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RAP-modulated Fluid Processes: First Passages and the Stationary Distribution

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Abstract

We construct a stochastic fluid process with an underlying piecewise deterministic Markov process (PDMP) akin to the one used in the construction of the rational arrival process (RAP) in [3], which we call the RAP-modulated fluid process. As opposed to the classic Markov-modulated fluid process driven by a Markov jump process, the underlying PDMP of a RAP-modulated fluid process has a continuous state space and is driven by matrix parameters which may not be related to an intensity matrix. Through novel techniques we show how well-known formulae associated to the Markov-modulated fluid process, such as first passage probabilities and the stationary distribution of its queue, translate to its RAP-modulated counterpart.

*Keywords:*Stochastic fluid process, rational arrival process, matrix-exponential distribution, first passage probability, stationary distribution

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1 Introduction

A *Markov-modulated fluid process* $(\mathcal{X},\varphi) = \{(X_t,\varphi_t)\}_{t\geq 0}$ is a Markov additive process in which the background component φ is a Markov jump process with finite state space and the additive component $\mathcal{X}$ is piecewise linear, with a rate that depends on the background state:

$$X_t = \int_0^t r_{\varphi_s}\mathrm{d}s, \quad t\geq 0.$$

The study of steady-state aspects of Markov-modulated fluid processes goes back to [15, 1, 13], and its literature has been prolific from both applied and theoretical perspectives. Latouche and Nguyen [14] provide a comprehensive survey on the topic; much of the existing analysis exploits the probabilistic interpretation of the background process φ with finite state space.

A related concept is a *Markovian arrival process* $(\mathcal{N},\mathcal{J}) = \{(N_t,J_t)\}_{t\geq 0}$, where $\mathcal{N}$ is a counting process and $\mathcal{J}$ is the underlying Markov jump process with initial distribution $\boldsymbol{\alpha}$, hidden jumps according to an intensity matrix C, and observable jumps according to a nonnegative intensity matrix D. Jump times of the latter kind correspond to the arrival epochs of $\mathcal{N}$. Asmussen and Bladt [3] introduce *rational arrival processes* (RAP), a generalization of Markovian arrival processes for which $\boldsymbol{\alpha}$, C and D are not necessarily related to the parameters of a Markov jump process. They show that the arrivals associated to this algebraic generalization are determined by an underlying continuous-state-space piecewise-deterministic Markov process which here we refer to as *an orbit process*. This orbit process evolves deterministically between its jump times, which occur according to a given space-dependent intensity function. The arrival epochs of the RAP can then be regarded as the jump times of the orbit process, allowing for a probabilistic analysis of the former using the physical interpretation of the latter.

In the present paper, we consider an extension of Markov-modulated fluid processes, which we call *RAP-modulated fluid processes*. A RAP-modulated fluid process $(\mathcal{R}, \boldsymbol{\mathcal{A}}) = \{(R_t, \boldsymbol{A}_t)\}_{t\geq 0}$ is a Markov additive process in which the additive component $\mathcal{R}$ is still piecewise linear but the background component $\boldsymbol{\mathcal{A}}$ is now an orbit process. A similar generalization was introduced in [6], from *Quasi-Birth-Death processes*, where the additive component lives on the integers and is modulated by a Markovian jump process, to *QBD-RAP*, where the additive component still lives on the integers but is now modulated by a RAP. The class of RAP-modulated fluid processes we introduce here goes beyond Markov-modulated fluid processes; for instance, one may use a Markov renewal process with matrix-exponential times in order to model the orbit process of a RAP-modulated fluid process.

Additional to defining the class of RAP-modulated fluid processes, the contributions of this paper include providing first passage probabilities, an expression for the stationary distribution of its queue, and an algorithm for computing the expected value of the orbit process $\boldsymbol{\mathcal{A}}$ at the first downcrossing times of the level process to level 0. While the expressions and pathwise analysis that we perform are well-known in the Markov-modulated fluid process framework (e.g. those in [7] and [10]), our main contribution in this respect is the introduction of the properly general mathematical description along with novel techniques to deal with this generalization.

The structure of the paper is as follows. We define in Section 2 the simplest RAP-modulated fluid process, which we call a *simple RAP-modulated fluid process*. Although simplistic in its nature, this process helps us introduce the physical analysis and discuss necessary conditions of a RAP-modulated fluid process. In Section 3, we give a precise definition of the *RAP-modulated fluid process with unit rates*, whose associated level process has either slope 1 or -1; we present closed-form formulae for some important descriptors of this process in Section 4. Then, in Section 5 we define and analyze an extension called the *RAP-modulated fluid process with unit and zero rates*, where the level process is now allowed to have piecewise-constant intervals. Later on, we define the *RAP-modulated fluid process with general rates* in Section 6 and discuss techniques to recover formulae for its descriptors using the results from Section 5. We conclude by providing a summary in Section 7.

Our work demonstrates that several results of Markov-modulated fluid processes translate into the framework of RAP-modulated fluid processes, although different techniques are required to analyze the latter.

2 Simple RAP-modulated fluid process

In the following we introduce the simplest non-trivial example of a RAP-modulated fluid process, $(\mathcal{R}, \boldsymbol{\mathcal{A}}) = \{(\mathcal{R}_t, \boldsymbol{A}_t)\}_{t\geq 0}$, which we refer to as *a simple RAP-modulated fluid process*. To that end, first we define $\boldsymbol{\mathcal{A}} = \{\boldsymbol{A}_t\}_{t\geq 0}$, the *(simple) orbit process*.

The process $\boldsymbol{\mathcal{A}}$ is a càdlàg piecewise-deterministic Markov process (PDMP) with state space $\cup_{k\in\{+,-\}}\mathfrak{Z}^k$, where for each $k \in \{+,-\}$, $\mathfrak{Z}^k$ is a subset of the affine hyperplane $\{\boldsymbol{x} \in \mathbb{R}^{m^k} : \boldsymbol{x}\mathbf{1} = 1\}$ for some fixed $m^k \geq 1$. Here, the elements of $\mathfrak{Z}^k$ are regarded as row vectors and $\mathbf{1}$ is a column vector of ones of appropriate dimension. Thus, $\boldsymbol{\mathcal{A}}$ is a row-vector process of varying dimension, either m^+ or m^- depending on whether it is in $\mathfrak{Z}^+$ or $\mathfrak{Z}^-$ at the given instant. In general $m^+ \neq m^-$, but even in the case $m^+ = m^-$ the subsets $\mathfrak{Z}^+$ and $\mathfrak{Z}^-$ will be considered to belong to different spaces, and thus they will always be disjoint. We let the initial point $\boldsymbol{A}_0 \in \mathfrak{Z}^+ \cup \mathfrak{Z}^-$ be arbitrary but fixed.

Each PDMP is characterized by its motion between jumps, jump intensity and transition mechanism at its jump epochs [11], properties which we define for $\boldsymbol{\mathcal{A}}$ next. During an interval without jumps, say $[t, t+h)$ with $h > 0$, the orbit process $\boldsymbol{\mathcal{A}}$ evolves according to the system of ordinary differential equations (ODE) given by

$$\frac{\mathrm{d}\boldsymbol{A}_s}{\mathrm{d}s} = \begin{cases} \boldsymbol{A}_s C^+ - (\boldsymbol{A}_s C^+ \mathbf{1})\boldsymbol{A}_s & \text{for } \boldsymbol{A}_s \in \mathfrak{Z}^+, \quad s \in (t, t+h), \\ \boldsymbol{A}_s C^- - (\boldsymbol{A}_s C^- \mathbf{1})\boldsymbol{A}_s & \text{for } \boldsymbol{A}_s \in \mathfrak{Z}^-, \quad s \in (t, t+h), \end{cases} \tag{2.1}$$

for some $C^+ \in \mathbb{R}^{m^+ \times m^+}$ and $C^- \in \mathbb{R}^{m^- \times m^-}$. The solution to the ODE (2.1) is

$$\boldsymbol{A}_{t+r} = \begin{cases} \dfrac{\boldsymbol{A}_t e^{C^+ r}}{\boldsymbol{A}_t e^{C^+ r}\mathbf{1}} \in \mathfrak{Z}^+ & \text{if } \boldsymbol{A}_t \in \mathfrak{Z}^+, \quad r \in [0, h), \\ \dfrac{\boldsymbol{A}_t e^{C^- r}}{\boldsymbol{A}_t e^{C^- r}\mathbf{1}} \in \mathfrak{Z}^- & \text{if } \boldsymbol{A}_t \in \mathfrak{Z}^-, \quad r \in [0, h). \end{cases} \tag{2.2}$$

Note that in (2.2) we implicitly assume that, for each initial point in $\mathfrak{Z}^+ \cup \mathfrak{Z}^-$, the system of ODEs (2.1) evolves entirely within $\mathfrak{Z}^+ \cup \mathfrak{Z}^-$. As $\boldsymbol{\mathcal{A}}$ evolves within $\mathfrak{Z}^+ \cup \mathfrak{Z}^-$, jump epochs occur according to a location-dependent intensity function $\lambda : \mathfrak{Z}^+ \cup \mathfrak{Z}^- \mapsto \mathbb{R}_+$ given by

$$\lambda(\boldsymbol{A}_t) = \begin{cases} \boldsymbol{A}_t D^{+-}\mathbf{1} & \text{if } \boldsymbol{A}_t \in \mathfrak{Z}^+, \\ \boldsymbol{A}_t D^{-+}\mathbf{1} & \text{if } \boldsymbol{A}_t \in \mathfrak{Z}^-, \end{cases} \tag{2.3}$$

for some $D^{+-} \in \mathbb{R}^{m^+ \times m^-}$ and $D^{-+} \in \mathbb{R}^{m^- \times m^+}$. Since $\lambda(\cdot)$ is assumed to be a nonnegative function, it indeed corresponds to a valid intensity function.

Condition 2.1. $C^+\mathbf{1} + D^{+-}\mathbf{1} = \mathbf{0}$ *and* $C^-\mathbf{1} + D^{-+}\mathbf{1} = \mathbf{0}$.

This condition implies that λ can alternatively be written as

$$\lambda(\boldsymbol{A}_t) = \begin{cases} -\boldsymbol{A}_t C^+\mathbf{1} & \text{if } \boldsymbol{A}_t \in \mathfrak{Z}^+, \\ -\boldsymbol{A}_t C^-\mathbf{1} & \text{if } \boldsymbol{A}_t \in \mathfrak{Z}^-. \end{cases} \tag{2.4}$$

Using (2.2) and (2.4) it can be readily verified that

$$\mathbb{P}\left(\boldsymbol{\mathcal{A}} \text{ has no jumps in } [t, t+h] \mid \boldsymbol{A}_t\right) = \begin{cases} \boldsymbol{A}_t e^{C^+ h}\mathbf{1} & \text{if } \boldsymbol{A}_t \in \mathfrak{Z}^+, \\ \boldsymbol{A}_t e^{C^- h}\mathbf{1} & \text{if } \boldsymbol{A}_t \in \mathfrak{Z}^-. \end{cases} \tag{2.5}$$

Indeed, by [11], the function $F(h) := \mathbb{P}\left(\boldsymbol{\mathcal{A}} \text{ has no jumps in } [t, t+h] \mid \boldsymbol{A}_t = \boldsymbol{\alpha}\right)$ with $\boldsymbol{\alpha} \in \mathfrak{Z}^k$ corresponds to the unique differentiable solution of

$$\log F(h) = -\int_0^h \lambda\left(\frac{\boldsymbol{\alpha} e^{C^k r}}{\boldsymbol{\alpha} e^{C^k r}\mathbf{1}}\right)\mathrm{d}r = \int_0^h \left[\frac{\boldsymbol{\alpha} e^{C^k r}}{\boldsymbol{\alpha} e^{C^k r}\mathbf{1}}\right] C^k\mathbf{1}\mathrm{d}r. \tag{2.6}$$

That $F(h) = \boldsymbol{\alpha} e^{C^k h}\mathbf{1}$ is a solution follows by differentiating both sides of (2.6).

Finally, given that a jump occurs at time t, the orbit process $\boldsymbol{\mathcal{A}}$ will directly jump to

$$\frac{\boldsymbol{A}_{t-}D^{+-}}{\boldsymbol{A}_{t-}D^{+-}\mathbf{1}} \in \mathfrak{Z}^{-} \text{ if } \boldsymbol{A}_{t-} \in \mathfrak{Z}^{+}, \quad \text{and} \quad \frac{\boldsymbol{A}_{t-}D^{-+}}{\boldsymbol{A}_{t-}D^{-+}\mathbf{1}} \in \mathfrak{Z}^{+} \text{ if } \boldsymbol{A}_{t-} \in \mathfrak{Z}^{-}, \tag{2.7}$$

so that a jump originating from $\boldsymbol{x} \in \mathfrak{Z}^{+}$ will land at a deterministic point in $\mathfrak{Z}^{-}$, and vice versa.

Note that the motion between jumps and jump behaviour of the PDMP $\boldsymbol{\mathcal{A}}$ depends only on the matrices C^{+}, C^{-}, D^{+-} and D^{-+}. Moreover, these matrices implicitly dictate some requirements on the state space $\mathfrak{Z}^{+} \cup \mathfrak{Z}^{-}$ through Equations (2.2), (2.3) and (2.7). Verifying if a set of matrices is *compatible* with a given state space is by no means trivial, which is a known issue in the context of matrix–exponential distributions and RAPs [3, 5]. Below we present two examples of *valid* orbit processes.

Example 2.2 (Markov jump process)**.** Let C^{+} and C^{-} be subintensity matrices, and let D^{+-} and D^{-+} be nonnegative matrices, in such a way that

$$\begin{pmatrix} C^{+} & D^{+-} \\ D^{-+} & C^{-} \end{pmatrix}$$

corresponds to the intensity matrix of a Markov jump process. For $k \in \{+,-\}$ choose

$$\mathfrak{Z}^{k} = \{\boldsymbol{x} \in \mathbb{R}^{m^k} : \boldsymbol{x} \geq \mathbf{0}, \boldsymbol{x}\mathbf{1} = 1\}.$$

Note that for all $\boldsymbol{b} \in \mathfrak{Z}^{k}$ and $h \geq 0$, $\boldsymbol{b}e^{C^k h}/\boldsymbol{b}e^{C^k h}\mathbf{1}$ corresponds to a normalized probability vector, and thus belongs to $\mathfrak{Z}^{k}$. The nonnegativity of D^{+-} and D^{-+} implies that the intensity function λ is always nonnegative. Finally, if $k \neq \ell$ with $k, \ell \in \{+,-\}$ and $\boldsymbol{b} \in \mathfrak{Z}^{k}$, then $\boldsymbol{b}D^{k\ell}/\boldsymbol{b}D^{k\ell}\mathbf{1}$ corresponds to an m^{ℓ}–dimensional normalized probability vector, and thus belongs to $\mathfrak{Z}^{\ell}$. Thus, any process defined by these parameters and with arbitrary initial point in $\mathfrak{Z}^{+} \cup \mathfrak{Z}^{-}$ is a valid orbit process.

Example 2.3 (Matrix-exponential renewal process)**.** For $k \in \{+,-\}$, let $(\boldsymbol{\alpha}^{k}, S^{k})$ be such that $\boldsymbol{\alpha}^{k}\mathbf{1} = 1$ and $F^{k}(x) := 1 - \boldsymbol{\alpha}^{k}e^{S^k x}\mathbf{1}$ is a distribution function. Such a class of distributions is commonly known as matrix-exponential, a generalization of phase–type distributions. For $\ell \neq k$, let

$$C^{k} = S^{k} \quad \text{and} \quad D^{k\ell} = (-S^{k}\mathbf{1})\boldsymbol{\alpha}^{\ell},$$

and define

$$\mathfrak{Z}^{k} = \left\{ \frac{\boldsymbol{\alpha}^{k}e^{C^k h}}{\boldsymbol{\alpha}^{k}e^{C^k h}\mathbf{1}} : h \geq 0 \right\}.$$

If $\boldsymbol{b} \in \mathfrak{Z}^{k}$, then $\boldsymbol{b} = \boldsymbol{\alpha}^{k}e^{C^k h}/\boldsymbol{\alpha}^{k}e^{C^k h}\mathbf{1}$ for some $h \geq 0$ and thus

$$\lambda(\boldsymbol{b}) = \boldsymbol{b}D^{k\ell}\mathbf{1} = \frac{\boldsymbol{\alpha}^{k}e^{S^k h}}{\boldsymbol{\alpha}^{k}e^{S^k h}\mathbf{1}}\left((-S^{k}\mathbf{1})\boldsymbol{\alpha}^{\ell}\right)\mathbf{1} = \frac{\boldsymbol{\alpha}^{k}e^{S^k h}(-S^{k}\mathbf{1})}{\boldsymbol{\alpha}^{k}e^{S^k h}\mathbf{1}}. \tag{2.8}$$

Since the numerator in the last expression of (2.8) corresponds to a probability density function and its denominator to a survival function, λ is indeed a nonnegative intensity function. Moreover, if a jump happens from some $\boldsymbol{b} \in \mathfrak{Z}^{k}$, it will land in

$$\frac{\boldsymbol{b}D^{k\ell}}{\boldsymbol{b}D^{k\ell}\mathbf{1}} = \frac{\boldsymbol{b}(-S^{k}\mathbf{1})\boldsymbol{\alpha}^{\ell}}{\boldsymbol{b}(-S^{k}\mathbf{1})\boldsymbol{\alpha}^{\ell}\mathbf{1}} = \frac{\boldsymbol{\alpha}^{\ell}}{\boldsymbol{\alpha}^{\ell}\mathbf{1}} = \boldsymbol{\alpha}^{\ell} \in \mathfrak{Z}^{\ell}, \quad \ell \neq k.$$

Thus, this set of parameters together with any arbitrary initial state in $\mathfrak{Z}^+ \cup \mathfrak{Z}^-$ corresponds to a valid orbit process whose inter-jump times follow a matrix-exponential distribution. Note that if the initial state $\boldsymbol{a}$ is in $\mathfrak{Z}^k$ for $k \in \{+,-\}$, then the time until the first jump follows the distribution $1 - \boldsymbol{a}e^{S^k x}\mathbf{1}$.

Now that the distributional characteristics of the orbit process have been completely described, we state two further conditions on $\boldsymbol{\mathcal{A}}$ and its state space.

Condition 2.4. *The sets $\mathfrak{Z}^+$ and $\mathfrak{Z}^-$ are bounded.*

In the RAP setting of [3], Condition 2.4 is needed for $\boldsymbol{\mathcal{A}}$ to correspond to the coefficients of a linear combination of probability measures that is itself a probability measure. In our setting, Condition 2.4 will enable us to rule out explosions of the orbit process. Indeed, if $\mathfrak{Z}^+$ and $\mathfrak{Z}^-$ are bounded, so is the jump intensity function $\lambda(\cdot)$ and thus, with probability 1, $\boldsymbol{\mathcal{A}}$ will have a finite amount of jumps on each compact time-interval. Although Condition 2.4 trivially holds in classic contexts such as the one of Example 2.2, in general such a condition has to be verified on a case by case basis; see [3, Section 3] for a nontrivial example.

Condition 2.5. *For $k \in \{+,-\}$, the set $\mathfrak{Z}^k$ is contained in a* minimal *(m^k-1)-dimensional affine hyperplane, meaning that $\mathfrak{Z}^k$ cannot be contained in any (m^k-2)–dimensional affine hyperplane.*

In our context, Condition 2.5 tells us that $\mathfrak{Z}^k$ is *rich enough* to guarantee a one-to-one correspondence between a matrix G and the collection $\{\boldsymbol{b}G : \boldsymbol{b} \in \mathfrak{Z}^k\}$, fact that is central to the study of orbit processes in [3, Proposition 2.1]. More precisely, Condition 2.5 implies the following.

Lemma 2.6. *For $k \in \{+,-\}$, there exist m^k linearly independent vectors contained in $\mathfrak{Z}^k$.*

Thus, Lemma 2.6 will allow us to determine uniqueness properties of matrices that are solutions to certain collections of equations (see the proof of Theorem 4.1 for an example). In the following we present a simple situation where Condition 2.5 fails to hold.

Example 2.7. Consider the matrices

$$C^+ = C^- = \begin{pmatrix} -q & 0 \\ 0 & -q \end{pmatrix} \quad \text{and} \quad D^{+-} = D^{-+} = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} \quad \text{for some } q > 0, \tag{2.9}$$

and the state space $\mathfrak{Z}^+ = \{(1,0)\}$ and $\mathfrak{Z}^- = \{(0,1)\}$. The orbit process associated to these parameters fulfils Conditions 2.1 and 2.4. However, it does not attain Condition 2.5, since both $\mathfrak{Z}^+$ and $\mathfrak{Z}^-$ can be embedded in a 0-dimensional affine hyperplane (i.e. a single point), then it does not attain Condition 2.5. In the context of Example 2.2, the parameters (2.9) are associated to a reducible Markov jump process which has states that are never visited, and thus can be disregarded; Condition 2.5 simply assumes that the parameters and state space are already *minimal* and no reduction can be made.

Definition 2.8. *A* simple RAP-modulated fluid process *is a Markov additive process $\{(R_t, \boldsymbol{A}_t)\}_{t\geq 0}$ with an orbit process $\boldsymbol{\mathcal{A}} = \{\boldsymbol{A}_t\}_{t\geq 0}$ with arbitrary but fixed initial point $\boldsymbol{A}_0 \in \mathfrak{Z}^+ \cup \mathfrak{Z}^-$, and additive component $\mathcal{R} = \{R_t\}_{t\geq 0}$ of the form*

$$R_t = \int_0^t \mathbb{1}\{\boldsymbol{A}_s \in \mathfrak{Z}^+\} - \mathbb{1}\{\boldsymbol{A}_s \in \mathfrak{Z}^-\}\mathrm{d}s. \tag{2.10}$$

We refer to the process $\mathcal{R}$, which takes values in $\mathbb{R}$, as the level process*.*

In other words, at time t the process $\mathcal{R}$ is either increasing at rate 1 or decreasing at rate -1, depending on the location of $\boldsymbol{A}_t$. The term *simple* stems from the facts that $\mathcal{R}$ is nowhere piecewise constant, and that $\boldsymbol{\mathcal{A}}$ is allowed to perform jumps from $\mathfrak{Z}^+$ to $\mathfrak{Z}^-$ or vice versa only. We relax these assumptions in Section 3. See Figure 1 for a visual description of the simple RAP-modulated fluid process.

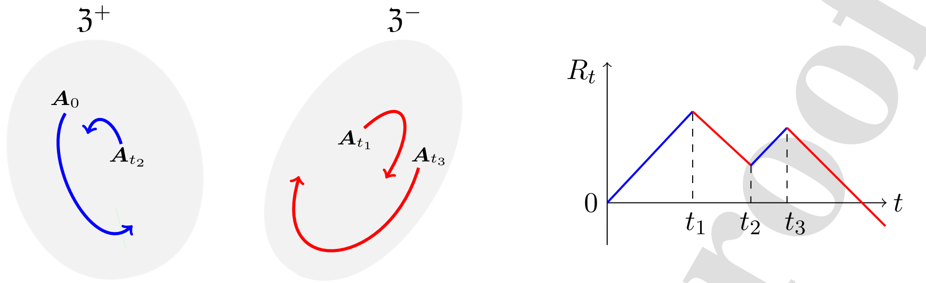

Figure 1: A sample path of the elements of the simple RAP-modulated fluid process $\{(R_t, \boldsymbol{A}_t)\}_{t\geq 0}$. The times t_1, t_2, t_3 correspond to jump epochs of the orbit process $\boldsymbol{\mathcal{A}}$. In between jumps, $\{\boldsymbol{A}_t\}_{t\geq 0}$ evolves deterministically, switching between states in $\mathfrak{Z}^+$ and $\mathfrak{Z}^-$ at t_1, t_2 and t_3.

Condition 2.9. *For* $k \in \{+, -\}$, $\lim_{t\to\infty} \mathbb{P}\left(\boldsymbol{A}_s \in \mathfrak{Z}^k \ \forall s \in [0,t] \,\middle|\, \boldsymbol{A}_0 = \boldsymbol{\alpha}\right) = 0$ *for all* $\boldsymbol{\alpha} \in \mathfrak{Z}^k$.

This implies that neither the orbit $\boldsymbol{\mathcal{A}}$ nor the level $\mathcal{R}$ are deterministic and thus trivial.

3 RAP-modulated fluid process with unit rates: Construction and basic properties

While the class of simple RAP-modulated fluid process considered in Section 2 is flexible enough to generalize the important fluid processes in Examples 2.2 and 2.3, its deterministic switching mechanism hinders its modelling capabilities; for instance, it would not be possible to describe a Markov renewal behaviour using this framework. The purpose of this section is to introduce the RAP-modulated fluid process, which allows for greater generality than the simple RAP-modulated fluid process, by additionally letting the orbit $\boldsymbol{\mathcal{A}}$ perform jumps that do not trigger a change of direction for the fluid component.

More specifically, $\mathfrak{Z}^+$ and $\mathfrak{Z}^-$ are now considered "macro sets", which are themselves unions of "micro sets" $\{\mathfrak{Z}_i^+\}$ and $\{\mathfrak{Z}_i^+\}$. During a stay in a given macro set, the orbit $\boldsymbol{\mathcal{A}}$ is allowed to perform jumps between different micro sets; the value of the orbit $\boldsymbol{\mathcal{A}}$ as a row vector then characterises the "micro" evolution, which follows an ordinary differential equation. Meanwhile, rates of $\mathcal{R}$ remain constant within each sojourn time in a macro set, meaning that jumps between micro sets during a given macro set sojourn have no visible effect in the level process, though they do affect its underlying orbit process. In a similar fashion to those of the simple orbit process of Section 2, jumps between micro sets depend on the orbit state before the transition via appropriate transition matrices, except that here the landing state is randomly chosen among micro sets (as opposed to entirely deterministic).

Besides allowing for a Markov renewal behaviour (see Example 3.3 below), the RAP-modulated fluid process introduced here enables us to study fluid processes with less constraints that those needed for the simple RAP-modulated fluid process. For instance, here we present an analogous to Condition 2.5 where the minimality needs to be checked, not for its macro sets, but for its micro sets. Additionally, the process introduced in this section paves the way towards an even more general process whose level rates are allowed to vary among micro sets; a discussion of such a model is included in Section 6.

The RAP-modulated fluid process defined here requires a slightly more sophisticated construction than the one considered in Section 2; however, as we see in Theorem 3.11 and Lemma 3.12, their probabilistic descriptors are very similar. Though this was an expected feature of the model, our main contribution in this section is providing a proper mathematical framework and rigorous proofs of these properties, some of which require the use of novel mathematical techniques.

3.1 Definition

Fix $k \in \mathcal{S}$. We suppose that the set $\mathfrak{Z}^k$ is contained in a collection of $n^k > 0$ orthogonal affine hyperplanes. Specifically, we assume that the set $\mathfrak{Z}^k$ can be partitioned in sets, say $\{\mathfrak{Z}_i^k\}_{i=1}^{n^k}$, with the following property: There exists a collection $\{m_i^k\}_{i=1}^{n^k} \subset \{1, 2, \dots\}$ such that each set $\mathfrak{Z}_i^k$ is contained in the affine hyperplane

$$\left\{ \boldsymbol{x} \in \mathbb{R}^{\eta^k} : \boldsymbol{x} = (\boldsymbol{0}_1^k, \dots, \boldsymbol{0}_{i-1}^k, \boldsymbol{y}, \boldsymbol{0}_{i+1}^k, \dots, \boldsymbol{0}_{n^k}^k) \text{ for some } \boldsymbol{y} \in \mathbb{R}^{m_i^k} \text{ with } \boldsymbol{y}\boldsymbol{1} = 1 \right\}, \tag{3.1}$$

where $\boldsymbol{0}_i^k$ denotes the row-vector of zeros with m_i^k elements and $\eta^k := \sum_{i=1}^{n^k} m_i^k$ corresponds to the dimension of the space in which $\mathfrak{Z}^k$ lives.

The *orbit process* $\boldsymbol{\mathcal{A}}$ is a càdlàg PDMP with state space $\mathfrak{Z} = \cup_{k\in\mathcal{S}}\mathfrak{Z}^k$ and some arbitrary but fixed initial state $\boldsymbol{A}_0 \in \mathfrak{Z}$. We describe its PDMP characteristics next. If $\boldsymbol{\mathcal{A}}$ has no jumps in $[t, t+h)$ for some $h > 0$, then $\boldsymbol{\mathcal{A}}$ follows the ODE

$$\frac{\mathrm{d}\boldsymbol{A}_s}{\mathrm{d}s} = \boldsymbol{A}_s\Gamma^k - (\boldsymbol{A}_s\Gamma^k\boldsymbol{1})\boldsymbol{A}_s \quad \text{for } \boldsymbol{A}_s \in \mathfrak{Z}^k, s \in [t, t+h], \tag{3.2}$$

with $\Gamma^k \in \mathbb{R}^{\eta^k\times\eta^k}$ of the form

$$\Gamma^k = \begin{pmatrix} C_{11}^k & & & \\ & C_{22}^k & & \\ & & \ddots & \\ & & & C_{n^kn^k}^k \end{pmatrix}$$

for some $C_{ii}^k \in \mathbb{R}^{m_i^k\times m_i^k}$, $1 \le i \le n^k$, where each empty block denotes a zero matrix of appropriate size. The block-diagonal structure of Γ^k guarantees that, if $\boldsymbol{A}_t \in \mathfrak{Z}_{i_0}^k$ for some $1 \le i_0 \le n^k$ and $\boldsymbol{\mathcal{A}}$ has no jumps in $[t, t+h]$, then $\{\boldsymbol{A}_s\}_{s=t}^{t+h}$ is contained in the affine hyperplane (3.1) with $i = i_0$. One can verify that the solution of (3.2) is given by

$$\boldsymbol{A}_{t+r} = \frac{\boldsymbol{A}_t e^{\Gamma^k r}}{\boldsymbol{A}_t e^{\Gamma^k r}\boldsymbol{1}} \in \mathfrak{Z}^k \quad \text{if } \boldsymbol{A}_t \in \mathfrak{Z}^k, r \in [0, h]. \tag{3.3}$$

If $\boldsymbol{A}_t \in \mathfrak{Z}_i^k$, then a jump to $\mathfrak{Z}_j^k$ with $j \neq i$ occurs with intensity $\boldsymbol{A}_t \widehat{C}_{ij}^k \mathbf{1} \geq 0$ with

$$\widehat{C}_{ij}^k := \begin{pmatrix} 0_{1,1}^k & \cdots & 0_{1,j-1}^k & & & & \\ \vdots & & \vdots & & & & \\ 0_{i-1,1}^k & \cdots & 0_{i-1,j-1}^k & & & & \\ & & & C_{ij}^k & & & \\ & & & & 0_{i+1,j+1}^k & \cdots & 0_{i+1,n^k}^k \\ & & & & \vdots & & \vdots \\ & & & & 0_{n^k,j+1}^k & \cdots & 0_{n^k,n^k}^k \end{pmatrix} \tag{3.4}$$

for some $C_{ij}^k \in \mathbb{R}^{m_i^k \times m_j^k}$, where $0_{a,b}^k$ denotes the zero matrix in $\mathbb{R}^{m_a^k \times m_b^k}$. If such a jump to $\mathfrak{Z}_j^k$ occurs at some time $s > t$, it will land at $\boldsymbol{A}_s = (\boldsymbol{A}_{s-}\widehat{C}_{ij}^k)/(\boldsymbol{A}_{s-}\widehat{C}_{ij}^k\mathbf{1}) \in \mathfrak{Z}_j^k$.

Similarly, if $\boldsymbol{A}_t \in \mathfrak{Z}_i^k$, then a jump to $\mathfrak{Z}_j^\ell$ with $\ell \neq k, j \leq n^\ell$, occurs with intensity $\boldsymbol{A}_t \widehat{D}_{ij}^{k\ell} \mathbf{1} \geq 0$ where

$$\widehat{D}_{ij}^{k\ell} = \begin{pmatrix} 0_{1,1}^{k\ell} & \cdots & 0_{1,j-1}^{k\ell} & & & & \\ \vdots & & \vdots & & & & \\ 0_{i-1,1}^{k\ell} & \cdots & 0_{i-1,j-1}^{k\ell} & & & & \\ & & & D_{ij}^{k\ell} & & & \\ & & & & 0_{i+1,j+1}^{k\ell} & \cdots & 0_{i+1,n^\ell}^{k\ell} \\ & & & & \vdots & & \vdots \\ & & & & 0_{n^k,j+1}^{k\ell} & \cdots & 0_{n^k,n^\ell}^{k\ell} \end{pmatrix} \tag{3.5}$$

for some $D_{ij}^{k\ell} \in \mathbb{R}^{m_i^k \times m_j^\ell}$. If such a jump occurs at some time $s > t$, it will land at $\boldsymbol{A}_s = (\boldsymbol{A}_{s-}\widehat{D}_{ij}^{k\ell})/(\boldsymbol{A}_{s-}\widehat{D}_{ij}^{k\ell}\mathbf{1}) \in \mathfrak{Z}_j^\ell$.

Condition 3.1.

$$\Gamma^k \mathbf{1} + \sum_{i=1}^{n^k} \sum_{\substack{j=1 \\ j \neq i}}^{n^k} \widehat{C}_{ij}^k \mathbf{1} + \sum_{\substack{\ell \in \mathcal{S} \\ \ell \neq k}} \sum_{i=1}^{n^k} \sum_{j=1}^{n^\ell} \widehat{D}_{ij}^{k\ell} \mathbf{1} = \mathbf{0}. \tag{3.6}$$

Remark 3.2. *Note that the collection $\{\ell \in \mathcal{S} : \ell \neq k\}$ considered in the sum in (3.6) contains only one element; this will no longer be the case in more general frameworks such as the ones analyzed later on in Sections 5 and 6.*

Since the jump intensities of a PDMP are additive, (3.6) implies that the jump intensity function of $\boldsymbol{\mathcal{A}}$ is given by $\lambda : \cup_{k \in \mathcal{S}} \mathfrak{Z}^k \mapsto \mathbb{R}_+$ of the form

$$\lambda(\boldsymbol{A}_t) = \boldsymbol{A}_t \left(\sum_{i=1}^{n^k} \sum_{\substack{j=1 \\ j \neq i}}^{n^k} \widehat{C}_{ij}^k \mathbf{1} + \sum_{\substack{\ell \in \mathcal{S} \\ \ell \neq k}} \sum_{i=1}^{n^k} \sum_{j=1}^{n^\ell} \widehat{D}_{ij}^{k\ell} \mathbf{1} \right) = -\boldsymbol{A}_t \Gamma^k \mathbf{1} \quad \text{for} \quad \boldsymbol{A}_t \in \mathfrak{Z}^k. \tag{3.7}$$

Analogous to (2.5), Equations (3.3) and (3.7) imply that

$$\mathbb{P}\left(\boldsymbol{\mathcal{A}} \text{ has no jumps in } [t, t+h] \mid \boldsymbol{A}_t, \boldsymbol{A}_t \in \mathfrak{Z}_i^k \right) = \boldsymbol{A}_t e^{\Gamma^k h} \mathbf{1}.$$

As in Section 2, note that the distributional properties of $\boldsymbol{\mathcal{A}}$ are completely determined by $\{C^k_{ij} : k \in \mathcal{S}, i \le n^k, j \le n^k\}$ and $\{D^{k\ell}_{ih} : k \in \mathcal{S}, \ell \ne k, i \le n^k, h \le n^\ell\}$. Similarly, here it is also difficult to assess if a collection of matrices is compatible with a given state space $\cup_{k\in\mathcal{S}}\mathfrak{Z}^k$, nonetheless, the following is an important example of a valid orbit process.

Example 3.3 (Markov renewal matrix-exponential process)**.** For $k, \ell \in \mathcal{S}$, let $P^{k\ell} = \{p^{k\ell}_{ij}\}_{ij} \in \mathbb{R}^{n^k \times n^\ell}$ be such that

$$\begin{pmatrix} P^{++} & P^{+-} \\ P^{-+} & P^{--} \end{pmatrix}$$

is a transition probability matrix. For $k \in \mathcal{S}$ let $\{(\boldsymbol{\alpha}^k_i, S^k_i)\}_{i=1}^{n^k}$ be a collection of matrix-exponential parameters. In a similar fashion to Example 2.3, one can verify that taking

$$C^k_{ii} := S^k_i, \quad C^k_{ij} := p^{kk}_{ij}\left(-S^k_i\mathbf{1}\right)\boldsymbol{\alpha}^k_j, \quad D^{k\ell}_{ih} := p^{k\ell}_{ih}\left(-S^k_i\mathbf{1}\right)\boldsymbol{\alpha}^\ell_h \quad \text{for} \quad \ell \ne k, i \ne j \le n^k, h \le n^\ell,$$

and defining

$$\mathfrak{Z}^k_i := \left\{ \boldsymbol{x} \in \mathbb{R}^{\eta^k} : \boldsymbol{x} = \left(\mathbf{0}^k_1, \ldots, \mathbf{0}^k_{i-1}, \frac{\boldsymbol{\alpha}^k e^{C^k_{ii}h}}{\boldsymbol{\alpha}^k e^{C^k_{ii}h}\mathbf{1}}, \mathbf{0}^k_{i+1}, \ldots, \mathbf{0}^k_{n^k} \right) \text{ for some } h \ge 0 \right\}$$

yields a valid orbit $\boldsymbol{\mathcal{A}}$ driven by a Markov renewal process with matrix-exponential jump times. Note that if the initial state is $\boldsymbol{a}$ in $\mathfrak{Z}^k_i$ for $k \in \{+,-\}$ and $i \le n^k$, then the distribution of the time until the first jump is given by $1 - \boldsymbol{a}e^{S^k_i x}\mathbf{1}$.

Condition 3.4. *For each $k \in \mathcal{S}$ and $i \le n^k$, the set $\mathfrak{Z}^k_i$ is bounded and is contained in a minimal $(m^k_i - 1)$-dimensional affine hyperplane.*

The advantage of the above condition with respect to Condition 2.5 is that, in general, it is easier to verify minimality of each $\mathfrak{Z}^k_i$ as oppossed to verifying the minimality of the set $\mathfrak{Z}^k$. For instance, in the context of Example 3.3 it is enough to have minimal matrix-exponential representations $(\boldsymbol{\alpha}^k_i, S^k_i)$, in the sense of [2], for each $i \le n^k$, $k \in \{+,-\}$ for Condition 3.4 to hold. In fact, the simple orbit in Example 2.7 can be easily repurposed into a Markov renewal matrix-exponential process which attains Condition 3.4, even if Condition 2.5 does not hold. Thus, Condition 3.4 is not only easier to verify than Condition 2.5, but also less restrictive.

Similarly to Lemma 2.6, Condition 3.4 implies the following.

Lemma 3.5. *For $k \in \mathcal{S}$, the set $\mathfrak{Z}^k$ contains η^k linearly independent vectors.*

Definition 3.6. *We define a* RAP-modulated fluid process with unit rates *to be a Markov additive process $(\mathcal{R}, \boldsymbol{\mathcal{A}}) = \{(R_t, \boldsymbol{A}_t)\}_{t\ge 0}$, where $\boldsymbol{\mathcal{A}}$ is an* orbit process *with state space $\mathfrak{Z}^+ \cup \mathfrak{Z}^-$ and arbitrary but fixed initial point $\boldsymbol{A}_0 \in \mathfrak{Z}^+ \cup \mathfrak{Z}^-$, and $\mathcal{R}$ is a* level process *(taking values in $\mathbb{R}$) of the form*

$$R_t := \int_0^t \mathbb{1}\{\boldsymbol{A}_s \in \mathfrak{Z}^+\} - \mathbb{1}\{\boldsymbol{A}_s \in \mathfrak{Z}^-\}\mathrm{d}s. \tag{3.8}$$

See Figure 2 for a visual description.

As in Section 2, in order to avoid trivial paths we impose the following.

Condition 3.7. *For any $k \in \mathcal{S}$,*

$$\lim_{t\to\infty} \mathbb{P}\left(\boldsymbol{A}_s \in \mathfrak{Z}^k \ \forall s \in [0,t] \,\middle|\, \boldsymbol{A}_0 = \boldsymbol{\alpha}\right) = 0 \quad \textit{for all } \boldsymbol{\alpha} \in \mathfrak{Z}^k. \tag{3.9}$$

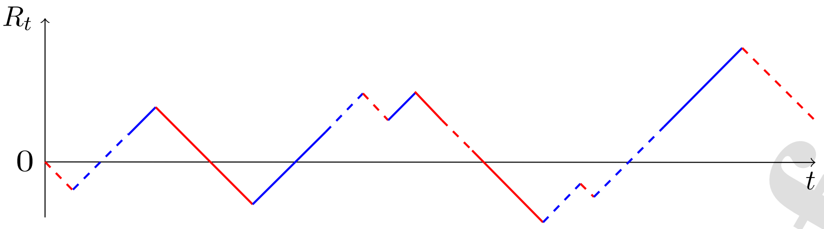

Figure 2: A sample path of the process $\{R_t\}_{t\geq 0}$ whose orbit process $\{\boldsymbol{A}_t\}_{t\geq 0}$ has state space $\mathfrak{Z}$ and intial state $\boldsymbol{A}_0 \in \mathfrak{Z}^-$, with $\mathfrak{Z}^+ = \mathfrak{Z}^+_{\text{solid}} \cup \mathfrak{Z}^+_{\text{dashed}}$ and $\mathfrak{Z}^- = \mathfrak{Z}^-_{\text{solid}} \cup \mathfrak{Z}^-_{\text{dashed}}$.

3.2 Preliminaries

For $\boldsymbol{\alpha} \in \mathfrak{Z}$, denote by $\mathbb{P}_{\boldsymbol{\alpha}}$ the probability law of $\mathcal{X} = \{(R_t, \boldsymbol{A}_t)\}_{t\geq 0}$ with the process $\boldsymbol{\mathcal{A}}$ starting in $\boldsymbol{\alpha}$, and denote by $\mathbb{E}_{\boldsymbol{\alpha}}$ its associated expectation. Let $(\Omega_*, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{\mathbb{P}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathfrak{Z}})$ be the canonical probability space associated to the Markov process $\{(R_t, \boldsymbol{A}_t)\}_{t\geq 0}$, and w.l.o.g. assume that $\{(R_t, \boldsymbol{A}_t)\}_{t\geq 0}$ is defined there. More specifically, for $\omega = \{(r_t, \boldsymbol{a}_t)\}_{t\geq 0} \in \Omega_*$,

$$R_t(\omega) = r_t \quad \text{and} \quad \boldsymbol{A}_t(\omega) = \boldsymbol{a}_t \qquad \text{for all} \quad t \geq 0. \tag{3.10}$$

Heuristically speaking, each $\{(r_t, \boldsymbol{a}_t)\}_{t\geq 0} \in \Omega_*$ corresponds to a feasible *path* of $\{(R_t, \boldsymbol{A}_t)\}_{t\geq 0}$.

Let Ω be the set of paths $\{(r_t, \boldsymbol{a}_t)\}_{t\geq 0} \in \Omega_*$ such that:

- $\{\boldsymbol{a}_t\}_{t\geq 0}$ has a finite number of jumps on each compact time interval,
- for all $s \geq 0$, neither $\boldsymbol{a}_t \in \mathfrak{Z}^+$ for all $t \geq s$, nor $\boldsymbol{a}_t \in \mathfrak{Z}^-$ for all $t \geq s$,
- there are no $s, t \geq 0$, $s \neq t$ such that $r_s = r_t$, $\boldsymbol{a}_{s-} \neq \boldsymbol{a}_s$ and $\boldsymbol{a}_{t-} \neq \boldsymbol{a}_t$; in other words, no two jumps of $\{\boldsymbol{a}_t\}_{t\geq 0}$ happen while at the same level of $\{r_t\}_{t\geq 0}$.

Note that these three properties hold with probability 1. Indeed, the first property follows by the boundedness of the jump intensity function $\lambda(\cdot)$, the second property follows from (3.9), and the final one from the fact that PDMPs with bounded jump intensities cannot jump at predetermined epochs.

Remark 3.8. *The paths in Ω are those that are regarded as* nice, *and are the ones that will be considered throughout all the arguments in this paper. Note that the event $\{\mathcal{X} \in \Omega_* \setminus \Omega\}$ is $\mathbb{P}_{\boldsymbol{\alpha}}$-null for all $\boldsymbol{\alpha} \in \mathfrak{Z}$; since the σ-algebra $\mathcal{F}$ is assumed to be complete, then the event $\{\mathcal{X} \in \Omega\}$ is an element of $\mathcal{F}$. For this reason, from here on we restrict the sample space of $\mathcal{X}$ to Ω and, with a slight abuse of notation, refer to its probability measure $\mathbb{P}_{\boldsymbol{\alpha}}|_{\{\mathcal{X}\in\Omega\}}$ as $\mathbb{P}_{\boldsymbol{\alpha}}$.*

For any $s \geq 0$, define the *shift operator* $\theta_s : \Omega \mapsto \Omega$ by

$$\theta_s\{(r_t, \boldsymbol{a}_t)\}_{t\geq 0} := \{(r_{t+s} - r_s, \boldsymbol{a}_{t+s})\}_{t\geq 0}.$$

In particular, according to (3.10)

$$R_t \circ \theta_s = R_{t+s} - R_s \quad \text{and} \quad \boldsymbol{A}_t \circ \theta_s = \boldsymbol{A}_{t+s} \qquad \text{for all} \quad t \geq 0, s \geq 0.$$

Now, let Z be an $\mathcal{F}_t$-stopping time. By the strong Markov property of PDMPs [11], it follows that $\mathcal{X}$ also has the strong Markov property. This implies that for any $\mathcal{F}$-measurable bounded function $f : \Omega \mapsto \mathbb{R}^k$, for $k \geq 1$, we have

$$\mathbb{E}_{\boldsymbol{\alpha}} \left[f(\mathcal{X} \circ \theta_Z) \mathbb{1}\{Z < \infty\} \mid \mathcal{F}_Z \right] = \mathbb{E}_{\boldsymbol{A}_Z} \left[f(\mathcal{X}) \right] \mathbb{1}\{Z < \infty\},$$

where $\mathcal{X} \circ \theta_Z(\omega) = \mathcal{X}(\theta_{Z(\omega)}(\omega))$.

Next we investigate a useful implication of the minimality property of $\boldsymbol{\mathcal{A}}$ and Lemma 3.5.

Corollary 3.9. *Fix $x \geq 0$, $k \in \mathcal{S}$, let $\mathcal{I}$ be an index set, and let $\{\Pi(s) : s \in \mathcal{I}\}$ be a collection of square matrices with identical dimensions. If $\{\boldsymbol{\alpha}\Pi(s) : s \in \mathcal{I}\}$ is uniformly entrywise-bounded, then so is $\{\Pi(s) : s \in \mathcal{I}\}$.*

Proof. Suppose $\exists \{s_n\}_{n=1}^{\infty} \subset \mathcal{I}$ and $j, \ell \in \{1, \dots, \eta^k\}$ such that $\{\Pi(s_n)_{j\ell}\}_{n=1}^{\infty}$ is unbounded. Let $\{\boldsymbol{g}_m\}_m \subset \mathfrak{Z}^k$ be a collection of η^k linearly independent vectors and let $\{c_m\}_m \subset \mathbb{R}$ be such that $\sum_m c_m \boldsymbol{g}_m = \boldsymbol{e}_j'$, where $\boldsymbol{e}_j$ denotes the unit column vector that is nonzero only at its jth entry. Then

$$\left\{ \sum_m c_m (\boldsymbol{g}_m \Pi(s_n))_\ell \right\}_{n=1}^{\infty} = \{\Pi(s_n)_{j\ell}\}_{n=1}^{\infty}$$

is unbounded, which is a contradiction. □

Next, we study the solution of a particular integral matrix-equation which is key to our analysis throughout this paper.

Theorem 3.10. *Let $\Pi : \mathbb{R}_+ \mapsto \mathbb{R}^{m \times m}$ and $A, B \in \mathbb{R}^{m \times m}$ be such that $\Pi(\cdot)$ is uniformly entrywise-bounded in compact intervals, and for all $x \geq 0$*

$$\Pi(x) = e^{Ax} + \int_0^x e^{As} B \Pi(x - s) \mathrm{d}s. \tag{3.11}$$

Then, $\Pi(x) = e^{(A+B)x}$ for all $x \geq 0$.

Proof. The assumptions of $\Pi(\cdot)$ imply that it is infinitely differentiable in $(0, \infty)$. Premultiplying (3.11) by e^{-Ax} and differentiating gives us

$$-e^{-Ax} A \Pi(x) + e^{-Ax} \Pi'(x) = e^{-Ax} B \Pi(x),$$

so that $\Pi'(x) = (A + B)\Pi(x)$, with initial condition $\Pi(0) = I$. Thus the result follows. □

3.3 Properties of the orbit process

In the following, we study the average behaviour of the process $\{\boldsymbol{A}_t\}_{t \geq 0}$ over time restricted to certain events. By (2.2), we have that if $\boldsymbol{\mathcal{A}}$ is simple and does not leave $\mathfrak{Z}^k$ by time t, for $k \in \mathcal{S}$, then the row vector $\boldsymbol{A}_t$ is proportional to $\boldsymbol{A}_0 e^{C^k t}$. Next we show an analogous result for the orbit process defined in Subsection 3.1. In the following and further proofs we use the infinitesimal notation $\mathrm{d}t$ to represent a fixed $\varepsilon > 0$ whose limit to 0 is eventually taken: this allows us to implicitly disregard any $o(\mathrm{d}t)$ functions in our calculations (see e.g. [9, Remark 1.1.5]).

Theorem 3.11. *For all $k \in \mathcal{S}$ and $\boldsymbol{\alpha} \in \mathfrak{Z}^k$,*

$$\mathbb{E}_{\boldsymbol{\alpha}} \left[\boldsymbol{A}_t \mathbb{1}\left\{ \boldsymbol{A}_s \in \mathfrak{Z}^k \ \forall s \in [0,t] \right\} \right] = \boldsymbol{\alpha} e^{C^k t} \quad \text{for } t \geq 0, \tag{3.12}$$

where $C^k \in \mathbb{R}^{\eta^k \times \eta^k}$ is of the form

$$C^k = \begin{pmatrix} C^k_{11} & \cdots & C^k_{1n^k} \\ \vdots & \ddots & \vdots \\ C^k_{n^k 1} & \cdots & C^k_{n^k n^k} \end{pmatrix}.$$

Proof. For $t \geq 0$, let

$$B := \{\exists s \in [t, t+\mathrm{d}t] \text{ such that } \boldsymbol{A}_{s-} \neq \boldsymbol{A}_s\}.$$

Then,

$$\begin{aligned} &\mathbb{E}_{\boldsymbol{\alpha}} \left[\boldsymbol{A}_{t+\mathrm{d}t} \mathbb{1}\left\{ \boldsymbol{A}_s \in \mathfrak{Z}^k \ \forall s \in [0,t+\mathrm{d}t] \right\} \mid \{\boldsymbol{A}_s\}_{s \leq t} \right] \\ &\quad = \mathbb{1}\left\{ \boldsymbol{A}_s \in \mathfrak{Z}^k \ \forall s \in [0,t] \right\} \times \left(\mathbb{E}_{\boldsymbol{\alpha}} \left[\boldsymbol{A}_{t+\mathrm{d}t} \mathbb{1}\left\{ B, \boldsymbol{A}_s \in \mathfrak{Z}^k \ \forall s \in [t,t+\mathrm{d}t] \right\} \mid \boldsymbol{A}_t \right] \right. \\ &\qquad \left. + \mathbb{E}_{\boldsymbol{\alpha}} \left[\boldsymbol{A}_{t+\mathrm{d}t} \mathbb{1}\left\{ B^c, \boldsymbol{A}_s \in \mathfrak{Z}^k \ \forall s \in [t,t+\mathrm{d}t] \right\} \mid \boldsymbol{A}_t \right] \right), \end{aligned}$$

where on $\{\boldsymbol{A}_t \in \mathfrak{Z}^k\}$

$$\begin{aligned} &\mathbb{E}_{\boldsymbol{\alpha}} \left[\boldsymbol{A}_{t+\mathrm{d}t} \mathbb{1}\left\{ B, \boldsymbol{A}_s \in \mathfrak{Z}^k \ \forall s \in [t,t+\mathrm{d}t] \right\} \mid \boldsymbol{A}_t \right] \\ &\quad = \sum_{1 \leq i \leq n^k} \sum_{\substack{1 \leq j \leq n^k, \\ j \neq i}} \frac{\boldsymbol{A}_t \widehat{C}^k_{ij}}{\boldsymbol{\alpha}^* \widehat{C}^k_{ij} \mathbf{1}} \left(\boldsymbol{A}_t \widehat{C}^k_{ij} \mathbf{1} \mathrm{d}t \right) \mathbb{1}\left\{ \boldsymbol{A}_t \in \mathfrak{Z}^k_i \right\} = \boldsymbol{A}_t \left(C^k - \Gamma^k \right) \mathrm{d}t, \end{aligned}$$

and

$$\mathbb{E}_{\boldsymbol{\alpha}} \left[\boldsymbol{A}_{t+\mathrm{d}t} \mathbb{1}\left\{ B^c, \boldsymbol{A}_s \in \mathfrak{Z}^k \ \forall s \in [t,t+\mathrm{d}t] \right\} \mid \boldsymbol{A}_t \right] = \frac{\boldsymbol{A}_t e^{\Gamma^k \mathrm{d}t}}{\boldsymbol{A}_t e^{\Gamma^k \mathrm{d}t} \mathbf{1}} \left(\boldsymbol{A}_t e^{\Gamma^k \mathrm{d}t} \mathbf{1} \right) = \boldsymbol{A}_t e^{\Gamma^k \mathrm{d}t} = \boldsymbol{A}_t \Gamma^k \mathrm{d}t.$$

Consequently,

$$\mathbb{E}_{\boldsymbol{\alpha}} \left[\boldsymbol{A}_{t+\mathrm{d}t} \mathbb{1}\left\{ \boldsymbol{A}_s \in \mathfrak{Z}^k \ \forall s \in [0,t+\mathrm{d}t] \right\} \mid \{\boldsymbol{A}_s\}_{s \leq t} \right] = (\boldsymbol{A}_t C^k \mathrm{d}t) \mathbb{1}\left\{ \boldsymbol{A}_s \in \mathfrak{Z}^k \ \forall s \in [0,t] \right\};$$

by taking expectations on both sides and defining

$$\boldsymbol{\Sigma}(t) = \mathbb{E}_{\boldsymbol{\alpha}} \left[\boldsymbol{A}_t \mathbb{1}\left\{ \boldsymbol{A}_s \in \mathfrak{Z}^k \ \forall s \in [0,t] \right\} \right]$$

we get that $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Sigma}(t) = \boldsymbol{\Sigma}(t) C^k$ with $\boldsymbol{\Sigma}(0) = \boldsymbol{\alpha}$, so that $\boldsymbol{\Sigma}(t) = \boldsymbol{\alpha} e^{C^k t}$ and (3.12) follows. □

Note that since $\boldsymbol{A}_t \mathbf{1} = 1$,

$$\mathbb{P}_{\boldsymbol{\alpha}} \left(\boldsymbol{A}_s \in \mathfrak{Z}^k \ \forall s \in [0,t] \right) = \boldsymbol{\alpha} e^{C^k t} \mathbf{1}.$$

While the matrices $\widehat{C}^k_{ij}$ have a deterministic physical meaning for $\boldsymbol{\mathcal{A}}$ given by (3.2), C^k does not. Nevertheless, Theorem 3.11 implies that C^k does have a role in the behaviour of $\boldsymbol{\mathcal{A}}$, not in a deterministic sense, but in an average sense instead. Lemma 3.12 elaborates on this further.

For $k \in \mathcal{S}$, define

$$\rho^k := \inf\{s \geq 0 : \boldsymbol{A}_s \notin \mathfrak{Z}^k\}, \tag{3.13}$$

the *first exit time* of $\boldsymbol{\mathcal{A}}$ from $\mathfrak{Z}^k$.

Lemma 3.12. *For $\ell, k \in \mathcal{S}$, $\ell \neq k$, and $t \geq 0$,*

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\rho^k}\mathbb{1}\left\{\rho^k \in [t, t+\mathrm{d}t],\ \boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^\ell\right\}\right] = \boldsymbol{\alpha}e^{C^k t}D^{k\ell}\mathrm{d}t, \tag{3.14}$$

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\rho^k}\mathbb{1}\left\{\boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^\ell\right\}\right] = \boldsymbol{\alpha}(-C^k)^{-1}D^{k\ell}, \tag{3.15}$$

where $\boldsymbol{\alpha} \in \mathfrak{Z}^k$ and $D^{k\ell} \in \mathbb{R}^{\eta^k \times \eta^\ell}$ is of the form

$$D^{k\ell} = \begin{pmatrix} D^{k\ell}_{11} & \cdots & D^{k\ell}_{1n^\ell} \\ \vdots & \ddots & \vdots \\ D^{k\ell}_{n^k 1} & \cdots & D^{k\ell}_{n^k n^\ell} \end{pmatrix}.$$

Proof. First, note that

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\rho^k}\mathbb{1}\left\{\rho^k \in [0, \mathrm{d}t],\ \boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^\ell\right\}\right] = \sum_{1\leq i\leq n^k}\sum_{1\leq j\leq n^\ell}\frac{\boldsymbol{\alpha}\widehat{D}^{k\ell}_{ij}}{\boldsymbol{\alpha}\widehat{D}^{k\ell}_{ij}\mathbf{1}}\left(\boldsymbol{\alpha}\widehat{D}^{k\ell}_{ij}\mathbf{1}\mathrm{d}t\right)\mathbb{1}\left\{\boldsymbol{\alpha} \in \mathfrak{Z}^k_i\right\} = \boldsymbol{\alpha}D^{k\ell}\mathrm{d}t.$$

Thus,

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\rho^k}\mathbb{1}\left\{\rho^k \in [t, t+\mathrm{d}t],\ \boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^\ell\right\}\right]\\
&= \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\rho^k}\mathbb{1}\left\{\rho^k \in [t, t+\mathrm{d}t],\ \boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^\ell\right\}\middle|\mathcal{F}_t\right]\mathbb{1}\left\{\boldsymbol{A}_s \in \mathfrak{Z}^k\ \forall s \in [0,t]\right\}\right]\\
&= \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{A}_t}\left[\boldsymbol{A}_{\rho^k}\mathbb{1}\left\{\rho^k \in [0, \mathrm{d}t],\ \boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^\ell\right\}\right]\mathbb{1}\left\{\boldsymbol{A}_s \in \mathfrak{Z}^k\ \forall s \in [0,t]\right\}\right]\\
&= \mathbb{E}_{\boldsymbol{\alpha}}\left[\left(\boldsymbol{A}_t D^{k\ell}\mathrm{d}t\right)\mathbb{1}\left\{\boldsymbol{A}_s \in \mathfrak{Z}^k\ \forall s \in [0,t]\right\}\right]\\
&= \boldsymbol{\alpha}e^{C^k t}D^{k\ell}\mathrm{d}t.
\end{aligned}$$

By Theorem 3.11 and Equation (3.9), we have $\lim_{t\to\infty}\boldsymbol{\alpha}e^{C^k t} = \mathbf{0}$. Since this holds for all $\boldsymbol{\alpha} \in \mathfrak{Z}^k$, Lemma 3.5 implies that $\lim_{t\to\infty} e^{C^k t} = 0$ and so the eigenvalue of maximum real part of C^k must have a strictly negative real part. Then

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\rho^k}\mathbb{1}\left\{\boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^\ell\right\}\right] = \int_0^\infty \boldsymbol{\alpha}e^{C^k t}D^{k\ell}\mathrm{d}t = \boldsymbol{\alpha}(-C^k)^{-1}D^{k\ell}.$$

□

Once again, while the matrices $\{\widehat{D}^{k\ell}_{ij}\}$ dictate in a deterministic way where $\boldsymbol{\mathcal{A}}$ lands after a jump, $D^{k\ell}$ describes such landings in an *average* sense.

4 RAP-modulated fluid process with unit rates: First passages and stationary distribution

Here, we exploit the fact that the process $\mathcal{R}$ has slopes that are 1 and -1 in order to compute descriptors for its first passage probabilities. Our analysis heavily relies on analysing the *up-down and down-up peaks* of $\mathcal{R}$, that is, the epochs at which $\boldsymbol{\mathcal{A}}$ jumps from $\mathfrak{Z}^+$ to $\mathfrak{Z}^-$ and from $\mathfrak{Z}^-$ to $\mathfrak{Z}^+$, respectively.

4.1 First-return probabilities

In this section we study the event in which the level process downcrosses its initial point. More specifically, define

$$\tau^- := \inf\{t > 0 : R_t \leq 0\}, \quad \tau^+ := \inf\{t > 0 : R_t \geq 0\},$$
$$\Omega^- := \{\omega \in \Omega : \boldsymbol{A}_0 \in \mathfrak{Z}^+, \tau^- < \infty\}, \quad \Omega^+ := \{\omega \in \Omega : \boldsymbol{A}_0 \in \mathfrak{Z}^-, \tau^+ < \infty\}.$$

The random variable τ^- (τ^+) corresponds to the first time the process $\mathcal{R}$ visits $[0,\infty)$ ($(-\infty, 0]$), and Ω^- (Ω^+) is the set of paths in Ω whose orbit starts in $\mathfrak{Z}^+$ ($\mathfrak{Z}^-$) and whose level eventually downcrosses (upcrosses) level 0 in finite time.

We are interested in computing $\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\tau^-}\mathbb{1}\{\tau^- < \infty\}\right]$ for $\boldsymbol{\alpha} \in \mathfrak{Z}^+$. To that end, we borrow ideas from the FP3 algorithm of [12], whose probabilistic interpretation was developed in [7] and requires a particular partition of all paths in Ω^-. Recall that $\{R_t\}_{t\geq 0}$ is piecewise linear with slopes ± 1, which implies that for all $s, t \geq 0$ and $k \in \{+,-\}$,

$$\left\{\boldsymbol{A}_t \in \mathfrak{Z}^k\right\} \cap \left\{|R_{t+s} - R_t| = s\right\} = \left\{\boldsymbol{A}_u \in \mathfrak{Z}^k \ \forall u \in [t, t+s)\right\}. \tag{4.1}$$

Let

$$\Omega_1 := \left\{\omega \in \Omega^- : \{\boldsymbol{A}_s\}_{s\geq 0} \text{ has exactly one jump from } \mathfrak{Z}^+ \text{ to } \mathfrak{Z}^- \text{ during } [0,\tau^-)\right\},$$

the set of sample paths whose level component returns to 0 after a single interval of increase and a single interval of decrease; in other words, there is no down-up peak before τ^-. Then, $\Omega^- \setminus \Omega_1$ is the set of sample paths whose level has one or more down-up peaks before τ^-. Define

$$p := \begin{cases} \inf\{y \geq 0 : \exists t \in [0,\tau^-) \text{ such that } R_t = y, \boldsymbol{A}_{t-} \in \mathfrak{Z}^-, \boldsymbol{A}_t \in \mathfrak{Z}^+\} & \text{if } \mathcal{X} \in \Omega^- \setminus \Omega_1, \\ \infty & \text{if } \mathcal{X} \in (\Omega^- \setminus \Omega_1)^c. \end{cases}$$

For every sample path in $\Omega^- \setminus \Omega_1$, p corresponds to the lowest level at which there is a down-up peak before time τ^-, and we decompose each path at the time such lowest level is attained, denoted by T_2. Let $T_1 := \inf\{t \geq 0 : R_t = p\}$ be the first time the level reaches p and $T_3 := \sup\{t \geq 0 : R_t = p, t < \tau^-\}$ be the last time it does so before returning to level zero. Then, $T_1 = p$ by the assumption of ± 1 rates, $T_2 = T_1 + \tau^- \circ \theta_{T_1}$, and $T_3 = T_2 + \tau^- \circ \theta_{T_2}$.

Now, we recursively define $\Omega'_n \subset \Omega^-$ and $\Omega_n \subset \Omega^-$. Let $\Omega'_1 = \Omega_1$. For $n \geq 2$, a path $\omega \in \Omega^-$ is an element of Ω'_n if and only if $\theta_{T_1}\omega \in \Omega'_{n-1} \cup \Omega_1$ and $\theta_{T_2}\omega \in \Omega'_{n-1} \cup \Omega_1$. The collection $\{\Omega'_n\}_{n\geq 1}$ may be thought as a collection of paths of possibly increasing *complexity*, in the sense that the *excursion above level p of a path in Ω'_n* can be decomposed, at time T_2, into two consecutive excursions above p (each increasing from p and returning to p) of paths in $\Omega'_{n-1} \cup \Omega_1$. Figure 3 exemplifies a path in Ω'_n.

Let $\Omega_n = \Omega'_n \cup \Omega_1$, then $\Omega_{n-1} \subset \Omega_n$ for $n \geq 2$ and $\Omega^- = \cup_{n=1}^{\infty}\Omega_n$. In Theorem 4.1, we show that restricted to paths in Ω_n, the mean value of $\boldsymbol{A}_{\tau^-}$ is characterized by a certain matrix Ψ_n.

Theorem 4.1. *For any given $\boldsymbol{\alpha} \in \mathfrak{Z}^+$ and for all $n \geq 1$,*

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\tau^-}\mathbb{1}\{\Omega_n\}\right] = \boldsymbol{\alpha}\Psi_n, \tag{4.2}$$

for unique matrices $\{\Psi_n\}_{n\geq 1}$ with

$$\Psi_0 = 0,$$
$$\Psi_n = \int_0^\infty e^{C^+ y}\left(D^{+-} + \Psi_{n-1}D^{-+}\Psi_{n-1}\right)e^{C^- y}\mathrm{d}y. \tag{4.3}$$

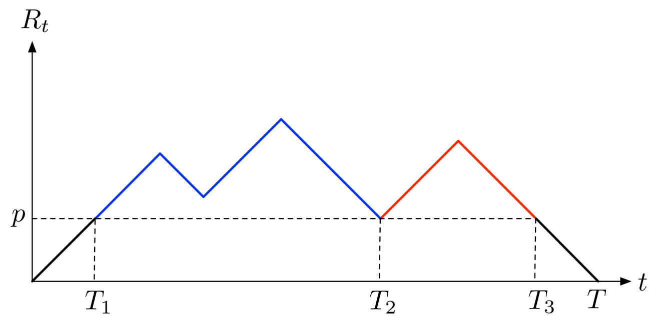

Figure 3: An example of a level process corresponding to $\omega \in \Omega_3'$. The level process corresponding to $\theta_{T_1}\omega, \theta_{T_2}\omega \in \Omega_2' \cup \Omega_1$ up to their downcrossing of level p are shown in blue and red, respectively. The blue and red segments, together, are also the sojourn above level p of $\mathcal{R}$ prior to returning to level 0.

Proof. We prove by induction.

Case $n = 1$. Let ρ^+ be defined by (3.13), so that ρ^+ corresponds to the epoch at which the first transition from $\mathfrak{Z}^+$ to $\mathfrak{Z}^-$ occurs. Fix $y \geq 0$ and define the stopping times

$$Z_1 := y, \qquad Z_2 := Z_1 + \tau^- \circ \theta_{Z_1}, \qquad Z_3 := Z_2 + y.$$

Then, $\mathbb{P}\left(\Omega_1 \cap \{\rho^+ \in (y, y + \mathrm{d}y)\}\right) = \mathbb{P}(E_1 \cap E_2 \cap E_3) + o(\mathrm{d}y)$, where

$$\begin{aligned}
E_1 &= \{\boldsymbol{A}_s \in \mathfrak{Z}^+ \text{ for all } s \in [0, Z_1)\},\\
E_2 &= \{\exists s \in (Z_1, Z_1 + \mathrm{d}y) \text{ such that } \boldsymbol{A}_{s-} \in \mathfrak{Z}^+ \text{ and } \boldsymbol{A}_s \in \mathfrak{Z}^-\},\\
E_3 &= \{\boldsymbol{A}_s \in \mathfrak{Z}^- \text{ for all } s \in [Z_2, Z_3)\}.
\end{aligned}$$

Moreover, on $\Omega_1 \cap \{\rho^+ \in (y, y + \mathrm{d}y)\}$ we have that $Z_3 = \tau^-$. Thus,

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\tau^-} \mathbb{1}\{\Omega_1 \cap \{\rho^+ \in (y, y + \mathrm{d}y)\}\}\right]\\
&= \mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{Z_3} \mathbb{1}\{E_1 \cap E_2 \cap E_3\}\right] = \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{Z_3} \mathbb{1}\{E_3\} \mid \mathcal{F}_{Z_2}\right] \mathbb{1}\{E_1 \cap E_2\}\right]\\
&= \mathbb{E}_{\boldsymbol{\alpha}}\left[\left(\boldsymbol{A}_{Z_2} e^{C^- y}\right) \mathbb{1}\{E_1 \cap E_2\}\right] = \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{Z_2} \mathbb{1}\{E_2\} \mid \mathcal{F}_{Z_1}\right] \mathbb{1}\{E_1\}\right] e^{C^- y}\\
&= \mathbb{E}_{\boldsymbol{\alpha}}\left[\left(\boldsymbol{A}_{Z_1} D^{+-} \mathrm{d}y\right) \mathbb{1}\{E_1\}\right] e^{C^- y} = \boldsymbol{\alpha} e^{C^+ y} D^{+-} e^{C^- y} \mathrm{d}y,
\end{aligned}$$

where the strong Markov property was used in the third and fifth equalities, Theorem 3.11 was used in the third and last equalities, and (3.14) in the fifth equality. Since this holds for all $\boldsymbol{\alpha} \in \mathfrak{Z}^+$, then Lemma 3.5 implies that

$$\Psi_1 = \int_0^\infty e^{C^+ y} D^{+-} e^{C^- y} \mathrm{d}y$$

is the only solution to (4.2) for $n = 1$.

Inductive part. Suppose (4.2) is true for some $n \geq 1$. Fix $y > 0$. Define the stopping times

$$S_1 := y, \quad S_2 := S_1 + \tau^- \circ \theta_{S_1}, \quad S_3 := S_2 + \tau^+ \circ \theta_{S_2}, \quad S_4 := S_3 + \tau^- \circ \theta_{S_3}, \quad S_5 := S_4 + y.$$

Note that

$$\mathbb{P}\left((\Omega_{n+1} \setminus \Omega_1) \cap \{p \in (y - \mathrm{d}y, y)\}\right) = \mathbb{P}\left(B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5\right) + o(\mathrm{d}y),$$

where

$$
\begin{aligned}
B_1 &= \{\boldsymbol{A}_s \in \mathfrak{Z}^+ \text{ for all } s \in [0, S_1)\},\\
B_2 &= \{\mathcal{X} \circ \theta_{S_1} \in \Omega_n\},\\
B_3 &= \{\exists s \in (S_2, S_2 + \mathrm{d}y) \text{ such that } \boldsymbol{A}_{s-} \in \mathfrak{Z}^- \text{ and } \boldsymbol{A}_s \in \mathfrak{Z}^+\},\\
B_4 &= \{\mathcal{X} \circ \theta_{S_3} \in \Omega_n\},\\
B_5 &= \{\boldsymbol{A}_s \in \mathfrak{Z}^- \text{ for all } s \in [S_4, S_5)\}.
\end{aligned}
$$

Moreover, on $(\Omega_{n+1} \setminus \Omega_1) \cap \{p \in (y - \mathrm{d}y, y)\}$ we have that $\{S_5 \neq \tau^-\}$ only occurs with a probability of order $o(\mathrm{d}y)$. Then,

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\tau^-}\mathbb{1}\{(\Omega_{n+1} \setminus \Omega_1) \cap \{p \in (y - \mathrm{d}y, y)\}\}\right]\\
&\quad= \mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{S_5}\mathbb{1}\left\{\cap_{i=1}^5 B_i\right\}\right] = \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{S_5}\mathbb{1}\{B_5\} \mid \mathcal{F}_{S_4}\right]\mathbb{1}\left\{\cap_{i=1}^4 B_i\right\}\right]\\
&\quad= \mathbb{E}_{\boldsymbol{\alpha}}\left[\left(\boldsymbol{A}_{S_4}e^{C^- y}\right)\mathbb{1}\left\{\cap_{i=1}^4 B_i\right\}\right] = \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{S_4}\mathbb{1}\{B_4\} \mid \mathcal{F}_{S_3}\right]\mathbb{1}\left\{\cap_{i=1}^3 B_i\right\}\right]e^{C^- y}\\
&\quad= \mathbb{E}_{\boldsymbol{\alpha}}\left[(\boldsymbol{A}_{S_3}\Psi_n)\mathbb{1}\left\{\cap_{i=1}^3 B_i\right\}\right]e^{C^- y} = \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{S_3}\mathbb{1}\{B_3\} \mid \mathcal{F}_{S_2}\right]\mathbb{1}\left\{\cap_{i=1}^2 B_i\right\}\right]\Psi_n e^{C^- y}\\
&\quad= \mathbb{E}_{\boldsymbol{\alpha}}\left[\left(\boldsymbol{A}_{S_2}D^{-+}\mathrm{d}y\right)\mathbb{1}\left\{\cap_{i=1}^2 B_i\right\}\right]\Psi_n e^{C^- y} = \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{S_2}\mathbb{1}\{B_2\} \mid \mathcal{F}_{S_1}\right]\mathbb{1}\{B_1\}\right]D^{-+}\Psi_n e^{C^- y}\mathrm{d}y\\
&\quad= \mathbb{E}_{\boldsymbol{\alpha}}\left[(\boldsymbol{A}_{S_1}\Psi_n)\mathbb{1}\{B_1\}\right]D^{-+}\Psi_n e^{C^- y}\mathrm{d}y = \boldsymbol{\alpha}e^{C^+ y}\Psi_n D^{-+}\Psi_n e^{C^- y}\mathrm{d}y, \qquad (4.4)
\end{aligned}
$$

where the strong Markov property was used in the third, fifth, seventh and ninth equalities, Theorem 3.11 in the third and last equalities, the induction hypothesis in the fifth and ninth equalities, and (3.14) in the seventh equality. Thus,

$$
\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\tau^-}\mathbb{1}\{\Omega_{n+1}\}\right] = \boldsymbol{\alpha}\int_0^\infty e^{C^+ y}(D^{+-} + \Psi_n D^{-+}\Psi_n e^{C^- y})\mathrm{d}y = \boldsymbol{\alpha}\Psi_{n+1}. \qquad (4.5)
$$

As (4.5) holds for all $\boldsymbol{\alpha} \in \mathfrak{Z}^+$, Lemma 3.5 implies that Ψ_{n+1} is uniquely determined by (4.3), which completes the proof. □

In Theorem 4.1 we provided a recursion and an integral equation for Ψ_n. Corollary 4.2 sets Ψ_n as the solution of a Sylvester equation, and links $\{\Psi_n\}_{n\geq 1}$ to the probability of $\{R_t\}_{t\geq 0}$ ever returning to 0.

Corollary 4.2. *The matrices $\{\Psi_n\}_{n\geq 0}$ (with $\Psi_0 := 0$) are the unique solutions of the recursive Sylvester equation*

$$
C^+\Psi_{n+1} + \Psi_{n+1}C^- = -D^{+-} - \Psi_n D^{-+}\Psi_n, \quad n \geq 0. \qquad (4.6)
$$

Furthermore,

$$
\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\tau^-}\mathbb{1}\left\{\tau^- < \infty\right\}\right] = \boldsymbol{\alpha}\Psi, \qquad (4.7)
$$

$$
\mathbb{P}_{\boldsymbol{\alpha}}(\tau^- < \infty) = \boldsymbol{\alpha}\Psi\mathbf{1}. \qquad (4.8)
$$

where $\Psi := \lim_{n\to\infty}\Psi_n$.

Proof. In the proof of (3.15) we checked that the eigenvalues of maximal real part of C^- and C^+ both have strictly negative real parts. Thus, premultiplying (4.3) by C^+ and integrating by parts we obtain

$$\begin{aligned} C^+\Psi_{n+1} &= \int_0^\infty C^+ e^{C^+y}(D^{+-}+\Psi_n D^{-+}\Psi_n)e^{C^-y}\mathrm{d}y \\ &= \left[e^{C^+y}(D^{+-}+\Psi_n D^{-+}\Psi_n)e^{C^-y}\right]_0^\infty - \int_0^\infty e^{C^+y}(D^{+-}+\Psi_n D^{-+}\Psi_n)e^{C^-y}C^-\mathrm{d}y \\ &= [0-(D^{+-}+\Psi_n D^{-+}\Psi_n)] - \Psi_{n+1}C^-, \end{aligned}$$

so that Ψ_{n+1} solves (4.6); since C^+ and $-C^-$ do not have overlapping eigenvalues, by general theory of Sylvester equations (for example, see [4]) such a solution is unique. Using the Bounded Convergence Theorem and the fact that $\Omega^- = \cup_{n=1}^\infty \Omega_n$,

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\tau^-}\mathbb{1}\{\tau^-<\infty\}\right] = \lim_{n\to\infty} \boldsymbol{\alpha}\Psi_n. \tag{4.9}$$

Since (4.9) holds for all $\boldsymbol{\alpha} \in \mathfrak{Z}^+$, Lemma 3.5 implies $\lim_{n\to\infty}\Psi_n =: \Psi$ exists and satisfies (4.7). Equation (4.8) follows by noticing that $\boldsymbol{A}_t\mathbf{1} = 1$ for all $t \geq 0$, a consequence of the affine nature of $\mathfrak{Z}^+$. □

The following is a property of the matrix Ψ in the case $\mathbb{P}(\Omega^-) = 1$.

Proposition 4.3. *If* $\mathbb{P}_{\boldsymbol{\alpha}}(\tau^- < \infty) = 1$ *for all* $\boldsymbol{\alpha} \in \mathfrak{Z}^+$, *then* $\Psi\mathbf{1} = \mathbf{1}$.

Proof. Since $\boldsymbol{\alpha}\Psi\mathbf{1} = 1 = \boldsymbol{\alpha}\mathbf{1}$ for all $\boldsymbol{\alpha} \in \mathfrak{Z}^+$, then Lemma 3.5 implies $\Psi\mathbf{1} = \mathbf{1}$. □

4.2 Downward record process

For $x \geq 0$, define $\tau_x^- := \inf\{t > 0 : R_t = -x, \boldsymbol{A}_t \in \mathfrak{Z}^-\}$, the first time the level process $\mathcal{R}$ downcrosses level $-x \leq 0$.

Define the *downward record process* $\{(\ell_x, \boldsymbol{O}_x)\}_{x\geq 0}$ by

$$(\ell_x, \boldsymbol{O}_x) = \begin{cases} (R_{\tau_x^-}, \boldsymbol{A}_{\tau_x^-}) & \text{if} \quad \tau_x^- < \infty \\ (\infty, \Delta) & \text{if} \quad \tau_x^- = \infty, \end{cases}$$

where Δ is some isolated cemetery state. If $\boldsymbol{O}_x \neq \Delta$ for $x \geq 0$, the vector $\boldsymbol{O}_x$ corresponds to the orbit value when the level process downcrosses $\ell_x = -x$ for the first time. We can see that $\{\boldsymbol{O}_x\}_{x\geq 0}$ is a (possibly absorbing) concatenation of orbits with state space $\mathfrak{Z}^- \cup \{\Delta\}$. In the following we compute the average value of $\boldsymbol{O}_x$ on the event that $\boldsymbol{O}_x \neq \Delta$.

Theorem 4.4. *For all* $\boldsymbol{\beta} \in \mathfrak{Z}^-$ *and* $x \geq 0$,

$$\mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{O}_x\mathbb{1}\{\tau_x^- < \infty\}\right] = \boldsymbol{\beta}e^{(C^-+D^{-+}\Psi)x}.$$

Proof. Let $\kappa_0 := 0$, and for $n \geq 1$ define

$$\kappa_n := \inf\{t \geq \kappa_{n-1} : R_t = \inf_{0\leq s\leq t} R_s, \text{ and } \boldsymbol{A}_t \in \mathfrak{Z}^+\};$$

$\{\kappa_n\}_{n\geq 0}$ form the successive time epochs at which the infimum process $\{\inf_{s\geq 0} R_s\}_{s\geq 0}$ stops decreasing. For $n \geq 0$, let $-\sigma_n := R_{\kappa_n}$, the values of $\mathcal{R}$ at these epochs. Note that the sequence $\{\kappa_n + \tau^- \circ \theta_{\kappa_n}\}_{n\geq 0}$ form the successive time epochs at which the infimum process $\{\inf_{s\geq 0} R_s\}_{s\geq 0}$

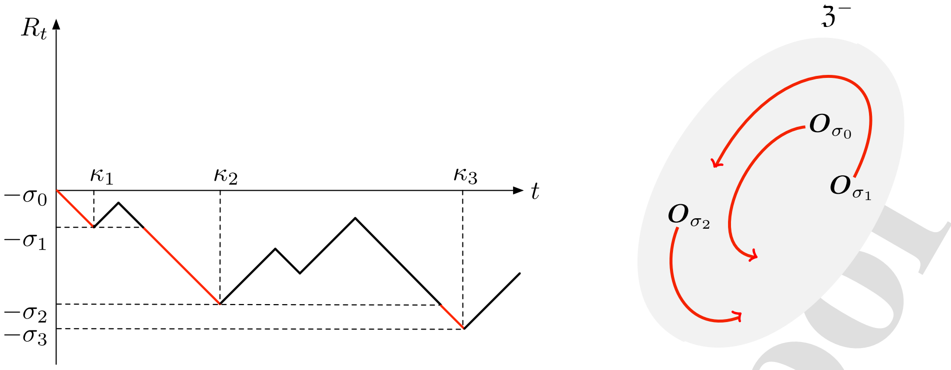

Figure 4: An example of the downward record process associated with $\mathcal{X}$. Left: Downward record levels are shown in red. Right: The concatenation $\{\boldsymbol{O}_x\}_{x\geq 0}$ of the corresponding orbit segments.

starts decreasing, and $\{\kappa_{n+1} - \kappa_n - \tau^- \circ \theta_{\kappa_n}\}_{n\geq 0}$ are the lengths of successive intervals in time at which $\mathcal{R}$ attains a new local minima. By (4.1), for $n \geq 0$

$$\sigma_n - \sigma_{n-1} = \kappa_{n+1} - \kappa_n - \tau^- \circ \theta_{\kappa_n}.$$

See Figure 4 for an illustration.

For $x \geq 0$, let

$$V_x := \inf\{n \geq 1 : \sigma_n > x\};$$

we call V_x the number of *record downcrossings up to level* $-x$. First, we prove by induction that for each $n \geq 1$, there exists a unique continuous matrix function $\Phi_n(\cdot)$ such that

$$\mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{A}_{\tau_x^-}\mathbb{1}\{V_x = n\}\right] = \boldsymbol{\beta}\Phi_n(x). \tag{4.10}$$

Case $n = 1$**.** On $\{V_x = 1\}$, $\tau_x^- = x$ by (4.1). Thus,

$$\mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{A}_{\tau_x^-}\mathbb{1}\{V_x = 1\}\right] = \mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{A}_x\mathbb{1}\{\boldsymbol{A}_s \in \mathfrak{Z}^-\}\ \forall s \in [0,x]\right] = \boldsymbol{\beta}e^{C^-x},$$

so that (4.10) holds with $\Phi_1(x) = e^{C^-x}$. Uniqueness follows from Lemma 3.5.

Inductive part. Suppose (4.10) holds for some $n \geq 1$. Fix $y \in [0,x]$ and define the stopping times

$$S_1 := y, \quad S_2 := S_1 + \tau^+ \circ \theta_{S_1}, \quad S_3 := S_2 + \tau^- \circ \theta_{S_2},$$
$$S_4 := S_3 + \inf\{t > 0 : R_t \circ \theta_{S_3} = -(x-y)\}.$$

Then, $\mathbb{P}(V_x = n+1, \sigma_1 \in (y, y+\mathrm{d}y)) = \mathbb{P}(B_1 \cap B_2 \cap B_3 \cap B_4) + o(\mathrm{d}y)$, where

$$B_1 = \{\boldsymbol{A}_s \in \mathfrak{Z}^- \text{ for all } s \in [0, S_1)\},$$
$$B_2 = \{\exists s \in (S_1, S_1 + \mathrm{d}y) \text{ such that } \boldsymbol{A}_{s-} \in \mathfrak{Z}^- \text{ and } \boldsymbol{A}_s \in \mathfrak{Z}^+\},$$
$$B_3 = \{\mathcal{X} \circ \theta_{S_2} \in \Omega^-\},$$
$$B_4 = \{V_{x-y} \circ \theta_{S_3} = n\}.$$

Moreover, on $\{V_x = n+1, \sigma_1 \in (y, y+\mathrm{d}y)\}$ we have that $\{S_4 \neq \tau_x^-\}$ occurs with a probability of order $o(\mathrm{d}y)$. Then,

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{A}_{\tau_x^-}\mathbb{1}\{V_x = n+1,\ \sigma_1 \in (-y-\mathrm{d}y, -y)\}\right] \\
&\quad= \mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{A}_{S_4}\mathbb{1}\{\cap_{i=1}^4 B_i\}\right] \\
&\quad= \mathbb{E}_{\boldsymbol{\beta}}\left[\mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{A}_{S_4}\mathbb{1}\{B_4\} \mid \mathcal{F}_{S_3}\right]\mathbb{1}\{\cap_{i=1}^3 B_i\}\right] \\
&\quad= \mathbb{E}_{\boldsymbol{\beta}}\left[(\boldsymbol{A}_{S_3}\Phi_n(x-y))\,\mathbb{1}\{\cap_{i=1}^3 B_i\}\right] \quad \text{(by the induction hypothesis)} \\
&\quad= \mathbb{E}_{\boldsymbol{\beta}}\left[\mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{A}_{S_3}\mathbb{1}\{B_3\} \mid \mathcal{F}_{S_2}\right]\mathbb{1}\{\cap_{i=1}^2 B_i\}\right]\Phi_n(x-y) \\
&\quad= \mathbb{E}_{\boldsymbol{\beta}}\left[(\boldsymbol{A}_{S_2}\Psi)\,\mathbb{1}\{\cap_{i=1}^2 B_i\}\right]\Phi_n(x-y) \quad \text{(by Corollary 4.2)} \\
&\quad= \mathbb{E}_{\boldsymbol{\beta}}\left[\mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{A}_{S_2}\mathbb{1}\{B_2\} \mid \mathcal{F}_{S_1}\right]\mathbb{1}\{B_1\}\right]\Psi\Phi_n(x-y) \\
&\quad= \mathbb{E}_{\boldsymbol{\beta}}\left[\left(\boldsymbol{A}_{S_1}D^{-+}\mathrm{d}y\right)\mathbb{1}\{B_1\}\right]\Psi\Phi_n(x-y) \quad \text{(by (3.14))} \\
&\quad= \boldsymbol{\beta}e^{C^-y}D^{-+}\Psi\Phi_n(x-y)\mathrm{d}y
\end{aligned}$$

by Theorem 3.11.

Thus, (4.10) recursively holds for $n \geq 2$ with

$$\Phi_n(x) = \int_0^x e^{C^-y}D^{-+}\Psi\Phi_{n-1}(x-y)\mathrm{d}y,$$

which is unique by Lemma 3.5. Since $\mathfrak{Z}^-$ is bounded, then

$$\mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{A}_{\tau_x^-}\mathbb{1}\{V_x \leq n+1\}\right] = \boldsymbol{\beta}\sum_{m=1}^{n+1}\Phi_m(x)$$

is uniformly bounded in $\boldsymbol{\beta} \in \mathfrak{Z}^-$, $n \geq 0$ and $x \geq 0$. Corollary 3.9 implies that the collection of matrices $\{\sum_{m=1}^{n+1}\Phi_m(x) : n \geq 1, x \geq 0\}$ is uniformly bounded, so that by the Bounded Convergence Theorem,

$$\mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{O}_x\mathbb{1}\{\tau_x^- < \infty\}\right] = \boldsymbol{\beta}\Phi(x)$$

where $\Phi(x) := \sum_{n=1}^{\infty}\Phi_n(x)$ statisfies the equation

$$\Phi(x) = e^{C^-y} + \int_0^x e^{C^-y}D^{-+}\Psi\Phi(x-y)\mathrm{d}y.$$

Theorem 3.10 implies that $\Phi(x) = e^{(C^-+D^{-+}\Psi)x}$, which completes the proof. □

The following corollary shows how the mean value of $\boldsymbol{O}_x$ relates to the probability that the process $\{R_t\}_{t\geq 0}$ ever downcrosses level $-x \leq 0$ given $\boldsymbol{A}_0 \in \mathfrak{Z}^+$.

Corollary 4.5. *For* $\boldsymbol{\alpha} \in \mathfrak{Z}^+$,

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{O}_x\mathbb{1}\{\tau_x^- < \infty\}\right] = \boldsymbol{\alpha}\Psi e^{(C^-+D^{-+}\Psi)x},$$
$$\mathbb{P}_{\boldsymbol{\alpha}}(\tau_x^- < \infty) = \boldsymbol{\alpha}\Psi e^{(C^-+D^{-+}\Psi)x}\mathbf{1}. \tag{4.11}$$

Proof. The strong Markov property, Corollary 4.2 and Theorem 4.4 imply that

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{O}_x\mathbb{1}\{\tau_x^- < \infty\}\right] &= \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{O}_x\mathbb{1}\{\tau_x^- < \infty\} \mid \mathcal{F}_{\tau^-}\right]\mathbb{1}\{\tau^- < \infty\}\right] \\
&= \mathbb{E}_{\boldsymbol{\alpha}}\left[\left(\boldsymbol{A}_{\tau^-}e^{(C^-+D^{-+}\Psi)x}\right)\mathbb{1}\{\tau^- < \infty\}\right] \\
&= \boldsymbol{\alpha}\Psi e^{(C^-+D^{-+}\Psi)x}.
\end{aligned}$$

Equation (4.11) follows because when $\tau_x^- < \infty$, $\boldsymbol{O}_x\mathbf{1} = 1$ due to the affine nature of $\mathfrak{Z}^-$. □

4.3 Stationary distribution of the queue

In this section we determine the limiting behaviour of the process $\mathcal{Q} = \{Q_t\}_{t\geq 0}$ defined by

$$Q_t := R_t + \max\left\{0, \sup_{0\leq s\leq t} -R_s\right\}.$$

The process $\mathcal{Q}$ is the process $\mathcal{R}$ regulated at the boundary in level 0, and thus, can only take values in $[0,\infty)$. We refer to $\{(Q_t, \boldsymbol{A}_t)\}_{t\geq 0}$ as the *RAP-modulated fluid queue*, and assume the following stability condition.

Condition 4.6. *The process $\{Q_t\}_{t\geq 0}$ is positive recurrent under the measure $\mathbb{P}_{\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha} \in \mathfrak{Z}_+ \cup \mathfrak{Z}_-$.*

Condition 4.6 implies that for all $\boldsymbol{\alpha} \in \mathfrak{Z}_+ \cup \mathfrak{Z}_-$,

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\alpha}}\left[\tau^-\right] &< \infty, \\ \mathbb{P}_{\boldsymbol{\alpha}}(\tau_x^- < \infty) &= 1 \quad \text{for } x \geq 0, \\ \Psi\mathbf{1} &= \mathbf{1}. \end{aligned} \tag{4.12}$$

Formulae and pathwise analysis found in this subsection may be compared to those of [10], which were used to compute the stationary distribution of a Markov-modulated fluid queue. In order to study the limiting behaviour of $\mathcal{Q}$, first let us compute

$$\lim_{t\to\infty} \mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t = 0,\ \boldsymbol{A}_t \in \mathfrak{Z}^-\}\right].$$

Define the time-change

$$z_t := \int_0^t \mathbb{1}\{Q_s = 0,\ \boldsymbol{A}_s \in \mathfrak{Z}^-\}\mathrm{d}s, \qquad t \geq 0.$$

A pathwise inspection reveals that on the event $\{Q_t = 0, \boldsymbol{A}_t \in \mathfrak{Z}^-\}$, we have $\boldsymbol{A}_t = \boldsymbol{O}_{z_t}$. Thus,

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t = 0,\ \boldsymbol{A}_t \in \mathfrak{Z}^-\}\right] &= \lim_{t\to\infty} \mathbb{E}\left[\boldsymbol{O}_{z_t} \mathbb{1}\{Q_t = 0,\ \boldsymbol{A}_t \in \mathfrak{Z}^-\}\right] \\ &= \lim_{y\to\infty} c_0 \mathbb{E}\left[\boldsymbol{O}_y\right], \end{aligned} \tag{4.13}$$

where $c_0 := \lim_{t\to\infty} \mathbb{P}(Q_t = 0, \boldsymbol{A}_t \in \mathfrak{Z}^-)$, the proportion of time $\{Q_t\}_{t\geq 0}$ spends in level 0. We will compute the precise value of c_0 in (4.24), for now let it be unknown.

Since, by Condition 4.6 $R_t \to -\infty$ as $t \to \infty$, this implies the first hitting time $\tau_y^- < \infty$ for all level $-y$, and therefore $\boldsymbol{O}_y\mathbf{1} = 1$ almost surely. Thus, Theorem 4.4 implies that under Condition 4.6 and in the case $\boldsymbol{\mathcal{A}}$ starts in $\boldsymbol{\beta} \in \mathfrak{Z}^-$,

$$\boldsymbol{\beta} e^{(C^- + D^{-+}\Psi)y}\mathbf{1} = \mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{O}_y \mathbf{1}\right] = 1 \quad \text{for all } y \geq 0. \tag{4.14}$$

Since $\boldsymbol{A}_0$ can be chosen to take any value $\boldsymbol{\beta}$ in $\mathfrak{Z}^-$, and $\mathfrak{Z}^-$ attains the minimality property, then by Lemma 3.5 and (4.14) the unique solution to $\boldsymbol{\beta x} = 1$ is $\boldsymbol{x} = e^{(C^- + D^{-+}\Psi)y}\mathbf{1}$. Since $\boldsymbol{x} = \mathbf{1}$ is also a solution, it follows that $e^{(C^- + D^{-+}\Psi)y}\mathbf{1} = \mathbf{1}$ for all $y \geq 0$. Differentiating and evaluating at $y = 0$ gives $(C^- + D^{-+}\Psi)\mathbf{1} = \mathbf{0}$, which implies that 0 is an eigenvalue of $C^- + D^{-+}\Psi$. The following is a stronger condition needed for our analysis.

Condition 4.7. *The eigenvalue* 0 *of the matrix* $C^- + D^{-+}\Psi$ *has multiplicity* 1 *and a normalised left eigenvector* $\boldsymbol{v}_0$ *(i.e.,* $\boldsymbol{v}_0\mathbf{1} = 1$*).*

Assume Condition 4.7. Then, by [3, Lemma 4.1(b)], $e^{(C^- + D^{-+}\Psi)y} = \mathbf{1}\boldsymbol{v}_0 + o(e^{-\varepsilon y})$ for some $\varepsilon > 0$. This implies that in the case $\boldsymbol{\mathcal{A}}$ starts in $\boldsymbol{\beta} \in \mathfrak{Z}^-$,

$$\lim_{y\to\infty} \mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{O}_y\right] = \boldsymbol{\beta}(\mathbf{1}\boldsymbol{v}_0) = (\boldsymbol{\beta}\mathbf{1})\boldsymbol{v}_0 = \boldsymbol{v}_0,$$

Similarly, using Corollary 4.5 and Eq. (4.12) we get that in the case $\boldsymbol{\mathcal{A}}$ starts in $\mathfrak{Z}^+$,

$$\lim_{y\to\infty} \mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{O}_y\right] = \boldsymbol{\beta}\Psi(\mathbf{1}\boldsymbol{v}_0) = \boldsymbol{\beta}(\Psi\mathbf{1})\boldsymbol{v}_0 = (\boldsymbol{\beta}\mathbf{1})\boldsymbol{v}_0 = \boldsymbol{v}_0.$$

Consequently, independently of the starting point of $\boldsymbol{\mathcal{A}}$, by (4.13),

$$\lim_{t\to\infty} \mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t = 0,\ \boldsymbol{A}_t \in \mathfrak{Z}^-\}\right] = c_0\boldsymbol{v}_0. \tag{4.15}$$

While (4.15) gives us a good indication of the expected behaviour of $\boldsymbol{A}_t$ for large values of t with $Q_t = 0$, we still need to analyse what happens to $\boldsymbol{\mathcal{A}}$ between the epochs at which $\mathcal{Q}$ leaves the boundary 0 and the epochs at which $\mathcal{Q}$ returns to 0. Thus, we now focus on the properties of $\boldsymbol{\mathcal{A}}$ while on a excursion of $\mathcal{Q}$ away from 0. In the following, we study the process $\mathcal{R}$ up to time τ^-, which is identical to the process $\mathcal{Q}$ up to time τ^-. For $x \ge 0$, let

$$\mathcal{U}^x := \{u \in [0, \tau^-) : R_u = x, \boldsymbol{A}_u \in \mathfrak{Z}^+\}, \quad \mathcal{D}^x := \{u \in (0, \tau^-] : R_u = x, \boldsymbol{A}_u \in \mathfrak{Z}^-\}.$$

Then $\mathcal{U}^x$ corresponds to the set of time epochs, $\{u_i^x\}_{i\ge1}$, at which the process $\mathcal{R}$ upcrosses level x before τ^-, while $\mathcal{D}^x$ corresponds to the set of time epochs, $\{d_i^x\}_{i\ge1}$, at which $\mathcal{R}$ downcrosses level x before τ^-. We compute the expected value of the sum of $\boldsymbol{\mathcal{A}}$ evaluated at each point in $\mathcal{U}^x$ and in $\mathcal{D}^x$ in Theorem 4.8 and Corollary 4.9, respectively.

Theorem 4.8. *Let* $x \ge 0$ *and* $\boldsymbol{\alpha} \in \mathfrak{Z}^+$*. Then,*

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{u_i^x \in \mathcal{U}^x} \boldsymbol{A}_{u_i^x}\right] = \boldsymbol{\alpha} e^{(C^+ + \Psi D^{-+})x}.$$

Proof. First, we classify the set of points $\{u_i^x\}_{i\ge1}$ in $\mathcal{U}^x$ according to their *complexity.* Define

$$\mathcal{U}_1^x := \begin{cases} \{u_1^x\} & \text{if } \boldsymbol{A}_s \in \mathfrak{Z}^+ \ \forall s \in [0, x], \\ \varnothing & \text{otherwise.} \end{cases}$$

By (4.1), the set $\mathcal{U}_1^x$ contains the *simplest* epoch at which $\mathcal{R}$ upcrosses x, in the sense that it upcrosses x before any change of directions; in that case, $u_1^x = x$.

Next, for $\mathcal{X} \in \Omega \setminus \Omega_1$ and for each $u_i^x \in \mathcal{U}^x \setminus \mathcal{U}_1^x$ define

$$\begin{aligned} \gamma_i^x &:= \inf\{y \in (0, x) : \exists t \in [0, u_i^x) \text{ s.t. } R_t = y,\ \boldsymbol{A}_{t-} \in \mathfrak{Z}^-,\ \boldsymbol{A}_t \in \mathfrak{Z}^+\}, \\ \xi_i^x &:= \arg\inf\{y \in (0, x) : \exists t \in [0, u_i^x) \text{ s.t. } R_t = y,\ \boldsymbol{A}_{t-} \in \mathfrak{Z}^-,\ \boldsymbol{A}_t \in \mathfrak{Z}^+\}; \end{aligned}$$

γ_i^x corresponds to the lowest level at which $\mathcal{R}$ has a down-up peak before time u_i^x, and ξ_i^x is the point in time of this down-up peak. For $n \ge 1$ recursively define

$$\mathcal{U}_{n+1}^x := \left\{u_i^x \in \mathcal{U}^x \setminus \mathcal{U}_1^x : u_i^x - \xi_i^x \in \mathcal{U}_n^{x-\gamma_i^x} \circ \theta_{\xi_i^x}\right\},$$

where $\mathcal{U}_n^{x-\gamma_i^x} \circ \theta_{\xi_i^x}$ is the set $U_n^{x-\gamma_i^x}$ of upcrossing times to level $x - \gamma_i^x$ of the $\theta_{\xi_i^x}$-shifted path. Thus, each epoch $u_i^x \in \mathcal{U}_{n+1}^x$ is such that $u_i^x - \xi_i^x$ is an upcrossing epoch in the set $\mathcal{U}_n^{x-\gamma_i^x}$ of the $\theta_{\xi_i^x}$-shifted path; the set $\mathcal{U}_n^{x-\gamma_i^x}$ is of a *lower complexity*[1]. This implies that the collection $\{\mathcal{U}_n^x\}_{n\ge 1}$ is a partition of $\mathcal{U}^x$. See Figure 5 for an illustration.

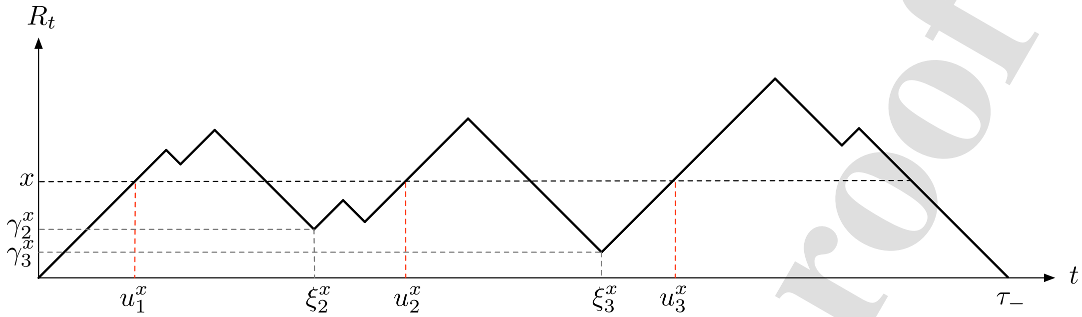

Figure 5: Upcrossing times $u_1^x, u_2^x, u_3^x \in \mathcal{U}^x$ before τ^-. Note that $u_1^x \in \mathcal{U}_1^x$, $u_2^x \in \mathcal{U}_3^x$, $u_3^x \in \mathcal{U}_2^x$.

First, we claim by induction that for all $n \ge 1$ and $x > 0$,

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{u_i^x \in \mathcal{U}_n^x} \boldsymbol{A}_{u_i^x}\right] = \boldsymbol{\alpha}\Upsilon_n(x), \tag{4.16}$$

where $\{\Upsilon_n(\cdot)\}_{n\ge1}$ are some continuous and unique matrices.

Case $n = 1$**.** By (4.1), we have that

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{u_1^x \in \mathcal{U}_1^x} \boldsymbol{A}_{u_i^x}\right] = \mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_x \mathbb{1}\{\boldsymbol{A}_s \in \mathfrak{Z}^+ \ \forall s \in [0,x]\}\right] = \boldsymbol{\alpha} e^{C^+ x},$$

so that (4.16) follows with $\Upsilon_1(x) = e^{C^+ x}$.

Inductive part. Suppose (4.16) holds for some $n \ge 1$. Let $y \in (0, x)$. Note that if $u_i^x, u_j^x \in \mathcal{U}_{n+1}^x$ and $\gamma_i^x, \gamma_j^x \in (y - \mathrm{d}y, y)$, then $\gamma_i^x = \gamma_j^x$ and $\xi_i^x = \xi_j^x$: this follows since each path in Ω there are no jumps that occur at the exact same level. This implies that

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{u_i^x \in \mathcal{U}_{n+1}^x} \boldsymbol{A}_{u_i^x} \mathbb{1}\{\gamma_i^x \in (y - \mathrm{d}y, y)\}\right] = \mathbb{E}_{\boldsymbol{\alpha}}\left[\left[\sum_{u_i^{x-y} \in \mathcal{U}_n^{x-y} \circ \theta_{S_3}} \boldsymbol{A}_{u_i^{x-y}}\right] \mathbb{1}\{B_1 \cap B_2 \cap B_3\}\right],$$

where

$$S_1 := y, \qquad S_2 := S_1 + \tau^- \circ \theta_{S_1}, \qquad S_3 := S_2 + \tau^+ \circ \theta_{S_2},$$

and

$$\begin{aligned} B_1 &= \{\boldsymbol{A}_s \in \mathfrak{Z}^+ \text{ for all } s \in [0, S_1)\}, \\ B_2 &= \{\mathcal{X} \circ \theta_{S_1} \in \Omega^-\}, \\ B_3 &= \{\text{There exists } s \in (S_2, S_2 + \mathrm{d}y) \text{ such that } \boldsymbol{A}_{s-} \in \mathfrak{Z}^- \text{ and } \boldsymbol{A}_s \in \mathfrak{Z}^+\}. \end{aligned}$$

[1]Another way to view this partition of sets: $u \in \mathcal{U}_n^x$ if and only if the time-reversed process $\{R_{u-s} - R_u\}_{s=0}^u$ reaches level $-x$ (at time u) with $V_x = n$.

Then,

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{u_i^x\in\mathcal{U}_{n+1}^x} \boldsymbol{A}_{u_i^x}\mathbb{1}\{\gamma_i^x\in(y-\mathrm{d}y,y)\}\right]\\
&=\mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{u_i^{x-y}\in\mathcal{U}_n^{x-y}\circ\theta_{S_3}} \boldsymbol{A}_{u_i^x}\,\middle|\,\mathcal{F}_{S_3}\right]\mathbb{1}\{B_1\cap B_2\cap B_3\}\right]\\
&=\mathbb{E}_{\boldsymbol{\alpha}}\left[(\boldsymbol{A}_{S_3}\Upsilon_n(x-y))\,\mathbb{1}\{B_1\cap B_2\cap B_3\}\right]\quad\text{(by the induction hypothesis)}\\
&=\mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{S_3}\mathbb{1}\{B_3\}\mid\mathcal{F}_{S_2}\right]\mathbb{1}\{B_1\cap B_2\}\right]\Upsilon_n(x-y)\\
&=\mathbb{E}_{\boldsymbol{\alpha}}\left[\left(\boldsymbol{A}_{S_2}D^{-+}\mathrm{d}y\right)\mathbb{1}\{B_1\cap B_2\}\right]\Upsilon_n(x-y)\quad\text{(by (3.14))}\\
&=\mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{S_2}\mathbb{1}\{B_2\}\mid\mathcal{F}_{S_1}\right]\mathbb{1}\{B_1\}\right]D^{-+}\Upsilon_n(x-y)\mathrm{d}y\\
&=\mathbb{E}_{\boldsymbol{\alpha}}\left[(\boldsymbol{A}_{S_1}\Psi)\,\mathbb{1}\{B_1\}\right]D^{-+}\Upsilon_n(x-y)\mathrm{d}y\quad\text{(by Corollary 4.2)}\\
&=\boldsymbol{\alpha}e^{C^+y}\Psi D^{-+}\Upsilon_n(x-y)\mathrm{d}y\quad\text{(by Theorem 3.11)}.
\end{aligned}
$$

Thus, (4.16) recursively holds with

$$\Upsilon_n(x)=\int_0^x e^{C^+y}\Psi D^{-+}\Upsilon_{n-1}(x-y)\mathrm{d}y,$$

with this matrix being unique by Lemma 3.5. By means similar to those in the proof of Theorem 4.4, using Corollary 3.9 and the ergodicity of $\{Q_t\}_{t\geq 0}$ we obtain

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{u_i^x\in\mathcal{U}^x}\boldsymbol{A}_{u_i^x}\right]=\boldsymbol{\beta}\Upsilon(x)$$

for $\Upsilon(x):=\sum_{n=1}^{\infty}\Upsilon_n(x)$ which is uniformly entrywise-bounded on compact intervals and satisfies

$$\Upsilon(x)=e^{C^+y}+\int_0^x e^{C^+y}\Psi D^{-+}\Upsilon(x-y)\mathrm{d}y.$$

Theorem 3.10 implies that $\Upsilon(x)=e^{(C^++\Psi D^{-+})x}$, which completes the proof. □

The expected value of the orbit at the points in $\mathcal{D}^x$ is computed as follows.

Corollary 4.9. *Let $x\geq 0$ and $\boldsymbol{\alpha}\in\mathfrak{Z}^+$. Then,*

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{d_i^x\in\mathcal{D}^x}\boldsymbol{A}_{d_i^x}\right]=\boldsymbol{\alpha}e^{(C^++\Psi D^{-+})x}\Psi.$$

Proof. Let $S = \tau^- \circ \theta_{u_i^x}$, the time elapsed from u_i^x until the next downcrossing of level x. Then

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{d_i^x \in \mathcal{D}^x} \boldsymbol{A}_{d_i^x}\right] &= \mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{u_i^x \in \mathcal{U}^x} \boldsymbol{A}_{u_i^x+S}\right] \\
&= \mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{u_i^x \in \mathcal{U}^x} \mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{u_i^x+S} \,\middle|\, \mathcal{F}_{u_i^x}\right]\right] \\
&= \mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{u_i^x \in \mathcal{U}^x} \boldsymbol{A}_{u_i^x}\Psi\right] \quad \text{(by Corollary 4.2)} \\
&= \boldsymbol{\alpha} e^{(C^+ + \Psi D^{-+})x}\Psi,
\end{aligned}$$

by Theorem 4.8. □

Remark 4.10. *Since* $\lim_{t\to\infty} R_t = -\infty$ *a.s., Theorem 4.8 implies that*

$$\lim_{x\to\infty} \boldsymbol{\alpha} e^{(C^+ + \Psi D^{-+})x} = \mathbf{0} \quad \textit{for all } \boldsymbol{\alpha} \in \mathfrak{Z}^+.$$

Lemma 3.5 implies that

$$\lim_{x\to\infty} e^{(C^+ + \Psi D^{-+})x} = 0,$$

so that the dominant eigenvalue of $C^+ + \Psi D^{-+}$ *has strictly negative real part.*

Now we are ready to state and prove the main result of this section regarding the limiting behaviour of $(\mathcal{Q}, \boldsymbol{\mathcal{A}})$.

Theorem 4.11. *For* $x > 0$ *define*

$$\Pi^+(x) = \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t \in (0,x),\ \boldsymbol{A}_t \in \mathfrak{Z}^+\}\right], \tag{4.17}$$

$$\Pi^-(x) = \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t \in (0,x),\ \boldsymbol{A}_t \in \mathfrak{Z}^-\}\right]. \tag{4.18}$$

Then,

$$\begin{aligned}
\Pi^+(x) &= c_0 \boldsymbol{v}_0 D^{-+} e^{(C^+ + \Psi D^{-+})x} \quad \textit{and} \\
\Pi^-(x) &= c_0 \boldsymbol{v}_0 D^{-+} e^{(C^+ + \Psi D^{-+})x}\Psi,
\end{aligned} \tag{4.19}$$

where $c_0 := \lim_{t\to\infty} \mathbb{P}(Q_t = 0, \boldsymbol{A}_t \in \mathfrak{Z}^-)$ *and* $\boldsymbol{v}_0$ *is defined as in Condition 4.7.*

Proof. For $t > 0$, let $\chi_t := \sup\{x > 0 : Q_r > 0 \text{ for all } r \in (t-x, t]\}$; see Figure 6 for a pathwise description of χ_t.

Then, for $h > 0$,

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t \in [x, x+h),\ \boldsymbol{A}_t \in \mathfrak{Z}^+\}\right] \\
&= \lim_{t\to\infty} \int_{s=0}^t \mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t \in (x, x+h),\ \boldsymbol{A}_t \in \mathfrak{Z}^+, \chi_t \in (s, s+\mathrm{d}s)\}\right] \\
&= \lim_{t\to\infty} \int_{s=0}^t \mathbb{E}\left[\mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t \in [x, x+h),\ \boldsymbol{A}_t \in \mathfrak{Z}^+, \chi_t \in (s, s+\mathrm{d}s)\} \,\middle|\, \mathcal{F}_{t-s}\right]\right].
\end{aligned}$$

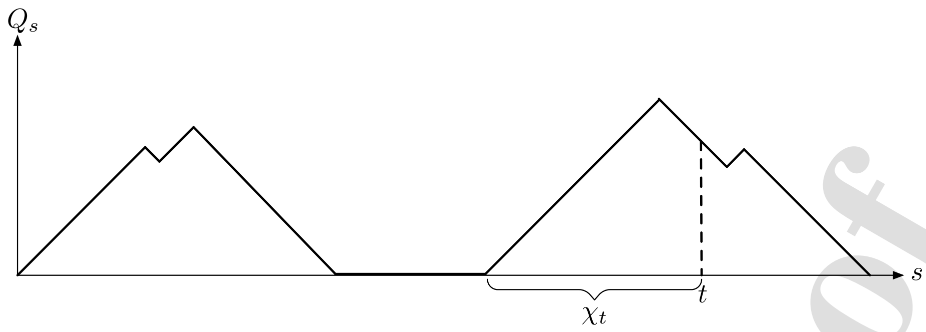

Figure 6: The random variable χ_t associated to the RAP-modulated fluid queue $\{Q_s\}_{s\geq 0}$. Note that it corresponds to the elapsed time since $\{Q_s\}_{s\geq 0}$ left 0 prior to t.

Now, focusing on the inner expectation, we have

$$\begin{aligned}
&\mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t \in [x, x+h),\ \boldsymbol{A}_t \in \mathfrak{z}^+, \chi_t \in (s, s+\mathrm{d}s)\} \,\middle|\, \mathcal{F}_{t-s}\right] \\
&= \mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t \in [x, x+h),\ \boldsymbol{A}_t \in \mathfrak{z}^+, Q_{t-s} = 0, \boldsymbol{A}_{t-s} \in \mathfrak{z}^-, \boldsymbol{A}_{t-s+\mathrm{d}s} \in \mathfrak{z}^+\} \,\middle|\, \mathcal{F}_{t-s}\right] \\
&= \mathbb{E}_{\boldsymbol{A}_{t-s}}\left[\boldsymbol{A}_s \mathbb{1}\{Q_s \in [x, x+h),\ \boldsymbol{A}_s \in \mathfrak{z}^+, \boldsymbol{A}_0 \in \mathfrak{z}^-, \boldsymbol{A}_{\mathrm{d}s} \in \mathfrak{z}^+\}\right] \mathbb{1}\{B_{t-s}\}. \qquad (4.20)
\end{aligned}$$

where $B_t := \{Q_t = 0, \boldsymbol{A}_t \in \mathfrak{z}^-\}$, $t \geq 0$. Thus, by Fubini's Theorem, we have for $h > 0$,

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t \in [x, x+h),\ \boldsymbol{A}_t \in \mathfrak{z}^+\}\right] \\
&= \lim_{t\to\infty} \mathbb{E}\left[\int_{s=0}^{\infty} \mathbb{1}\{s \leq t\} \cdot \mathbb{E}_{\boldsymbol{A}_{t-s}}\left[\boldsymbol{A}_s \mathbb{1}\{Q_s \in [x, x+h),\ \boldsymbol{A}_s \in \mathfrak{z}^+, \boldsymbol{A}_0 \in \mathfrak{z}^-, \boldsymbol{A}_{\mathrm{d}s} \in \mathfrak{z}^+\}\right] \mathbb{1}\{B_{t-s}\}\right]. \\
&\qquad (4.21)
\end{aligned}$$

In (4.21), the Bounded Convergence Theorem allows us to switch the limit with the expectation and integral operators, change the variable $t-s$ to t in the integrand (which is valid since both variables converge to ∞), and switch the limit with the expectation and integral operators once again. Thus,

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t \in [x, x+h),\ \boldsymbol{A}_t \in \mathfrak{z}^+\}\right] \\
&= \lim_{t\to\infty} \mathbb{E}\left[\int_{s=0}^{\infty} \mathbb{E}_{\boldsymbol{A}_t}\left[\boldsymbol{A}_s \mathbb{1}\{Q_s \in [x, x+h),\ \boldsymbol{A}_s \in \mathfrak{z}^+, \boldsymbol{A}_0 \in \mathfrak{z}^-, \boldsymbol{A}_{\mathrm{d}s} \in \mathfrak{z}^+\}\right] \mathbb{1}\{B_t\}\right]. \qquad (4.22)
\end{aligned}$$

Since the integrand in the expression above is computed on the event $\{\boldsymbol{A}_t \in \mathfrak{z}^-\}$, w.l.o.g. suppose that $\boldsymbol{A}_t \in \mathfrak{z}_i^-$ for some $i \in \mathcal{N}$. Then,

$$\begin{aligned}
&\int_{s=0}^{\infty} \mathbb{E}_{\boldsymbol{A}_t}\left[\boldsymbol{A}_s \mathbb{1}\{Q_s \in [x, x+h),\ \boldsymbol{A}_s \in \mathfrak{z}^+, \boldsymbol{A}_0 \in \mathfrak{z}^-, \boldsymbol{A}_{\mathrm{d}s} \in \mathfrak{z}^+\}\right] \mathbb{1}\{B_t\} \\
&= \int_{s=0}^{\infty} \sum_{j\in\mathcal{N}} \mathbb{E}_{\boldsymbol{\alpha}_t}\left[\boldsymbol{A}_s \mathbb{1}\{Q_s \in [x, x+h),\ \boldsymbol{A}_s \in \mathfrak{z}^+\}\right] \mathbb{1}\{B_t\}(\boldsymbol{A}_t \widehat{D}_{ij}^{-+} \mathbf{1} \mathrm{d}s),
\end{aligned}$$

where $\boldsymbol{\alpha}_t := \frac{\boldsymbol{A}_t \widehat{D}_{ij}^{-+}}{\boldsymbol{A}_t \widehat{D}_{ij}^{-+}\mathbf{1}}$,

$$
\begin{aligned}
&= \sum_{j\in N} \mathbb{E}_{\boldsymbol{\alpha}_t}\left[\int_{s=0}^{\infty} \boldsymbol{A}_s \mathbb{1}\{Q_s \in [x, x+h),\ \boldsymbol{A}_s \in \mathfrak{Z}^+\}\mathrm{d}s\right] \mathbb{1}\{B_t\}(\boldsymbol{A}_t \widehat{D}_{ij}^{-+}\mathbf{1}) \\
&= \sum_{j\in N} \mathbb{E}_{\boldsymbol{\alpha}_t}\left[\sum_{u\in\mathcal{U}^x} \boldsymbol{A}_u h + o(h)\right] \mathbb{1}\{B_t\}(\boldsymbol{A}_t \widehat{D}_{ij}^{-+}\mathbf{1}). \qquad (4.23)
\end{aligned}
$$

Substituting (4.23) into (4.22) gives

$$
\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t \in [x, x+h),\ \boldsymbol{A}_t \in \mathfrak{Z}^+\}\right] \\
&= \lim_{t\to\infty} \mathbb{E}\left[\sum_{j\in N} \mathbb{E}_{\boldsymbol{\alpha}_t}\left[\sum_{u\in\mathcal{U}^x} \boldsymbol{A}_u h + o(h)\right] \mathbb{1}\{B_t\}(\boldsymbol{A}_t \widehat{D}_{ij}^{-+}\mathbf{1})\right] \\
&= \lim_{t\to\infty} \mathbb{E}\left[\sum_{j\in N} \boldsymbol{\alpha}_t (e^{C^+\Psi D^{-+}x} h)\mathbb{1}\{B_t\}(\boldsymbol{A}_t \widehat{D}_{ij}^{-+}\mathbf{1}) + o(h)\right] \\
&= \lim_{t\to\infty} \mathbb{E}\left[\boldsymbol{A}_t D^{-+}(e^{C^+\Psi D^{-+}x} h)\mathbb{1}\{B_t\} + o(h)\right] \\
&= c_0 \boldsymbol{v}_0 D^{-+} e^{(C^+ + \Psi D^{-+})x} h + o(h),
\end{aligned}
$$

where (4.15) is used in the last equality. Equation (4.19) follows by analogous steps and arguments in the proof of Corollary 4.9. □

To compute c_0, note that since $\boldsymbol{A}_t \mathbf{1} = 1$ for all $t \geq 0$,

$$
\begin{aligned}
1 &= \lim_{t\to\infty} \mathbb{E}[\boldsymbol{A}_t]\,\mathbf{1} \\
&= \left(\lim_{t\to\infty} \mathbb{E}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t = 0,\ \boldsymbol{A}_t \in \mathfrak{Z}^-\}\right]\mathbf{1}\right) + \int_0^\infty \Pi^+(x)\mathbf{1}\mathrm{d}x + \int_0^\infty \Pi^-(x)\mathbf{1}\mathrm{d}x \\
&= c_0 + c_0 \boldsymbol{v}_0 \left[D^{-+}\int_0^\infty e^{(C^+ + \Psi D^{-+})x}\mathrm{d}x\mathbf{1} + D^{-+}\int_0^\infty e^{(C^+ + \Psi D^{-+})x}\mathrm{d}x(\Psi\mathbf{1})\right] \\
&= c_0\left(1 - 2\boldsymbol{v}_0 D^{-+}(C^+ + \Psi D^{-+})^{-1}\mathbf{1}\right), \quad \text{since } \Psi\mathbf{1} = \mathbf{1}. \qquad (4.24)
\end{aligned}
$$

By solving (4.24) for c_0 and using Theorem 4.11 we arrive at the following.

Corollary 4.12. *Let*

$$
\pi(x) = \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(Q_t \in (0, x)), \qquad x \geq 0, \qquad (4.25)
$$

which we call the stationary density *of* $\{Q_t\}_{t\geq 0}$. *Then,*

$$
\begin{aligned}
\pi(x) &= 2c_0 \boldsymbol{v}_0 D^{-+} e^{(C^+ + \Psi D^{-+})x}\mathbf{1}, \\
\lim_{t\to\infty} \mathbb{P}(Q_t = 0, \boldsymbol{A}_t \in \mathfrak{Z}^-) &= c_0
\end{aligned}
$$

with $c_0 = \left(1 - 2\boldsymbol{v}_0 D^{-+}(C^+ + \Psi D^{-+})^{-1}\mathbf{1}\right)^{-1}$.

5 RAP-modulated fluid process with unit and zero rates

Here we consider a modification of the RAP-modulated fluid process with unit rates by allowing the orbit to transition back and forth to an additional space, $\mathfrak{Z}^0$, and by letting $\mathcal{R}$ be piecewise-constant during the sojourn times of $\boldsymbol{\mathcal{A}}$ at $\mathfrak{Z}^0$. More specifically, let us we augment the index set $\{+,-\}$ with 0, so that $\mathcal{S} := \{+,-,0\}$ from now on. The orbit process $\boldsymbol{\mathcal{A}}$ is then a PDMP transitioning between three disjoint sets, $\mathfrak{Z}^+$, $\mathfrak{Z}^-$ and $\mathfrak{Z}^0$. Each set $\mathfrak{Z}^k$, $k \in \mathcal{S}$, is partitioned in disjoint affine hyperplanes $\{\mathfrak{Z}_i^k\}_{i=1}^{n_k}$, with $\boldsymbol{\mathcal{A}}$ evolving in $\mathfrak{Z} := \cup_{k\in\mathcal{S}}\mathfrak{Z}$ according to (3.2), and with a jump mechanism described by the matrices (3.4) and (3.5).

Definition 5.1. *The* RAP-modulated fluid process with unit and zero rates *is the Markov additive process* $(\mathcal{R}, \boldsymbol{\mathcal{A}}) = \{(R_t, \boldsymbol{A}_t)\}_{t\geq 0}$*, where* $\boldsymbol{\mathcal{A}}$ *is an* orbit process *with state space* $\mathfrak{Z}^+ \cup \mathfrak{Z}^- \cup \mathfrak{Z}^0$ *and an arbitrary but fixed initial point* $\boldsymbol{A}_0 \in \mathfrak{Z}^+ \cup \mathfrak{Z}^- \cup \mathfrak{Z}^0$*, and* $\mathcal{R}$ *is a* level process *of the form*

$$R_t := \int_0^t \mathbb{1}\{\boldsymbol{A}_s \in \mathfrak{Z}^+\} - \mathbb{1}\{\boldsymbol{A}_s \in \mathfrak{Z}^-\}\mathrm{d}s. \tag{5.1}$$

Even though $\mathcal{R}$ defined by (5.1) has the same form as in (3.8), the sojourn times of $\boldsymbol{\mathcal{A}}$ in $\mathfrak{Z}^0$ translate to piecewise-constant intervals of $\mathcal{R}$ in Definition 5.1; such a behaviour is not present in Definition 3.6. See Figure 7 for a visual description of a RAP-modulated fluid process with unit and zero rates.

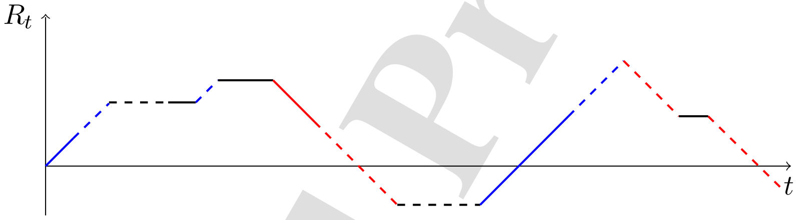

Figure 7: A sample path of the process $\{R_t\}_{t\geq 0}$ whose orbit process $\{\boldsymbol{A}_t\}_{t\geq 0}$ has state space $\mathfrak{Z} = \mathfrak{Z}^+ \cup \mathfrak{Z}^- \cup \mathfrak{Z}^0$ and intial state $\boldsymbol{A}_0 \in \mathfrak{Z}^+$, with $\mathfrak{Z}^+ = \mathfrak{Z}^+_{\text{solid}} \cup \mathfrak{Z}^+_{\text{dashed}}$, $\mathfrak{Z}^- = \mathfrak{Z}^-_{\text{solid}} \cup \mathfrak{Z}^-_{\text{dashed}}$ and and $\mathfrak{Z}^0 = \mathfrak{Z}^0_{\text{solid}} \cup \mathfrak{Z}^0_{\text{dashed}}$.

From now on we assume that Conditions 3.1 and 3.4 hold for the case of RAP-modulated fluid processes with unit and zero rates. To guarantee non-triviality of the fluid process (i.e. the path $\mathcal{R}$ has both piecewise downwards and upwards intervals a.s.), Condition 3.7 needs to be updated to the following.

Condition 5.2. *For any* $k \in \{+,-\}$*,*

$$\lim_{t\to\infty} \mathbb{P}\left(\boldsymbol{A}_s \in \mathfrak{Z}^k \cup \mathfrak{Z}^0 \ \forall s \in [0,t] \,\middle|\, \boldsymbol{A}_0 = \boldsymbol{\alpha}\right) = 0 \quad \textit{for all } \boldsymbol{\alpha} \in \mathfrak{Z}^k.$$

Likewise, let $(\Omega_*, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{\mathbb{P}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathfrak{Z}})$ be the canonical probability space associated to the Markov process $\{(R_t, \boldsymbol{A}_t)\}_{t\geq 0}$ and let Ω be the set of paths in $\omega = \{(r_t, \boldsymbol{a}_t)\}_{t\geq 0} \in \Omega_*$ such that

- $\{\boldsymbol{a}_t\}_{t\geq 0}$ has a finite number of jumps on each compact time interval,
- for all $s \geq 0$, neither $\boldsymbol{a}_t \in \mathfrak{Z}^+ \cup \mathfrak{Z}^0$ for all $t \geq s$, nor $\boldsymbol{a}_t \in \mathfrak{Z}^- \cup \mathfrak{Z}^0$ for all $t \geq s$,

- there are no $s, t \geq 0$, $s \neq t$ such that $r_s = r_t$, $\boldsymbol{a}_{s-} \in \mathfrak{Z}^k$, $\boldsymbol{a}_s \neq \boldsymbol{a}_{s-}$, $\boldsymbol{a}_{t-} \in \mathfrak{Z}^\ell$, $\boldsymbol{a}_t \neq \boldsymbol{a}_{t-}$, for $k, \ell \in \{+, -\}$.

As in Section 4, Ω is the set of nice paths which occur with probability 1: w.l.o.g. we restrict the forthcoming analysis to this set. Furthermore, note that the results in Subsection 3.3 translate verbatim to the case of RAP-modulated processes with unit and zero rates; in particular, Theorem 3.11 and Lemma 3.12 hold.

Next, we develop analogous results to those in Section 4 for RAP-modulated fluid processes with unit and zero rates. Since in this framework $\mathcal{R}$ may have some piecewise-constant intervals, the concepts of up-down and down-up peaks used throughout Section 4 are not sufficient. To address this issue, for $k \in \{+, -\}$ define

$$\rho_*^k := \rho^k + \rho^0 \circ \theta_{\rho^k},$$

where $\rho^\ell := \inf\{s \geq 0 : \boldsymbol{A}_s \notin \mathfrak{Z}^\ell\}$ for $\ell \in \{+, -, 0\}$. Then ρ_*^k corresponds to the first exit time from $\mathfrak{Z}^0$ following an exit from $\mathfrak{Z}^k$. Note that $\rho_*^k = \rho^k$ if and only if $\boldsymbol{A}_{\rho^k} \notin \mathfrak{Z}^0$.

Lemma 5.3. *Let $k, \ell \in \{+, -\}$ and let $\boldsymbol{\alpha} \in \mathfrak{Z}^k$. Then, for all $x \geq 0$.*

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\rho_*^k} \mathbb{1}\left\{\rho^k \in (x, x + \mathrm{d}x),\ \boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^0,\ \boldsymbol{A}_{\rho_*^k} \in \mathfrak{Z}^\ell\right\}\right] = \boldsymbol{\alpha} e^{C^k x} D^{k0} (-C^0)^{-1} D^{0\ell} \mathrm{d}x$$

Proof. Using the strong Markov property and Lemma 3.12, we obtain

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\rho_*^k} \mathbb{1}\left\{\rho^k \in (x, x + \mathrm{d}x),\ \boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^0,\ \boldsymbol{A}_{\rho_*^k} \in \mathfrak{Z}^\ell\right\}\right] \\
&\quad = \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\rho_*^k} \mathbb{1}\left\{\boldsymbol{A}_{\rho_*^k} \in \mathfrak{Z}^\ell\right\} \,\middle|\, \mathcal{F}_{\rho^k}\right] \mathbb{1}\left\{\rho^k \in (x, x + \mathrm{d}x),\ \boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^0\right\}\right] \\
&\quad = \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{A}_{\rho^k}}\left[\boldsymbol{A}_{\rho^0} \mathbb{1}\left\{\boldsymbol{A}_{\rho^0} \in \mathfrak{Z}^\ell\right\}\right] \mathbb{1}\left\{\rho^k \in (x, x + \mathrm{d}x),\ \boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^0\right\}\right] \\
&\quad = \mathbb{E}_{\boldsymbol{\alpha}}\left[\left(\boldsymbol{A}_{\rho^k} (-C^0)^{-1} D^{0\ell}\right) \mathbb{1}\left\{\rho^k \in (x, x + \mathrm{d}x),\ \boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^0\right\}\right] \\
&\quad = \mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\rho^k} \mathbb{1}\left\{\rho^k \in (x, x + \mathrm{d}x),\ \boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^0\right\}\right] (-C^0)^{-1} D^{0\ell} \\
&\quad = \boldsymbol{\alpha} e^{C^k x} D^{k0} (-C^0)^{-1} D^{0\ell} \mathrm{d}x.
\end{aligned}$$

□

Lemma 5.3 is analogous to *censoring* in the context of Markov-modulated fluid processes. In general terms, it concerns the inspection of certain aspects of a process before and after some interval; in the case of Lemma 5.3 such an interval is (ρ^k, ρ_*^k), during which $\boldsymbol{A}_t \in \mathfrak{Z}^0$, and the aspects inspected are ρ^k and $\boldsymbol{A}_{\rho_*^k}$.

When $n^0 > 0$, we say that an *up-down peak* of $\mathcal{R}$ occurs at time $t > 0$ if there exists some $s \in (0, t)$ such that $\rho_*^+ \circ \theta_s = t - s$ and $\boldsymbol{A}_t \in \mathfrak{Z}_-$. That is, an up-down peak results from $\{\boldsymbol{A}_t\}_{t \geq 0}$ exiting $\mathfrak{Z}^+$ and either going directly to $\mathfrak{Z}^-$, or going through $\mathfrak{Z}^0$ and jumping later to $\mathfrak{Z}^-$. Analogously, a *down-up peak* happens at time $t > 0$ if there exists some $s \in (0, t)$ such that $\rho_*^- \circ \theta_s = t - s$ and $\boldsymbol{A}_t \in \mathfrak{Z}_+$.

In the following we compute the expected value of the orbit at a peak for the general setting.

Corollary 5.4. *Let $k, \ell \in \{+, -\}$, $k \neq \ell$, and $\boldsymbol{\alpha} \in \mathfrak{Z}^k$. Then,*

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\rho_*^k} \mathbb{1}\left\{\rho^k \in (x, x + \mathrm{d}x),\ \boldsymbol{A}_{\rho_*^k} \in \mathfrak{Z}^\ell\right\}\right] = \boldsymbol{\alpha} e^{C^k x} D^{k\ell *} \mathrm{d}x, \tag{5.2}$$

where $D^{k\ell *} = D^{k\ell} + D^{k0} (-C^0)^{-1} D^{0\ell}$.

Proof. Condition the indicator function on the LHS of (5.2) on the events $\{\boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^\ell\}$ and $\{\boldsymbol{A}_{\rho^k} \in \mathfrak{Z}^0\}$. The result follows by Lemma 3.12 and Lemma 5.3. □

Now that we have a notion for *peaks*, we develop next a way to handle the intervals between peaks in the case $n^0 > 0$. For instance, suppose that $\boldsymbol{A}_0 \in \mathfrak{Z}_+$. A first step would be to compute the expected value of $\boldsymbol{\mathcal{A}}$ at the instant $\mathcal{R}$ reaches some level $x \geq 0$ given that no peak has occured until that epoch, similar in spirit to Theorem 3.11. Note that up to the time before the first up-down peak, the orbit process can perform jumps from/to $\mathfrak{Z}^+$ and $\mathfrak{Z}^0$, so the analysis differs from that of Section 4. However, it turns out that the solution to this problem has a structure similar to that of Theorem 3.11 once we censor the occupation times in $\mathfrak{Z}^0$, as we will see next.

For $t \geq 0$, define

$$W_t := \int_0^t \mathbb{1}\{\boldsymbol{A}_s \in \mathfrak{Z}^+ \cup \mathfrak{Z}^-\}\mathrm{d}s, \quad \zeta_x := \inf\{t \geq 0 : W_t > x\}.$$

Thus, W_t corresponds to the *occupation time of* $\{\boldsymbol{A}_t\}_{t\geq 0}$ *in* $\mathfrak{Z}^+ \cup \mathfrak{Z}^-$ *up to time* t, while ζ_x corresponds to the *necessary time to reach* x *units of occupation time in* $\mathfrak{Z}^+ \cup \mathfrak{Z}^-$. The process $\{W_t\}_{t\geq 0}$ is continuous nondecreasing and $\{\zeta_x\}_{x\geq 0}$ is càdlàg. A process analogous to $\{W_t\}_{t\geq 0}$ has been used in the Markov-modulated fluid processes literature, it being regarded as the total amount of fluid that has flowed into or out of the system [8].

Since the slopes of $\mathcal{R}$ are either 1,−1 or 0, for $t, x \geq 0$ and $k \in \{+,-\}$

$$\{\boldsymbol{A}_t \in \mathfrak{Z}^k \cup \mathfrak{Z}^0\} \cap \{|R_{t+\zeta_x\circ\theta_t} - R_t| = x\} = \left\{\boldsymbol{A}_u \in \mathfrak{Z}^k \cup \mathfrak{Z}^0 \;\forall u \in [t, t+\zeta_x \circ \theta_t)\right\}, \tag{5.3}$$

a relation similar to (4.1). In the following we investigate more about the event (5.3).

Theorem 5.5. *Let* $\boldsymbol{\alpha} \in \mathfrak{Z}^k$, $k \in \{+,-\}$ *and* $x \geq 0$. *Then*

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\zeta_x}\mathbb{1}\left\{\boldsymbol{A}_s \in \mathfrak{Z}^k \cup \mathfrak{Z}^0 \;\forall s \in [0,\zeta_x]\right\}\right] = \boldsymbol{\alpha}e^{C^{k*}x},$$

where $C^{k*} = C^k + D^{k0}(-C^0)^{-1}D^{0k}$.

Proof. Let $s_0^* := \inf\{y > 0 : \boldsymbol{A}_{\zeta_y} \in \mathfrak{Z}^0\} = \inf\{t > 0 : \boldsymbol{A}_t \in \mathfrak{Z}^0\}$, and for $x \geq 0$ let

$$L_x^* := \#\{s \in (0,\zeta_x] : \boldsymbol{A}_{s-} \in \mathfrak{Z}^k, \boldsymbol{A}_s \in \mathfrak{Z}^0\}.$$

We claim that there exist continuous matrices $\{\Sigma_n^*(\cdot)\}_{n\geq 0}$ such that for all $n \geq 0$,

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\zeta_x}\mathbb{1}\left\{\boldsymbol{A}_s \in \mathfrak{Z}^k \cup \mathfrak{Z}^0 \;\forall s \in [0,\zeta_x],\; L_x^* = n\right\}\right] = \boldsymbol{\alpha}\Sigma_n^*(x). \tag{5.4}$$

Case $n = 0$**.** As $\zeta_x = x$ on $\{L_x^* = 0\}$, we have

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\zeta_x}\mathbb{1}\left\{\boldsymbol{A}_s \in \mathfrak{Z}^k \cup \mathfrak{Z}^0 \;\forall s \in [0,\zeta_x],\; L_x^* = 0\right\}\right] = \mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\zeta_x}\mathbb{1}\left\{\boldsymbol{A}_s \in \mathfrak{Z}^k \;\forall s \in [0,x]\right\}\right] = \boldsymbol{\alpha}e^{C^k x},$$

where Theorem 3.11 is used in the last equality. Thus, (5.4) follows by choosing $\Sigma_0^*(x) = e^{C^k x}$, and this solution is unique by Lemma 3.5.

Inductive part. Suppose that (5.4) holds for some $n \geq 0$. Then, for $r \in (0, x)$

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\zeta_x}\mathbb{1}\left\{\boldsymbol{A}_s \in \mathfrak{Z}^k \cup \mathfrak{Z}^0\ \forall s \in [0,\zeta_x],\ L_x^* = n+1, s_0^* \in (r-\mathrm{d}r, r)\right\}\right] \\
&\quad = \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\zeta_x}\mathbb{1}\left\{\boldsymbol{A}_s \in \mathfrak{Z}^k \cup \mathfrak{Z}^0\ \forall s \in [0,\zeta_x],\ L_x^* = n+1\right\} \,\middle|\, \mathcal{F}_{\zeta_r}\right]\mathbb{1}\left\{s_0^* \in (r-\mathrm{d}r, r),\ \boldsymbol{A}_{\zeta_r} \in \mathfrak{Z}^k\right\}\right] \\
&\quad = \mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{E}_{\boldsymbol{A}_{\zeta_r}}\left[\boldsymbol{A}_{\zeta_{x-r}}\mathbb{1}\left\{\boldsymbol{A}_s \in \mathfrak{Z}^k \cup \mathfrak{Z}^0\ \forall s \in [0,\zeta_{x-r}],\ L_{x-r}^* = n\right\}\right]\mathbb{1}\left\{s_0^* \in (r-\mathrm{d}r, r),\ \boldsymbol{A}_{\zeta_r} \in \mathfrak{Z}^k\right\}\right] \\
&\quad = \mathbb{E}_{\boldsymbol{\alpha}}\left[\left(\boldsymbol{A}_{\zeta_r}\Sigma_n^*(x-r)\right)\mathbb{1}\left\{s_0^* \in (r-\mathrm{d}r, r),\ \boldsymbol{A}_{\zeta_r} \in \mathfrak{Z}^k\right\}\right] \\
&\quad = \mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\zeta_{s_0}}\mathbb{1}\left\{s_0^* \in (r-\mathrm{d}r, r),\ \boldsymbol{A}_{\zeta_{s_0}} \in \mathfrak{Z}^k\right\}\right]\Sigma_n^*(x-r) \\
&\quad = \boldsymbol{\alpha}e^{C^k r}D^{k0}(-C^0)^{-1}D^{0k}\Sigma_n^*(x-r)\mathrm{d}r,
\end{aligned}$$

where Lemma 5.3 was used in the last equality.

Thus, (5.4) recursively holds for $n \geq 0$ as

$$\Sigma_{n+1}^*(x) = \int_0^x e^{C^k r}D^{k0}(-C^0)^{-1}D^{0k}\Sigma_n^*(x-r)\mathrm{d}r,$$

which is unique by Lemma 3.5. Employing similar uniform-boundedness arguments to those in the proof of Theorem 4.4, together with Corollary 3.9, the Bounded Convergence Theorem and Theorem 3.10, we get

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\zeta_x}\mathbb{1}\left\{\boldsymbol{A}_s \in \mathfrak{Z}^k \cup \mathfrak{Z}^0\ \forall s \in [0,\zeta_x]\right\}\right] = \boldsymbol{\alpha}\Sigma^*(x),$$

where $\Sigma^*(x) = e^{C^{k*}x}$. □

Using Corollary 5.4 and Theorem 5.5 and the generalized concept of peaks we can develop the theory of first passages for RAP-modulated fluid processes with unit and zero rates in virtually the same way as in Sections 4.1 and 4.2 and part of Section 4.3. We present the final formulae below.

Theorem 5.6. *Let $\mathcal{X} = (\mathcal{R}, \boldsymbol{\mathcal{A}})$ be a RAP-modulated fluid process with $n^0 > 0$. Let $\boldsymbol{\alpha} \in \mathfrak{Z}^+$. Then*

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{A}_{\tau^-}\mathbb{1}\{\tau^- < \infty\}\right] = \boldsymbol{\alpha}\Psi^*$$

where $\Psi^* = \lim_{n\to\infty}\Psi_n^*$ and $\{\Psi_n^*\}_{n\geq 0}$ are recursively computed by setting $\Psi_0^* = 0$ and solving

$$C^{+*}\Psi_{n+1}^* + \Psi_{n+1}^*C^{-*} = -D^{+-*} - \Psi_n^*D^{-+*}\Psi_n^*.$$

Furthermore, for $x \geq 0$

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\alpha}}\left[\boldsymbol{O}_x\mathbb{1}\{\tau_x^- < \infty\}\right] &= \boldsymbol{\alpha}\Psi^*e^{(C^{-*}+D^{-+*}\Psi^*)x}, \\
\mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{u\in\mathcal{U}^x}\boldsymbol{A}_u\right] &= \boldsymbol{\alpha}e^{(C^{+*}+\Psi^*D^{-+*})x}, \\
\mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{u\in\mathcal{D}^x}\boldsymbol{A}_u\right] &= \boldsymbol{\alpha}e^{(C^{+*}+\Psi^*D^{-+*})x}\Psi^*.
\end{aligned}$$

We can compute $\Pi^+(\cdot)$ and $\Pi^-(\cdot)$ as defined in (4.17) and (4.18) by using analogous arguments to the ones used in the proof of Theorem 4.11, however, there is a considerable difference. In (4.20) we used that the event $\{Q_t = 0, \boldsymbol{A}_t \in \mathfrak{Z}^+\}$ implies $\{\boldsymbol{A}_{t-} \in \mathfrak{Z}^-\}$, however, this is no longer true in the case $n^0 > 0$. In fact, $\{Q_t = 0, \boldsymbol{A}_t \in \mathfrak{Z}^+\}$ implies that $\boldsymbol{A}_{t-} \in \mathfrak{Z}^- \cup \mathfrak{Z}^0$, so that a distinction needs to be made between occupations of $\{Q_t\}_{t\geq 0}$ in $(0,\infty)$ due to jumps coming from $\mathfrak{Z}^-$ or from $\mathfrak{Z}^0$. The first case was already addressed in the proof of Theorem 4.11, while the second one follows by analogous arguments and by noticing that

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t = 0,\ \boldsymbol{A}_t \in \mathfrak{Z}^0\}\right] \\
&\quad = \lim_{t\to\infty} \mathbb{E}_{\boldsymbol{\beta}}\left[\int_{s=0}^{\infty} \mathbb{E}_{\boldsymbol{A}_t}\left[\boldsymbol{A}_s \mathbb{1}\{\rho^- \in (0, \mathrm{d}s),\ \boldsymbol{A}_r \in \mathfrak{Z}^0\ \forall r \in [\rho^-, s]\}\right] \mathbb{1}\{Q_t = 0,\ \boldsymbol{A}_t \in \mathfrak{Z}^-\}\right] \\
&\quad = \lim_{t\to\infty} \mathbb{E}_{\boldsymbol{\beta}}\left[\left(\int_{s=0}^{\infty} \boldsymbol{A}_t D^{-0} e^{C^0 s}\mathrm{d}s\right) \mathbb{1}\{Q_t = 0,\ \boldsymbol{A}_t \in \mathfrak{Z}^-\}\right] \\
&\quad = \lim_{t\to\infty} \mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t = 0,\ \boldsymbol{A}_t \in \mathfrak{Z}^-\}\right] D^{-0}(-C^0)^{-1}.
\end{aligned}$$

Thus, the average limit orbit values in $\mathfrak{Z}^0$ on the event $\{Q_t = 0\}$ can be computed once we compute the average limit orbit values in $\mathfrak{Z}^-$ on the event $\{Q_t = 0\}$ as $t \to \infty$. The latter is computed similarly to (4.13): for any $\boldsymbol{\beta} \in \mathfrak{Z}^-$,

$$\lim_{t\to\infty} \mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t = 0,\ \boldsymbol{A}_t \in \mathfrak{Z}^-\}\right] = \lim_{y\to\infty} c_0^* \mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{O}_y\right] = c_0^* \boldsymbol{v}_0^*,$$

where $c_0^* := \lim_{t\to\infty} \mathbb{P}(Q_t = 0, \boldsymbol{A}_t \in \mathfrak{Z}^-)$ and $\boldsymbol{v}_0^*$ is the left eigenvector with $\boldsymbol{v}_0^* \mathbf{1} = 1$ associated to the eigenvalue 0 of $C^{+*} + \Psi^* D^{-+*}$. Analogous to Condition 4.7, we assume that such an eigenvector $\boldsymbol{v}_0^*$ is unique.

We now consider one final component in order to compute the stationary distribution of $\{Q_t\}_{t\geq 0}$, which is

$$\Pi^0(x) := \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}x} \mathbb{E}_{\boldsymbol{\beta}}\left[\boldsymbol{A}_t \mathbb{1}\{Q_t \in (0, x),\ \boldsymbol{A}_t \in \mathfrak{Z}^0\}\right], \quad x > 0. \tag{5.5}$$

This can be computed using the following result whose proof is similar to that of Theorem 4.11.

Theorem 5.7. *Let $\boldsymbol{\alpha} \in \mathfrak{Z}^+$ and $h > 0$. Then*

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\alpha}}\left[\int_{s=0}^{\infty} \boldsymbol{A}_s \mathbb{1}\{s < \tau^-,\ Q_s \in [x, x+h),\ \boldsymbol{A}_s \in \mathfrak{Z}^0\}\mathrm{d}s\right] \\
&\quad = \boldsymbol{\alpha} e^{(C^{+*} + \Psi^* D^{-+*})x}[D^{+0} + \Psi^* D^{-0}](-C^0)^{-1} h + o(h).
\end{aligned}$$

The previous leads to the following characterization of the stationary behaviour of $\{Q_t\}_{t\geq 0}$.

Theorem 5.8. *Let Π^+, Π^- and Π^0 be defined as in (4.17), (4.18) and (5.5), respectively. Then, for $x > 0$*

$$\begin{aligned}
\Pi^+(x) &= c_0^* \boldsymbol{v}_0^* D^{-+*} e^{(C^{+*} + \Psi^* D^{-+*})x} \\
\Pi^-(x) &= c_0^* \boldsymbol{v}_0^* D^{-+*} e^{(C^{+*} + \Psi^* D^{-+*})x} \Psi^* \\
\Pi^0(x) &= c_0^* \boldsymbol{v}_0^* D^{-+*} e^{(C^{+*} + \Psi^* D^{-+*})x} [D^{+0} + \Psi^* D^{-0}](-C^0)^{-1},
\end{aligned}$$

where $c_0^* := \lim_{t\to\infty} \mathbb{P}(Q_t = 0, \boldsymbol{A}_t \in \mathfrak{Z}^-)$ is of the form

$$c_0^* = \left(1 - \boldsymbol{v}_0^* D^{-+*}(C^{+*} + \Psi^* D^{-+*})^{-1}(2\mathbf{1} + [D^{+0} + \Psi^* D^{-0}](-C^0)^{-1}\mathbf{1})\right)^{-1}.$$

Furthermore, the stationary density function of $\{Q_t\}_{t\geq 0}$ as defined in (4.25) is given by

$$\pi(x) = \Pi^+(x)\mathbf{1} + \Pi^-(x)\mathbf{1} + \Pi^0(x)\mathbf{1} = 2\Pi^+(x)\mathbf{1} + \Pi^0(x)\mathbf{1}, \quad x > 0.$$

6 RAP-modulated fluid process with general rates

A further generalization of the RAP-modulated fluid process with unit and zero rates can be made by considering instead a level process with arbitrary (but fixed) slopes which depend only on the subset $\mathfrak{Z}_i^k$ in which the orbit is evolving. More precisely, we define the following.

Definition 6.1. *For each* $i \leq n^+$ *and* $j \leq n^-$, *fix* $\nu_i^+ \in (0, \infty)$ *and* $\nu_j^- \in (-\infty, 0)$. *A* RAP-modulated fluid process with general rates *is a Markov additive process* $(\mathcal{R}, \boldsymbol{\mathcal{A}}) = \{(R_t, \boldsymbol{A}_t)\}_{t\geq 0}$, *where* $\boldsymbol{\mathcal{A}}$ *is an* orbit process *with state space* $\mathfrak{Z}^+ \cup \mathfrak{Z}^- \cup \mathfrak{Z}^0$ *and an arbitrary but fixed initial point* $\boldsymbol{A}_0 \in \mathfrak{Z}^+ \cup \mathfrak{Z}^- \cup \mathfrak{Z}^0$, *and* $\mathcal{R}$ *is a* level process *of the form*

$$R_t := \int_0^t \left(\sum_{j=1}^{n^+} \nu_j^+ \mathbb{1}\left\{ \boldsymbol{A}_s \in \mathfrak{Z}_j^+ \right\} + \sum_{j=1}^{n^-} \nu_j^- \mathbb{1}\left\{ \boldsymbol{A}_s \in \mathfrak{Z}_j^- \right\} \right) \mathrm{d}s.$$

In principle, the results of Section 5 can be extended to the case of RAP-modulated fluid process with general rates. One way to do so relies on constructing a *total fluid rates process* as in [8, Section 3], and then replicating the analysis in Section 5 with the time component replaced by such a process. An alternative approach is to regard the RAP-modulated fluid process with general rates as a time-changed version of the unit and zero rates case, technique exploited in [15] for the case of Markov-modulated fluid processes. Even though both methods are straightforward to implement, they lead to cumbersome notation and provide little mathematical insight, and thus, we omit further details.

7 Concluding remarks

In this paper we prove that a number of classic matrix-analytic results associated to Markov-modulated fluid processes extend naturally to stochastic fluid processes driven by an orbit process, called the RAP-modulated fluid process. These results include first return probabilities, downward record probabilities, and stationary distribution of the reflected level process at 0. Our work relies on novel techniques which exploit the physical interpretation of the orbit and its interplay with the level process. We explicitly compute the aforementioned descriptors for RAP-modulated fluid processes with unit rates, extending it later to the case of unit and zero rates. We also provide directions on how to compute the same descriptors for the case of a RAP-modulated fluid process with general rates in a straightforward way. Overall, our work provides a rigorous framework for stochastic fluid processes which no longer have an underlying finite or countable state space process, as well as algorithms and formulae to compute a number of its descriptors.

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References

[1] S. Asmussen. Stationary distributions for fluid flow models with or without Brownian noise. *Stochastic Models*, 11(1):21–49, 1995.

[2] S. Asmussen and M. Bladt. Renewal theory and queueing algorithms for matrix-exponential distributions. In *Matrix-analytic methods in stochastic models*, pages 313–341. Dekker, New York, 1997.

[3] S. Asmussen and M. Bladt. Point processes with finite-dimensional probabilities. *Stochastic Processes and their Applications*, 82(1):127–142, 1999.

[4] R. H. Bartels and G. W. Stewart. Solution of the matrix equation $AX + XB = C$. *Communications of the ACM*, 15(9):820–826, 1972.

[5] N. G. Bean, M. Fackrell, and P. Taylor. Characterization of matrix-exponential distributions. *Stochastic Models*, 24(3):339–363, 2008.

[6] N. G. Bean and B. F. Nielsen. Quasi-birth-and-death processes with rational arrival process components. *Stochastic Models*, 26(3):309–334, 2010.

[7] N. G. Bean, M. M. O'Reilly, and P. G. Taylor. Algorithms for return probabilities for stochastic fluid flows. *Stochastic Models*, 21(1):149–184, 2005.

[8] N. G. Bean, M. M. O'Reilly, and P. G. Taylor. Hitting probabilities and hitting times for stochastic fluid flows. *Stochastic Processes and their Applications*, 115(9):1530–1556, 2005.

[9] M. Bladt and B. F. Nielsen. *Matrix-Exponential Distributions in Applied Probability*, volume 81. Springer, 2017.

[10] A. da Silva Soares and G. Latouche. Matrix-analytic methods for fluid queues with finite buffers. *Performance Evaluation*, 63(4-5):295–314, 2006.

[11] M. H. Davis. Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 353–388, 1984.

[12] C.-H. Guo. Nonsymmetric algebraic Riccati equations and Wiener–Hopf factorization for M-matrices. *SIAM Journal on Matrix Analysis and Applications*, 23(1):225–242, 2001.

[13] R. L. Karandikar and V. G. Kulkarni. Second-order fluid flow models: reflected Brownian motion in a random environment. *Operations Research*, 43(1):77–88, 1995.

[14] G. Latouche and G. T. Nguyen. Analysis of fluid flow models. *Queueing Models and Service Management*, 1(2):1–29, 2018.

[15] L. C. G. Rogers. Fluid models in queueing theory and Wiener-Hopf factorization of Markov chains. *The Annals of Applied Probability*, 4(2):390–413, 1994.