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Conservation of currents in reduced full-F electromagnetic kinetic and fluid models

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Abstract. In this paper we present an analysis of the conservation of currents in a full-F electromagnetic gyro-kinetic model in the long-wavelength limit. This equation corresponds to what is usually named "vorticity equation", which is not strictly correct as it cannot be formulated as the curl of a velocity equation. In the paper we will therefore use the term "current conservation equation" instead. Our results are relevant to reduced plasma descriptions like gyro-kinetic, drift-kinetic, gyro-fluid and drift-fluid models for tokamaks and stellarators. The equation describes the change of the polarization charge density (often called "vorticity") in terms of the polarization stress due to the $\boldsymbol{E} \times \boldsymbol{B}$ flow, external sources and three currents: the parallel current, the curvature current and a current related to the magnetic field fluctuations. We compare this equation with previous drift- and gyro-fluid equations and find general agreement except in the vorticity source terms where previous drift-fluid models fail to capture the heating and density sources. We discuss the role of the currents in the dynamics of diamagnetic and $\boldsymbol{E} \times \boldsymbol{B}$ flow shear. The possible connection between these currents with phenomena observed in experiments that influence the radial electric field in the edge of tokamak plasmas, like resonant magnetic perturbations, different magnetic field configurations and shapes, is presented.

Keywords: vorticity, gyro-fluid, transport barrier, radial electric field, L-H transition, edge current, shear, conservation of currents, reduced model

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1. Introduction

Plasma confinement inside of tokamaks is crucial for the development of self sustained nuclear fusion reactions. One of the considered scenarios for the plasma operation inside of tokamaks is called high confinement mode or H-Mode. The H-mode is characterized by a transport barrier and a radial electric field well in the edge that reduce the turbulent transport losses and increase the confinement [1, 2]. This is due to the shear of the poloidal flow and the decorrelation of turbulence, producing a density pedestal close to the Last Closed Flux Surface (LCFS) where the pressure gradient is increased [3, 4]. Recent studies in ASDEX-Upgrade (AUG) have pointed to the maximum value of the $\boldsymbol{E} \times \boldsymbol{B}$ plasma velocity of the transport barrier as the key parameter for the transition from Low to High confinement (L-H transition) [5]. However, experiments in EAST show that the shear of the perpendicular flows (strongly coupled with $\boldsymbol{E} \times \boldsymbol{B}$ flows) is the trigger for the L-H transition [6]. These investigations exhibit the crucial role that the radial electric field driven flows and its shear play in the transition, and the necessity of understanding in depth its dynamical behaviour. A review of this crucial role can be found in [1].

In general, when the radial electric field related with this transport barrier is studied, it is obtained using the radial force balance, relating the radial electric field E_r to the ion pressure gradient $\frac{\partial P_i}{\partial r}$ and the poloidal and toroidal velocities (v_θ and v_ϕ respectively) [7, 8]:

$$\begin{aligned} E_r &= \frac{1}{n_i Z_i e}\frac{\partial P_i}{\partial r} - (\boldsymbol{v}_\perp \times \boldsymbol{B}) \cdot \hat{\boldsymbol{e}}_{\boldsymbol{r}} \\ &= \frac{1}{n_i Z_i e}\frac{\partial P_i}{\partial r} - v_\theta B_\phi + v_\phi B_\theta \end{aligned} \tag{1}$$

With n_i and Z_i the density and the charge of the species studied, e the absolute value of the electron charge and B_ϕ and B_θ the toroidal and poloidal components of the magnetic field. The radial force balance expression assumes a steady state plasma, and thus fails to describe transitions of the radial electric field between two different steady states, like the L-H transition and the formation of the transport barrier, or when the plasma conditions change drastically as in edge localized modes (ELMs). Furthermore, this equation does not show how the radial electric field depends on other plasma parameters like magnetic field configuration [9, 10, 11, 12] or the magnetic field perturbations [13] including Resonant Magnetic Perturbations (RMP). In addition, there are more magnetic field parameters that affect the power threshold to the H-mode that are not involved in Eq. (1) for the electric field, like the ∇B drift direction with respect to the X-point, the X-point height or triangularity [14, 15, 16, 17].

Some theoretical analyses have proposed an effect of the Pfirsch-Schlüter current on the radial electric field in the edge that could qualitatively explain the dependency of the power threshold of the L-H transition on some of the magnetic field parameters previously mentioned [18]. At the same time, currents like the bootstrap current that are enhanced in the edge due to density gradients even in L mode [19] can produce significant current profiles in the edge region.

All of these experimental observations and theoretical analyses necessitate a dynamical description of the electric field and the pressure gradients in the edge of the plasma taking into account the currents in the region, similarly to the description presented in [20]. The equation that describes the dynamics of currents is the conservation of currents, usually called "vorticity equation".

The so-called "vorticity equation" is an integral part of any drift-fluid model, where it results from the conservation of charge plus quasineutrality as the conservation of currents, $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$ [21, 22, 23] and is used to compute the electric potential ϕ and therefore the electric field $E = -\nabla\phi$. During the paper, we will name the "vorticity equation" the currents conservation equation as this name is physically more correct, and we will give some arguments to support this change of nomenclature.

In other models like gyro-kinetic or gyro-fluid, the "vorticity equation" is more involved to derive, even though its physical meaning is still the conservation of currents and its function is analogous to the drift-fluid models. Previous formulations of "vorticity equations" in gyro-fluid models are limited to either reduced dimensionality, simplified magnetic field configurations, or delta-f models [24, 25, 26, 27, 28]. In the recent work [29] we derive a flux-surface averaged "vorticity equation" of a gyro-kinetic model applying drift-ordering, which we relate to the toroidal angular momentum equation and rotation. The downside of that approach is that the flux-surface average is hard to obtain experimentally and a full "vorticity equation" is preferable. "Vorticity equations" are further derived in short-wavelength limits and with flux surface average quantities in [30].

In this paper we work with a gyro-kinetic model taking the long-wavelength approximation to the gyro-average operator. This makes our results relevant not only for gyro-kinetic, but also for gyro-fluid and in extension drift-kinetic and drift-fluid models (with a proper fluid closure to the distribution function). The extension to arbitrary-wavelength gyro-kinetic and -fluid theory (including tractable closures) is presented in Ref. [31].

The paper is structured as follows: in section 2 the currents conservation equation from full-F electromagnetic gyro-kinetics in the long-wavelength limit is derived, we discuss the terms that appear in it, we relate those terms with experimental evidence and discuss the terminology of "vorticity equation" on the derived equation. In section 3 we compare the equation derived with other gyro-fluid "vorticity equations" obtained in more constrained conditions and with some drift-fluid "vorticity equations" from well known codes, specifically, to GDB [21], SOLEDGE [22] and GRILLIX [23]. In the last section 4, we present conclusions and future steps.

2. Currents conservation equation in reduced gyro-kinetics

2.1. Model presentation

Our derivation is based on the full-F, electromagnetic gyro-kinetic model in the long wavelength limit that is also applied in [29] and all the equations presented in this subsection can be found there. The long wavelength limit allows to keep in the derivation Finite Larmor Radius (FLR) effects up to second order. As in every gyro-kinetic model, we work with the transformed phase-space coordinates $\boldsymbol{Z} := \{\boldsymbol{X}, w_\parallel, \mu, \theta\}$, with gyro-centre coordinate $\boldsymbol{X}$, parallel canonical moment $w_\parallel$, defined in terms of the parallel particle velocity $v_\parallel$ and the magnetic gyro-centre potential $\mathcal{A}_{1,\parallel}$ as $mw_\parallel := mv_\parallel + q\mathcal{A}_{1,\parallel}$, magnetic moment μ and gyro-angle θ. We also work with the gyro-kinetic particle distribution function $F(\boldsymbol{Z}, t) \equiv F(\boldsymbol{X}, w_\parallel, \mu, t)$ and a particle source function $S = S(\boldsymbol{X}, w_\parallel, \mu, t)$ (independent of gyro-angle θ, averaged out), which allow the definition of the velocity space moment operators:

$$\|\zeta\| := \int \mathrm{d}w_\parallel \mathrm{d}\mu \mathrm{d}\theta m^2 B F(\boldsymbol{Z}, t)\zeta \tag{2}$$

$$\|\zeta\|_S := \int \mathrm{d}w_\parallel \mathrm{d}\mu \mathrm{d}\theta m^2 B S(\boldsymbol{Z}, t)\zeta \tag{3}$$

where $\zeta(\boldsymbol{X}, w_\parallel, \mu, t)$ is any function defined on the phase-space, B is the magnitude of the magnetic field and m is the species mass. This gives rise to the different gyro-fluid moments: $N \equiv \|1\|$ the gyro-fluid density, $NU_\parallel \equiv \|v_\parallel\|$ the gyro-fluid parallel momentum density, the gyro-fluid perpendicular pressure $P_\perp = NT_\perp \equiv \|\mu B\|$ and the second order gyro-fluid moment $(Q_\perp + U_\parallel P_\perp) \equiv \|\mu B v_\parallel\|$, with $Q_\perp$ being the perpendicular component of the parallel heat flux. In complete analogy we name $S_N \equiv \|1\|_S$ the gyro-fluid density source, $S_{NU_\parallel} \equiv \|v_\parallel\|_S$ the parallel momentum source and similarly $S_{P_\perp} \equiv \|\mu B\|_S$ and $S_{Q_\perp} + U_\parallel S_{P_\perp} \equiv \|\mu B v_\parallel\|_S$.

In this paper, we start the derivation from the velocity moment operator conservation equation:

$$\frac{\partial}{\partial t}\|\zeta\| + \boldsymbol{\nabla} \cdot \|\zeta \dot{\boldsymbol{X}}\| = \left\|\frac{\mathrm{d}\zeta}{\mathrm{dt}}\right\| + \|\zeta\|_S, \tag{4}$$

which is strictly related to the Vlasov Equation, together with the equation of motion of the gyro-centre position:

$$\begin{aligned}\dot{\boldsymbol{X}} &= \frac{1}{B}(\boldsymbol{B^*} v_\parallel + \frac{1}{q}\hat{\boldsymbol{b}} \times \nabla H) \\ &= \frac{1}{B}(v_\parallel \boldsymbol{B} + \frac{mv_\parallel^2}{q}\nabla \times \hat{\boldsymbol{b}} + \frac{\mu B}{q}\hat{\boldsymbol{b}} \times \nabla \ln B + \\ &\quad + v_\parallel \nabla \times \mathcal{A}_{1,\parallel}\hat{\boldsymbol{b}} + \hat{\boldsymbol{b}} \times \nabla\Psi)\end{aligned} \tag{5}$$

and the Maxwell system equations:

$$\sum_{\mathrm{s}} qN - \boldsymbol{\nabla} \cdot \boldsymbol{P}_{\mathrm{gy}} = 0, \tag{6}$$

$$\begin{aligned}\sum_{\mathrm{s}} qNU_\parallel + \boldsymbol{\nabla} \cdot (\boldsymbol{M}_\perp^{\mathrm{gy}} \times \hat{\boldsymbol{b}}) &= -\frac{1}{\mu_0}\Delta_\perp A_{1,\parallel} \\ &= J_\parallel + J_{mag}\end{aligned} \tag{7}$$

with $\sum_{\mathrm{s}}$ representing the sum over the different species that affect to the species quantities like the charge q, the mass m or the gyrofluid moments. $\hat{\boldsymbol{b}}$ is the magnetic field unit vector, μ_0 the vacuum permeability, $J_\parallel = \sum_{\mathrm{s}} qNU_\parallel$ being the guiding center parallel current and $J_{mag} = \boldsymbol{\nabla} \cdot (\boldsymbol{M}_\perp^{\mathrm{gy}} \times \hat{\boldsymbol{b}})$ being the magnetization current. The conservation of charge presented in equation 6 can also be derived for the gyro-fluid density sources S_N, obtaining the conservation of charge for sources with the definition of the gyro-fluid polarization source $\boldsymbol{S_P}$ as:

$$\sum_{\mathrm{s}} qS_n = \sum_{\mathrm{s}} qS_N - \boldsymbol{\nabla} \cdot \boldsymbol{S_P} = 0 \tag{8}$$

$$\boldsymbol{S_P} := -\sum_{\mathrm{s}} \left[\boldsymbol{\nabla}_\perp \left(\frac{mS_{P_\perp}}{2qB^2}\right) + \frac{mS_N \boldsymbol{\nabla}_\perp \phi}{B^2}\right] \tag{9}$$

We also need the definitions of the modified magnetic field $\boldsymbol{B^*}$:

$$q\boldsymbol{B^*} = q\boldsymbol{B} + mw_\parallel \nabla \times \hat{\boldsymbol{b}} \tag{10}$$

the effective gyro-centre magnetic $\mathcal{A}_{1,\parallel}$ and electric potentials Ψ, where the FLR effects on the long wavelength limit are present in the gradient $(\nabla_\perp)$ and laplacian $(\Delta_\perp)$ terms:

$$q\mathcal{A}_{1,\parallel} := qA_{1,\parallel} + \frac{m\mu}{2qB}\Delta_\perp A_{1,\parallel} \tag{11}$$

$$q\Psi := q\phi + \frac{m\mu}{2qB}\Delta_\perp \phi - \frac{1}{2}m\left(\frac{\boldsymbol{\nabla}_\perp \phi}{B}\right)^2 \tag{12}$$

the gyro-kinetic polarization $\boldsymbol{P}_{\rm gy}$ and perpendicular magnetization $\boldsymbol{M}_\perp^{\rm gy}$ densities:

$$\boldsymbol{P}_{\rm gy} := -\sum_{\rm s}\left[\boldsymbol{\nabla}_\perp\left(\frac{mP_\perp}{2qB^2}\right)+\frac{mN\boldsymbol{\nabla}_\perp\phi}{B^2}\right] \tag{13}$$

$$\boldsymbol{M}_\perp^{\rm gy} := \sum_{\rm s}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\left(\frac{m(Q_\parallel+U_\parallel P_\perp)}{2qB^2}\right) \tag{14}$$

and the system Hamiltonian:

$$H := \frac{(mw_\parallel - q\mathcal{A}_{1,\parallel})^2}{2m}+\mu B+q\Psi = \frac{1}{2}mw_\parallel^2+\mu B+H_f \tag{15}$$

These definitions contain the electromagnetic field perturbations ϕ and $A_{1,\parallel}$, and the field Hamiltonian $H_f \equiv q\Psi - qw_\parallel\mathcal{A}_{1,\parallel} + q^2\mathcal{A}_{1,\parallel}^2/2m$. The use of $\nabla_\perp$ is common during the derivation, meaning the perpendicular component of the gradient of a quantity $\nabla_\perp f = \hat{\boldsymbol{b}}\times(\nabla f\times\hat{\boldsymbol{b}})$.

2.2. Currents conservation equation derivation

As discussed in the introduction, the currents conservation equation is the often called "vorticity equation". We will discuss in subsection 2.5 how the currents conservation equation derived is related to the definition of vorticity and if this nomenclature is correct. In plasma theory, the "vorticity equation" is derived from the conservation of charge in different ways. In our case, we derive it from the moment operator conservation (4), summing over all the species using $\zeta = q$. Using the definitions of the gyro-fluid moments introduced in subsection 2.1:

$$\sum_{\rm s}\left(\frac{\partial}{\partial t}(qN)+\boldsymbol{\nabla}\cdot q\|\dot{\boldsymbol{X}}\| - qS_N\right)=0 \tag{16}$$

Making use of the quasineutrality of density (6) and density sources (8), we can rewrite (16) as:

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\boldsymbol{P}_{\rm gy})+\sum_{\rm s}\boldsymbol{\nabla}\cdot q\|\dot{\boldsymbol{X}}\| - \boldsymbol{\nabla}\cdot\boldsymbol{S_P}=0 \tag{17}$$

The gyro-kinetic polarization density $\boldsymbol{P}_{\rm gy}$ (13) has the term related with $P_\perp$ divided by 2. This 1/2 factor makes the polarization density different from the usual definition of the general "vorticity" used in most drift-fluid models [21, 22, 23]. In fluid models, this term arises from $\boldsymbol{E}\times\boldsymbol{B}$ and diamagnetic drift as will be explained in subsection 2.5. We define the gyro-fluid polarization density $\boldsymbol{w}$ following the fluid criteria for "vorticity" in Eq. (18), multiplying by 2 the diamagnetic term of the gyro-kinetic polarization density $\boldsymbol{P}_{\rm gy}$, which leads to the definition of the gyro-fluid polarization charge density $\mathcal{W}$, which is what could be called "gyro-fluid vorticity" as it follows the usual definition of "vorticity" in drift-fluid models but with gyro-fluid quantities. The gyro-fluid polarization charge density source is defined analogously as $\mathcal{W}_{\mathcal{S}}$ in Eq. (19), where we introduce the term $\boldsymbol{J_S}$ as the gyro-fluid polarization current source that appears due to sources thanks to the polarization:

$$\mathcal{W} := -\boldsymbol{\nabla}\cdot\boldsymbol{w} = \sum_{\rm s}\boldsymbol{\nabla}\cdot\left[\boldsymbol{\nabla}_\perp\left(\frac{mP_\perp}{qB^2}\right)+\frac{mN\boldsymbol{\nabla}_\perp\phi}{B^2}\right] \tag{18}$$

$$\begin{aligned}\mathcal{W}_S &:= -\boldsymbol{\nabla}\cdot\boldsymbol{J_S}\\ &= \sum_{\rm s}\boldsymbol{\nabla}\cdot\left[\boldsymbol{\nabla}_\perp\left(\frac{mS_{P_\perp}}{qB^2}\right)+\frac{mS_N\boldsymbol{\nabla}_\perp\phi}{B^2}\right]\end{aligned} \tag{19}$$

Counting with these definitions, and taking into account that the time derivative commutes with the divergences and gradients, we can rewrite (17) as:

$$\begin{aligned}&\frac{\partial\mathcal{W}}{\partial t}-\mathcal{W}_S-\sum_{\rm s}\boldsymbol{\nabla}\cdot q\|\dot{\boldsymbol{X}}\|\\ &-\sum_{\rm s}\boldsymbol{\nabla}\cdot\left[\boldsymbol{\nabla}_\perp\left(\frac{m}{2qB^2}\left(\frac{\partial P_\perp}{\partial t}-S_{P_\perp}\right)\right)\right]=0\end{aligned} \tag{20}$$

Considering the conservation law for a moment variable (4) applied to $\|\mu B\| = P_\perp$, we can write:

$$\begin{aligned}\frac{\partial P_\perp}{\partial t}-S_{P_\perp} &= \frac{\partial}{\partial t}\|\mu B\| - \|\mu B\|_S = -\boldsymbol{\nabla}\cdot\|\mu B\dot{\boldsymbol{X}}\| + \left\|\frac{d(\mu B)}{dt}\right\| =\\ &= -\boldsymbol{\nabla}\cdot\|\mu B\dot{\boldsymbol{X}}\| + \|\mu B\dot{\boldsymbol{X}}\cdot\boldsymbol{\nabla}\ln B\|\end{aligned} \tag{21}$$

where we have used the dynamical pressure equation $d(\mu B)/dt = \mu B\dot{\boldsymbol{X}}\cdot\boldsymbol{\nabla}\ln B$. This allows us to transform (20) into:

$$\begin{aligned}&\frac{\partial\mathcal{W}}{\partial t}-\mathcal{W}_S-\sum_{\rm s}\boldsymbol{\nabla}\cdot\Bigg[\,q\|\dot{\boldsymbol{X}}\| +\\ &+\boldsymbol{\nabla}_\perp\left(\frac{m}{2qB^2}\left(\|\mu B\dot{\boldsymbol{X}}\cdot\nabla\ln B\| - \boldsymbol{\nabla}\cdot\|\mu B\dot{\boldsymbol{X}}\|\right)\right)\Bigg]=0\end{aligned} \tag{22}$$

Once we have this equation, we apply the following ordering to avoid a complex final expression and be able to compare with drift-kinetic models. We follow the same ordering as in [29]:

- The frequency of turbulent fluctuations ω_t compared to the ion gyro-frequency ω_i are very slow (order δ^2), $\omega_t/\omega_i \sim \partial_t\ln F/\omega_i \sim \partial_t\ln\phi/\omega_i \sim \delta^2 \ll 1$, where $\omega_i = q_iB/m_i$. This ordering also applies to the sources.
- The derivatives ∇_k of the dynamical fields as perpendicular derivatives ($\nabla_\perp$) are small (order δ), $\rho_i|\nabla_k\ln F| \sim \rho_i|\nabla_k\ln\phi| \sim \rho_i|\nabla_k\ln A_{1,\parallel}| \sim \rho_i k_\perp \sim \delta$, with ion thermal gyro-radius $\rho_i = \sqrt{m_iT_i}/q_iB$.

- All derivatives on the magnetic field as $L_B^{-1} \sim |\nabla \ln B| \sim 1/R$, where R is the major radius, are very small (order δ^3), $\rho_i/L_B \sim \delta^3$.
- All parallel derivatives are very small, on the same scale as the magnetic field variation (order δ^3), $\rho_i|\nabla_\parallel \ln\phi| \sim \rho_i|\nabla_\parallel \ln A_{1,\parallel}| \sim \rho_i|\nabla_\parallel \ln F| \sim \rho_i k_\parallel \sim \delta^3$.

This ordering can be linked to the usual ordering presented in [32] as $\epsilon_\perp = \delta$, $\epsilon_\delta = \delta^2$ and $\epsilon_B = \delta^3$. This is an intermediate ordering between gyro-kinetics with $\epsilon_\perp \sim O(1)$ and the usual drift-kinetic ordering $\epsilon_\perp \sim \delta^2 \ll 1$. The concept of long-wavelength approximation applied already in equations (11) and (12) of keeping terms up to order $O(\epsilon_\perp^2)$ is perfectly coherent with our ordering. Keeping all the orders linked to a same parameter δ simplifies the derivation with the expense of having less freedom on the range of magnitudes that the model can describe.

Keeping up to order δ^4 in (22), we only keep the terms $\sum_s \boldsymbol{\nabla}\cdot q\|\dot{\boldsymbol{X}}\|$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_\perp(m\boldsymbol{\nabla}\cdot\|\mu B\dot{\boldsymbol{X}}\|/2qB^2)$ up to this order. The first one can be written using the definition of the Hamiltonian (15):

$$\sum_s \boldsymbol{\nabla}\cdot q\|\dot{\boldsymbol{X}}\| = \sum_s \boldsymbol{\nabla}\cdot\left[q\|v_\parallel\|(\hat{\boldsymbol{b}} + A_{1,\parallel}\frac{\nabla\times\hat{\boldsymbol{b}}}{B}) + \|mv_\parallel^2\|\frac{\nabla\times\hat{\boldsymbol{b}}}{B} + \|\mu B\|\frac{\hat{\boldsymbol{b}}\times\nabla\ln B}{B} + \frac{\hat{\boldsymbol{b}}\times\|\nabla H_f\|}{B}\right] \tag{23}$$

For the last term of (23), we will use the variational derivative of the field Hamiltonian on $\zeta \in \{\phi, A_{1,\parallel}\}$, $H = H(\zeta, \boldsymbol{\nabla}_\perp\zeta, \Delta_\perp\zeta)$. Using also the properties of the action as described in [29], this term can be described as follows:

$$\begin{aligned}\sum_s \boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{b}}\times\|\nabla H_f\|}{B}\right) &= \boldsymbol{\nabla}\cdot\left(\frac{\Delta_\perp A_{1,\parallel}}{\mu_0}\frac{\hat{\boldsymbol{b}}\times\nabla A_{1,\parallel}}{B}\right)\\
&+ \sum_s \boldsymbol{\nabla}\cdot\Bigg[\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}_\perp\left(\frac{m\|\mu B v_\parallel\|}{qB^2}\right)\frac{\hat{\boldsymbol{b}}\times\nabla A_{1,\parallel}}{B}\right)\\
&- \boldsymbol{\nabla}\cdot\left(\left(\frac{mN\boldsymbol{\nabla}_\perp\phi}{B^2} + \boldsymbol{\nabla}_\perp\left(\frac{mP_\perp}{qB^2}\right)\right)\frac{\hat{\boldsymbol{b}}\times\nabla\phi}{B}\right)\\
&+ \Delta_\perp\left(\frac{m}{2qB^2}(\|\mu B\|\frac{\hat{\boldsymbol{b}}\times\nabla\phi}{B} - \|\mu B v_\parallel\|\frac{\hat{\boldsymbol{b}}\times\nabla A_{1,\parallel}}{B})\right)\Bigg]\end{aligned} \tag{24}$$

With $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{u}\boldsymbol{v}) = \boldsymbol{\nabla}\cdot((\boldsymbol{\nabla}\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\cdot\boldsymbol{v} + \boldsymbol{\nabla}\cdot\boldsymbol{u}\boldsymbol{\nabla}\cdot\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{u}:\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\cdot\boldsymbol{u} = u^i_{|i}v^k_{|k} + u^i v^k_{|ki} + v^i u^k_{|ik} + u^i_{|k}v^k_{|i} = \boldsymbol{\nabla}\cdot((\boldsymbol{\nabla}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{u}) = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{v}\boldsymbol{u})$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are general vectors and $u^k_{|i}$ the covariant derivative. Now, we can describe the term $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_\perp(m\|\mu B\dot{\boldsymbol{X}}\|/2qB^2)$ up to order δ^4:

$$\begin{aligned}&\sum_s \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_\perp(\frac{m}{2qB^2}\boldsymbol{\nabla}\cdot\|\mu B\dot{\boldsymbol{X}}\|)\\
&= \sum_s \Delta_\perp(\frac{m}{2qB^2}\boldsymbol{\nabla}\cdot\|\frac{\mu B\hat{\boldsymbol{b}}\times\nabla H_f}{qB}\|)\\
&= \sum_s \Delta_\perp(\boldsymbol{\nabla}\cdot(\frac{m}{2qB^2}(\|\mu B\|\frac{\hat{\boldsymbol{b}}\times\nabla\phi}{B}\\
&- \|\mu B v_\parallel\|\frac{\hat{\boldsymbol{b}}\times\nabla A_{1,\parallel}}{B})))\end{aligned} \tag{25}$$

As we are in the long-wavelength limit, the operators divergence ($\boldsymbol{\nabla}\cdot$) and the perpendicular Laplacian ($\Delta_\perp$) commute, so when we introduce equations (24) and (25) in equation (22), the last terms in equations (24) and (25) cancel each other, so we can describe the final equation using (7) as:

$$\frac{\partial\mathcal{W}}{\partial t} - \mathcal{W}_S - \boldsymbol{\nabla}\cdot\left(J_\parallel(\hat{\boldsymbol{b}} - \boldsymbol{b}_\perp) + \boldsymbol{J_{curv}} - J_{mag}\boldsymbol{b}_\perp\right) + - \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{w}\boldsymbol{u_E}) - \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{M^{em}}\boldsymbol{b}_\perp) = 0 \tag{26}$$

Having used the following definitions for the $\boldsymbol{E}\times\boldsymbol{B}$ velocity $\boldsymbol{u_E}$ and the magnetic field perturbation $\boldsymbol{b}_\perp$ (27), the gyro-fluid curvature current $\boldsymbol{J_{curv}}$ (28) and the gyro-fluid electromagnetic magnetization density $\boldsymbol{M^{em}}$ (29):

$$\boldsymbol{u_E} := \frac{\hat{\boldsymbol{b}}\times\nabla\phi}{B}; \qquad \boldsymbol{b}_\perp := \frac{\hat{\boldsymbol{b}}\times\nabla A_{1,\parallel}}{B} \tag{27}$$

$$\boldsymbol{J_{curv}} := \left[\sum_s(P_\parallel + mNU_\parallel^2) + J_\parallel A_{1,\parallel}\right]\frac{\nabla\times\hat{\boldsymbol{b}}}{B} + \sum_s P_\perp\frac{\hat{\boldsymbol{b}}\times\nabla\ln B}{B} \tag{28}$$

$$\begin{aligned}\boldsymbol{M^{em}} &:= \sum_s \boldsymbol{\nabla}_\perp\left(\frac{m\|\mu B v_\parallel\|}{qB^2}\right)\\
&= \sum_s \boldsymbol{\nabla}_\perp\left(\frac{m(Q_\perp + P_\perp U_\parallel)}{qB^2}\right)\\
&= 2\boldsymbol{M}_\perp^{\rm gy}\times\hat{\boldsymbol{b}}\end{aligned} \tag{29}$$

2.3. Conservation of currents description

Following the origin of each term in Eq. (26), we can define different currents to describe the currents conservation equation only with currents: first, we have the term $\boldsymbol{b}_\perp$, which is simply the magnetic field perturbation vector that arises from $A_{1,\parallel}$, the fluctuating magnetic potential. This means that any term containing $A_{1,\parallel}$ or $\boldsymbol{b}_\perp$ appears if we have magnetic fluctuations. This is why we merge every term that depends on $\boldsymbol{b}_\perp$ in a term that we will call $\boldsymbol{J_{b_\perp}}$ or gyro-fluid magnetic perturbation current:

$$\begin{aligned}\boldsymbol{J_{b_\perp}} &:= -(J_\| + J_{mag})\boldsymbol{b}_\perp + \boldsymbol{\nabla}\cdot(\boldsymbol{M^{em}}\boldsymbol{b}_\perp) = \\ &= (\frac{\Delta_\perp A_{1,\|}}{\mu_0} + \boldsymbol{\nabla}\cdot\boldsymbol{M^{em}} + \boldsymbol{M^{em}}\cdot\nabla)\boldsymbol{b}_\perp\end{aligned} \tag{30}$$

Second, we have the gyro-fluid curvature current term $\boldsymbol{J_{curv}}$ (28), where the curvature drift, the ∇B drift and a final curvature effect on the parallel current that depends on magnetic fluctuations are included.

Thirdly, we could define the gyro-fluid polarization current following the usual dielectric definition as the time derivative of the gyro-fluid polarization density $\boldsymbol{J_{pol}} = \partial_t \boldsymbol{w}$. This definition is strictly from solid dielectrics, where $\mathcal{W}$ is usually called "bound charge density" as the charges that produce the polarization are bounded to the medium and cannot move, which is not the case in a plasma. In some drift-fluid models, what we will call the gyro-fluid polarization density tensor term $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{w}\boldsymbol{u_E})$ is included in the definition of the polarization current as $\boldsymbol{J_{pol}} = \partial_t\boldsymbol{w} + \boldsymbol{\nabla}\cdot(\boldsymbol{w}\boldsymbol{u_E})$[22]. This has a certain physical sense, since the gyro-fluid polarization density tensor term $-\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{w}\boldsymbol{u_E})$ contains the advection by $\boldsymbol{E}\times\boldsymbol{B}$ velocity of the gyro-fluid polarization density charge $\boldsymbol{\nabla}\cdot(\mathcal{W}\boldsymbol{u_E})$. However, the gyro-fluid polarization density tensor term contains also $\boldsymbol{\nabla}\cdot(\boldsymbol{w}\cdot\nabla\boldsymbol{u_E})$, which in the ordering, is not negligible, and cannot be interpreted as the advection of the polarization density charge. Still, the definition of the gyro-fluid polarization current following the drift-fluid definition does not leave out this component of the gyro-fluid polarization density tensor term. This gives good arguments to define the gyro-fluid polarization current following the drift-fluid criteria as the total time derivative of the gyro-fluid polarization density, defined by the time derivative of the gyro-fluid polarization density plus the divergence of the gyro-fluid polarization density stress:

$$\boldsymbol{J_{pol}} := \frac{d\boldsymbol{\omega}}{dt} = \frac{\partial\boldsymbol{w}}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{w}\boldsymbol{u_E}) \tag{31}$$

This gyro-fluid polarization density tensor term is neglected in the derivations in [30] but is common in drift-fluid models (simplified as the advection of "vorticity") as will be presented in section 3.

With these definitions we could rewrite the currents conservation equation (26) in terms of currents (the gyro-fluid polarization current $\boldsymbol{J}_{pol}$ (31), the gyro-fluid parallel current $J_\|\hat{\boldsymbol{b}}$ (7), the gyro-fluid curvature current $\boldsymbol{J}_{curv}$ (28), the gyro-fluid magnetic perturbation current $\boldsymbol{J}_{b_\perp}$ (30) and the gyro-fluid polarization current source $\boldsymbol{J}_S$ (19), this last one with a negative sign as it is a source and in the right hand side would be positive):

$$\boldsymbol{\nabla}\cdot\boldsymbol{J} = \boldsymbol{\nabla}\cdot(\boldsymbol{J_{pol}} + J_\|\hat{\boldsymbol{b}} + \boldsymbol{J_{curv}} + \boldsymbol{J_{b_\perp}} - \boldsymbol{J_S}) = 0 \tag{32}$$

Without using the definition of the gyro-fluid polarization current and the gyro-fluid polarization current source, the equation can be rewritten as a transport equation of gyro-fluid polarization charge density:

$$\frac{\partial\mathcal{W}}{\partial t} - \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{w}\boldsymbol{u_E}) = \boldsymbol{\nabla}\cdot(J_\|\hat{\boldsymbol{b}} + \boldsymbol{J_{curv}} + \boldsymbol{J_{b_\perp}}) + \mathcal{W}_S \tag{33}$$

The interpretation of this equation is as follows: the change in time of the gyro-fluid polarization charge density $\partial_t\mathcal{W}$ (which is usually considered as "vorticity"), is being produced by the gyro-fluid polarization density stress due to $\boldsymbol{E}\times\boldsymbol{B}$ flow $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{w}\boldsymbol{u_E})$ (that contains in it the polarization charge density advection), by the gyro-fluid polarization charge density sources $\mathcal{W}_S$ (related with the sources of gyro-fluid density and pressure) and the divergence of the gyro-fluid currents.

2.4. Transformation to fluid quantities

We have derived the gyro-fluids currents conservation equation (33) in terms of N, $U_\|$, $P_\perp$ or $P_\|$, up to order δ^4. In order to compare to previous drift-fluid results a final step is to transform the gyro-centre quantities to fluid quantities. Following the transformation from gyro-fluid to fluid quantities for our model described in [29] and presented in equation 34:

$$\|\xi\|_v = \|\zeta\| + \Delta_\perp\left(\frac{m\|\mu B\zeta\|}{2qB^2}\right) + \boldsymbol{\nabla}\cdot\left(\frac{m\|\zeta\|\nabla_\perp\phi}{B^2}\right) \tag{34}$$

where ξ is ζ transformed to particle coordinates and $\|\eta\|_v = \int d^3v\eta f$ the particle phase-space moment operator, making $\|\xi\|_v$ the fluid moment analogous to the gyro-fluid moment $\|\zeta\|$. Keeping the equation up to order δ^4, all this FLR corrections from the transformation can be ignored, as they make their terms become higher order than δ^4. This implies that within our ordering we can transform from gyro-fluid to fluid quantities easily, like $N\approx n$, $U_\|\approx u_\|$, $P_\perp\approx p_\perp, P_\|\approx p_\|$ and $Q_\perp\approx q_\perp$. This allows us to rewrite Eq. (33) as follows

$$\partial_t\Omega - \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{\omega}\boldsymbol{u_E}) = \boldsymbol{\nabla}\cdot(j_\|\hat{\boldsymbol{b}} + \boldsymbol{j_{curv}} + \boldsymbol{j_{b_\perp}}) + \Omega_S \tag{35}$$

with the definition of the polarization charge density Ω (usually called vorticity), the polarization density $\boldsymbol{\omega}$, the polarization charge density source Ω_S and the polarization current sources j_s (all of them with subscripts E or D if we want to describe the electric or diamagnetic component of each of them). We transform the rest of the gyro-fluid quantities from capital letters to lower-case letters to describe them with fluid quantities, but their names are still the same (without gyro-fluid in front of them).

$$\begin{aligned}\Omega :=& \boldsymbol{\nabla}\cdot\sum_s\left[\frac{mn}{B^2}\left(\frac{\nabla_\perp p_\perp}{nq}+\nabla_\perp\phi\right)\right]\\ =& -\boldsymbol{\nabla}\cdot\boldsymbol{\omega} = -\boldsymbol{\nabla}\cdot(\boldsymbol{\omega_D}+\boldsymbol{\omega_E}) = \Omega_D+\Omega_E\end{aligned} \tag{36}$$

$$\begin{aligned}\Omega_S :=& \boldsymbol{\nabla}\cdot\sum_s\left[\frac{m}{B^2}\left(\frac{\nabla_\perp S_p}{q}+S_n\nabla_\perp\phi\right)\right]\\ =& \boldsymbol{\nabla}\cdot\boldsymbol{j_S} = \boldsymbol{\nabla}\cdot(\boldsymbol{j_{S,D}}+\boldsymbol{j_{S,E}}) = \Omega_{S,D}+\Omega_{S,E}\end{aligned} \tag{37}$$

$$j_\parallel := \sum_s qnu_\parallel \tag{38}$$

$$\boldsymbol{m^{em}} := \sum_s \frac{m}{qB^2}\boldsymbol{\nabla}_\perp\left(q_\perp+u_\parallel p_\perp\right) \tag{39}$$

$$j_{mag} := \boldsymbol{\nabla}\cdot(\boldsymbol{m^{em}}/2) \tag{40}$$

$$\boldsymbol{j_{b_\perp}} := -(j_\parallel+j_{mag})\boldsymbol{b}_\perp+\boldsymbol{\nabla}\cdot(\boldsymbol{m^{em}}\boldsymbol{b}_\perp) \tag{41}$$

$$\begin{aligned}\boldsymbol{j_{curv}} :=& \left[\sum_s(p_\parallel+mnu_\parallel^2)+j_\parallel A_{1,\parallel}\right]\frac{\nabla\times\hat{\boldsymbol{b}}}{B}\\ &+\sum_s p_\perp\frac{\hat{\boldsymbol{b}}\times\nabla\ln B}{B}\end{aligned} \tag{42}$$

Eq. (35) is our final result and forms the basis of the following discussion.

It is also possible to describe Eq. (35) only in terms of the time change of the electric component of the polarization charge density Ω_E in the following form:

$$\begin{aligned}&\partial_t\Omega_E-\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{\omega_E}\boldsymbol{u_E}) = \boldsymbol{\nabla}\cdot(j_\parallel\hat{\boldsymbol{b}}+\boldsymbol{j_{b_\perp}}+\boldsymbol{j_{curv}})\\ &+\Omega_{S,E}+\boldsymbol{\nabla}\cdot\Bigg[\boldsymbol{\nabla}\cdot(\boldsymbol{\omega_D}\boldsymbol{u_E})\\ &-\sum_s\nabla_\perp\left(\boldsymbol{\nabla}\cdot\left(\frac{mp_\perp}{qB^2}\boldsymbol{u_E}-\frac{m(q_\perp+p_\perp u_\parallel)}{qB}\boldsymbol{b}_\perp\right)\right)\Bigg]\end{aligned} \tag{43}$$

This equation would be interesting as we would only have the time change of the component proportional to the gradient of the electric field. Unfortunately, not keeping $\partial_t\Omega_D$ and $\Omega_{S,D}$ makes the term in equation (25) to not cancel with the last term in (24), but add up to give the last term in (43). This expression, although correct, is harder to interpret and less clear than the total currents conservation equation (35). For the remainder of this work we will thus focus on equation (35).

2.5. Interpretation of the currents conservation equation as vorticity

Vorticity is defined as the curl of a velocity field $\nabla\times\boldsymbol{v}$. This makes vorticity a vectorial quantity. However, the polarization charge density Ω is a scalar and often called "vorticity", because it can be related under our ordering to the parallel projection of the curl of the addition of diamagnetic and $\boldsymbol{E}\times\boldsymbol{B}$ density flows of the plasma species:

$$\begin{aligned}\Omega =& \sum_s\boldsymbol{\nabla}\cdot\left(\frac{mn}{B}(\boldsymbol{u_E}+\boldsymbol{u_D})\times\hat{\boldsymbol{b}}\right)\\ \approx& \sum_s\frac{m}{B}\hat{\boldsymbol{b}}\cdot\nabla\times\Big(n(\boldsymbol{u_E}+\boldsymbol{u_D})\Big)\end{aligned} \tag{44}$$

$$\boldsymbol{u_D} := \frac{\hat{\boldsymbol{b}}\times\nabla p_\perp}{qnB} \tag{45}$$

This definition of Ω still contains the density inside of the curl, making it in any case proportional to the parallel projection of the $\boldsymbol{E}\times\boldsymbol{B}$ (electric) and diamagnetic vorticity *density*, and not simply the vorticity of these velocities.

The approximation in equation 44 implies ignoring the term proportional to $\nabla\times\hat{\boldsymbol{b}}$ from taking $\hat{\boldsymbol{b}}$ out of the curl. This term is of higher order than δ^3 (in the equation would be higher than δ^4 as it also has a time derivative) and would be ignored in our equation. This same reason can be applied to moving B from inside to outside of the curl, which leaves the approximation in (44) as correct. One must remember that this last approximation does not imply that B is constant, but simply that its gradient is small compared with the terms that we keep. This is important to consider, as many models (we will see some in section 3) only keep the magnetic field dependence on $\boldsymbol{u}_E$ and $\boldsymbol{u}_D$ and assume this magnetic field out of the curl as constant B_0, dividing the whole term by the gyrofrequency $\omega_i = q_iB_0/m_i$. This makes a difference in the currents conservation equation as the same change in time of electric and diamagnetic vorticity density in the high field side (HFS) and the low field side (LFS) will produce a larger change of polarization charge density in the LFS than in the HFS. Here we find the first difference between the concept of polarization charge density and the usually called "vorticity".

Secondly, the vorticity equation is usually obtained from applying the curl over a velocity equation. As our polarization charge density Ω can be related with the parallel projection of $\boldsymbol{E}\times\boldsymbol{B}$ and diamagnetic vorticity density, one could test if the currents conservation equation can be obtained analogously from applying the curl over a velocity equation. For this to be possible, we need to describe every term in the currents conservation equation (35) as $\boldsymbol{\nabla}\cdot(mn\hat{\boldsymbol{b}}\times\boldsymbol{v}/B)$, being $\boldsymbol{v}$ any vectorial quantity, in the same way that the "vorticity" is described in terms of velocities in (44).

This description is possible for the polarization density stress term, resulting in a fluid stress term $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{\omega}\boldsymbol{u_E}) = \boldsymbol{\nabla}\cdot(nm\hat{\boldsymbol{b}}\times(\boldsymbol{u_E}+\boldsymbol{u_D})\boldsymbol{u_E}/B)$. This term represents the advection and Reynolds stress in the flow equation. The description is also possible for the curvature current $\boldsymbol{j_{curv}}$ and the magnetic fluctuation current $\boldsymbol{j_{b_\perp}}$, as these terms are perpendicular to $\hat{\boldsymbol{b}}$. They would become the external forces applied to the fluid flow (working in isotropic assumption $p_\perp = p_\|$, part of the curvature current recovers the pressure gradient force $-\nabla p$ from fluid equations (46)).

$$\boldsymbol{\nabla}\cdot\left[p_\|\frac{\nabla\times\hat{\boldsymbol{b}}}{B}+p_\perp\frac{\hat{\boldsymbol{b}}\times\nabla\ln B}{B}\right]\approx-\boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{b}}\times\nabla p}{B}\right) \tag{46}$$

Ω_S can also be described in this way as it is also defined in perpendicular direction with $\nabla_\perp$. This term would become the sources of $\boldsymbol{E}\times\boldsymbol{B}$ and diamagnetic momentum due to external sources of density and pressure, as discussed in [29].

Unfortunately, the divergence of the parallel current term $\boldsymbol{\nabla}\cdot(j_\|\hat{\boldsymbol{b}})$ cannot be described as the divergence of a quantity perpendicular to the magnetic field, and therefore, it cannot arise from the parallel projection of the curl of the flow equation. This is a second reason that points out that the "vorticity" and the "vorticity equation" are not truly that. If we work in regimes where the effect of parallel currents is neglected (as in 2D slab geometry), this term would not appear and therefore, a flow interpretation of the equation would be reasonable.

Thirdly, one can observe that vorticity has units of $[\nabla\times\boldsymbol{v}]=s^{-1}$, while the polarization charge density has units of Cm^{-3}. These units are coherent with the derivation of the conservation of currents as it comes from quasineutrality. This gives us a third reason to do not call Ω a "vorticity".

These three facts lead to the conclusion that the "vorticity equation" is not a vorticity equation, although under our ordering, the equation describes the dynamical behaviour of a term related with the parallel projection of the vorticity density of $\boldsymbol{E}\times\boldsymbol{B}$ and diamagnetic flows. It would be preferable to describe this equation as conservation of currents, as has been done through the paper, considering that the units of the complete equation are $Cm^{-3}s^{-1}$, and to call the "vorticity" Ω as the polarization charge density, as it has units of Cm^{-3} and it comes from the divergence of the polarization density. This nomenclature gives a more correct description of what the equation describes.

2.6. Physical interpretation

Obviously, the currents conservation equation is in the end an equation that expresses the conservation of charge and implies that it is impossible that in any point we have charge entering and not leaving (which is why the equation can be expressed as a divergence equal to 0 $\boldsymbol{\nabla}\cdot\boldsymbol{j}=0$). This is why we have on the left hand side of equation 35 the time change of the polarization charge density $\partial_t\Omega$ and the polarization density stress term $-\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{\omega u_E})$, that represents the total change in time of this charge density $d\Omega/dt$, or what is the same, the flux of polarization current $\boldsymbol{\nabla}\cdot\boldsymbol{j_{pol}}$, and that must balance the flux of currents of the right hand side of equation 35 and the flux of polarization current sources $\Omega_S=\boldsymbol{\nabla}\cdot\boldsymbol{j_S}$, to ensure that no charges stuck in any point. This reaction of the plasma to the charge accumulation is what can be understood as polarization, and is reasonable that can produce a change in the electric field and pressure gradients.

For a more detailed physical interpretation of each term in the currents conservation equation (35), we make two more assumptions: we assume only one ion species and that the electron mass is negligible compared with the ion mass ($m_e/m_i\ll 1$). The first assumption is not correct for many experiments, and one must remember that in case of multiple species, the interpretations discussed in this subsection will be more complex due to the mixture and influenced by all isotope species dynamics. The second assumption is a more realistic one that is usually done in many physical reduced models. These assumptions simplify the addition over species of terms like the polarization charge density Ω, the polarization density $\boldsymbol{\omega}$ or the electromagnetic magnetization density $\boldsymbol{m^{em}}$ to only be represented by the single ion species present.

2.6.1. Time derivative of the polarization charge density ($\partial_t\Omega$)

On the left hand side of (35), the first term is the time change of the polarization charge density Ω (36). The definition of Ω allows to compute the electric potential ϕ with this equation as explained in the introduction. The definition of Ω also allows to interpret it as a quantity proportional to the shear of $\boldsymbol{E}\times\boldsymbol{B}$ and diamagnetic flows. This relation is easily seen in a slab geometry (x,y) with a magnetic field $\boldsymbol{B}=B\hat{\boldsymbol{z}}$ and the perpendicular plane defined with $\hat{x}$ direction as it would be the radial direction $\hat{\boldsymbol{r}}$ in a toroidal geometry and the $\hat{y}$ direction as it would be the poloidal direction in toroidal geometry. With this description, we simplified the geometry factors of the gradients and the polarized charge density term can be described in a simpler way:

$$\begin{aligned}\Omega = &-\partial_x\left[\frac{m_i}{B}n_i(u_{E,y}+u_{D,y})\right]\\ &+\partial_y\left[\frac{m_i}{B}n_i(u_{E,x}+u_{D,x})\right]\end{aligned} \tag{47}$$

The first term of the right hand side of Eq. (47) is the shear of the $\boldsymbol{E}\times\boldsymbol{B}$ and diamagnetic poloidal flow momentum (divided by B), which is usually interpreted as the turbulence decorrelation mechanism [3]. The second term would be the poloidal change of the radial

$\boldsymbol{E}\times\boldsymbol{B}$ flow, which is basically related with the poloidal distribution of radial transport, which is also crucial for transport research. Under the assumption of no radial transport used in experimental flow analysis [33], the second term is ignored, making the polarized charge density only proportional to the shear of perpendicular velocity.

2.6.2. Polarization density stress $(-\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{\omega u_E}))$

The second term in (35) is the polarization density stress term $-\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{\omega u_E})$. This term keeps a minus as it corresponds for the definition of polarization charge density in Eq. (36). This term contains partially the advection of polarization charge density $\boldsymbol{\nabla}\cdot(\Omega u_E)$, that represents that polarization charge density can be transferred and moved by the $\boldsymbol{E}\times\boldsymbol{B}$ flow to other regions. So if, for example, the polarization density is increased in the outboard midplane, it will be transported by the $\boldsymbol{E}\times\boldsymbol{B}$ flow to other parts of the poloidal section of the tokamak. However, the terms corresponding to $-\boldsymbol{\nabla}\cdot(\boldsymbol{\omega}\cdot\nabla\boldsymbol{u_E})$ are physically more complex to interpret, but are an integral part of the polarization density stress $\boldsymbol{\omega u_E}$, and will require a more detailed analysis with simulations to clarify their relevance in the currents conservation equation. As we work in a full-F model, one must notice that this stress contains both the mean component of velocity and polarization density as well as the fluctuations related with turbulence. As discussed in [27, 29], the polarization density term $\boldsymbol{\omega}$ contains density fluctuations and a Favre decomposition is preferable over a classical Reynolds decomposition to analyze the fluctuations effects. However, the Favre decomposition presented in [29] works with flux surface averages, which would imply losing the local dependence that this equation presents. In this case, a Favre decomposition in time similar to the one presented in [34] is preferable. With time windows on a time scale much larger than the gyro-frequency of particles, $\Delta t >> \omega_i$, we can define the following time average and temporal Favre decomposition (with f any quantity and n as density):

$$\overline{f}(t) = \frac{1}{\Delta t}\int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} f(t')dt' \tag{48}$$

$$f(t) := \overline{f}(t) + f'(t) \qquad \overline{f'(t)} = 0 \tag{49}$$

$$f(t) := \widetilde{f}(t) + f''(t) \quad \widetilde{f}(t) := \frac{\overline{nf(t)}}{\overline{n}} \quad \widetilde{f''}(t) = 0 \tag{50}$$

With $\tilde{f}$ and $\bar{f}$ the Favre and the time average of f, respectively. With these definitions, the time average of the polarization density can be described as $\bar{\boldsymbol{\omega}} = \overline{n\boldsymbol{\omega_*}} = \bar{n}(\widetilde{\boldsymbol{\omega_*}} + \boldsymbol{\omega_*''})$ with $\boldsymbol{\omega_*} = \boldsymbol{\omega}/n$. The same can be applied to the polarization density stress $\overline{\boldsymbol{\omega u_E}} = \overline{n}(\widetilde{\boldsymbol{\omega_*}}\widetilde{\boldsymbol{u_E}} + \widetilde{\boldsymbol{\omega_*'' u_E''}})$. The first term represents the mean component of the polarization density stress and the second term represents the effects that the fluctuating polarization density and $\boldsymbol{E}\times\boldsymbol{B}$ velocity can produce on the time averaged quantities.

2.6.3. Divergence of parallel, curvature and magnetic fluctuation currents $(\boldsymbol{\nabla}\cdot(j_\parallel\hat{\boldsymbol{b}}),\ \boldsymbol{\nabla}\cdot\boldsymbol{j_{curv}},\ \boldsymbol{\nabla}\cdot\boldsymbol{j_{b_\perp}})$

On the right hand side of equation 35 we have the divergence of the currents. All of them can work as sources and sinks of charge density and therefore, influence the polarization charge density, and it is important to take all of them into consideration as possible mechanisms of increasing or reducing the shear of flows, as seen at the beginning of this subsection.

The first term is the divergence of the parallel current term $\boldsymbol{\nabla}\cdot(j_\parallel\hat{\boldsymbol{b}})$. One must notice that this current, as it is in the derivation, does not contain the equilibrium currents for the creation of the magnetic field, but the parallel currents produced by the plasma flows. This term under our ordering has two components: the parallel gradient of the parallel current $(\nabla_\parallel j_\parallel)$ and a curvature effect $(j_\parallel\boldsymbol{\nabla}\cdot\hat{\boldsymbol{b}})$. Which term is the dominant one is something that will depend on the plasma conditions, but changing the magnetic configuration can modify the second component, and therefore, keeping the whole description of this term is important to capture the effect of different magnetic configurations in the equation. Usually, the whole parallel current term $(\boldsymbol{\nabla}\cdot(j_\parallel\hat{\boldsymbol{b}}))$ is considered one of the dominant terms in the equation and is balanced with the curvature current [21, 35].

The second current term is the divergence curvature term $\boldsymbol{\nabla}\cdot\boldsymbol{j_{curv}}$. This term has two components with the fluid definitions (42): the ∇B drift current (proportional to $p_\perp$) and the parallel pressure curvature current, which have a component that arises from the magnetic field in the gyro-kinetic description with the canonical parallel momentum $(p_\parallel + mnu_\parallel^2)$ (10) and one that accounts for the magnetic fluctuation effect $(j_\parallel A_{1,\parallel}\nabla\times\hat{\boldsymbol{b}}/B)$. The reader should notice that under an isotropic assumption $(p_\perp = p_\parallel = p)$, the ∇B drift current and the parallel pressure curvature current component add up to the classic diamagnetic drift in fluid models. All of these terms are described in $\boldsymbol{j_{curv}}$ as they are all related to the curvature, and therefore, they will be very dependent on the magnetic configuration. The relevance of each term for the dynamics will be addressed in future works.

The third current term is the divergence of the magnetic fluctuation current $\boldsymbol{\nabla}\cdot\boldsymbol{j_{b_\perp}}$. This term has all of the magnetic fluctuations effects in the currents conservation equation except for the term proportional to $A_{1,\parallel}$ that appears in the curvature current. It is composed of two terms (30): The first one corresponds to the component of the parallel and the magnetic current that work on the fluctuating field direction $-(j_{\parallel}+j_{mag})\boldsymbol{b}_{\perp}$ (which comes from using the relation between currents and $\Delta_{\perp}A_{1,\parallel}/\mu_0$ (7)) and will transform into the Maxwell Stress under flux surface average [29] and the second one is a correction due to the magnetization $\boldsymbol{\nabla}\cdot(\boldsymbol{m^{em}b}_{\perp})$.

2.6.4. Polarization charge density sources (Ω_S)

The last term in equation 35 is the polarization charge density source Ω_S, which includes external sources of density S_n and pressure $S_{p_\perp}$ (heating). The form of the source terms is noteworthy and surprising because it contradicts the form of the source term in drift-fluid models given in for example [35]. Several recent publications independently reach these source terms in gyro-fluid/-kinetic models [28, 29, 36] strengthening this result. Reference [28] neglects the density source and highlights that a heat source can generate vorticity. Our own work [29] points out that both density and pressure source generate angular momentum and that a density source on the high-field side of the tokamak is an effective source of zonal flow energy (and thus poloidal rotation). The term Ω_S has the form of the polarization charge density Ω but with density source instead of density in the electric component and with perpendicular pressure source gradient instead of perpendicular pressure gradient in the diamagnetic one. The polarization can be interpreted as an effect directly linked with the change in the gyro-center of motion of electrons and ions produced by drifts [37]. This makes reasonable that both Ω (36) and Ω_S (37) have a mass dependence in their definitions. With this interpretation, it makes sense that the sources of particles S_n produce a source of polarization charge density, as every atom ionized (source of electrons and ions), with the electric field $\nabla_\perp\phi$, will suffer a polarization shift in their gyro-motion compared with the absence of electric field (that will mainly affect the ion and not the electron, as its gyro-radius is much larger). Following the momentum interpretation of this term from [29], if there is an ionization or an ion or electron appears in the magnetized plasma with an electric field, it instantaneously gains the $\boldsymbol{E}\times\boldsymbol{B}$ velocity and momentum, independently of the previous neutral particle momentum, that would transform into gyro-motion rotation $v_\perp$ and parallel momentum $v_\parallel$, but not perpendicular momentum of the species, produced by the drifts. This might seem as creation/destruction of momentum from nowhere, but the reader must consider that this momentum appears due to the drift thanks to the magnetic field, so a reaction would appear in the source of the magnetic field.

The term proportional to the gradient of perpendicular pressure source $\nabla_\perp S_{p_\perp}$ represents a source of polarization charge density with the gradient of heating. With the orbit interpretation, if we would have a constant heating, the gyro-radius would increase constantly in a rotation, and therefore, would follow a spiral trajectory with the gyro-center in the same point, producing neither polarization nor drift. However, if the heating is not uniform on the gyration, the perpendicular velocity $v_\perp$ will increase differently in the different points of the gyro-orbit, increasing in different rates the gyroradius, producing a drift perpendicular to the heating gradient and the magnetic field, that can be interpreted as a source of momentum from [29], and a shift of the gyro-centre in the direction of the heating gradient.

These sources terms come from a particle source distribution function $S(\boldsymbol{Z},t)$ (3) and through it some neoclassical effects can be added, like collisions. One can assume neoclassical collision as sources of heating $S_{p_\perp}$ at the same time that close to the last closed flux surface (LCFS), a collisional ion orbit loss mechanism could be included as a sink of ions, linking these loses to flows and currents [38, 39, 40, 41]. These reasoning of polarization charge density and drifts present the direct relation between these two concepts, probing the capacity of the polarization charge density of describing also the drifts and flows of the plasma.

2.6.5. Steady state analysis

If we look at a steady state situation with Eq. (35), we should work with the time average of the equation. In steady state, the time derivative ∂_t must be 0. This leaves the time average of equation (35) in steady state as:

$$-\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\overline{n}(\widetilde{\boldsymbol{\omega}_*}\widetilde{\boldsymbol{u}_{\boldsymbol{E}}}+\widetilde{\boldsymbol{\omega}_*''\boldsymbol{u}_{\boldsymbol{E}}''})) = \boldsymbol{\nabla}\cdot(\overline{j_\parallel\hat{\boldsymbol{b}}}+\overline{\boldsymbol{j}_{\boldsymbol{curv}}}+\overline{\boldsymbol{j}_{\boldsymbol{b}_\perp}})+\overline{\Omega_S} \tag{51}$$

This equation tells us that, in a steady state, there must be a balance between the currents, the polarization charge density source and the mean and fluctuating parts of the polarization density stress. The simulation and analysis of this equation will be the topic of future research, but one can conclude that in steady state, when all these terms are balanced, increasing or decreasing any of them (like increasing the parallel current) would unbalance the equation and produce a change in time in the polarization charge density, or what is related, in the shear of $\boldsymbol{E}\times\boldsymbol{B}$ and diamagnetic flow, leaving the steady state until a new one is achieved.

At the same time, from this equation, one can understand that the local electric field in a steady state plasma is the one necessary in the polarization tensor term to compensate the unbalance of the RHS currents. This is still unfortunate for the experimental measurements, as many of these terms are hard to measure locally, and even with very simplified assumptions (like mainly poloidal $\boldsymbol{E} \times \boldsymbol{B}$ flow) require poloidal gradients, which imply measurements in different poloidal positions.

2.7. Experimental effects on edge dynamics and its relation with the currents conservation equation

One of the most common approaches to L-H transition power threshold prediction is with empirical laws [42] where, taking all machines' data for the transitions, a relation of the power threshold with the machines' global operation parameters (major radius, magnetic field, electron density...) is obtained. This kind of prediction might be accurate under certain restrictions, but does not rely on any physical principle nor holds with more specific experimental dependencies for the transition (like isotopic dependence [43, 44, 45], magnetic field parameters like q95 [45, 46, 47], X-point height [15, 48, 49, 50], triangularity [16, 47], ∇B drift direction [14], RMP [13, 51] and more).

Even though the ordering applied to the derivation might not be applicable to the H-mode pedestal pressure gradients, some of these dependencies are related with specific terms in equation (35). One example can be resonant magnetic perturbations (RMP). In the reference [13], the effects that RMP have on the L-H transition power threshold is interpreted as an effect due to the excitation of magnetic islands in the region. With the currents conservation equation, these magnetic perturbations could be described by $\boldsymbol{j_{b_\perp}}$, changing the balance of currents and therefore, producing a change in the shear of flows in the edge. The effects produced by the change in the magnetic field shape on the L-H transition threshold and on the dynamics of the radial electric field in the edge could be accounted with the curvature current $\boldsymbol{j_{curv}}$ and the parallel current $j_\parallel$ for the Pfirsch-Schlüter current. For example, when the magnetic field shape is changed from Lower Single Null (LSN) to Upper Single Null (USN), it is the same as changing the direction of ∇B drift from towards the X-point to away from the X-point. This changes the direction of part of the curvature current $\boldsymbol{J_{curv}}$, producing a change in the equilibrium of currents. A similar effect can happen if the triangularity of the plasma is changed, changing the curvature current $\boldsymbol{J_{curv}}$ and therefore, affecting the steady state.

Following this reasoning and experimental indications, one could produce changes in the shear of flows in the edge producing changes in the terms of the equilibrium equation (51). For example, invoking a change in the polarization fluctuating tensor term $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot (\widetilde{\overline{n}\boldsymbol{\omega}_*''\boldsymbol{u}_E''})$ can be complex, but it could be addressed by some eddy tilting technique. Then, the parallel current can be modified in the edge like in D-IIID with off-axis ECCD driven current [52] or with other mechanisms like Lower Hybrid Current Drive [53]. The curvature currents can be edited changing the magnetic shape, but its effect would be hard to predict, as the parallel current is the one that connects the top and bottom parts of the plasma shape, which are the regions where the magnetic configuration changes are larger. The effect of the magnetic field perturbations is also hard to predict, but experimentally would be possible to test with RMP techniques. Finally, it would be possible to produce changes in the shear with the source terms, which could be experimentally reproduced with pellets, gas puffing or heating gradient, preferably ICRH (as the mass dependence point to the ions as main drivers of polarization charge density source and it is preferable to avoid counting on effects like collisions between electrons and ions, that are not strictly included in the model).

Even though many of theses perturbations in the edge also affect the equilibrium of the plasma, which is not described under our model, simulations are required to test the magnitude of each of the terms of our equation in steady state and predict which perturbations are most relevant for the dynamics. At the same time, it will be interesting to test how the equilibrium of currents changes between different plasma scenarios like different magnetic configurations.

3. Comparison with other models

During this section, we will get back to the common nomenclature of "vorticity" to compare with other models that use this terminology, but the reader must keep in mind that the terminology used in the rest of the paper is more correct and should be implemented for physical consistency.

3.1. 2D gyro-fluid vorticity equations

Gyro-fluid models result from taking moments of gyrokinetic equations and are a common approach to study the edge [54, 55] and the scrape-off layer (SOL) [24, 25, 26, 56, 57] of magnetized plasmas, including dynamics like zonal flows [27, 58] or transport barriers [59, 60]. The simulations of these models are interesting due to their reduced computational cost compared to gyrokinetic simulations and a simplicity comparable to drift-fluid models.

Vorticity equations have been derived for gyrofluid models in previous research [25, 26, 27, 60, 61], usually in 2D simplified magnetic fields and in the LWL limit. Those equations in the references are described by equation (35) applying the simplifications of those models. If we remove magnetic fluctuations ($A_{1,\parallel} = 0$, $\boldsymbol{\nabla} \cdot \boldsymbol{j_{b_\perp}} = 0$), discard parallel dynamics with

2D slab geometry ($\boldsymbol{\nabla}\cdot(j_\parallel\hat{b}) = m_s n_s u_{\parallel,s}^2 \nabla\times\hat{\boldsymbol{b}}/B = 0$), neglect components of the advective term that were small in the 2D simulations in [26] ($\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{\omega u_E}) \approx \boldsymbol{u_E}\cdot\boldsymbol{\nabla}\Omega + \boldsymbol{\nabla}_\perp\boldsymbol{\omega_D} : \boldsymbol{\nabla}_\perp\boldsymbol{u_E}$), ignore sources ($\Omega_S = 0$) and assume isotropic pressure ($p_\perp = p_\parallel = p$), we obtain the following equation:

$$\partial_t\Omega + \boldsymbol{u_E}\cdot\boldsymbol{\nabla}\Omega = \boldsymbol{\nabla}\cdot\left(\frac{\nabla(p_e+p_i)\times\hat{\boldsymbol{b}}}{B}\right) - \boldsymbol{\nabla}_\perp\boldsymbol{\omega_D} : \boldsymbol{\nabla}_\perp\boldsymbol{u_E} \tag{52}$$

This simplified equation (52) represents the same in our notation as equation (21) from [26], where the advection of the vorticity is defined inside of a total time derivative, the curvature effects on electrons and ions described by the diamagnetic drift, the extra component of the advective term $\boldsymbol{\nabla}_\perp\boldsymbol{\omega_D} : \boldsymbol{\nabla}_\perp\boldsymbol{u_E}$ and then a diffusive term added by hand. This reference adopts the bracket notation of Ref.[62] as a placeholder for the advection term, where $[f,g] = \partial_x f\partial_y g - \partial_x y\partial_y g$ with f and g are general scalar functions. However, we do not adopt this notation here since the bracket notation is not well defined for vectorial quantities like $\nabla_\perp\phi$.

3.2. Flux surface average on vorticity equation

Upon volume integration of equation (35) we obtain an equation of the flux surface average (FSA) toroidal momentum, as it appears in [29]. In our model, and using the definitions of flux surface average and magnetic coordinates from the reference, we would obtain:

$$\begin{aligned}&\partial_t\langle(\boldsymbol{B}\times\boldsymbol{\omega})_\varphi\rangle + \partial_v\langle(\boldsymbol{B}\times\boldsymbol{\omega})_\varphi u_E^v\rangle = \\ &\langle(\boldsymbol{B}\times(\boldsymbol{j_{b_\perp}} + \boldsymbol{j_{curv}}))_\varphi\rangle + \langle(\boldsymbol{B}\times\boldsymbol{\omega_S})_\varphi\rangle\end{aligned} \tag{53}$$

The subindex φ corresponds to the covariant toroidal component. This equation represents equation 126 from [29]. The main difference between both equations is that in the reference, the definition used for $\boldsymbol{j_{b_\perp}}$ is the second identity of (30), using $\Delta_\perp A_{1,\parallel}/\mu_0$ and giving rise to the Maxwell stress. $(\boldsymbol{B}\times\boldsymbol{\omega})_\varphi$ represents the toroidal $\boldsymbol{E}\times\boldsymbol{B}$ and diamagnetic angular momentum, while the next term is its advection by the electric field. The effect of the currents (except for the parallel current that vanishes under the FSA) is interpreted as a Lorentz force density and the term $(\boldsymbol{B}\times\boldsymbol{\omega_S})_\varphi$ is discussed in depth in the reference as the mechanism of creating momentum from heating and density sources. The fact that the parallel current term vanishes under flux surface average opens the possibility of parallel current affecting locally to the shear of flows in the flux surface that vanishes under the flux surface average, but that can produce poloidal asymmetries inside a flux surface. This will be tested in future research.

3.3. SOLEDGE model

SOLEDGE is a drift-fluid model that can be found in [22] and references therein. A vorticity equation is presented (54) in terms of currents as (35), with a vorticity definition only differing from ours (36) due to viscosity and collision terms, that we do not consider in this work.

$$\partial_t\Omega + \boldsymbol{\nabla}\cdot\boldsymbol{F_\Omega} = \boldsymbol{\nabla}\cdot(j_\parallel\hat{\boldsymbol{b}} + \boldsymbol{j_d} + \boldsymbol{j_c} + \boldsymbol{j_{col}}) + \boldsymbol{\nabla}\cdot(\xi\nabla_\perp\Omega) \tag{54}$$

On their left hand side, they present the time derivative of the vorticity and the advection of vorticity, as we do. One important difference between the advective term $\boldsymbol{\nabla}\cdot\boldsymbol{F_\Omega}$ in (54) (Eq. (28) in [22]) and the advection $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(\boldsymbol{\omega u_E})$ is that the first one has the velocity component only formed by the fluctuations, while in (35) both components are present. On the right hand side they have the divergence of the parallel current, the diamagnetic $\boldsymbol{j_d} = \sum_s \hat{\boldsymbol{b}}\times\nabla p_\perp/B$ and the curvature current $\boldsymbol{j_c} = \sum_s(mnu_\parallel^2 + p_\parallel - p_\perp)\nabla\times\hat{\boldsymbol{b}}/B$ that correspond to $\boldsymbol{j_{curv}}$ with their assumption of $p = (p_\parallel + 2p_\perp)/3$ but without the fluctuating component proportional to $j_\parallel A_{1,\parallel}$, and a collisional current $\boldsymbol{j_{col}}$, which we do not have in our model. In addition, we have the extra current related to magnetic fluctuations, $\boldsymbol{j_{b_\perp}}$. The last term in their expression is an ad-hoc stabilization term for their simulations, with a classical diffusive constant ξ.

It is interesting to point out that this model describes the advective term of vorticity in the most similar way to (35), as the flux of vorticity $\boldsymbol{F_\Omega}$ is defined in terms of the divergence of a tensor. Other models often represent the advection terms like $\boldsymbol{\nabla}\cdot(\Omega\boldsymbol{u_E})$ or directly $\boldsymbol{\nabla}\Omega\cdot\boldsymbol{u_E}$ (expressed with our notation).

3.4. GDB model

The GDB model is a drift reduction of Braginskii equations found in [21]. Their vorticity equation is the following (Eq.(5) in [21]):

$$\begin{aligned}&\frac{\partial\Omega}{\partial t} + \nabla_\perp\cdot n\left[\phi, \nabla_\perp\phi + \alpha_d\frac{\nabla_\perp p_i}{n}\right] + \epsilon_G C(G) = \\ &= -C(p_e+p_i) + \nabla_\parallel j_\parallel\end{aligned} \tag{55}$$

Vorticity in this model is defined as $\Omega = \nabla_\perp\cdot n(\nabla_\perp\phi + \alpha_d\nabla_\perp p_e/n)$, with α_d a dimensionless constant. With this definition, the effects of the magnetic field magnitude in the vorticity term discussed in subsection 2.5 are neglected. Similar to the model presented in section 3.1, the convection term is described with the bracket operator on vectorial quantities although this time it is defined as $[f,g] = \hat{\boldsymbol{b}}\cdot(\nabla f\times\nabla g)$. The curvature effect is defined with the operator $C(f) = -\hat{\boldsymbol{b}}\times\hat{\kappa}\cdot\nabla f = \nabla f\cdot(\nabla\times\hat{\boldsymbol{b}}/B)$. It is used to define a curvature effect on magnetic pumping ($\epsilon_G C(G)$) that we miss but that it is concluded to be irrelevant for the dynamics, and on a pressure curvature term as $-C(p_e + p_i)$, that corresponds only to part of $\boldsymbol{j_{curv}}$ with isotropic assumption. There is also the parallel derivative of the current, that compared to $\boldsymbol{\nabla} \cdot (j_\parallel \hat{\boldsymbol{b}})$ misses the divergence $\boldsymbol{\nabla} \cdot \hat{\boldsymbol{b}}$. In their description, the magnetic field vector contains the electromagnetic contribution, which would be part of $\boldsymbol{j_{b_\perp}}$, but as it is included in the parallel gradient definition, its relevance cannot be split between the main magnetic field and the fluctuations.

3.5. GRILLIX model

The *GRILLIX* code model [23] is a global full-F drift-reduced Braginskii model that allows to study the plasma dynamics and turbulence in the edge and SOL with realistic magnetic field configurations like ASDEX-Upgrade. This model is similar to GDB, but it still differs from our model in some terms.

$$\begin{aligned} \boldsymbol{\nabla} \cdot \left[\frac{n}{B^2} \left(\frac{d}{dt} + u_\parallel \nabla_\parallel \right) \left(\nabla_\perp \varphi + \xi \frac{\nabla_\perp p_i}{n} \right) \right] = \\ - C(p_e + \xi p_i) + \boldsymbol{\nabla} \cdot (j_\parallel \hat{\boldsymbol{b}}) - \frac{\xi}{6} C(G) + \mathcal{D}_\Omega(\Omega) \end{aligned} \tag{56}$$

This model presents an advection inside of the total time derivative that only contains the component $\boldsymbol{u}_E \cdot \nabla \boldsymbol{\omega}$ at the same time it contains a parallel advective term, that we order out. The equation also differs from (35) in the definition of the vorticity, as in this equation, the density is outside of the time derivative. The model also contains the complete parallel current term and the pressure components of the curvature current (42) with isotropic assumption yielding Eq. (46), missing the curvature component proportional to $mnu_\parallel^2$ and $j_\parallel A_{1,\parallel}$. This time the curvature operator is defined as $C(f) = -\delta_0 (\nabla \times \frac{\boldsymbol{B}}{B^2}) \cdot \nabla f$, with the drift-scale dimensionless factor $\delta_0 = R_0/\rho_{s,0}$ relating the machine major radius R_0 and the species gyro-radius $\rho_{s,0}$ and the ion-electron temperature ratio $\xi = T_{i0}/T_{e0}$. The magnetic fluctuations are not implemented in this equation for the moment, while effects of ion viscous stress G and artificial dissipation $\mathcal{D}$ are present. Recently, a source effect in the vorticity equation has been added in simulations with neutral density [63]. However, our form of the source terms Eq. (37) results in different expressions for the plasma-neutral interaction terms, in particular, the heating source $\Omega_{S,D}$ is not present in [63].

4. Conclusions

We have obtained the often called "vorticity equation" in a full-F electromagnetic gyro-kinetic model in the long wavelength limit (35) without any assumptions on the magnetic field geometry, which implies that it can be applied to both tokamaks and stellarators. Comparison between the derived model and other fluid models [21, 22, 23] probe the consistency between fluid and gyro-fluid models under similar ordering, although some differences still remain, mostly on the definitions of the advection, certain components of the curvature effects that fluid models cannot describe and the effects that sources have on the "vorticity equation". Whether these differences are important for the dynamics will be the topic of future research. At the same time, we have improved previous "vorticity equations" in gyro-fluids [25, 26, 27, 60, 61], working in 3 dimensions and a generic magnetic field.

We have presented several reasons on why this equation should not be defined as a "vorticity equation", but the currents conservation equation. The equation describes the influence that parallel currents $j_\parallel \hat{\boldsymbol{b}}$, curvature currents $\boldsymbol{j}_{curv}$, magnetic field perturbation currents $\boldsymbol{j}_{b_\perp}$ and external density sources and gradient of heating Ω_S have in the dynamics of the polarization charge density ("vorticity") Ω, related with the shear of electric field and pressure gradients, under the model constraints. How these terms could explain experimental evidence on how RMP [13] or magnetic field configuration [9, 10, 11, 12] affect the edge electric field structure is discussed. Experiment proposals with the goal of changing the shear of the electric field in the edge are presented following the equation description. Experiments looking at the flow shear structure in the edge in different poloidal positions like in [33, 64, 65] could help to probe the dependence of the electric field shear on currents like Pfirsch-Schlüter or curvature currents. In addition, experimental techniques to measure the plasma current in the edge like the one presented in [66] could be extremely useful to correlate currents change with flows shear.

This new equation allows us to analyse in detail a steady state like it is done in [22] using the FELTOR gyro-fluid code [67]. In this steady-state, we will be able to look at the currents that play a fundamental role on the edge dynamics, to split effects between fluctuations and main parallel current in a realistic magnetic field configurations (upper single null, lower single null or double null), and to study the poloidal dependence of these currents and the electric field and pressure gradients, being able to analyze the effectiveness of the flux surface average quantities.

In conclusion, the "vorticity equation" is actually the currents conservation equation and the nomenclature related with polarization used in this paper is more correct to describe this equation. The equation, derived in this paper from first principles, presents the relevance of parallel current, curvature currents, currents associated with magnetic fluctuations and polarization charge density sources on the temporal dynamics of $\boldsymbol{E} \times \boldsymbol{B}$ and diamagnetic flow shear in the edge, which are key parameters to understand the formation of transport barriers and can address known experimental dependencies.

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