



## Computational and Theoretical Challenges for Computing the Minimum Rank of a Graph

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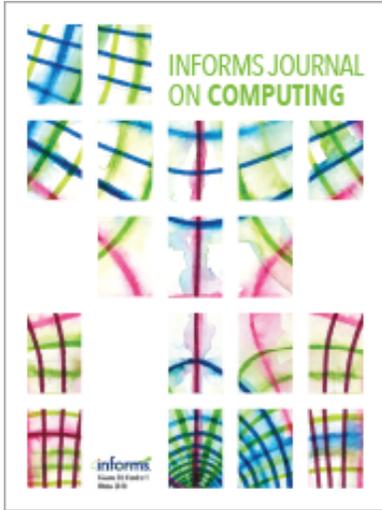
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# Computational and Theoretical Challenges for Computing the Minimum Rank of a Graph

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**Abstract.** The minimum rank of a graph  $G$  is the minimum of the ranks of all symmetric adjacency matrices of  $G$ . We present a new combinatorial bound for the minimum rank of an arbitrary graph  $G$  based on enumerating certain subsets of vertices of  $G$  satisfying matroid theoretic properties. We also present some computational and theoretical challenges associated with computing the minimum rank. This includes a conjecture that this bound on the minimum rank actually holds with equality for all graphs.

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**Keywords:** minimum rank • maximum nullity • matroid • zero forcing • forts

## 1. Introduction

Let  $S_n(\mathbb{R})$  denote the set of real symmetric  $n \times n$  matrices. For a matrix  $A \in S_n(\mathbb{R})$ ,  $\mathcal{G}(A)$  denotes the graph with vertex set  $\{1, \dots, n\}$  and edge set  $\{\{i, j\} : A_{ij} \neq 0, 1 \leq i < j \leq n\}$ . The diagonal of  $A$  is not used when constructing  $\mathcal{G}(A)$ . The set of *symmetric matrices associated with a graph*  $G$  is defined to be  $\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}$ . The *minimum rank* of  $G$  is defined as the minimum rank over all symmetric matrices associated with  $G$ , that is,  $\text{mr}(G) = \min\{\text{rank}(A) : A \in \mathcal{S}(G)\}$ . Similarly, the *maximum nullity* of  $G$  is defined as the maximum nullity over all symmetric matrices associated with  $G$ , that is,  $M(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G)\}$ . Because for any symmetric  $n \times n$  matrix,  $\text{rank}(A) + \text{null}(A) = n$ , it follows that  $\text{mr}(G) + M(G) = n$ . Thus, the minimum rank and maximum nullity problems are essentially equivalent. Because the maximum multiplicity of the eigenvalue 0 is the same as the maximum multiplicity of any eigenvalue (by translation by a scalar matrix), the maximum nullity problem is also equivalent to the problem of determining the maximum multiplicity of an eigenvalue of the matrices in  $\mathcal{S}(G)$ .

The minimum rank problem was first studied in 1996 by Nylen (Nylen 1996), who gave an algorithm for computing the minimum rank of trees; this algorithm was later improved (Johnson and Duarte 1999, Wei and Weng 2001, Johnson and Saiago 2002) and

generalized to block-cycle graphs in Barioli et al. (2005a). The graphs having very large and very small minimum ranks have been characterized in Barrett et al. (2004, 2005; 2009b), Hogben and van der Holst (2007), and Johnson et al. (2009). Decomposition formulas have been derived for computing the minimum ranks of graphs with cut vertices (Hsieh 2001, Barioli et al. 2004) and joins of graphs (Barioli and Fallat 2007) in terms of the minimum ranks of certain subgraphs. The effects of edge subdivisions (Barrett et al. 2009a; 2014), edge deletions (Edholm et al. 2012), and graph complements (Hogben 2008, Barioli et al. 2012) on the minimum rank have also been explored. Upper and lower bounds for the minimum rank of a graph can be obtained using graph theoretic parameters such as the zero-forcing number and its variants (AIM Special Work Group 2008, Huang et al. 2010, Barioli et al. 2013, Gentner et al. 2016), Colin de Verdière type parameters (Barioli et al. 2005b, 2013; Hogben and van der Holst 2007), ordered and induced subgraphs (Mitchell et al. 2010), and other methods. Techniques for computing the minimum rank of small graphs are described in DeLoss et al. (2010) and are combined in DeLoss et al. (2008b) with the bounds mentioned previously to compute the minimum ranks of all graphs on up to seven vertices. Although there exists a finite time algorithm to compute the minimum rank of an arbitrary graph (Brimkov and Scherr 2019), the

literature still lacks an efficient (and hopefully polynomial time) algorithm for this problem. The minimum rank problem is a special case of the matrix completion problem and has numerous theoretical and practical applications such as the million-dollar Netflix challenge (Koren 2009), collaborative filtering in recommender systems used by companies like Amazon and Walmart (Candes and Plan 2010), and recovery of the topology of social networks (Mahindre et al. 2019). It is also related to the inverse eigenvalue problem (Hogben 2005), quantum controllability on graphs (Godsil and Severini 2010), and various other problems in spectral graph theory and combinatorial matrix theory.

In this paper, we present a new bound on the minimum rank of a graph, based on enumerating certain subsets of vertices of  $G$  satisfying matroid theoretic properties. Our bound can be combined with the current state-of-the-art software for the minimum rank problem (DeLoss et al. 2008a), which implements other known bounds and performs reduction techniques. We also explore both theoretical and computational challenges related to this new bound. Finally, we conjecture that our bound on the minimum rank holds with equality for all graphs.

## 2. Preliminaries

Let  $A$  be a matrix and  $x$  be a vector.  $A_i$  will denote the  $i$ th row of  $A$ ,  $A_{ij}$  will denote the entry in the  $i$ th row and the  $j$ th column of  $A$ , and  $x_i$  will denote the  $i$ th entry of  $x$ . Given a vector  $x$ , the *support* of  $x$ , denoted  $\text{supp}(x)$ , is the set of nonzero entries of  $x$ . A *null vector* of  $A$  is a vector  $w$  such that  $Aw = 0$ . A *null support* of  $A$  is the support of a null vector. A null support of  $A$  is *nontrivial* if it is not empty. An *irreducible* nontrivial null support of  $A$  is a nontrivial null support of  $A$  that does not properly contain any nontrivial null support of  $A$ .

Given a graph  $G$  and vertices  $u$  and  $v$  in  $G$ ,  $u \sim v$  means  $u$  is adjacent to  $v$ . A *fort* of  $G$  is a nonempty set  $F \subset V(G)$  such that no vertex outside  $F$  has a single neighbor in  $F$ . A collection  $\mathcal{C}$  of forts is *compatible* if  $|\mathcal{C}| = 1$ , or if for every distinct  $F_1, F_2 \in \mathcal{C}$  and  $v \in F_1 \cap F_2$ , there exists some  $F \in \mathcal{C}$  with  $F \subseteq (F_1 \cup F_2) \setminus \{v\}$ . The *hitting number* of a family of sets  $\mathcal{S}$  is defined as  $\tau(\mathcal{S}) = \min\{|X| : \forall S \in \mathcal{S}, X \cap S \neq \emptyset\}$ .

A *matroid* is an ordered pair  $\mathcal{M} = (S, \mathcal{I})$  where  $S$  is a finite set and  $\mathcal{I}$  is a family of subsets of  $S$  with the following properties:

- $\mathcal{I} \neq \emptyset$ , we have  $\emptyset \in \mathcal{I}$
- If  $J' \subset J \in \mathcal{I}$ , then  $J' \in \mathcal{I}$
- If  $J \in \mathcal{I}$ ,  $J' \in \mathcal{I}$ , and  $|J| > |J'|$ , then there exists  $x \in J \setminus J'$  such that  $J' \cup \{x\} \in \mathcal{I}$

The elements of  $\mathcal{I}$  are called *independent sets*. A *circuit* of  $\mathcal{M}$  is a minimal dependent subset of  $S$ , that is, a dependent set whose proper subsets are all independent. If  $S$  is a finite collection of vectors,  $A$  is a matrix whose columns are the vectors in  $S$ , and  $\mathcal{I}$  is the collection of linearly independent subsets of  $S$ , then  $\mathcal{M}(A) = (S, \mathcal{I})$  is a matroid called the *linear matroid represented by*

A. Given a matrix  $A$ , the set of irreducible nontrivial null supports of  $A$  correspond to the circuits of  $\mathcal{M}(A)$ .

## 3. Main Result

Before we prove the main theorem, we need two helpful lemmas.

**Lemma 1.** *Let  $G = (V, E)$  be a graph of order  $n$  and let  $x \in \mathbb{R}^n$ . There exists a matrix  $A \in \mathcal{S}(G)$  such that  $Ax = 0$  if and only if  $\text{supp}(x)$  is a fort of  $G$ .*

**Proof.** First suppose there exists  $A \in \mathcal{S}(G)$  such that  $Ax = 0$ , and let  $F = \text{supp}(x)$ . We will show that  $F$  is a fort. Suppose for contradiction that there exists a vertex  $u \in V \setminus F$  such that  $N(u) \cap F = \{w\}$ . Since  $w \in F = \text{supp}(x)$ , it follows that  $x_w \neq 0$ . Then,

$$0 = (Ax)_u = \sum_{v \in V} A_{uv}x_v = A_{uw}x_w,$$

since for all other terms in the sum, either  $A_{uv} = 0$  or  $x_v = 0$ . Because  $A \in \mathcal{S}(G)$  and  $u \sim w$ , we have that  $A_{uw} \neq 0$ . Therefore,  $x_w = 0$ , a contradiction.

Now, suppose that  $F := \text{supp}(x)$  is a fort of  $G$  and let  $f = |F|$ . Let  $A \in S_{n-f}(\mathbb{R})$  and consider  $M \in S_n(\mathbb{R})$  written in block form as follows:

$$M = \begin{pmatrix} C & B \\ B^T & C' \end{pmatrix}.$$

Let  $\pi(x)$  be a permutation of the entries of  $x$  such that  $\pi(x)_1 = \dots = \pi(x)_{n-f} = 0$ , and let  $\bar{x} = \pi(x)$ . Let

$$\bar{x} = \begin{pmatrix} 0 \\ x' \end{pmatrix},$$

where every entry of  $x'$  is nonzero. Written in this form, we have that

$$M\bar{x} = \begin{pmatrix} Bx' \\ Cx' \end{pmatrix}.$$

Therefore, to show that we can pick  $M \in \mathcal{S}(G)$  such that  $M\bar{x} = 0$ , we must show we can pick  $B$  and  $C$  so that  $Bx' = 0$  and  $Cx' = 0$  in a manner such that  $M$  is in  $\mathcal{S}(G)$ . Note that  $A$  does not affect whether  $M\bar{x} = 0$  holds, so we can choose it to be any symmetric matrix that does not violate the conditions for  $M \in \mathcal{S}(G)$ .

For  $i \in V \setminus F$ , consider

$$(Bx')_i = \sum_{j \in N(i) \cap F} B_{ij}x'_j. \tag{1}$$

There are either zero or at least two terms in this sum. In the former case where the sum is necessarily zero, we can pick the entries of the  $i$ th row of  $B$  arbitrarily so that they meet the conditions for  $M$  to be in  $\mathcal{S}(G)$ . In the latter case, let  $\ell$  and  $k$  be distinct elements of  $N(i) \cap F$ . Set  $B_{ij} = 1$  for all  $j \in (N(i) \cap F) \setminus \{\ell, k\}$ , and let

$$s = \sum_{j \in (N(i) \cap F) \setminus \{\ell, k\}} B_{ij}x'_j = \sum_{j \in (N(i) \cap F) \setminus \{\ell, k\}} x'_j.$$

Now let  $a$  be an arbitrary nonzero number different from  $-s$ , and set  $B_{i\ell} := a/x'_\ell$ . Then  $a + s \neq 0$  and we can let

$B_{ik} = -(a+s)/x'_k$ , which ensures that (1) holds. Doing this for every  $i \in V \setminus F$ , we have that  $Bx' = 0$ .

Now pick arbitrary values for the nondiagonal entries of  $C$  so that it is symmetric and satisfies  $C_{ij} \neq 0$  if and only if  $i \sim j$ . We then let

$$C_{ii} = -\frac{1}{x'_i} \sum_{j \in N(i) \cap F} C_{ij}x'_j,$$

for all  $i \in F$ . This is well defined because  $x'_i \neq 0$  for all  $i \in F$ , and it is easy to see that this choice guarantees that

$$(Cx')_i = C_{ii}x'_i + \sum_{j \in N(i) \cap F} C_{ij}x'_j = 0.$$

Therefore, we have that  $Cx' = 0$ , and we have chosen the entries of  $M$  so that  $M \in \mathcal{S}(G)$  as desired.  $\square$

The conditions in the proof are still satisfied with minimal forts. Hence, we have the following.

**Corollary 1.** For  $A \in \mathcal{S}(G)$ , let  $\mathcal{F}_A$  be the set of irreducible nontrivial null supports of  $A$ .  $\mathcal{F}_A$  is a collection of minimal forts.

Next, we will use the hitting number of  $\mathcal{F}_A$  to define the nullity of the associated matrix.

**Lemma 2.** Let  $G$  be a graph. Then  $\text{null}(A) = \tau(\mathcal{F}_A)$  for each  $A \in \mathcal{S}(G)$ .

**Proof.** Let  $S$  be a subset of  $V(G)$  that intersects every fort in  $\mathcal{F}_A$ . Consider the columns of  $A$  corresponding to  $V(G) \setminus S$ . Since  $V(G) \setminus S$  cannot contain any fort in  $\mathcal{F}_A$ , it follows from Lemma 1 that these columns must be linearly independent.

Conversely, if  $T$  is a subset of vertices such that the corresponding columns of  $A$  are linearly independent, then  $T$  cannot contain any fort in  $\mathcal{F}_A$ , and thus  $V(G) \setminus T$  intersects every fort in  $\mathcal{F}_A$ . Thus, if  $S$  is a set of minimum size intersecting every fort in  $\mathcal{F}_A$ , then  $V(G) \setminus S$  corresponds to a maximum size set of linearly independent columns of  $A$ . Thus, it follows that  $\tau(\mathcal{F}_A) = n - \text{rank}(A) = \text{null}(A)$ .  $\square$

Now, we can prove our main result.

**Theorem 1.** Let  $G$  be a graph on  $n$  vertices and  $C_1, \dots, C_t$  be all the compatible sets of minimal forts of  $G$ . Then,  $\text{mr}(G) \geq n - \max_{1 \leq i \leq t} \{\tau(C_i)\}$ .

**Proof.** We will prove Theorem 1 by two claims.

**Claim 1.** Let  $G$  be a graph. Then  $M(G) = \max\{\tau(\mathcal{F}_A) : A \in \mathcal{S}(G)\}$ .

**Proof of Claim 1.** Notice that  $M(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G)\} = \max\{\tau(\mathcal{F}_A) : A \in \mathcal{S}(G)\}$ , where the second equality follows from Lemma 2.  $\square$

**Claim 2.** Let  $G$  be a graph. Then  $\max\{\tau(\mathcal{F}_A) : A \in \mathcal{S}(G)\} \leq \max\{\tau(\mathcal{C}) : \mathcal{C} \text{ is a compatible collection of minimal forts of } G\}$ .

**Proof of Claim 2.** Let  $A^* \in \mathcal{S}(G)$  be a matrix such that  $\tau(\mathcal{F}_{A^*}) = \max\{\tau(\mathcal{F}_A) : A \in \mathcal{S}(G)\}$ . Then,  $\mathcal{F}_{A^*}$  is the set of irreducible nontrivial null supports of  $A^*$ , and hence the set of circuits of the linear matroid  $\mathcal{M}(A^*)$ . The set of circuits of  $\mathcal{M}(A^*)$  satisfies elimination compatibility. Moreover, by Corollary 1,  $\mathcal{F}_{A^*}$  is a set of minimal forts of  $G$ . Thus,  $\mathcal{F}_{A^*}$  is a compatible collection of minimal forts of  $G$ , say  $\mathcal{C}^*$ . Thus,  $\max\{\tau(\mathcal{F}_A) : A \in \mathcal{S}(G)\} = \tau(\mathcal{F}_{A^*}) = \tau(\mathcal{C}^*) \leq \max\{\tau(\mathcal{C}) : \mathcal{C} \text{ is a compatible collection of minimal forts of } G\}$ .  $\square$

By Claims 1 and 2,  $M(G) \leq \max\{\tau(\mathcal{C}) : \mathcal{C} \text{ is a compatible collection of minimal forts of } G\}$ , so  $\text{mr}(G) \geq n - \max_{1 \leq i \leq t} \{\tau(C_i)\}$ .  $\square$

Figure 1 offers an example graph  $G$  to consider along with the theoretical result. For this example graph, the hitting number for the compatible minimal forts is two;  $M(G) = 2$ ;  $\text{mr}(G) = 8$  (Figure 2); but  $Z(G) = 3$ . Hence, zero forcing is too restrictive; a zero-forcing set transverses every minimal fort. The compatible minimal forts in Figure 1 form a circuit family for a matroid; we refer to this matroid as a *fort matroid* of  $G$ .

**Figure 1.** (Color online) Example Graph and a Set of Compatible Minimal Forts for the Graph

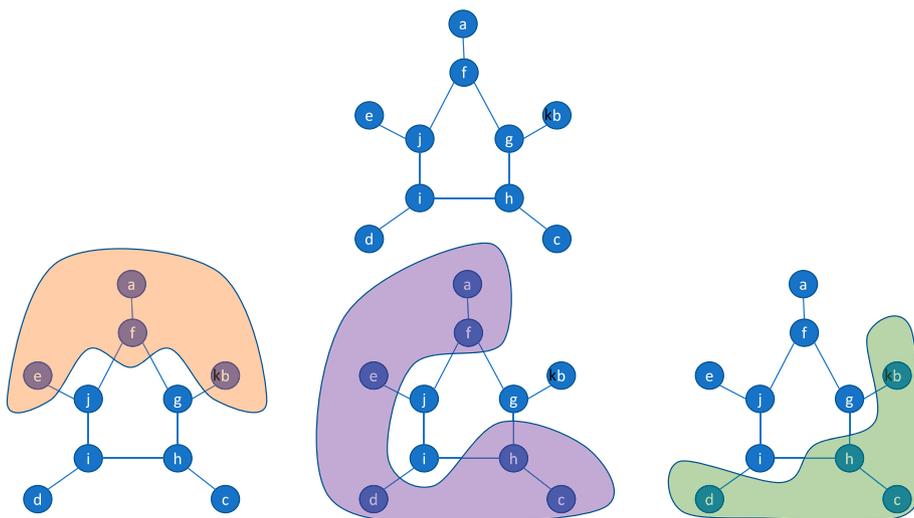
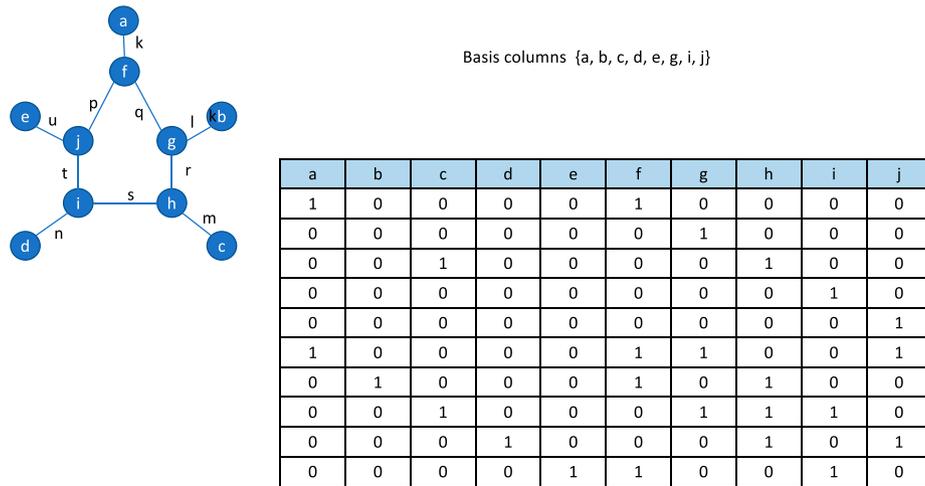


Figure 2. (Color online) Optimal Realization for the Example Graph in Figure 1, Where  $\{a, b, c, d, e, g, i, j\}$  Serve as Basis Columns



### 4. Computability of the Bound

In this section, we explore how quickly the bound in Theorem 1 can be computed. We first note that for any graph  $G$ , we can compute the bound by brute force in doubly exponential time.

**Proposition 1.** Let  $G$  be a graph on  $n$  vertices and  $\mathcal{C}_1, \dots, \mathcal{C}_t$  be all the compatible sets of forts of  $G$ . Then,  $\max_{1 \leq i \leq t} \{\tau(\mathcal{C}_i)\}$  can be computed in  $O(2^{2^n + 5n} n^3)$  time.

**Proof.** It can be checked whether a set  $S \subset V(G)$  is a fort in  $O(n^2)$  time. There are at most  $O(2^n)$  forts in  $G$ , so a list of all forts of  $G$  can be obtained in  $O(2^n n^2)$  time. It can be checked by brute force whether a collection of forts  $\mathcal{C}$  is compatible in  $O\left(\binom{|C|}{2} n^2\right) = O(2^{3n} n)$  time. There are at most  $O(2^{2^n})$  distinct collections of forts of  $G$ , so the list of compatible collections of forts  $\mathcal{C}_1, \dots, \mathcal{C}_t$  can be found in  $O(2^{2^n} 2^{3n} n)$  time. For a compatible collection of forts  $\mathcal{C}$ , by brute force,  $\tau(\mathcal{C})$  can be found in  $O(|\mathcal{C}| 2^n n^2) = O(2^{2n} n^2)$  time. Thus,  $\max_{1 \leq i \leq t} \{\tau(\mathcal{C}_i)\}$  can be found in  $O(2^{2^n} 2^{3n} n 2^{2n} n^2) = O(2^{2^n + 5n} n^3)$  time.  $\square$

Clearly, deriving necessary and sufficient conditions under which a family of minimal forts is compatible can significantly speed up this computation. We can also explore the computational complexity for finding minimal and minimum forts. The following are computational complexity problems and theorems related to finding forts. The proofs of the theorems are given in the online appendix.

**Problem:** MIN-FORT

**Instance:** Graph  $G$ , integer  $k$

**Question:** Does  $G$  contain a fort of cardinality at most  $k$ ?

**Problem:** RESTRICTED MIN-FORT

**Instance:** Graph  $G$ , Vertex  $v$ , Integer  $k$

**Question:** Does  $G$  contain a fort of cardinality less than  $k$  that contains  $v$ ?

**Theorem 2.** MIN-FORT is NP-complete.

**Theorem 3.** RESTRICTED MIN-FORT is NP-complete.

### 5. Challenges

The following are open questions in this area:

Question 1. In which graphs are there polynomially-many compatible sets of minimal forts?

Question 2. In which graphs can the collection of compatible sets of minimal forts be found in polynomial time?

Question 3. Is every fort matroid (circuit family corresponds to a compatible set of minimal forts) of a graph a linear matroid? Can we characterize this class of matroids?

Question 4. What is the true complexity for computing the minimum rank?

Question 5. Is there an integer programming formulation for computing the minimum rank?

Question 6. Are there connections between the minimum rank problem and network viability analysis, infeasibility analysis, or sparse solutions for linear systems? The most important challenge is the last open question.

Question 7. For any graph  $G$  on  $n$  vertices, is it true that  $\text{mr}(G) = n - \max_{1 \leq i \leq t} \{\tau(\mathcal{C}_i)\}$ , where  $\mathcal{C}_1, \dots, \mathcal{C}_t$  are all the compatible sets of forts of  $G$ ?

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