l = kr j_l(kr)$, where j_l is a spherical Bessel function of order l . In so far as the scattering length is finite, the phase shift δ will decrease with energy $E \sim k^2$ because of decreasing wave number k and we can substitute δ for $\sin(\delta)$ to get an integral representation of the scattering length [37] with $j_0(kr) = \frac{\sin(kr)}{kr}$,

$$b = \int j_0(kr) U(r) f_0(r) |_{k \rightarrow 0} r^2 dr, \quad f_0(r) = \frac{y_0(r)}{r}. \quad (\text{A.8})$$

Ross and Shaw [36] solve this to give an implicit expression

$$b = \frac{r_e I}{1 + I}$$

and have

$$f_0|_{k \rightarrow 0} = e^{i\delta} \frac{\sin(kr + \delta)}{kr} |_{k \rightarrow 0}, \quad (\text{A.9})$$

where

$$I = (r_e f_0(r_e) |_{k \rightarrow 0})^{-1} \int U f_0(r) |_{k \rightarrow 0} r^2 dr. \quad (\text{A.10})$$

Using (A.7) and a power expansion of the sine we have [37]

$$f_0|_{k \rightarrow 0} = e^{i\delta} \frac{\sin(kr + \delta)}{kr} |_{k \rightarrow 0} = 1 - \frac{b}{r}. \quad (\text{A.11})$$

We insert the last expression in (A.10) and find

$$I = \frac{1}{r_e - b} \int_0^\infty U(r) \left(1 - \frac{b}{r}\right) r^2 \, dr = \frac{2m_n V_0}{\hbar^2} \frac{\frac{1}{3} r_e^3 - \frac{1}{2} b r_e^2}{r_e - b} \quad (\text{A.12})$$

Solving (A.9) for the scattering length we get for $|b| \ll r_e$

$$b \approx \frac{1}{3} r_e^3 \frac{2m_n}{\hbar^2} V_0. \quad (\text{A.13})$$

The implied proportionality between b_{ne} and V_0 agrees with the conversion used by Krohn and Ringo [8] although they do not state the above relation explicitly.

⁵Provided that V and $\hat{V}$ go quicker than $1/r$ to zero at infinity and have no singularity stronger than $1/r^2$ at $r = 0$ [36]. These requirements are met by the square well potentials we consider here.

Appendix B: metric components for curvature calculation. – We follow [35] and find the metric coefficients of the square of the line element ds ,

$$\begin{aligned} ds^2 &= E d\theta_3^2 + F d\theta_3 d\theta_1 + G d\theta_1^2, \\ E &\equiv \mathbf{x}_{\theta_3} \cdot \mathbf{x}_{\theta_3} = h^2(a + a \cos e_q \theta_1)^2, \\ F &\equiv \mathbf{x}_{\theta_3} \cdot \mathbf{x}_{\theta_1} = 0, \\ G &\equiv \mathbf{x}_{\theta_1} \cdot \mathbf{x}_{\theta_1} = a^2 e_q^2. \end{aligned} \quad (\text{B.1})$$

Using [35]

$$K = \frac{-1}{\sqrt{EG}} \left[\left(\frac{(\sqrt{E})_{\theta_1}}{\sqrt{G}} \right)_{\theta_1} + \left(\frac{(\sqrt{G})_{\theta_3}}{\sqrt{E}} \right)_{\theta_3} \right] \quad (\text{B.2})$$

the result in (38) follows.

Irons [35] uses a non-singular embedding with two different torus radii c and a where $c > a$. In case the singularity at $\cos e_q \theta_1 = -1$ in (38) is of concern one may choose $c = a + \epsilon$ and let $\epsilon \rightarrow 0$. For the average curvatures (39) the singularity does not matter since the set of singular points are of zero measure in the $(\theta_1, \theta_2, \theta_3)$ eigenangle space. Thus (38) integrates to final results (39) over the domain of integration.

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