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Abstract:

Effective properties of mechanical systems under high-frequency excitation at multiple frequencies

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Abstract

Effects of strong high-frequency excitation at multiple frequencies (multi-HFE) are analyzed for a class of generally nonlinear systems. The effects are illustrated for a simple pendulum system with a vibrating support, and for a parametrically excited flexible beam. For the latter, theoretical predictions are supported by experimental observations, providing good agreement for a wide range of excitation conditions. The main effect of strong multi-HFE is to change the effective or apparent stiffness in a manner similar to that of mono-HFE, provided the HFE frequencies are well separated and non-resonant. Then the change in effective stiffness is proportional to the sum of squared excitation velocities, and the corresponding changes in equilibria, equilibrium stability, and natural frequencies can be computed as for the mono-HFE case. When there are two or more close excitation frequencies, an additional contribution of slowly oscillating stiffness appears. This may cause strong parametrical resonance at conditions that might not appear obvious, i.e. when the difference in two HFE-frequencies is near twice an effective system natural frequency, which due to the HFE itself is shifted away from the natural frequency without HFE. Also, it is shown that strong multi-HFE can stabilize otherwise unstable equilibria, but only if the frequencies are well separated; thus continuous broadband and random HFE does not have a stabilizing effect paralleling that of mono-HFE, or multi-HFE with non-close frequencies. The general results may be used to investigate or utilize general effects, or as a shortcut to calculate effective properties for specific systems, or to calculate averaged equations of motion that may be much faster to simulate numerically.

Keywords: high-frequency excitation, multiple frequency excitation, dynamic stiffening, effective properties.

1. Introduction

The paper shows how effects of strong high-frequency excitation (HFE) having multiple frequencies (multi-HFE) can be conveniently analyzed, and concludes that most results for mono-HFE readily extend to the multi-HFE case. To illustrate basic principles, a simple example is first given (pendulum on a two-frequency oscillating support). Then a generalized nonlinear model is considered which covers a broad range of applications. Finally, to illustrate an application of the general theory, a structural example (base-excited elastic string) is analyzed and tested experimentally. The theoretical results have been presented in a conference paper [1], along with preliminary experimental tests of qualitative features only. This present paper repeats the theoretical developments for completeness, and adds new experimental results providing good quantitative agreement for the effects of concern.

Strong high-frequency excitation can change the effective properties of elastic structures in a manner that may not be obvious, or may be overlooked or misinterpreted when using pure numerical simulation or laboratory experiments. Even a zero-mean small-amplitude HFE may change structural features that may be essential for structural operation and safety, such as effective stiffness, natural frequencies, equilibria, equilibrium stability, and bifurcation paths, as has been known for many years [2-6]. Much has been learnt in recent years on such effects – for specific systems and for quite general classes of linear and nonlinear mathematical models – and well-proven methods exist for analyzing them (e.g. [7-13]). In

most studies the HFE is assumed to be a stationary harmonic mono-frequency function of time. However, in applications multi-HFE is not uncommon – either with a few frequencies (e.g., structures connected to multiple imbalanced rotors), with many discrete frequencies (e.g. higher harmonics of non-harmonic time-functions), or with a continuous spectrum of high frequencies (e.g. structures connected to rocket or jet engines).

There are many studies of various effects of multifrequency excitation, of which in particular the possibility of combination resonances for non-linear systems is important, and thoroughly treated by Nayfeh and coworkers in e.g. [14,15]. Also, [16] analyzes the response of a single-degree-of-freedom system with cubic nonlinearity to slowly modulated HFE, effectively corresponding to a three-frequency excitation with close frequencies. In these studies the multifrequency excitation is not supposed to be strong and of high frequency, and thus effects such as apparent stiffening, of concern in the present study, do not show up.

There seems to be rather few studies dealing with effects of strong multi-HFE: Chelomei [17] considered the effect of multi-HFE for a general class of systems, focusing on its effect on the stability of equilibria of linear systems. Bogdanoff and Citron [18] studied linear systems with multi-HFE, and showed that a pendulum on a vibrating support can be stabilized in the inverted position by random multi-HFE, provided the frequencies of the HFE are not close. Hemp and Sethna [19] contributed a thorough, mathematically oriented investigation of general nonlinear dynamical systems in Hamiltonian form, with particular focus to effects of simultaneous fast and slow parametric excitation. Kovaleva [20] examined the stability of discrete dynamical systems in a general Hamiltonian form and showed, generally and by physical examples, that multi-HFE having random and quasiperiodic components can be used to generate new, stable equilibria. These studies [17-20] do not include consideration to the influence of resonance, which is important for the analysis of continuous elastic structures.

The analysis of continuous elastic structures with HFE introduces special problems. This is because asymptotic analysis of HF-excited systems usually assumes the excitation frequencies are far beyond the linear natural frequencies of the system, leading to approximate expressions for the fast components of motion that ignores any influence of possible resonances. With HF-excited continuous elastic structures, as e.g. the beams, columns, and spinning discs investigated in [21-26], this assumption is questionable, since the modal density is typically so high that resonances are affecting the response in most of the relevant frequency range. Thus care must be taken to restrict the frequency range of application to non-resonant regimes, or the influence of resonance must explicitly be taken into account, as in [26] for a beam system. This present work incorporates general consideration of resonant influence (but not exact or *sharp* resonance), in a manner that adds only marginally to computational burden and complexity.

The general class of systems covered by this work extends [12] in two respects: First the type of HFE under consideration is extended from mono-frequency with a short period, to generally multifrequency with any period (including infinite, corresponding to quasiperiodic HFE). Secondly, consideration of weakly resonant effects is directly incorporated, which means that the multi-HFE frequencies do not need to be much larger than the linear resonances of the system. This makes the results directly applicable also to discretised models of continuous elastic systems.

Next we present an example to illustrate, in a simple setting, various effects that can occur with multi-HFE. Then a very general class of systems is considered, and correspondingly general results are derived using the method of direct separation of motions [7]. Finally the general results are employed to predict how the apparent stiffness of an elastic string changes in response to multi-HFE, and the predictions are tested against laboratory experiments.

2. Example I: Illustrating the main effects of multi-HFE

Consider a pendulum in gravity g (Fig. 1), whose support vibrates vertically and harmonically in time $\tilde{t}$, at multiple frequencies $\tilde{\Omega}_j$ being much higher than the natural frequency $\omega_0 = \sqrt{g/l}$, with corresponding amplitudes $A_j l$ much smaller than the pendulum length l . The pendulum swing $\theta(t)$ is governed by the non-dimensional equation of motion:

$$\ddot{\theta} + 2\beta\dot{\theta} + (1 + \ddot{q}(t)) \sin \theta = 0, \quad q(t) = \sum_{j=1}^m A_j \sin(\Omega_j t + \alpha_j), \quad (1)$$

$$A_j \ll 1, \quad \Omega_j \gg 1, \quad \Omega_j A_j = O(1)$$

where time and frequencies have been normalized by ω_0 , i.e. $t = \omega_0 \tilde{t}$ and $\Omega_j = \tilde{\Omega}_j / \omega_0$, and overdots denotes derivatives with respect to t .

The system (1) is a specific example belonging to a general class of discrete dynamical systems, to be analyzed in Section 3. In this present section we focus on illustrating effects, thus bypassing intermediate calculations and just using final results to be derived in Section 3. The correspondence of variables and parameters between the general system (16) (and subclass (27)) and the pendulum system (1) is:

$$n = 1, \quad \mathbf{u} = \theta, \quad \mathbf{M} = 1, \quad \mathbf{K} = \omega^2 = 0, \quad \mathbf{s}(\theta, \dot{\theta}, t) = 2\beta\dot{\theta} + \sin \theta, \quad \mathbf{h}_j = \mathbf{0},$$

$$\mathbf{f}_j(\theta, t) = \Omega A_j \sin \theta, \quad \xi_j(t, \tau) = \sin\left(\frac{\Omega_j}{\Omega} \tau + \alpha_j\right), \quad \tau = \Omega t, \quad \Omega = m^{-1} \sum_{j=1}^m \Omega_j \gg 1. \quad (2)$$

To accuracy-order Ω^{-1} of approximation, the motions θ are then given by (18), which in terms of pendulum variables gives:

$$\theta(t) = z(t) + \Omega^{-1} \varphi(t, \Omega t), \quad \Omega^{-1} \ll 1, \quad (3)$$

where $z(t)$ holds the slow or average component of θ governed by (34), which gives:

$$\ddot{z} + 2\beta\dot{z} + \left(1 + \Omega^2 \sum_{i,j=1}^m \sigma_{ij}^2(t) A_i A_j \cos z\right) \sin z = 0, \quad (4)$$

where σ_{ij}^2 are mean-square derivatives of the HFE given by (35), while $\Omega^{-1} \varphi$ is a small overlay of HF-oscillations given by (20) with (32):

$$\Omega^{-1} \varphi(t, \tau) = -\sin z \sum_{j=1}^m A_j \sin(\Omega_j t + \alpha_j) + O(\Omega^{-1}). \quad (5)$$

The initial conditions $z(0)$ and $\dot{z}(0)$ for (4) are given by (26):

$$z(0) - \sin z(0) \sum_{j=1}^m A_j \sin(\alpha_j) = \theta(0); \quad \dot{z}(0) = \dot{\theta}_0 + \sin z(0) \sum_{j=1}^m A_j \Omega_j \cos(\alpha_j), \quad (6)$$

Fig. 1. Pendulum with a support that vibrates at multiple high frequencies

which completes the separation of the full motions θ into slow and fast components z and φ . Of primary interest here is (4) for the slow pendulum motions z , while the fast but small motions $\Omega^{-1}\varphi$ are usually interesting only by their effect on z .

The main effects of multi-HFE depends on whether two or more of the excitation frequencies Ω_j are close, or if they are all well separated; we consider each of these cases.

2.1. Case: Multi-HFE with non-close frequencies

When all the excitation frequencies Ω_j are high and well separated, $|\Omega_i - \Omega_j| \gg 1$, the mean-square derivatives σ_{ij}^2 are given by the upper expression in (36), which inserted into (4) gives:

$$\ddot{z} + 2\beta\dot{z} + (1 + v^2 \cos z) \sin z = 0, \quad (7)$$

where the constant v^2 expresses the intensity of the HFE, which is proportional to the sum of squared excitation velocities:

$$v^2 = \frac{1}{2} \sum_{j=1}^m (\Omega_j A_j)^2. \quad (8)$$

Equation (7), governing the slow component z of θ , is similar to Equation (1) for the full pendulum motions θ , though with the non-autonomous excitation terms replaced by an averaged effect of these. Also, (7) is identical to the averaged equation for the case of mono-HFE [11,13], where $m=1$ and $v=(\Omega A)^2/2$. Thus the presence of multiple non-close frequencies is accounted for solely by the definition (8) of the excitation intensity.

As appears when letting $\ddot{z}=\dot{z}=0$ in (7), there are at least two equilibria for the slow pendulum motions: $z=0$ and $z=\pi$, corresponding to $\theta=0$ (down-pointing pendulum) and $\theta=\pi$ (up-pointing pendulum). To study motions near these we linearise (7) near each:

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = 0 \quad \text{for } z \approx 0 \quad (9)$$

$$\ddot{z} + 2\beta\dot{z} + \omega_\pi^2 (z - \pi) = 0 \quad \text{for } z \approx \pi, \quad (10)$$

where the linear stiffness coefficients are:

$$\omega_0^2 = v^2 + 1, \quad \omega_\pi^2 = v^2 - 1. \quad (11)$$

The linear stiffness coefficient ω_0^2 for (9) increases with the HFE intensity v^2 , and is always positive; hence the equilibrium $z=0$ is always stable to small disturbances. Also, as v is increased, the (non-dimensional) frequency ω_0 of small oscillations near $z=0$ increases from unity to $\sqrt{1+v^2}$, so that the pendulum oscillates faster about its down-pointing equilibrium in the presence of HFE.

In (10) the linear stiffness coefficient ω_π^2 also increases with the HFE intensity v^2 . When $v^2 < 1$ the stiffness coefficient is negative, so that the up-pointing equilibrium is unstable, while if $v^2 > 1$ the stiffness is positive and the pendulum is stable in the up-pointing position (as reported in numerous studies since [2,3]). In the latter case small oscillations near $z=\pi$ occur at frequency $\omega_\pi = \sqrt{v^2 - 1}$, which increases from zero as the HFE intensity is increased beyond the critical value $v^2=1$.

In addition to the equilibria at $z=0$ and $z=\pi$, when $v^2 > 1$ two new equilibria emerge at $z = \pm \arccos(-v^{-2})$, symmetrically about the vertical axis of gravity, and moving smoothly from $z = \pi$ towards $\pi \pm \pi/2$ as v^2 is increased from 1 towards infinity; These equilibria are always unstable, and act as potential barriers between the equilibria at $z=0$ and $z=\pi$, which are both stable when $v^2 > 1$.

Thus, the HFE causes several changes in the effective properties of the pendulum system: New equilibria are created, the effective natural frequencies are changed, and the stability of existing equilibria is changed. All these effects are typical for HF-excited systems, and originate from a change in effective stiffness.

To illustrate a quantitative application of the above results, Fig. 2 shows a time series obtained by numerical simulation (using the MATLAB-function *ode45*) of the original system (1) with two-frequency HFE, with parameters as given in the figure legend. The pendulum is initially positioned at $\theta=\pi$, and at $t=0$ given a small velocity disturbance $\dot{\theta}(0)$. The simulated time series shows damped slow oscillations towards the inverted equilibrium $\theta=\pi$, with an average period of $T=13.39$, and a small HF-overlay of the magnitude order of the HFE amplitudes. With the present parameters (8) gives $v^2=1.22>1$, so that the equilibrium $z=\pi$ should be stable to small disturbances (while if only one of the two excitations were applied then $v^2<1$, and $z=\pi$ would be unstable). Furthermore, small oscillations about the stable equilibrium should occur at angular frequency $\omega_\pi=\sqrt{v^2-1}=0.4690$, corresponding to a period of $T=2\pi/0.4690=13.40$, thus agreeing rather accurately with the simulation result $T=13.39$. The agreement will decrease as the assumption ceases to be fulfilled, i.e. as the excitation frequencies becomes lower and closer, and the HFE amplitude becomes larger.

It appears that for non-close high frequencies, the slow effects of multi-HFE does not differ from the case of mono-frequency excitation, if just the excitation intensity is calculated as a sum of squares of the intensities at each frequency, cf. (8).

2.2. Case: Multi-HFE with close frequencies

When two or more of the high excitation frequencies are close, i.e. $O(1)$ or smaller, the mean-square derivatives σ_{ij}^2 are given by the lower expression in (36), which inserted into (4) gives:

$$\ddot{z} + 2\beta\dot{z} + \left(1 + (v^2 + p(t))\cos z\right)\sin z = 0, \quad (12)$$

where v^2 is still given by (8), while:

Fig. 2. Damped pendulum oscillations $\theta(t)$ (upper time series) towards the inverted position $\theta=\pi$, which is stabilized by small-amplitude support motions $q(t)$ (lower time series) at two non-close, high frequencies, as obtained by numerical simulation of the equation of motion (1). Parameters: $\beta=0.05$, $\Omega_1=20$, $\Omega_2=40$, $A_1=0.05$, $A_2=0.03$, $\alpha_1=\alpha_2=0$, $\theta(0)=\pi$, $\dot{\theta}(0)=0.05$.

$$p(t) = \sum_{i=1}^{m-1} \sum_{j=i+1}^m \Omega^2 A_i A_j \cos((\Omega_i - \Omega_j)t + \alpha_i - \alpha_j). \quad (13)$$

Comparing to the similar expression (7) for the case of non-close excitation frequencies, we see that the averaged effect of the presence of close frequencies corresponds to a parametric excitation, oscillating at the small difference frequencies $\Omega_i - \Omega_j$.

The interaction between the change in effective stiffness (the v^2 -term) and parametric excitation may create slow effects that may not be obvious when considering only Equation (1) for the full motions: As is well-known [14], primary parametric resonance may occur when a parametric excitation frequency is near twice the linear natural frequency of a system. For the present system this means any $\Omega_i - \Omega_j$ coming near twice the *effective* natural frequency, and this natural frequency may be shifted away (by the v^2 -term) from the corresponding natural frequency of the unexcited pendulum.

To study motions near the static equilibria $z=0$ and $z=\pi$, we linearise (12) near each of them and obtain:

$$\ddot{z} + 2\beta\dot{z} + (\omega_0^2 + p(t))z = 0 \quad \text{for } z \approx 0 \quad (14)$$

$$\ddot{z} + 2\beta\dot{z} + (\omega_\pi^2 + p(t))(z - \pi) = 0 \quad \text{for } z \approx \pi, \quad (15)$$

where the linear stiffness coefficients ω_0^2 and ω_π^2 are still given by (11).

With (14) being a damped Hill-equation, the down-pointing equilibrium $z=0$ of the pendulum may be destabilized by parametric resonance when $|\Omega_i - \Omega_j| \approx 2\omega_0/k$, $k=1,2,\dots$, and the corresponding excitation amplitude $\Omega^2 A_i A_j$ is beyond a critical value. The nearness to resonance and critical amplitude for this to occur can be calculated using standard results for the Hill-equation as given in many texts (e.g. [14]), as can the resulting nonlinear response under resonant conditions (in [11] this is done for a system similar to (12)). Though, when using results from perturbation analysis, care should be taken to check the accuracy, since the excitation amplitudes $\Omega^2 A_i A_j$ are $O(1)$, and thus not small as compared to the stiffness coefficients ω_0^2 and ω_π^2 .

To illustrate the effect of parametric resonance at a shifted frequency, Fig. 3 shows a time series obtained by numerical simulation of the original system (1) with two-frequency HFE, with parameters as given in the figure legend, and initially transients discarded. The resulting stationary oscillations of the pendulum (upper time series) have a slow period $T=4.833$, and is resonant in character with a large ($O(1)$) amplitude compared to the small-amplitude ($O(\Omega^{-1})$) input. Considering (14) and the remarks below, this could be expected, since with the present parameters the frequency-difference $|\Omega_1 - \Omega_2| = 2.6$ is close to twice the effective natural frequency $2\omega_0 = 2\sqrt{v^2 + 1} = 2\sqrt{1 + \frac{1}{2}(\Omega_1 A_1)^2 + \frac{1}{2}(\Omega_2 A_2)^2} = 2.831$. The pendulum should then perform parametrically resonant oscillations at half the difference in excitation frequencies, i.e. $(\Omega_2 - \Omega_1)/2 = 1.3$, corresponding to a period of $T = 2\pi/1.3 = 4.833$, which agrees with the simulated time series. Considering just the original equation of motion (1), one might not think that large responses could occur under the above conditions, while this appears more readily from the averaged system (12).

As for the up-pointing equilibrium $z=\pi$ of (15), it cannot be stabilized as for the case of non-close frequencies, even if $\omega_\pi > 0$. The only possibility for this would be if the stiffness coefficient $\tilde{\omega}_\pi^2(t) = \omega_\pi^2 + p(t)$ in (15) be positive for all t , in which case small oscillations about $z=\pi$ would occur at a slowly modulated frequency $\tilde{\omega}_\pi$. But due to the definitions of ω_π and p this would require a frequency-difference $|\Omega_i - \Omega_j| = O(A_{i,j}^{-1}) \gg 1$, which contradicts the present assumption of close frequencies. Hence, stabilization of the inverted pendulum position requires either strong mono-HFE, or strong multi-HFE with frequency separations

being larger in magnitude order than the unforced natural frequency; this conclusion agrees with [18,19].

3. Results for a general class of systems

With the above specific example in mind to ease interpretation, we next consider a quite general class of discrete dynamical systems with multi-HFE, covering a broad variety of applications. The purpose of the analysis is to determine the slow or averaged components of motions, which will be those apparent to lowpass-filtering observers such as measuring instruments and human senses. The analysis can be performed using different perturbation techniques, considering Ω^{-1} as a small parameter where Ω is a characteristic large excitation frequency. Standard averaging can be used as in [10], or multiple time scaling methods as in [22,24,27], or the method of direct separation of motions (MDSM) as in a large number of studies (e.g. see [7-13] and references cited there). In [10,13] these techniques are briefly compared, and shown to give identical or similar results, though with the MDSM being computationally more convenient.

3.1. The general system

Consider discrete mechanical systems modeled by:

$$\mathbf{M}(\mathbf{u}, t)\ddot{\mathbf{u}} + \Omega^2 \mathbf{K}\mathbf{u} + \mathbf{s}(\mathbf{u}, \dot{\mathbf{u}}, t) + \sum_{j=1}^m \left(\mathbf{h}_j(\mathbf{u}, \dot{\mathbf{u}}, t) + \Omega \mathbf{f}_j(\mathbf{u}, t) \right) \frac{\partial^2 \xi_j(t, \tau)}{\partial \tau^2} = \mathbf{0} \quad (16)$$

$$\mathbf{u} = \mathbf{u}(t); \quad \mathbf{u}(0) = \mathbf{u}_0; \quad \dot{\mathbf{u}}_0(0) = \dot{\mathbf{u}}_0; \quad \tau = \Omega t, \quad \Omega \gg 1,$$

which covers numerous applications, see e.g. [11,13], and is an extension of the one analyzed in [12]. Here $\mathbf{u}(t) \in D \subset \mathbb{R}^n$ describes positional state with corresponding velocities $\dot{\mathbf{u}} = d\mathbf{u}/dt$, $\mathbf{M}$ is a positive definite mass matrix, $\Omega^2 \mathbf{K}$ is a constant matrix describing linear

Fig. 3. Large stationary pendulum motions $\theta(t)$ (upper time series) induced by small-amplitude support motions $q(t)$ (lower time series) at two close high frequencies, as obtained by numerical simulation of the equation of motion (1). (Parameters: $\beta = 0.05$, $\Omega_1 = 18.7$, $\Omega_2 = 21.3$, $A_1 = A_2 = 0.05$, $\alpha_1 = \alpha_2 = 0$, $\theta(0) = 0.01$, $\dot{\theta}(0) = 0$.)

stiffness or restoring force, $\mathbf{s}$ holds the ‘slow’ forces, and $\mathbf{h}_j$, $\mathbf{f}_j$, and ξ_j jointly describe the multi-HFE of characteristic frequency $\Omega \gg 1$, i.e. the ‘fast’ forces. The sum of fast forces, along with its first and second derivative with respect to τ , is supposed to have zero fast-time average (any non-zero average being included in $\mathbf{s}$ or $\mathbf{Ku}$), and to be T -periodic and square-integrable in τ , where $T \in]T^*; \infty[$, $T^* = O(2\pi\Omega)$, with the fast-time average of a function $\mathbf{g}(t, \tau)$ defined by:

$$\langle \mathbf{g}(t, \tau) \rangle_\tau \equiv \frac{1}{T} \int_0^T \mathbf{g}(t, \tau) d\tau, \quad (17)$$

where the integration is performed with the slow time t considered fixed. All functions in (16) are generally nonlinear and of magnitude order unity $O(1)$ or lower, ξ_j are bounded on $[0; T]$, and $\mathbf{f}_j$ and $\mathbf{M}$ are bounded on D with continuous first derivatives with respect to $\mathbf{u}$ (whereas $\mathbf{h}_j$ and $\mathbf{s}$ need not to be continuous). Terms multiplied by the large parameter Ω are considered ‘strong’.

The decision of whether to consider the linear stiffness strong depends on whether the HFE could possibly excite linear resonances in the system: If the excitation frequencies are not well separated from the linear natural frequencies, then the linear stiffness must be considered strong and should be put in the $\Omega^2 \mathbf{K}$ term; if not, it should be represented by the $\mathbf{s}$ -term. There would be no formal difficulties, for the following calculations, in letting the stiffness term be generally nonlinear and dependent of slow time, i.e. $\mathbf{Ku} = \mathbf{k}(\mathbf{u}, t)$; however, since in applications a possible nonlinear part of the restoring force is typically much weaker than the linear contribution, the nonlinear part could be represented by the $\mathbf{s}$ -term in (16).

The notions of ‘slow’ and ‘fast’ here refer to two distinct characteristic time scales or frequencies characterizing motions of the system: There is a time scale t and a characteristic frequency ω describing motions of the system when $\xi_j = 0$ for all j , e.g. one can take as ω the largest natural frequency of the linearized, unexcited system. It is assumed that time t has been normalized such that $\omega = O(1)$. Then there is fast time scale $\tau = \Omega t$, which describes fluctuations imposed by the external loading of characteristic high frequency $\Omega \gg \omega$, the time variation of such loads are given by ξ_j . The ‘fast’ loading can contain terms oscillating at many different frequencies, but some of them are high, and Ω describe their order magnitude, e.g. their average. For cases where $\Omega \gg \omega$ is not satisfied, the linear stiffness is ‘strong’ and should be represented by the $\Omega^2 \mathbf{K}$ -term as explained above.

For the general system analyzed in [12], $T = 2\pi$, and the strong term $\Omega^2 \mathbf{K}$ was not included. For the extension (16), the fast excitations are assumed to be generally T -periodic or quasiperiodic ($T \rightarrow \infty$) and to be square-integrable in τ (to justify Fourier expansion), and the strong stiffness term $\Omega^2 \mathbf{K}$ extends the applicability to discrete models that arise from mode shape expansion of continuous models of elastic structures, where the influence of resonances cannot be ignored. Many of the below results can be found also in [11, 12], but is given here with special emphasis to the analysis of various stiffening effects of multi-HFE, taking also resonant influences into account.

3.2. An approximate set of equations governing the slow or average motions

Using the method of direct separation of motions [7] (see also [9-11]), one separates solutions $\mathbf{u}(t)$ to (16) into slow and fast components as follows:

$$\mathbf{u} = \mathbf{u}(t, \tau) = \mathbf{z}(t) + \Omega^{-1} \boldsymbol{\phi}(t, \tau), \quad (18)$$

where $\mathbf{z}(t)$ is the slow or average motion, and $\Omega^{-1} \boldsymbol{\phi}$ is a rapidly oscillating overlay which has small amplitude, is T -periodic in the fast time $\tau = \Omega t$, and zero fast-time averages:

$$\langle \boldsymbol{\varphi} \rangle_\tau = \langle \boldsymbol{\varphi}' \rangle_\tau = \langle \boldsymbol{\varphi}'' \rangle_\tau = 0, \quad (19)$$

where primes denote differentiation with respect to τ . Inserting into (16) and solving to first order of accuracy for $\boldsymbol{\varphi}$, one finds (see [11,12] for further details):

$$\boldsymbol{\varphi}(t, \tau) = \hat{\boldsymbol{\varphi}}(t, \tau) + O(\Omega^{-1}), \quad (20)$$

where the first approximation $\hat{\boldsymbol{\varphi}}$ for the fast motions $\boldsymbol{\varphi}$ is a particular solution of:

$$\mathbf{M}(\mathbf{z}, t) \hat{\boldsymbol{\varphi}}'' + \mathbf{K} \hat{\boldsymbol{\varphi}} = - \sum_{j=1}^m \mathbf{f}_j(\mathbf{z}, t) \xi_j''(t, \tau). \quad (21)$$

Averaging (16) with (18)-(20) inserted then gives, to first order of accuracy, a set of equations for the slow or averaged motions $\mathbf{z}$:

$$\mathbf{M}(\mathbf{z}, t) \ddot{\mathbf{z}} + \Omega^2 \mathbf{K} \mathbf{z} + \mathbf{s}(\mathbf{z}, \dot{\mathbf{z}}, t) + \mathbf{v}(\mathbf{z}, \dot{\mathbf{z}}, t) = \mathbf{0}, \quad (22)$$

which is similar to (16), but with the HFE terms replaced by $\mathbf{v}(\mathbf{z}, \dot{\mathbf{z}}, t)$, the so-called *vibrational forces* [7]:

$$\begin{aligned} \mathbf{v}(\mathbf{z}, \dot{\mathbf{z}}, t) = & \langle \mathbf{s}(\mathbf{z}, \dot{\mathbf{z}} + \hat{\boldsymbol{\varphi}}', t) - \mathbf{s}(\mathbf{z}, \dot{\mathbf{z}}, t) \rangle_\tau + \sum_{j=1}^m \langle \mathbf{h}_j(\mathbf{z}, \dot{\mathbf{z}} + \hat{\boldsymbol{\varphi}}', t) \xi_j''(t, \tau) \rangle_\tau \\ & + \sum_{j=1}^m \nabla \mathbf{f}_j(\mathbf{z}, t) \langle \hat{\boldsymbol{\varphi}} \xi_j''(t, \tau) \rangle_\tau + \langle \nabla \mathbf{m}(\mathbf{z}, t, \hat{\boldsymbol{\varphi}}) \rangle_\tau, \end{aligned} \quad (23)$$

where overdots denote differentiation with respect to t , and ∇ denote positional derivatives:

$$\nabla \mathbf{f}_j(\mathbf{u}, t) \equiv \frac{\partial \mathbf{f}_j}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial \mathbf{f}_j}{\partial \mathbf{u}_{(1)}} & \dots & \frac{\partial \mathbf{f}_j}{\partial \mathbf{u}_{(n)}} \end{bmatrix} \quad (24)$$

$$\nabla \mathbf{m}_{(i)}(\mathbf{u}, t, \boldsymbol{\varphi}) \equiv (\boldsymbol{\varphi}'')^T \nabla \mathbf{M}_{(i)} \boldsymbol{\varphi} \quad \text{where} \quad \nabla \mathbf{M}_{(i)}(\mathbf{u}, t) \equiv \frac{\partial \mathbf{M}_{(i)}}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial \mathbf{M}_{(i)}}{\partial \mathbf{u}_{(1)}} & \dots & \frac{\partial \mathbf{M}_{(i)}}{\partial \mathbf{u}_{(n)}} \end{bmatrix}, \quad (25)$$

where a subscript in parenthesis denotes a vector element or matrix column. The vibrational forces $\mathbf{v}$ correspond to real physical forces having the same effect as the HFE, on the average. To an observer or measuring instrument filtering out small-amplitude HF vibrations, the response $\mathbf{u}$ from (16) is identical (within the order of approximation) to the response $\mathbf{z}$ from (22).

The initial conditions needed to solve (22) for $\mathbf{z}$ is obtained from the original initial conditions in (16), which turns into:

$$\mathbf{z}(0) = \mathbf{u}_0 - \Omega^{-1} \hat{\boldsymbol{\varphi}}(0, 0); \quad \dot{\mathbf{z}}(0) = \dot{\mathbf{u}}_0 - \hat{\boldsymbol{\varphi}}'(0, 0) - \Omega^{-1} \hat{\boldsymbol{\varphi}}(0, 0), \quad (26)$$

which completes the separation of the full motions $\mathbf{u}$ into slow and fast components $\mathbf{z}$ and $\boldsymbol{\varphi}$.

Since in (22) any explicit dependence on the fast time τ has been averaged out, this equation for $\mathbf{z}$ is far easier to solve than the original equation (16) for $\mathbf{u}$. This holds for analytical solutions, but also for numerical simulation where much larger time steps can be used, since there is no need to keep track of the rapid oscillations at frequency order of magnitude Ω .

Based on the averaged system (22), a number of general results regarding HFE effects can be derived, without any need to actually solve the equations; see [12] for $\mathbf{K}=\mathbf{0}$, and [26] for a case corresponding to $\mathbf{K} \neq \mathbf{0}$. In [12] three main effects were identified and discussed: 1) *Stiffening*, which is an apparent change in the linear stiffness associated with an equilibrium, along with derived quantities such as stability and natural frequencies; 2) *biasing*, by which a

system is biased towards a particular state, static or dynamic, which does not exist or is unstable in the absence of the HFE; and 3) *smoothing*, referring to a tendency for discontinuities to be effectively “smeared out” by HFE. Though these results are valid also for multi-frequency excitation ($m > 1$ in (16)), no attempts was made in [12] to consider particular effects associated with this, and all specific examples were given in terms of mono-frequency harmonic excitation. Below we focus on effects of multi-HFE for an important subclass of (16).

3.3. Results for an important subclass

To illustrate the influence of multi-HFE for a simpler subclass of systems, still covering a great many applications, we assume from now a unitary mass matrix $\mathbf{M} = \mathbf{I}$ (since this could be obtained anyway by multiplying with $\mathbf{M}^{-1}$), that $\mathbf{s}$ is linear in the velocities $\dot{\mathbf{u}}$, and that the fast forces are independent of $\dot{\mathbf{u}}$. We also assume $\Omega^2 \mathbf{K} = \Omega^2 \boldsymbol{\omega}^2$, where $\boldsymbol{\omega}^2$ is a diagonal matrix holding the linear undamped natural frequencies of the system, normalized by Ω^2 . Note that the matrix $\Omega^2 \boldsymbol{\omega}^2$ should only be given in case its components are comparable in magnitude to the excitation frequencies of the HFE (i.e. $|\boldsymbol{\omega}| = O(1)$), otherwise the corresponding force $\mathbf{K}\mathbf{u}$ or acceleration $\boldsymbol{\omega}^2 \mathbf{u}$ should be included in the $\mathbf{s}$ -term. Then (16) becomes:

$$\ddot{\mathbf{u}} + \Omega^2 \boldsymbol{\omega}^2 \mathbf{u} + \mathbf{s}(\mathbf{u}, \dot{\mathbf{u}}, t) + \sum_{j=1}^m \Omega \mathbf{f}_j(\mathbf{u}, t) \frac{\partial^2 \xi_j(t, \tau)}{\partial \tau^2} = \mathbf{0}, \quad \tau = \Omega t, \quad \Omega \gg 1. \quad (27)$$

The splitting of motions into slow and fast components is still given by (18), and the fast motions $\hat{\boldsymbol{\phi}}$ by (20)-(21), while the averaged system (22)-(23) for the slow motions $\mathbf{z}$ becomes:

$$\ddot{\mathbf{z}} + \Omega^2 \boldsymbol{\omega}^2 \mathbf{z} + \mathbf{s}(\mathbf{z}, \dot{\mathbf{z}}, t) + \sum_{j=1}^m \nabla \mathbf{f}_j(\mathbf{z}, t) \langle \hat{\boldsymbol{\phi}} \xi_j''(t, \tau) \rangle_\tau = \mathbf{0}. \quad (28)$$

In (28) the HF-terms of the original system (27) is replaced by the average effect of these, which takes the form of *parametrical stiffening* [11,12]. Thus the main characteristic of the subclass (27) of systems (16) is that the *smoothing* effects of HFE and certain forms of *biasing* are not covered.

Next, since the HFE assumed to be periodic and square-integrable, it can be Fourier-expanded and written in a form where all the m HFE-terms are time-harmonics with frequencies Ω_j and phases α_j , i.e.:

$$\xi_j(t, \tau) = \tilde{\xi}_j(t, \tau) \equiv \sin(\Omega_j t + \alpha_j) = \sin\left(\frac{\Omega_j}{\Omega} \tau + \alpha_j\right), \quad (29)$$

where the characteristic high frequency Ω is taken as the average excitation frequency:

$$\Omega = \frac{1}{m} \sum_{j=1}^m \Omega_j \gg 1. \quad (30)$$

Then (21) becomes:

$$\hat{\boldsymbol{\phi}}'' + \boldsymbol{\omega}^2 \hat{\boldsymbol{\phi}} = \sum_{j=1}^m \mathbf{f}_j(\mathbf{z}, t) \left(\Omega_j / \Omega\right)^2 \tilde{\xi}_j(t, \tau) \quad (31)$$

with particular solution:

$$\hat{\boldsymbol{\phi}} = -\sum_{j=1}^m \mathbf{D}_j^{-1} \mathbf{f}_j(\mathbf{z}, t) \tilde{\xi}_j(t, \tau), \quad (32)$$

where $\mathbf{D}_j$ is the j th $n \times n$ diagonal *resonance matrix* corresponding to the HFE at frequency Ω_j :

$$\mathbf{D}_j = \mathbf{I} - \left(\Omega / \Omega_j \right)^2 \boldsymbol{\omega}^2. \quad (33)$$

where it should be recalled that $\Omega^2 \boldsymbol{\omega}^2$ holds the squared natural frequencies. For excitation frequencies Ω_j much higher than the highest natural frequency, the second term in (33) is negligible so that $\mathbf{D}_j^{-1} \rightarrow \mathbf{I}$. But as an excitation frequency Ω_j approaches any of the system natural frequencies, then $|\mathbf{D}_j^{-1}| \rightarrow \infty$, and resonant oscillations of the fast motions $\hat{\boldsymbol{\phi}}$ will occur. Since $|\hat{\boldsymbol{\phi}}| = O(1)$ has been assumed, the following results are thus only valid when the HFE is not *sharply* resonant to the high-frequency natural frequencies.

Inserting (32) and (29) into (28), and using the definition (17), integration by parts, and the T -periodicity of $\tilde{\xi}_j$ and $\tilde{\xi}'_j$ one obtains:

$$\ddot{\mathbf{z}} + \Omega^2 \boldsymbol{\omega}^2 \mathbf{z} + \mathbf{s}(\mathbf{z}, \dot{\mathbf{z}}, t) + \sum_{i,j=1}^m \sigma_{ij}^2(t) \nabla \mathbf{f}_i(\mathbf{z}, t) \mathbf{D}_j^{-1} \mathbf{f}_j(\mathbf{z}, t) = \mathbf{0}, \quad (34)$$

where σ_{ij}^2 holds the slowly varying mean-square derivatives of the HFE terms:

$$\sigma_{ij}^2(t) = \left\langle \tilde{\xi}'_i(t, \tau) \tilde{\xi}'_j(t, \tau) \right\rangle_{\tau}. \quad (35)$$

When $\boldsymbol{\omega}^2 = \mathbf{0}$ (i.e. the HFE-frequencies are much higher than the natural frequencies), then the averaged equation (34) with (35) holds not just when $\xi_j = \tilde{\xi}_j$ as given by (29), but for any set of zero-mean T -periodic and square-integrable functions ξ_i ; for that case only the mean-square derivatives count. The ξ_j could even be random signals, in which case $T \rightarrow \infty$ and $\sigma_{\xi_j}^2$ would be the covariance matrix of the HF-derivatives, and indeed results agreeing with (34)-(35) has been obtained for multi-HFE with randomly varying phase [18,20].

The calculation of mean-square derivatives σ_{ij}^2 depends on whether some of the frequencies Ω_j are close. Inserting (29) into (35) and employing (17) with $T \rightarrow \infty$ gives:

$$\sigma_{ij}^2(t) = \begin{cases} \frac{\Omega_i \Omega_j}{2\Omega^2} \delta_{ij}, & \text{for } |\Omega_i - \Omega_j| \gg 1, \quad i, j = 1, \dots, m \\ \frac{1}{2} \cos((\Omega_i - \Omega_j)t + \alpha_i - \alpha_j), & \text{for } |\Omega_i - \Omega_j| \leq O(1), \end{cases} \quad (36)$$

where δ_{ij} is the Kronecker delta, and the first case (non-close frequencies) is calculated by directly inserting (29) into (35) and using (17), while the second case (any two close frequencies) is calculated by inserting into (35) an equivalent form of (29), written in terms of the small frequency-difference $\Delta\Omega_j$:

$$\tilde{\xi}_j(t, \tau) = \sin(\tau + \Delta\Omega_j t + \alpha_j), \quad \Delta\Omega_j = \Omega_j - \Omega, \quad |\Delta\Omega_j| \leq O(1). \quad (37)$$

As appears, for non-close frequencies the mean-square derivatives can be considered constants, while for any two close frequencies Ω_i and Ω_j , the corresponding mean-square derivative σ_{ij}^2 is slowly oscillating in time.

3.4. Changes in effective properties of quasistatic equilibria

Suppose the averaged system (34) has a quasistatic equilibrium at $\mathbf{z} = \tilde{\mathbf{z}}$, i.e. :

$$\Omega^2 \omega^2 \tilde{\mathbf{z}} + \mathbf{s}(\tilde{\mathbf{z}}, 0, t) + \sum_{i,j=1}^m \sigma_{ij}^2(t) \nabla \mathbf{f}_i(\tilde{\mathbf{z}}, t) \mathbf{D}_j^{-1} \mathbf{f}_j(\tilde{\mathbf{z}}, t) = \mathbf{0}, \quad (38)$$

at which the slowly varying derivatives wrt. state variables are:

$$\mathbf{K}^s(t) = \left. \frac{\partial \mathbf{s}}{\partial \mathbf{u}} \right|_{(\mathbf{u}, \dot{\mathbf{u}}) = (\tilde{\mathbf{z}}, \mathbf{0})}, \quad \mathbf{C}(t) = \left. \frac{\partial \mathbf{s}}{\partial \dot{\mathbf{u}}} \right|_{(\mathbf{u}, \dot{\mathbf{u}}) = (\tilde{\mathbf{z}}, \mathbf{0})} \quad (39)$$

$$\mathbf{K}_{ij}^f(t) = \left. \frac{\partial (\nabla \mathbf{f}_i \mathbf{D}_j^{-1} \mathbf{f}_j)}{\partial \mathbf{u}} \right|_{\mathbf{u}=\tilde{\mathbf{z}}} = (\nabla \mathbf{f}_i \mathbf{D}_j^{-1} \nabla \mathbf{f}_j + \mathbf{f}_j^T \otimes \nabla^2 \mathbf{f}_i) \Big|_{\mathbf{u}=\tilde{\mathbf{z}}}, \quad (40)$$

where the last term is an $m \times m$ matrix whose k 'th column is given by

$$(\mathbf{f}_j^T \otimes \nabla^2 \mathbf{f}_i)_{(k)} = \mathbf{f}_j^T \mathbf{D}_j^{-1} (\nabla^2 \mathbf{f}_i)_{(k)}, \quad (41)$$

and $(\nabla^2 \mathbf{f}_i)_{(k)}$ is the Hessian matrix corresponding to $\mathbf{f}_{i(k)}$:

$$(\nabla^2 \mathbf{f}_i)_{(k)} = \begin{bmatrix} \frac{\partial^2 \mathbf{f}_{i(k)}}{\partial \mathbf{u}_{(1)}^2} & \cdots & \frac{\partial^2 \mathbf{f}_{i(k)}}{\partial \mathbf{u}_{(1)} \partial \mathbf{u}_{(m)}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathbf{f}_{i(k)}}{\partial \mathbf{u}_{(m)} \partial \mathbf{u}_{(1)}} & \cdots & \frac{\partial^2 \mathbf{f}_{i(k)}}{\partial \mathbf{u}_{(m)}^2} \end{bmatrix}. \quad (42)$$

Then (34) can be Taylor-expanded at $\mathbf{z}=\tilde{\mathbf{z}}$ and rearranged into:

$$\ddot{\mathbf{z}} + \Omega^2 \omega^2 \mathbf{z} + \mathbf{C}(t) \dot{\mathbf{z}} + (\mathbf{K}^s(t) + \Delta \mathbf{K}^f(t)) (\mathbf{z} - \tilde{\mathbf{z}}) + O(|\mathbf{z} - \tilde{\mathbf{z}}|^2) + O(|\dot{\mathbf{z}}|^2) = \mathbf{0}, \quad (43)$$

where

$$\Delta \mathbf{K}^f(t) = \sum_{i,j=1}^m \sigma_{ij}^2(t) \mathbf{K}_{ij}^f(t), \quad (44)$$

which governs motions near the quasi-equilibrium $\tilde{\mathbf{u}}$. Here the matrix $\mathbf{C}$ is seen to be a damping matrix (in general slowly changing) for the system linearized near $\tilde{\mathbf{u}}$, $\Omega^2 \omega^2 + \mathbf{K}^s$ is the corresponding stiffness matrix for the system without HFE, and $\Delta \mathbf{K}^f$ is an apparent or effective change in linear stiffness caused by the average effect of the HFE.

As appears from (44) and the definition of $\mathbf{K}^f$ in (40), for systems (27) there are two terms contributing to stiffening by HFE: The first of the two summed terms in (40) exists only if there is at least one $\mathbf{f}_j$ which is linearizable with a nonzero gradient at zero. Since this term disappears for functions that are essentially nonlinear, we term its contribution *linearly induced parametrical stiffness* [12]. If $\langle \xi'_i \xi'_j \rangle = 0$ for $j \neq k$, as is often the case in applications, then this contribution is positive definite if all the HFE-frequencies are above the highest natural frequency of oscillations, since then $\omega^2 = \mathbf{0}$, which gives $\mathbf{D} = \mathbf{I}$, so that $|\nabla \mathbf{f}_j \mathbf{D}_j^{-1} \nabla \mathbf{f}_j| = |\nabla \mathbf{f}_j| |\mathbf{D}_j^{-1} \nabla \mathbf{f}_j| > 0$ for $i = j$. However, if some of the HFE-frequencies are within the range of natural frequencies, then $|\mathbf{D}_j^{-1}|$ may be negative, and the stiffness contribution is negative, i.e. the HFE makes the system appear *less* stiff than without HFE. (Indeed this was observed experimentally in [26], where a piano-string in mono-HFE showed clear drops in stiffness just below resonances.)

The second term in (40) exists only when $\mathbf{f}_j$ has both a constant and a quadratic part when

Taylor-expanded near the equilibrium; we therefore term it *nonlinearly induced parametrical stiffness*, since only nonlinear functions $\mathbf{f}_j$ can contribute to it. Terms of order three and higher of the Taylor-expansion of $\mathbf{f}_j$, does not contribute to stiffness (to the level of approximation employed), since their second derivatives vanish at the equilibrium

A change in effective stiffness implies changes in derived properties, as described next.

3.4.1. Change of natural frequency for oscillations near an equilibrium

The natural frequencies associated with small oscillations near an equilibrium $\tilde{\mathbf{z}}$ of the averaged system (34) are determined by the eigenvalues $\lambda = \lambda_r$, $r = 1, \dots, 2m$, which are roots of the characteristic polynomial corresponding to the linear part of (43):

$$\left| \lambda^2 + \lambda \mathbf{C}(t) + \lambda \left(\Omega^2 \boldsymbol{\omega}^2 + \mathbf{K}^s(t) + \Delta \mathbf{K}^f(t) \right) \right| = 0, \quad (45)$$

where the time-variation of the matrices are supposed to be quasistatic. The natural frequencies (also quasistatic) are then given by $|\text{Im}(\lambda_r)|$; They will generally differ from the values obtained when there is no HFE, i.e. when $\Delta \mathbf{K}^f = \mathbf{0}$ in (45).

3.4.2. Shift in equilibrium position

Suppose (27) has an equilibrium at $\mathbf{u} = \tilde{\mathbf{u}}$ when there is no HFE, i.e. :

$$\boldsymbol{\omega}^2 \tilde{\mathbf{u}} + \mathbf{s}(\tilde{\mathbf{u}}, \mathbf{0}, t) = \mathbf{0}. \quad (46)$$

Then $\tilde{\mathbf{u}}$ may be different from the equilibrium(s) $\mathbf{z} = \tilde{\mathbf{z}}$ for the averaged system under HFE. The condition for an equilibrium to be unchanged under HFE is obtained by letting $\tilde{\mathbf{z}} = \tilde{\mathbf{u}}$ in (38) and inserting (46), which gives:

$$\sum_{i,j=1}^m \sigma_{ij}^2(t) \nabla \mathbf{f}_i(\tilde{\mathbf{u}}, t) \mathbf{f}_j(\tilde{\mathbf{u}}, t) = \mathbf{0} \quad (\text{when } \tilde{\mathbf{u}} = \tilde{\mathbf{z}}), \quad (47)$$

from which it appears that a necessary condition for a shift in equilibrium to occur is that at least one of the functions $\mathbf{f}_j$ should be nonzero with a nonzero gradient at the equilibrium.

A small shift $\Delta \tilde{\mathbf{u}}$ in equilibrium position can be determined by Taylor-expanding (38) for $\tilde{\mathbf{z}} = \tilde{\mathbf{u}} + \Delta \tilde{\mathbf{u}}$ and solving for $\Delta \tilde{\mathbf{u}}$; this gives:

$$\Delta \tilde{\mathbf{u}} = - \left(\Omega^2 \boldsymbol{\omega}^2 + \mathbf{K}^s(t) + \Delta \mathbf{K}^f(t) \right)^{-1} \sum_{i,j=1}^m \sigma_{ij}^2(t) \nabla \mathbf{f}_i(\tilde{\mathbf{u}}, t) \mathbf{D}_j^{-1} \mathbf{f}_j(\tilde{\mathbf{u}}, t) + O(|\tilde{\mathbf{z}} - \tilde{\mathbf{u}}|^2). \quad (48)$$

If two or more of the HFE-frequencies are close, then according to (36) σ_{ij}^2 will oscillate at the slow frequency-difference, and so will the equilibrium in (48).

3.4.3. Change of equilibrium stability

An equilibrium $\tilde{\mathbf{z}}$ of the averaged system (34) is stable to small disturbances only if $\text{Re}(\lambda_r) < 0$ for all r , with λ_r being the solutions of (45). Even if the HFE does not change the position of an equilibrium, i.e. $\tilde{\mathbf{z}} = \tilde{\mathbf{u}}$, the additional term $\Delta \mathbf{K}^f$ in (45) caused by the HFE, may imply changing real parts of the eigenvalues λ_r , and thus a stable equilibrium may turn unstable or *vice versa* in the presence of HFE.

3.4.4. Creation of new equilibria

If solutions $\tilde{\mathbf{z}}$ to (38) exist that are not equal to or close to $\tilde{\mathbf{u}}$, we consider them as being ‘new’, i.e. they are created by the HFE. The stability of such equilibria can be calculated as described above.

4. Example II: Elastic String with multi-HFE

This example illustrates how the above general results can be used to analyze multi-HFE effects for a specific continuous, elastic system. The system is the horizontal piano string in Fig. 4, whose clamped base at the left end is vibrated horizontally at high frequency and small amplitude by an electromagnetic shaker. As appears from the figure, the string lifts in gravity in response to the HFE; this is a clear sign of the change in effective stiffness. In [26] theoretical predictions of the string lift were derived, and shown to agree with experimental results with good accuracy for a broad range of mono-HFE frequencies and amplitudes. Below we reconsider this system, but with multi-HFE. First a few experimental observations with a simple setup are presented, to illustrate qualitative features of string lift. Then follows theoretical predictions of string lift, derived from the general results of Section 3. Finally these predictions are tested experimentally, using a setup that allows accurate measurement of string lift and accurate control of the input multi-HFE.

4.1. Initial experimental observations with a simple setup

The initial observations were made for the string in Fig. 4, using a two-frequency harmonic HF-signal whose displacement amplitude A and frequency $f=\Omega/2\pi$ is indexed 1 and 2 below, while in all cases the phase α was zero. The excitation frequency f was controlled by software, and the excitation amplitudes A were calculated as $a/(2\pi f)^2$, with a being the acceleration amplitude measured by an accelerometer at the shaker base. Lift of the string end was determined visually, by inspecting the string end against a non-magnetic ruler fixed immediately behind the string. The latter measurement is inaccurate, with reading errors in the range 10-25% due to the fast and slow movements of the string, and parallax, but will suffice for a first test of the theoretical predictions.

4.1.1. Experiment A: Two frequency-excitation at non-close high frequencies

With signal 1 ($f_1=62\text{Hz}$, $a_1=332\text{m/s}^2$, $A_1=2.2\text{mm}$), the string end lifted off $u_1\approx 6\text{mm}$ from the unexcited static equilibrium in gravity, this state being reached after slow oscillations of the string at its fundamental natural frequency had decayed, while small-amplitude oscillations corresponding to the HFE remained. With signal 2 ($f_2=138\text{Hz}$, $a_2=298\text{m/s}^2$, $A_2=0.40\text{mm}$), the corresponding lift of the string end was $u_2\approx 2\text{mm}$. Removing either signal, the

Fig. 4. Piano string (0.5×500mm) excited by horizontal vibrations at the base. Two images overlaid: One with gravity as the only excitation (lower string image), and one with the base vibrating horizontally at 62 Hz and 3.3 mm displacement, 500 m/s² acceleration amplitude (upper image). The string tip is seen to lift about 2 cm. Camera shutter speed was 1/50 s, so the slightly blurred image of the upper string reflects positions traced during a little more than a full vibration cycle. (Adapted from [26])

string returned to its previous, unexcited static equilibrium. Adding signals 1 and 2, the string end lifted off $u_{1+2} \approx 8.5\text{mm}$, with beating occurring only for the small-amplitude oscillations corresponding to the HFE. We note that $u_1 + u_2 \approx u_{1+2}$, indicating that the equilibrium shift due to stiffening could be approximately additive for each frequency component; indeed this is confirmed below. Also, the observation that no slow oscillations in the lifted equilibrium position remained, after initial transients had decayed, agrees with the general theoretical prediction (48) for multi-HFE with non-close frequencies.

4.1.2. Experiment B: Two frequency-excitation at close high frequencies

With signal 1 equal to signal 2 ($f_1=f_2=138\text{Hz}$, $a_1=a_2=200\text{m/s}^2$, $A_1=A_2=2.7\text{mm}$), the string end lifted off $u_{1+2} \approx 4.5\text{mm}$, which was twice as much as for signal 1 or 2 alone. However giving signal 2 a small frequency shift of 0.5Hz ($f_2=138.5\text{Hz}$, $a_2=200\text{mm}$, $A_2=2.6\text{mm}$), the string started performing stationary slow oscillations at about 0.5Hz , repeatedly tracing all positions between the static, unexcited equilibrium $u_{1+2} \approx 0\text{mm}$, and the maximum lift $u_{1+2} \approx 4.5\text{mm}$. These observations again support the indication of a linear additive effect, as well as the general prediction under (48) that multi-HFE with close frequencies induce effective stiffness that oscillates at the small frequency-difference of the close frequencies.

4.1.3. Experiment C: Band-limited HFE

Using broadband random excitation (center frequency $f=100\text{Hz}$, frequency span $\Delta f=50\text{Hz}$, digitally generated with frequency spacing 0.031Hz), it was not possible by eye to detect an average raise of the string end from its static equilibrium. At stronger excitation amplitudes, it started performing violent vibrations, modulated by low-frequency oscillations with seemingly random components. A similar observation was noted with narrowband excitation ($f=62\text{Hz}$, $\Delta f=3\text{Hz}$). These observations further support the theoretical prediction that (quasi-) stationary stiffening requires frequencies that are well separated and not sharply resonant.

4.2. Theoretical model predictions

4.2.1. Mathematical model

The string is modeled as a viscously damped clamped-free Bernoulli-Euler beam (Fig. 5) of length l , mass per unit length ρA , and bending stiffness EI . The beam vibrates in a vertical plane with configuration $u(x,t)$ in response to external forces from gravity g , and time-harmonic multi-HFE base excitation with high frequencies Ω_j , small displacement amplitudes A_j , and phases α_j , $j=1, \dots, m$. The equation of motion and boundary conditions are [26]:

$$\begin{aligned} \ddot{u} + c\dot{u} + \omega_0^2 l^4 u_{xxxx} - ((l-x)u_x)_x \sum_{j=1}^m \Omega_j^2 A_j \sin(\Omega_j t + \alpha_j) &= -g; \\ \omega_0^2 &= \frac{EI}{\rho A l^4}, \quad \Omega = \frac{1}{m} \sum_{j=1}^m \Omega_j \gg \omega_0, \quad A_j \ll l, \\ u(0,t) = u_x(0,t) = u_{xx}(l,t) = u_{xxx}(l,t) &= 0, \end{aligned} \quad (49)$$

Fig. 5. Beam model of the base-excited piano string

where subscripts x denote partial derivatives with respect to x , ω_0 is a characteristic system frequency, and Ω a characteristic excitation frequency.

As in [26] (where $m=1$) we use mode shape expansion for discretizing (49), i.e. let:

$$u(x, t) = \mathbf{u}(t)^T \boldsymbol{\phi}(x), \quad (50)$$

where the n -vector $\mathbf{u}$ holding the modal coefficients $u_i(t)$ is the new dependent variable, and the n -vector $\boldsymbol{\phi}$ holds the eigenfunctions $\phi_i(x)$ for a clamped-free beam [11]:

$$\phi_i(x) = \cosh\left(\frac{\lambda_i x}{l}\right) - \cos\left(\frac{\lambda_i x}{l}\right) - \frac{\cosh(\lambda_i) + \cos(\lambda_i)}{\sinh(\lambda_i) + \sin(\lambda_i)} \left(\sinh\left(\frac{\lambda_i x}{l}\right) - \sin\left(\frac{\lambda_i x}{l}\right) \right), \quad (51)$$

where λ_i is a solution of the frequency equation $\cos(\lambda_i)\cosh(\lambda_i) + 1 = 0$ (the first six approximations are $\{1.8751041, 4.6940911, 7.8547574, 10.995541, 14.137168, 17.278760\}$; for $i \geq 6$ the deviation from the asymptotic value $\lambda_i \rightarrow (2i-1)\pi/2$ is less than 10^{-7}). The corresponding natural frequencies are:

$$\omega_i = \left(\frac{\lambda_i}{l}\right)^2 \sqrt{\frac{EI}{\rho A}} = \lambda_i^2 \omega_0 \quad (52)$$

Next insert (50) into (49), multiply by ϕ_j , integrate over the beam length, employ integration by parts, exploit the orthogonality properties of ϕ , and obtain a set of n ordinary differential equations for the modal coefficients u :

$$\ddot{u}_k + c\dot{u}_k + \omega_k^2 u_k + \sum_{i=1}^n \gamma_{ki} u_i \sum_{j=1}^m \Omega_j^2 A_j \sin(\Omega_j t + \alpha_j) = -\beta_k g, \quad k = 1, n \quad (53)$$

where the modal constants γ and β are given by:

$$\beta_k = \alpha_k^{-1} \int_0^l \phi_k dx, \quad \gamma_{ki} = \alpha_k^{-1} \int_0^l (l-x) \phi_{kx} \phi_{ix} dx, \quad \text{where} \quad \alpha_k \equiv \int_0^l \phi_k^2 dx. \quad (54)$$

4.2.2. Predictions of multi-HFE effects

In [26] the model (53) with $m=1$ was used to predict the stiffening effect of mono-HFE for the piano string, using the method of direct separation of motions in a manner that specifically took into account the effect of resonances; This proved necessary to obtain good agreement with experimental results. Here, instead, we can use the general results of Section 3 directly, with consideration to the influence of resonances taken into account by the matrix $\Omega^2 \boldsymbol{\omega}^2$.

The discretised beam model has the general form (27) with (29), with the following correspondences of variables and parameters:

$$\mathbf{u} = \{u_1 \quad \dots \quad u_n\}^T, \quad \Omega^2 \boldsymbol{\omega}^2 = \text{diag}_n(\omega^2), \quad \mathbf{s}(\mathbf{u}, \dot{\mathbf{u}}, t) = c\dot{\mathbf{u}} + g\boldsymbol{\beta}, \quad \mathbf{f}_j(\mathbf{u}, t) = -\Omega A_j \boldsymbol{\gamma} \mathbf{u}, \quad (55)$$

where $\Omega^2 \boldsymbol{\omega}^2$ is a diagonal matrix holding the squared natural frequencies ω_i , the vector $\boldsymbol{\beta}$ holds β_i , and the matrix $\boldsymbol{\gamma}$ holds γ_{ij} , $i, j = 1, \dots, n$. Then the fast component $\Omega^{-1} \hat{\boldsymbol{\phi}}$ of the modal coefficients $\mathbf{u}$ is given by (32), while the slow or averaged component $\mathbf{z}$ is given by (34), with mean-square derivatives σ_{ij}^2 calculated by (36).

Focusing here on calculating string lift, which is an easily measurable characteristic of stiffness, we first use (50) and (18) to calculate motions of the string end:

$$u(l, t) = \mathbf{u}^T(t) \boldsymbol{\phi}(l) = \mathbf{z}(t)^T \boldsymbol{\phi}(l) + \Omega^{-1} \boldsymbol{\phi}(t, \tau)^T \boldsymbol{\phi}(l) \quad (56)$$

where the slow and fast components $\mathbf{z}$ and $\Omega^{-1} \boldsymbol{\phi}$ are governed by (34) and (32), respectively;

with (55) and (33) these are governed by:

$$\ddot{\mathbf{z}} + \Omega^2 \boldsymbol{\omega}^2 \mathbf{z} + c\dot{\mathbf{z}} + \Omega^2 \sum_{i,j=1}^m \sigma_{ij}^2 A_i A_j \boldsymbol{\gamma} \mathbf{D}_j^{-1} \boldsymbol{\gamma} \mathbf{z} = -g\boldsymbol{\beta}, \quad (57)$$

and

$$\Omega^{-1} \hat{\boldsymbol{\phi}} = \sum_{j=1}^m A_j \mathbf{D}_j^{-1} \boldsymbol{\gamma} \tilde{\boldsymbol{\zeta}}_j, \quad (58)$$

where $\mathbf{D}_j$ is the resonance matrix (33). The stationary equilibrium $\tilde{\mathbf{z}}$ for the slow motions $\mathbf{z}$ is calculated by letting $\dot{\mathbf{z}} = \ddot{\mathbf{z}} = 0$ in (57) and solving for $\mathbf{z}$, which gives:

$$\tilde{\mathbf{z}} = -g \left(\Omega^2 \boldsymbol{\omega}^2 + \Omega^2 \sum_{i,j=1}^m \sigma_{ij}^2 A_i A_j \boldsymbol{\gamma} \mathbf{D}_j^{-1} \boldsymbol{\gamma} \right)^{-1} \boldsymbol{\beta}, \quad (59)$$

In the absence of HFE (i.e. all $A_i=0$) this gives $\tilde{\mathbf{z}} = -g(\Omega\boldsymbol{\omega}^2)^{-1}$, so that the time-averaged lift of the string end becomes, using (56):

$$\begin{aligned} \Delta \tilde{u}(l) &= \langle \tilde{u}(l, t) - \tilde{u}_0(l, t) \rangle = \left(\tilde{\mathbf{z}}|_{A_i \neq 0} - \tilde{\mathbf{z}}|_{A_i = 0} \right)^T \boldsymbol{\phi}(l) \\ &= g\boldsymbol{\beta}^T \left(\left(\Omega^2 \boldsymbol{\omega}^2 \right)^{-1} - \left(\Omega^2 \boldsymbol{\omega}^2 + \Omega^2 \sum_{i,j=1}^m \sigma_{ij}^2 A_i A_j \boldsymbol{\gamma} \mathbf{D}_j^{-1} \boldsymbol{\gamma} \right)^{-1} \right) \boldsymbol{\phi}(l). \end{aligned} \quad (60)$$

4.3. Case I: Non-close HFE-frequencies

Inserting (36) into (60) gives, for the case of *non-close* HFE-frequencies:

$$\Delta \tilde{u}(l) = g\Omega^{-2} \boldsymbol{\beta}^T \left(\left(\boldsymbol{\omega}^2 \right)^{-1} - \left(\boldsymbol{\omega}^2 + \Delta \boldsymbol{\omega}^2 \right)^{-1} \right) \boldsymbol{\phi}(l) \quad \text{for} \quad |\Omega_i - \Omega_j| \gg 1, \quad (61)$$

where $\Omega^2 \Delta \boldsymbol{\omega}^2$ holds the apparent changes in squared natural frequencies due to the HFE:

$$\Omega^2 \Delta \boldsymbol{\omega}^2 = \frac{1}{2} \sum_{j=1}^m (\Omega_j A_j)^2 \boldsymbol{\gamma} \mathbf{D}_j^{-1} \boldsymbol{\gamma} \quad (62)$$

With $m=1$ (61)-(62) becomes identical to the expression derived under similar assumptions in [26] for mono-HFE.

If the HFE is not resonant to the system, i.e. $\mathbf{D}_j$ is not close to being singular, then $\Delta \boldsymbol{\omega}^2$ (representing HFE-generated change in stiffness) is small as compared to $\boldsymbol{\omega}^2$ (representing static structural stiffness), and it makes sense to Taylor-expand (61) to first order:

$$\Delta \tilde{u}(l) = g\boldsymbol{\beta}^T \left(\Omega^2 \boldsymbol{\omega}^2 \right)^{-2} \Omega^2 \Delta \boldsymbol{\omega}^2 \boldsymbol{\phi}(l) \quad \text{for} \quad |\Omega_i - \Omega_j| \gg 1, \quad (63)$$

where $(\Omega^2 \boldsymbol{\omega}^2)^{-2} = \text{diag}(\omega_i^{-4})$. Thus, it appears from (62)-(63), to the first order of approximation, the stiffening effect and corresponding string lift with non-close multi-HFE is simply linear additive: The lift resulting from a sum of non-close HFEs equals the sum of the lifts for each separate HFE. Also, when all of the HFE frequencies Ω_j , $j=1, m$, are far from the resonances ω_i , $i=1, n$, then $\mathbf{D}_j \rightarrow \mathbf{I}$ and the stiffening effect $\Omega^2 \Delta \boldsymbol{\omega}^2$ – and therefore the string lift $\Delta \tilde{u}(l)$ – becomes simply proportional to the sum of squared HFE velocities $\sum_j (\Omega_j A_j)^2$. This agrees with the experimental observation for test A in Section 4.1.1.

4.4. Case II: Some close HFE-frequencies

Inserting (36) into (60) gives, when there are *close* HFE-frequencies:

$$\Delta \tilde{u}(l) = g \Omega^{-2} \beta^T \left((\omega^2)^{-1} - (\omega^2 + \Delta \omega^2 + Q(t))^{-1} \right) \phi(l) \quad \text{for } |\Omega_i - \Omega_j| \leq O(1), \quad (64)$$

$$Q(t) = \frac{1}{2} \sum_{i,j=1}^{m,j \neq i} \cos((\Omega_i - \Omega_j)t + \alpha_i - \alpha_j) A_i A_j \gamma D_j^{-1} \gamma. \quad (65)$$

Taylor-expanding again to first order gives, for a weak stiffening effect and responses that are not sharply resonant:

$$\Delta \tilde{u}(l) = g \beta^T (\Omega^2 \omega^2)^{-2} (\Omega^2 \Delta \omega^2 + \Omega^2 Q(t)) \phi(l) \quad \text{for } |\Omega_i - \Omega_j| \gg 1, \quad (66)$$

Comparing to (63), it appears that for close frequencies, to first order of approximation, the string lifts off to the position determined as for non-close frequencies, and on top of that slow oscillations occur at an amplitude determined by the maximum of the function $Q(t)$, which depends on the particular values of the parameters in (65). This agrees qualitatively with the experimental observation for test B in Section 4.1.2. However, the experimental observation that the slow oscillations is ‘... tracing all positions between the static, unexcited equilibrium ... and the maximum lift ...’ may be particular to the parameters of that experiment, since according to (66) with (65), the string end could even dip below its static equilibrium during each period the slow oscillations.

5. Quantitative experimental testing

5.1. Experimental setup

The setup is similar to the one used in [26], including the same piano string of length 550 mm and diameter 1 mm (i.e. much stiffer than the one in Fig. 4), density 7819 kg/m, and Young’s modulus 195 GPa, measured lowest natural frequencies $\{2.4, 14.6, 40.4, 79.1, 131.0, 195.6\}$ Hz deviating less than 1% (though 3.9% for the lowest frequency) from Bernoulli-Euler theoretical values, and linear damping ratios less than 0.3% (though 4.5% for the lowest frequency).

The string is clamped and excited horizontally by a vibration shaker at one end, as in Fig. 4. The resulting string base acceleration is measured by an accelerometer mounted at the shaker, and the time-varying string tip lift is measured by an optical tracker with an accuracy better than 0.1 mm. Measured signals are conditioned and sampled (typically at 4 kHz) by a B&K PULSE front-end unit and analyzed using B&K PULSE software. Details of the experimental setup and procedures are as described in [26], except that for the present study the newest version of B&K PULSE front-end and software were used, that this was also used for controlling the multi-frequency input to the vibration shaker, that average string lift was obtained from the zero-frequency component of the FFT of the optical tracker signal, and that root-mean-square accelerations at each input frequency was obtained from the FFT of the accelerometer signal.

5.2. Control parameters

A two-frequency excitation was used for the input horizontal displacement $A(t)$ at the shake base, i.e. $m=2$ in the equations of Section 4.2, so that:

$$A(t) = A_1 \sin(\Omega_1 t) + A_2 \sin(\Omega_2 t); \quad \Omega_1 \neq \Omega_2. \quad (67)$$

The experimental setup allows measurements of the resulting acceleration $a(t)$ at the shaker base, as well as the root-mean-square accelerations at each input frequency $a_{i,\text{rms}}$, $i=1,2$ so that the corresponding shaker displacements can be calculated as:

$$A_i = \sqrt{2} a_{i,\text{rms}} / \Omega_i^2, \quad i=1,2, \quad (68)$$

For the presentation of results, it is convenient to express the excitation in terms of an *input acceleration* parameter a_{rms} , which is the total rms acceleration at the shaker base, and a continuous *frequency mixing* parameter $\eta \in [0;1]$, which is zero (unity) when all of the input energy is at frequency Ω_1 (Ω_2) so that $(a_{1,\text{rms}}/a_{\text{rms}})^2 = 1-\eta$ and $(a_{2,\text{rms}}/a_{\text{rms}})^2 = \eta$, i.e.:

$$a_{\text{rms}} = \sqrt{a_{1,\text{rms}}^2 + a_{2,\text{rms}}^2}; \quad \eta = (a_{2,\text{rms}}/a_{\text{rms}})^2, \quad (69)$$

We also define a *specific input energy* parameter $\bar{E} = \frac{1}{2} \sum_i (\Omega_i A_i)^2$, where $\Omega_i A_i$ is the velocity amplitude at input frequency Ω_i , i.e., by (68)-(69):

$$\bar{E} = \left(\frac{a_{1,\text{rms}}}{\Omega_1} \right)^2 + \left(\frac{a_{2,\text{rms}}}{\Omega_2} \right)^2 = \left(\frac{1-\eta}{\Omega_1^2} + \frac{\eta}{\Omega_2^2} \right) a_{\text{rms}}^2, \quad (70)$$

5.3. Procedures

In comparing theoretical predictions with experimental measurements, we focus on the prediction of (quasi-)stationary average string tip lift $\Delta \tilde{u}(l)$, since this represents the change in effective stiffness of interest here. Theoretical predictions for this are calculated using (61)-(62) for non-close excitation frequencies, and (64)-(65) for close frequencies. Corresponding experimental values for given excitation frequencies are obtained by 1) calibrating the optical tracker to give zero reading when the string is loaded only by gravity, i.e. without HFE, 2) imposing HFE through the shaker base, 3) waiting for stationary conditions to occur, 4) recording imposed acceleration levels at each excitation frequency from the FFT of the shaker base acceleration signal, 5) recording corresponding average string tip lift from the zero frequency component of the FFT of the optical tracker signal, 6) regularly removing the HFE to check if string tip lift reading returns to zero (recalibrating and repeating measurement if not).

Predictions for the trivial HF overlay on average tip lift can readily be obtained from the second term in (56), with $\hat{\phi}$ from (32) substituted for the fast motions ϕ . However this prediction is not tested separately, because the fast string motions ϕ are considered interesting only by their effect on the change in effective stiffness, i.e. only their effect “on the average” on string lift need to be correctly predicted. (Still, in [26], dealing with the mono-frequency case, excellent agreement between predicted and measured fast motions were noted).

5.4. Results

Fig. 6 shows examples of measured *time series* in form of string tip lift $\tilde{u}(l,t) - \tilde{u}_0(l,t)$ versus time at constant input acceleration level a_{rms} , along with measured and theoretically and predicted averages (solid and dashed horizontal lines), and with parameters as given in the legend. Figs. (a) and (b) are for mono-frequency excitation at 61 and 110 Hz, respectively, in both cases displaying averages lifting off from zero; this reflects the stiffening effect of mono-frequency HFE as described already in [26]. The measured and predicted averages are

almost indistinguishable. For Fig. (c) the excitations of (a) and (b) has been added into a two-frequency HFE, while keeping the same input rms-acceleration level. As appears the string tip still lifts off to a fixed value, in-between the values of case (a) and (b), and the multi-HFE is only apparent through the overlay of two-frequency oscillations; the measured average is seen to agree with the prediction (61)-(62) for the case of non-close excitation frequencies. For Fig. (d) the highest of the two HFE frequencies in (c) has been lowered to being only 0.5 Hz from the lowest excitation frequency. The string tip then does not settle down to a stationary average value, but lifts and drops at the slow rate of the difference frequency, still with a HF-overlay corresponding to the excitation frequencies. This is in full accordance with the prediction (64)-(65) for close frequencies, and the measured average still agrees with what is theoretically predicted.

Fig. 7 shows sets of *input level responses*, i.e. average string tip lifts versus (a) input acceleration a_{rms} and (b) input energy $\bar{E}$, for two-frequency horizontal base excitation with non-close and non-resonant frequencies $(\Omega_1, \Omega_2)/2\pi = (60, 110)$ Hz, and for five different frequency mixings η . In Fig. (a) the set of curves from pure 60 Hz excitation ($\eta=0$), over increasing mixings with 110 Hz excitation, to pure 110 Hz excitation ($\eta=1$) shows decreasing lift with increased η (reflecting that at a higher frequency the velocity amplitude and therefore energy input must decrease to keep the acceleration level fixed), and also a nearly quadratic increase of string tip lift with acceleration level (reflecting the quadratic term in (62)). It also appears that the experimentally measured values (symbol markers) are in good agreement with the theoretical predictions. Fig. (b) shows the same theoretical and experimental data as Fig. (b), but plotted instead with the specific input energy $\bar{E}$ as the input level variable. As appears, all the data clusters around the same almost straight line. This reflects the prediction stated under (63), that for non-close and non-resonant HFE, stiffening (and thus string lift) grows simply with the sum of squared HFE velocities $\sum_i (\Omega_i A_i)^2 = 2\bar{E}$, while the particular mixing of excitation frequencies is less relevant.

Fig. 6. Measured string tip lift $\tilde{u}(l,t) - \tilde{u}_0(l,t)$ versus time for the 1×550 mm horizontal string at input base acceleration $a_{\text{rms}} = 300/\sqrt{2}$ m/s² and (a) monofrequency excitation ($\eta=0$) at $\Omega_1/2\pi=61$ Hz; (b) monofrequency excitation ($\eta=1$) at $\Omega_2/2\pi=110$ Hz; (c) two-frequency excitation ($\eta=0.5$) at non-close frequencies $(\Omega_1, \Omega_2)/2\pi=(61, 110)$ Hz; (d) two-frequency excitation ($\eta=0.5$) at close frequencies $(\Omega_1, \Omega_2)/2\pi=(61, 61.5)$ Hz. Curves: experimentally measured time series (stationary conditions); Solid horizontal line: times series average; Dashed horizontal line: theoretically predicted average $\Delta\tilde{u}(l)$ by (61)-(62). System parameters: $L=0.55$ m, $E=195\times 10^9$ Pa, $g=9.82$ m/s², $\rho A=6.1410\times 10^{-3}$ kg/m, $El=9.5720\times 10^{-3}$ Nm², $\rho A c=8\times 10^{-3}$ kg/m/s, $n=6$ modes.

Fig. 8 shows two *frequency responses*, i.e. average string lift versus input excitation frequency Ω_1 . The curve in thin line is for mono-frequency HFE ($a_{2,\text{rms}}=0$), while the one in thick line is for two-frequency excitation with $\eta=0.5$. In the later case, as Ω_2 is kept constant at 110Hz, Ω_1 is varied (quasi-statically) over the range 0-240 Hz, keeping the total acceleration level at $a_{\text{rms}} = 300/\sqrt{2} \text{ m/s}^2$. The swept range includes the natural frequencies at 40, 79, 131, and 196 Hz, and thus tests the capability of the theoretical predictions to include effects of resonance on stiffening. It should be emphasized that the figure does *not* show the amplitude

Fig. 7. Average string tip lift $\Delta\tilde{u}(l)$ for the 1×550 mm horizontal string with two-frequency horizontal base excitation ($\Omega_1, \Omega_2/2\pi = (60, 110)$ Hz), as a function of (a) input acceleration a_{rms} and (b) input energy $\bar{E}$, for five different frequency mixings η . Lines: theoretical predictions by (61)-(62); Marker symbols: experimentally measured. System parameters as for Fig. 6.

Fig. 8. Average string tip lift $\Delta\tilde{u}(l)$ for the 1×550 mm horizontal string with constant horizontal input acceleration $a_{\text{rms}} = 300/\sqrt{2} \text{ m/s}^2$, constant second excitation frequency $\Omega_2/2\pi = 110$ Hz, and quasi-statically varied first excitation frequency $\Omega_1/2\pi \in [0; 240]$ Hz. Lines: theoretical predictions by (61)-(62); Thick line: $\eta=0.5$ (i.e. 2-frequency excitation with $a_{1,\text{rms}}=a_{2,\text{rms}}=150 \text{ m/s}^2$); Thin line: $\eta=0$ (i.e. mono-frequency excitation with $a_{1,\text{rms}} = 300/\sqrt{2} \text{ m/s}^2$, $a_{2,\text{rms}}=0$); Marker symbols: experimentally measured (with up-pointing arrows indicating large resonant response). Horizontal arrow lines above the graph span theoretically predicted ranges of parametric resonance. System parameters as for Fig. 6.

of string vibrations, as would a usual frequency response, but shows the *average lift* of the string tip (undergoing HF oscillations) under stationary conditions. As appears, for both the mono- and the two-frequency case, the string generally lifts off from zero (i.e. is stiffened) by the HFE, most pronounced just above the natural frequencies. However, at excitation frequencies just below the natural frequencies, the string tip *drops down* under zero, reflecting a *negative stiffening* effect at these frequencies, where the string appears more flexible with than without HFE. The main effect of two-frequency excitation, as compared to the mono-frequency case, appears to be a *flattening* of the non-resonant parts of the response curve. But this just reflects, that in the two-frequency case some of the input energy goes to Ω_2 , leaving less for the sweep frequency Ω_1 ; ultimately, as $\eta \rightarrow 1$ and all energy goes to Ω_2 , the frequency response will be all flat, taking on the constant value of the string lift corresponding to the HFE at Ω_2 . As also appears, the experimentally measured values (symbol markers) for both the mono- and the two-frequency case are in good agreement with the theoretical predictions, even close to natural frequencies. At some excitation frequencies the string exhibits strongly resonant behavior (up-pointing arrows in the Fig.); These frequencies are either very close to natural frequencies of the string (curve spikes in the Fig.), or within or close to theoretically predicted frequency ranges of *parametric* resonance (horizontal arrow lines above the Fig.); see [26] for more details on parametric resonances for this string system.

The results presented in this section demonstrates how the effective stiffness for a flexible structure is affected by mono- and two-frequency HFE, illustrates how to employ the general theoretical results of Section 3 for predicting effective stiffening, and proves good agreement between theory and experimental measurements for a rather wide range of excitation conditions.

6. Conclusions

- Strong parametrical high-frequency excitation at multiple frequencies has a stiffening effect on mechanical systems, which is similar to that of mono-frequency excitation, provided the excitation frequencies are well separated.
- For non-close and non-resonant excitation frequencies, the change in effective static stiffness is proportional to the sum of squared excitation velocities, i.e. to the input energy level.
- With two or more close excitation frequencies, there is an additional contribution of slowly oscillating stiffness, having magnitude order similar to the change in static stiffness, and frequencies equal to the differences in non-close frequencies.
- With close excitation frequencies, strong parametrical resonance can occur at conditions that might not appear obvious, that is: When the difference in any two excitation frequencies comes near $2\hat{\omega}/k$, where k is an integer (with $k=1$ corresponding to primary parametric resonance), and $\hat{\omega}$ is an effective natural frequency of the system, which due to the HF-excitation is shifted away from the natural frequency ω without HF-excitation.
- Strong multifrequency HF-excitation can stabilize unstable quasistatic equilibria, but only if the frequencies are well separated, i.e. with differences much larger than the lowest natural frequencies of the system. Thus continuous broadband and random excitation does not have a stabilizing effect paralleling that of monofrequency HF-excitation, or multifrequency HF-excitation with non-close frequencies.

- The above conclusions were derived for a class of generally nonlinear systems subjected to strong multifrequency HF-excitation of arbitrary period, and illustrated for a simple pendulum system with a vibrating support, and for a parametrically excited horizontal flexible beam in gravity. For the later example, comparison of theoretical predictions with experimental measurement for a wide range of excitation conditions showed good agreement on all aspects of concern.
- The general results may be used to investigate or utilize general effects, or as a shortcut for calculating effective properties for specific systems, or for calculating averaged equations of motion that may be much faster to simulate numerically.

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